به نام خدا
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1 2.8 Polyhedra

Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x|Ax \preceq b, Fx = g\}$.

1. $S = \{y_1 a_1 + y_2 a_2 | 1 \leq y_1 \leq 1, 1 \leq y_2 \leq 1\}$, where $a_1, a_2 \in R^n$.
2. $S = \{x \in R^n | x \succeq 0, 1^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, \dots, a_n \in R$ and $b_1, b_2 \in R$.
3. $S = \{x \in R^n | x \succeq 0, x^T y \leq 1$ for all y with $||y||_2 = 1\}$.
4. $S = \{x \in R^n | x \succeq 0, x^T y \leq 1$ for all y with $\sum_{i=1}^n |y_i| = 1\}$.

Solution:

1. It is polyhedra. It is the Affine mapping of $\hat{S} = \{y | 1 \leq y_1 \leq 1, 1 \leq y_2 \leq 1\}$. For more illustration:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \\ \vdots & \vdots \\ a_n^1 & a_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

   Now, we should figure out how conditions on y act on z.

2. It is polyhedra. For S form, $A = -I_{n\times n}$, $b = 0_{n\times 1}$, and

$$F = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}$$

3. It is not a polyhedra. $x \succeq 0, x^T y \leq 1$ for all y with $||y||_2 = 1 \Leftrightarrow ||x||_2 \leq 1$. So the final result is the intersection of the unit ball and the nonnegative orthant R_n^+. We can not see S as the intersection of finite halfspaces and hyperplane.

4. It is a polyhedra. First we should prove that $x^T y \leq 1$ for all y with $\sum_{i=1}^n |y_i| = 1 \Leftrightarrow |x_i| \leq 1$ for $i = 1, 2, \dots n$.
   Right to left:

$$x^T y = \sum_i x_i y_i \leq \sum_i |x_i||y_i| \leq \sum_i |y_i| = 1$$

   Left to right:

$$x_k = \max(|x_i|) \Rightarrow y = e_k \times sign(x_k) \Rightarrow x^T y = \sum_i x_i y_i = x_k y_k = |x_k| \Rightarrow |x_i| \leq 1 \text{ for } i = 1, 2, \dots n.$$

This can be defined by the intersection of nonnegative orthant R_n^+ with $\{x| -1 \preceq x \preceq 1\}$. For S form

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \\ -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2 2.9 Voronoi sets and polyhedral decomposition

Let $x_0, x_1, \dots, x_K \in R^n$ Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in R^n | \; ||x - x_0||_2 \leq ||x - x_i||_2, i = 1, \dots, K\}.$$

V is called the Voronoi region around x_0 with respect to $x_1, \dots, x_K$.

1. Show that V is a polyhedron. Express V in the form $V = \{x|Ax \preceq b\}$.
2. Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \dots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \dots, x_K$.
3. We can also consider the sets

$$V_k = \{x \in R^n | \; ||x - x_k||_2 \leq ||x - x_i||_2, i \neq k\}.$$

   The set V_k consists of points in R^n for which the closest point in the set $\{x_0, \dots, x_K\}$ is x_k.
   The sets V_0, ..., V_K give a polyhedral decomposition of R^n. More precisely, the sets V_k are polyhedra, $\bigcup_{k=1}^{K} V_k = R^n$, and int $V_i \cap$ int $V_j = \emptyset$ for $i \neq j$, i.e., V_i and V_j intersect at most along a boundary.
   Suppose that $P_1, \dots, P_m$ are polyhedra such that $\bigcup_{i=1}^{m} P_i = R^n$, and int $P_i \cap$ int $P_j = \emptyset$ for $i \neq j$. Can this polyhedral decomposition of R_n be described as the Voronoi regions generated by an appropriate set of points?

Solution:

1. For every $i \neq j$, we have a hyperspace where every point on the hyperplane has less or equal distance to x_i than x_j. This hyperspace includes the point $\frac{x_i + x_j}{2}$ and its normal vector is $x_j - x_i$. So for every point of $\{x_1, x_2, \dots, x_k\}$, we have a hyperspace $\{h_1, h_2, \dots, h_k\}$ that in every hyperspace like h_i, the distance of any points to x_0 is less than its distance to x_i. Each hyperspace can be written in the form below:

$$\begin{cases} h_i : \{x \in R^n | a_i^T x \leq b_i\} \\ a_i = x_i - x_0 \\ b_i = a_i^T (\dfrac{x_0 + x_i}{2}) \end{cases}$$

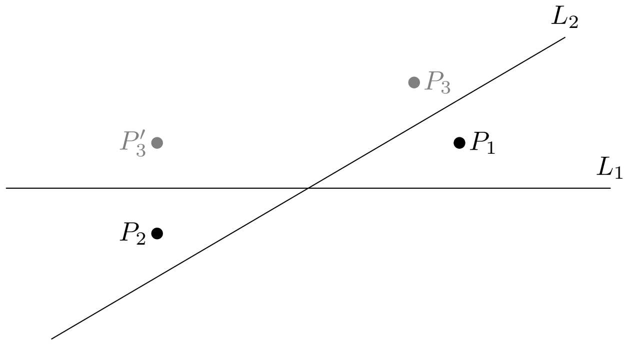
The intersection of these hyperspace is our final result. So we can write it as below:

$$V = \{x \in R^n | Ax \preceq b\}$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

2. Suppose we have $V = \{x | Ax \preceq b\}$. This polyhedron is the intersection of K hyperspaces. Each hyperspace has the form $a_i^T x \leq b_i$ which a_i^T is the i'th row of A and b_i is the i'th element of b. x_0 can be any point in V and each x_i is the mirror image of x_0 with respect to the hyperplane $a_i^T x = b_i$.

3. No, this can be a counter example.



By selecting P_1 in hyperspace between of L_1 and L_2 and R_{++}, P_2 should be the mirror point of P_1 with respect to the origin. The third point can not be computed because the mirror image of P_1 with respect to L_2 (P_3) is not the same of the mirror image of P_2 with respect to L_1 (P_3').

3 2.15 Some sets of probability distributions

Some sets of probability distributions. Let x be a real-valued random variable with $\mathbb{P}(x = a_i) = p_i, i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in R^n$ lies in the standard probability simplex $P = \{p | 1^T p = 1, p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

1. $\alpha \leq \mathbb{E}f(x) \leq \beta$, where $\mathbb{E}f(x)$ is the expected value of $f(x)$, i.e., $\mathbb{E}f(x) = \sum_{i=1}^n p_i f(a_i)$. (The function $f : R \to R$ is given.)

2. $\mathbb{P}(x > \alpha) \leq \beta$.

3. $\mathbb{E}|x^3| \leq \alpha E|x|$.

4. $\mathbb{E}x^2 \leq \alpha$.

5. $\mathbb{E}x^2 \geq \alpha$.

6. var(x) $\geq \alpha$, where var(x) = $\mathbb{E}(x - \mathbb{E}x)^2$ is the variance of x.

7. var(x) $\leq \alpha$.

8. quartile(x) $\geq \alpha$, where quartile(x) = $\inf\{\beta | \mathbb{P}(x \leq \beta) \geq 0.25\}$.

9. quartile(x) $\leq \alpha$.

Solution:

1. $\mathbb{E}f(x) = \sum_{i=1}^{n} p_i f(a_i)$ is a weighted sum of p_i, so $\mathbb{E}f(x) \geq \alpha$ and $\mathbb{E}f(x) \leq \beta$ are two hyperspaces. So the final result is the intersection of two halfspaces and a polyhedron. As a result, it is still convex.

2. Imagine $\alpha < a_k$ and $\alpha \geq a_{k-1}$. So $\mathbb{P}(x > \alpha) \leq \beta$ is equal to $\sum_{i=k}^{n} p_i \leq \beta$.
$$\begin{bmatrix} 0_{1\times(k-1)} & 1_{1\times(n-k+1)} \end{bmatrix} P \leq \beta$$
The answer is the intersection of P and the halfspace described above. So it is still convex.

3. $\mathbb{E}|x^3| \leq \alpha E|x|$ is equal to $\mathbb{E}(|x^3| - \alpha|x|) \leq 0$. $\mathbb{E}(|x^3| - \alpha|x|) \leq 0$ is a weighted sum of p_i, so $\mathbb{E}(|x^3| - \alpha|x|) \leq 0$ is a hyperspace. So the final result is the intersection of a halfspace and a polyhedron. As a result, it is still convex.

4. $\mathbb{E}x^2 = \sum_{i=1}^{n} p_i a_i^2$ is a weighted sum of p_i, so $\mathbb{E}x^2 \leq \alpha$ is a hyperspace. So the final result is the intersection of a halfspace and a polyhedron. As a result, it is still convex.

5. $\mathbb{E}x^2 = \sum_{i=1}^{n} p_i a_i^2$ is a weighted sum of p_i, so $\mathbb{E}x^2 \geq \alpha$ is a hyperspace. So the final result is the intersection of a halfspace and a polyhedron. As a result, it is still convex.

6. Counter example: with $n = 2, a_1 = 0, a_2 = 1$
$$\begin{aligned} \text{Case1}: p_1 = 1, p_2 = 0 \Rightarrow var(x) = \tfrac{1}{4} \\ \text{Case2}: p_1 = 0, p_2 = 1 \Rightarrow var(x) = \tfrac{1}{4} \\ \text{Case3}: p_1 = 0.5, p_2 = 0.5 \Rightarrow var(x) = 0 \end{aligned}$$
So by taking $\alpha < \frac{1}{4}$, the convex combination of Case1 and Case2 is not in the set. So it is not convex.

7. It is convex
$$var(x) = \sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 \Rightarrow var(x) \leq \alpha \Rightarrow \sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 \leq \alpha$$
$$b = [a_1^2, a_2^2, \ldots, a_n^2]_{n\times 1}, A = aa^T \Rightarrow b^T p - p^T A p \leq \alpha$$
$$\forall p_1, p_2 \in \{P | var(P) \leq \alpha\} \Rightarrow b^T p_1 - p_1^T A p_1 \leq \alpha \text{ and } b^T p_2 - p_2^T A p_2 \leq \alpha$$
$$p3 = \theta_1 p_1 + \theta_2 p_2 \text{ with } \theta_1, \theta_2 \geq 0, \theta_1 + \theta_2 = 1$$
$$\begin{aligned} var(p_3) &= b^T p_3 - p_3^T A p_3 = b^T(\theta_1 p_1 + \theta_2 p_2) - (\theta_1 p_1 + \theta_2 p_2)^T A (\theta_1 p_1 + \theta_2 p_2) \\ &= \theta_1 b^T p_1 + \theta_2 b^T p_2 - \theta_1^2 p_1^T A p_1 - \theta_1\theta_2 p_1^T A p_2 - \theta_1\theta_2 p_2^T A p_1 - \theta_2^2 p_2^T A p_2 \\ &\leq \theta_1(\alpha + p_1^T A p_1) + \theta_2(\alpha + p_2^T A p_2) - \theta_1^2 p_1^T A p_1 - \theta_1\theta_2 p_1^T A p_2 - \theta_1\theta_2 p_2^T A p_1 - \theta_2^2 p_2^T A p_2 \\ &= \alpha + \theta_1 p_1^T A p_1 + \theta_2 p_2^T A p_2 - \theta_1^2 p_1^T A p_1 - \theta_1\theta_2 p_1^T A p_2 - \theta_1\theta_2 p_2^T A p_1 - \theta_2^2 p_2^T A p_2 \leq \alpha \end{aligned}$$
The last in equality comes from the fact that:
$$z = \max(p_1^T A p_1, p_2^T A p_2, p_1^T A p_2)$$
$$\theta_1 p_1^T A p_1 + \theta_2 p_2^T A p_2 - \theta_1^2 p_1^T A p_1 - \theta_1\theta_2 p_1^T A p_2 - \theta_1\theta_2 p_2^T A p_1 - \theta_2^2 p_2^T A p_2 \leq \theta_1 z + \theta_2 z - \theta_1^2 z - \theta_1\theta_2 z - \theta_1\theta_2 z - \theta_2^2 z \leq 0$$
This is obvious by putting $\theta_1 = 1 - \theta_2$.

8. This means that $\mathbb{P}(x \leq \beta) < 0.25$ for all $\beta < \alpha$. if $\alpha < a_1, \mathbb{P}(x \leq \beta) = 0$ and it is always true. Otherwise we have a set like $\{a_1, a_2, \ldots, a_k\}$ that its member are all less that α. So for the condition $\mathbb{P}(x \leq \beta) < 0.25$ for all $\beta < \alpha$ we should have:
$$\sum_{i=1}^{k} \mathbb{P}(p_i) < 0.25$$
which is a linear inequality. So it is convex.

9. Same as previous, this time we need
$$\sum_{i=k+1}^{n} \mathbb{P}(p_i) > 0.25$$
which is a linear inequality. So it is convex.

4 2.17 Image of polyhedral sets under perspective function

In this problem we study the image of hyperplanes, halfspaces, and polyhedra under the perspective function $P(x,t) = x/t$, with dom $P = R^n \times R_{++}$. For each of the following sets C, give a simple description of
$$P(C) = \{v/t | (v,t) \in C, t > 0\}$$

1. The polyhedron $C = \text{conv}\{(v_1, t_1), \dots, (v_K, t_K)\}$ where $v_i \in R^n$ and $t_i > 0$.
2. The hyperplane $C = \{(v,t) | f^T v + gt = h\}$ (with f and g not both zero).
3. The halfspace $C = \{(v,t) | f^T v + gt \le h\}$ (with f and g not both zero).
4. The polyhedron $C = \{(v,t) | Fv + gt \preceq h\}$.

Solution:

1. Every $(x,y) \in C$ can be written in form below:
$$x = a_1 v_1 + a_2 v_2 + \dots + a_K v_K$$
$$y = a_1 t_1 + a_2 t_2 + \dots + a_K t_K$$
$$P(x,y) = \frac{x}{y} = \frac{\sum_{i=1}^K a_i v_i}{\sum_{i=1}^K a_i t_i} = \sum_{i=1}^K \frac{a_i t_i}{\sum_{i=1}^K a_i t_i} \frac{v_i}{t_i}$$
$$\sum_{i=1}^K \frac{a_i t_i}{\sum_{i=1}^K a_i t_i} = 1 \Rightarrow \text{ Convex combination of } \{\frac{v_i}{t_i}\}$$
$$P(x,y) \subseteq \text{conv}\{\frac{v_1}{t_1}, \frac{v_2}{t_2}, \dots, \frac{v_K}{t_K}\}$$
We need to show that $\text{conv}\{\frac{v_1}{t_1}, \frac{v_2}{t_2}, \dots, \frac{v_K}{t_K}\} \subseteq P(x,y)$ too.
$$\frac{x'}{y'} = b_1 \frac{v_1}{t_1} + b_2 \frac{v_2}{t_2} + \dots + b_K \frac{v_K}{t_K}$$
We can see that $P(x,y) = \frac{x'}{y'}$ with x and y to be:
$$b_i = \frac{a_i t_i}{\sum_{i=1}^K a_i t_i} \Rightarrow a_i = \frac{b_i}{t_i} \sum_{i=1}^K a_i t_i = \frac{b_i}{t_i \sum_{i=1}^K \frac{b_i}{t_i}}$$
We can see that x and y can be written in form below.
$$x = a_1 v_1 + a_2 v_2 + \dots + a_K v_K$$
$$y = a_1 t_1 + a_2 t_2 + \dots + a_K t_K$$
So $\text{conv}\{\frac{v_1}{t_1}, \frac{v_2}{t_2}, \dots, \frac{v_K}{t_K}\} \subseteq P(x,y)$. As a result $P(C) = \text{conv}\{\frac{v_1}{t_1}, \frac{v_2}{t_2}, \dots, \frac{v_K}{t_K}\}$

2.
$$f^T v + gt = h \Rightarrow f^T \frac{v}{t} + g = \frac{h}{t} \Rightarrow f^T x + g = \frac{h}{t}$$
$$P(C) = \begin{cases} \{x | f^T x + g > 0\} & h > 0 \\ \{x | f^T x + g = 0\} & h = 0 \\ \{x | f^T x + g < 0\} & h < 0 \end{cases}$$

3.

$$f^T v + gt \le h \Rightarrow f^T \frac{v}{t} + g \le \frac{h}{t} \Rightarrow f^T x + g \le \frac{h}{t}$$

$$P(C) = \begin{cases} R^n & h > 0 \\ \{x | f^T x + g \le 0\} & h = 0 \\ \{x | f^T x + g < 0\} & h < 0 \end{cases}$$

4.

$$Fv + gt \preceq h \Rightarrow F\frac{v}{t} + g \preceq \frac{h}{t} \Rightarrow Fx + g \preceq \frac{h}{t}$$

For this, $(Fx+g)_i$ and h_i should be both negative or both positive. Then we can find a t to reach our inequality.

$$P(C) = \{x | Fx + g \preceq \frac{h}{t} \text{ for a } t > 0\}$$

$$\begin{cases} (Fx+g)_i \le 0 & h_i = 0 \\ (Fx+g)_i < 0 & h_i < 0 \end{cases}$$

Another important point is to find a t. For negative h_i we tend to have big t and for positive h_i we tend to have small t. So we need to have another constrain.

$$\frac{(Fx+g)_i}{h_i} \le \frac{(Fx+g)_j}{h_j} \text{ for every } i \text{ with } h_i > 0 \text{ and every } j \text{ with } h_j < 0$$

5 2.31 Properties of dual cones

Let K^* be the dual cone of a convex cone K, as defined in (2.19). Prove the following.

1. K^* is indeed a convex cone.
2. $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.
3. K^* is closed.
4. The interior of K^* is given by int $K^* = \{y | y^T x > 0 \text{ for all } x \in \text{cl } K\}$.
5. If K has nonempty interior then K^* is pointed.
6. K^{**} is the closure of K. (Hence if K is closed, $K^{**} = K$.)
7. If the closure of K is pointed then K^* has nonempty interior.

Solution:

1. for every $y_1, y_2 \in K^*$, we need to prove $w = \theta_1 y_1 + \theta_2 y_2$ for every $\theta_1, \theta_2 > 0$ is in K^*.

$$w^T x = (\theta_1 y_1 + \theta_2 y_2)^T x = \theta_1 y_1^T x + \theta_2 y_2^T x > 0 \Rightarrow w \in K^*$$

2. We should prove that every $y \in K_2^*$ is in K_1^*.

$$y \in K_2^* \Rightarrow y^T x \ge 0 \text{ for all } x \in K_2 \Rightarrow y^T x \ge 0 \text{ for all } x \in K_1 \Rightarrow y \in K_1^*$$

3. K^* is the intersection of a set of homogeneous halfspaces including the origin. Hence it is a closed convex cone.

4. y is in the interior of a convex space K^* if every point in $B(r,y)$ is in K^* with some $r>0$. ($B(r,y)$ is a norm ball with radius r around center y)
if $y^Tx > 0$ for all $x \in$ cl K, we can find a $c>0$ which c is argmin y^Tx for all $x \in$ cl K. Then $(y+v)^Tx > 0$ for all $x \in$ cl K for every $|v| < \frac{c}{|x|}$ for all $x \in$ cl K(Using Cauchy Schwarz inequality). So $y \in$ intK^*.
Also if $y \in$ intK^* and $y^Tx = 0$, for every $\epsilon > 0$ we will have $(y+\epsilon v)^Tx < 0$ for $v = -x$ So it can't be in the int K^*.
So int $K^* \subseteq \{y|y^Tx > 0$ for all $x \in$ cl $K\}$ and $\{y|y^Tx > 0$ for all $x \in$ cl $K\} \subseteq$ int $K^* \Rightarrow$ int $K^* = \{y|y^Tx > 0$ for all $x \in$ cl $K\}$

5. Suppose K^* is not pointed. So we can have a point like $w \in K^*$ that $-w \in K^*$ too. So for every x in K we have $w^Tx > 0$ and $-w^Tx > 0$ which can not be true. So K^* must be pointed.

6. The intersection of all homogeneous halfspaces containing a convex cone K is the closure of K. We know that the normal vector (v) of every halfspace containing K is and only if $v \in K^*$ because

$$v^Tx \geq 0 \text{ for all } x \in K \Rightarrow v \in K^*$$
$$v \in K^* \Rightarrow v^Tx \geq 0 \text{ for all } x \in K \Rightarrow v \text{ is a normal vactor of a halfspce containing } K$$
$$v^Tx \geq 0 \text{ for all } x \in K \Leftrightarrow v \in K^*$$

So the closure of K is:

$$cl(K) = \{x|v^Tx \geq 0 \text{ for every } v \in K^*\} = K^{**}$$

7. If the closure of K is K^{**}. If K^* doesn't have nonempty interior, so there will be a vector v that $v^Tx = 0$ for every $x \in K^*$. On the other hand $-v^Tx = 0$ for every $x \in K^*$. So $v, -v \in K^{**}$. So K^{**} can not be pointed!.

6 2.4 Dual of exponential cone

The exponential cone $K_{exp} \subseteq R^3$ is defined as

$$K_{exp} = \{(x,y,z)|y > 0, ye^{\frac{x}{y}} \leq z\}$$

Find the dual cone K_{exp}.
We are not worried here about the fine details of what happens on the boundaries of these cones, so you really needn't worry about it. But we make some comments here for those who do care about such things. The cone K_{exp} as defined above is not closed. To obtain its closure, we need to add the points

$$\{(x,y,z)|x \leq 0, y = 0, z \geq 0\}$$

(This makes no difference, since the dual of a cone is equal to the dual of its closure.)
Solution:
We need to find a set $V = \{(v_1,v_2,v_3)\}$ that $(v_1,v_2,v_3)^T(x,y,z) = v_1x + v_2y + v_3z \geq 0$ for all $(x,y,z) \in K_{exp}$.
First, $v_3 \geq 0$ because z can be extremely big. For $v_3 = 0$, we have $v_1x + v_2y \geq 0$ which we know should hold for any $y > 0$ and x so $v_2 \geq 0$ and $v_1 = 0$.
For $v_3 > 0$, we new to minimize $v_1x + v_2y + v_3z$ over x, y, z. The best z would be $ye^{\frac{x}{y}}$ (because $v_3 > 0$).
So we should minimize $v_1x + v_2y + v_3ye^{\frac{x}{y}}$ over x and y. If $v_1 > 0$, then we can lessen our function by limiting x towards $-\infty$. If $v_1 = 0$, we need to minimize $v_2y + v_3ye^{\frac{x}{y}}$ which would be clear that is v_2y when $x \to -\infty$. v_2y is always non-negative when $v_2 \geq 0$
If $v_1 < 0$, we have

$$\frac{\partial}{\partial x}[v_1x + v_2y + v_3ye^{\frac{x}{y}}] = v_1 + v_3e^{\frac{x}{y}} = 0 \Rightarrow x = y\log(\frac{-v_1}{v_3})$$

So we will have $v_1x + v_2y + v_3ye^{\frac{x}{y}} = v_1y\log(\frac{-v_1}{v_3}) + v_2y - v_1y = y(v_1\log(\frac{-v_1}{v_3}) + v_2 - v_1)$. For this to be always non-negative for all $y > 0$, we need $v_1\log(\frac{-v_1}{v_3}) + v_2 - v_1 \geq 0$

So the final result is this

$$\begin{cases} v_1\log(\frac{-v_1}{v_3}) + v_2 - v_1 \geq 0 \text{ and } v_1 < 0 \text{ and } v_3 > 0 \\ v_1 = \text{ and } 0v_2 \geq 0 \text{ and } v_3 > 0 \\ v_1 = 0 \text{ and } v_2 \geq 0 \text{ and } v_3 = 0 \end{cases}$$

So the K^*_{exp} is the union of these tree spaces.

7 2.5 Dual of intersection of cones.

Let C and D be closed convex cones in R^n. In this problem we will show that

$$(C \cap D)^* = C^* + D^*$$

when $C^* + D^*$ is closed. Here, $+$ denotes set addition: $C^* + D^*$ is the set $\{u + v | u \in C^*, v \in D^*\}$. In other words, the dual of the intersection of two closed convex cones is the sum of the dual cones. (A sufficient condition for of $C^* + D^*$ to be closed is that $C \cap \text{int } D \neq \emptyset$. The general statement is that $(C \cap D)^* = \text{cl }(C^* + D^*)$, and that the closure is unnecessary if $C \cap \text{int } D \neq \emptyset$, but we won't ask you to show this.)

1. Show that $C \cap D$ and $C^* + D^*$ are convex cones.

2. Show that $(C \cap D)^* \supseteq C^* + D^*$.

3. Now let's show $(C \cap D)^* \subseteq C^* + D^*$ when $C^* + D^*$ is closed. You can do this by first showing

$$(C \cap D)^* \subseteq C^* + D^* \Leftrightarrow C \cap D \supseteq (C^* + D^*)^*$$

   You can use the following result:

   If K is a closed convex cone, then $K^{**} = K$.

   Next, show that $C \cap D \supseteq (C^* + D^*)^*$ and conclude $(C \cap D)^* = C^* + D^*$.

4. Show that the dual of the polyhedral cone $V = \{x | Ax \succeq 0\}$ can be expressed as

$$V^* = \{A^T v | v \succeq 0\}.$$

Solution:

1. $\forall v, w \in C \cap D \Rightarrow v, w \in C$ and $D \Rightarrow av + bw \in C$ and $D \forall a, b \geq 0 \Rightarrow av + bw \in C \cap D$
   $\forall v, w \in C^* + D^* \Rightarrow$ There is $v_1, w_1 \in C^*$ and $v_2, w_2 \in D^*$ that $v = v_1 + v_2$ and $w = w_1 + w_2 \Rightarrow av + bw = a(v_1 + v_2) + b(w_1 + w_2) = (av_1 + bw_1) + (av_2 + bw_2) \in C^* + D^* \forall a, b \geq 0 \Rightarrow av + bw \in C^* + D^*$

2. $\forall v \in C^* + D^* \Rightarrow$ There is $v_1 \in C^*$ and $v_2 \in D^*$ that $v = v_1 + v_2 \Rightarrow v_1^T x \geq 0$ for all $x \in C$ and $v_2^T x \geq 0$ for all $x \in D \Rightarrow (v_1 + v_2)^T x = v_1^T x + v_2^T x \geq 0$ for all $x \in C \cap D \Rightarrow v \in (C \cap D)^* \Rightarrow (C \cap D)^* \supseteq C^* + D^*$

3. $\forall v \in (C^* + D^*)^* \Rightarrow v^T x \geq 0 \ \forall x \in C^* + D^*$. Each $x \in C^* + D^*$ can be written in form $x = x_1 + x_2$ that $x_1 \in C^*$ and $x_2 \in D^*$. As x_1 and x_2 can be zero, $v^T x \geq 0$ for all $x \in C^*$ and $v^T x \geq 0$ for all $x \in D^* \Rightarrow v^T x \geq 0$ for all $x \in C^* \cap D^* \Rightarrow (C^* + D^*)^* \subseteq C \cap D \Rightarrow (C \cap D)^* \subseteq C^* + D^*$
   So we have $(C \cap D)^* = C^* + D^*$

4.

$$V^* = (\{x | a_1^T x \geq 0\} \cap \{x | a_2^T x \geq 0\} + \cdots + \{x | a_n^T x \geq 0\})^* = \{x | a_1^T x \geq 0\}^* + \{x | a_2^T x \geq 0\}^* + \cdots + \{x | a_n^T x \geq 0\}^*$$

As it is obvious, the dual of $\{x|a^T x \geq 0\}$ is $\{ka|k \geq 0\}$. So

$$V^* = \{ka_1|k \geq 0\} + \{ka_2|k \geq 0\} + \cdots + \{ka_n|k \geq 0\} = \{k_1a_1 + k_2a_2 + \cdots + k_na_n|k_i \geq 0\} = \{A^T k|k \succeq 0\}$$