دانشگاه صنعتی شریف
دانشکده مهندسی برق

به نام خدا
دکتر مجتبی تفاق ـ بهینه‌سازی در علوم داده

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1 4.5 Equivalent convex problems

Show that the following three convex problems are equivalent. Carefully explain how the solution of each problem is obtained from the solution of the other problems. The problem data are the matrix $A \in R^{m\times n}$ (with rows a_i^T), the vector $b \in R^m$, and the constant $M > 0$.

1. The robust least-squares problem

$$\min \sum_{i=1}^{m} \phi(a_i^T x - b_i)$$

   with variable $x \in R^n$, where $\phi : R \to R$ is defined as

$$\phi(u) = \begin{cases} u^2 & |u| \geq M \\ M(2|u| - M) & |u| < M \end{cases}$$

   (This function is known as the Huber penalty function; see §6.1.2.)

2. The least-squares problem with variable weights

$$\min \sum_{i=1}^{m} (a_i^T x - b_i)^2/(w_i + 1) + M^2 1^T w$$
$$\text{subject to : } w \succeq 0$$

   with variables $x \in R^n$ and $w \in R^m$, and domain $D = \{(x, w) \in R^n \times R^m | w \succ -1\}$.
   Hint. Optimize over w assuming x is fixed, to establish a relation with the problem in part (a).
   (This problem can be interpreted as a weighted least-squares problem in which we are allowed to adjust the weight of the ith residual. The weight is one if $w_i = 0$, and decreases if we increase w_i. The second term in the objective penalizes large values of w, i.e., large adjustments of the weights.)

3. The quadratic program

$$\min \sum_{i=1}^{m} (u_i^2 + 2Mv_i)$$
$$\text{subject to : } -u - v \preceq Ax - b \preceq u + v$$
$$0 \preceq u \preceq M1$$
$$v \succeq 0.$$

Solution:

For showing this equivalency, we only need to show problem (1,2) and (1,3) are equal.

- problem (1,2)

$$f(z,w) = \frac{z^2}{1+w} + M^2 w \Rightarrow \frac{\partial}{\partial w} f(z,w) = 0$$
$$\Rightarrow -\frac{z^2}{(1+w)^2} + M^2 = 0 \Rightarrow w^* = \frac{|z|}{M} - 1$$

if $w^* = \frac{|z|}{M} - 1 > 0$, this w^* is optimum. Otherwise, we should consider $w^* = 0$.
By taking $z = a_i^T x - b_i$, we will have

$$w^* = \max(0, \frac{|a_i^T x - b_i|}{M} - 1)$$

So the second problem reduce to

$$\min \frac{(a_i^T x - b_i)^2}{1+w} + M^2 w = \begin{cases} M(2|a_i^T x - b_i| - M) & |a_i^T x - b_i| \geq M \\ (a_i^T x - b_i)^2 & o.w \end{cases}$$

By summation over i, we will see the first problem.

- problem (1,3) we first should understand the relation between v and u. For all i, we should have $v_i + u_i = |a_i^T x + b_i|$. If not:

  1. $v_i + u_i > |a_i^T x + b_i|$. By reducing v_i or u_i, we can reduce the objective function. This can be done because v_i and u_i can not be both zero. (Because $|a_i^T x + b_i| \geq 0$)

  2. $v_i + u_i < |a_i^T x + b_i|$. Because of the first constrain, $a_i^T x + b_i \leq 0$. So this will say that $-v_i - u_i > a_i^T x + b_i$ which is against the first constrain.

  So we will have $v_i = |a_i^T x - b_i| - u_i$

$$\min \sum_{i=1}^{m} (u_i^2 - 2Mu_i + 2M|a_i^T x - b_i|)$$
$$\text{subject to : } 0 \leq u_i \leq \min\{M, |a_i^T x - b_i|\}$$

If $M < |a_i^T x - b_i|$, so $0 \leq u_i \leq M$. Then the best option for u_i is M and the ith term of objective function will be $2M|a_i^T x - b_i| - M^2$. Otherwise if $M > |a_i^T x - b_i|$, so $0 \leq u_i \leq |a_i^T x - b_i|$. Then the best option for u_i is $|a_i^T x - b_i|$ and the ith term of objective function will be $|a_i^T x - b_i|^2$. So the third problem is equal to the first one.

2 4.26 Hyperbolic constraints as SOC constraints.

Verify that $x \in R^n$, $y, z \in R$ satisfy

$$x^T x \leq yz, \qquad y \geq 0, \qquad z \geq 0$$

if and only if

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z, \qquad y \geq 0, \qquad z \geq 0.$$

Use this observation to cast the following problems as SOCPs.

1. Maximizing harmonic mean

$$\max(\sum_{i=1}^{m} 1/(a_i^T x - b_i))^{-1}$$

with domain $\{x|Ax \succ b\}$, where a_i^T is the ith row of A.

2. Maximizing geometric mean.

$$\max(\prod_{i=1}^{m} (a_i^T x - b_i))^{1/m}$$

with domain $x|Ax \succeq b$, where a_i^T is the ith row of A.

Solution:

$$x^T x \leq yz \Leftrightarrow 4x^T x \leq 4yz = (y+z)^2 - (y-z)^2 \Leftrightarrow 4x^T x + (y-z)^2 \leq (y+z)^2 \Leftrightarrow \left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z$$

1.

$$\max(\sum_{i=1}^{m} 1/(a_i^T x - b_i))^{-1} \equiv \min \sum_{i=1}^{m} 1/(a_i^T x - b_i)$$

By taking $t_i = 1/(a_i^T x - b_i)$, we can rewrite this problem like below

$$\min 1^T t$$

$$\text{subject to : } t_i(aTix + bi) \geq 1, \; i = 1, \ldots, m$$

Second-order cone programming version is:

$$\min 1^T t$$

$$\text{subject to : } \left\| \begin{bmatrix} 2 \\ a_i^T x + b_i - t \end{bmatrix} \right\|_2 \leq a_i^T x + b_i + t, \; i = 1, \ldots, m$$

$$t_i 0; a_i^T x + b_i \geq 0, \; i = 1, \ldots, m$$

2.

$$\max(\prod_{i=1}^{m} (a_i^T x - b_i))^{1/m} \equiv \max \prod_{i=1}^{m} (a_i^T x - b_i)$$

Without loss of generality, we assume that $m = 2^n$. We can add some term that will not affect the objective, like $a_i^T = 0, b_i = 1$.

For $m = 4$, we will have

$$\max y_1 y_2 y_3 y_4$$

$$\text{sibject to : } \begin{cases} y = Ax - b \\ y \succeq 0 \end{cases}$$

By taking $y_1y_2 = t_1$ and $y_3y_4 = t_2$, we will have

$$\max t_1t_2$$

$$\text{sibject to : } \begin{cases} y = Ax - b \\ y_1y_2 \geq t_1^2 \\ y_3y_4 \geq t_2^2 \\ t_1t_2 \geq t^2 \\ y \succeq 0, t_1, t_2, t \geq 0 \end{cases}$$

By taking $t = t_1t_2$, we will have the SOCP version of problem as followed:

$$\min -t$$

$$\text{sibject to : } \begin{cases} \left\| \begin{bmatrix} 2t_1y_1 - y_2 \end{bmatrix} \right\| \leq y_1 + y_2, \quad y_1 \geq 0, y_2 \geq 0 \\ \left\| \begin{bmatrix} 2t_2y_3 - y_4 \end{bmatrix} \right\| \leq y_3 + y_4, \quad y_3 \geq 0, y_4 \geq 0 \\ \left\| \begin{bmatrix} 2tt_1 - t_2 \end{bmatrix} \right\| \leq t_1 + t_2, \quad t_1 \geq 0, t_2 \geq 0 \\ y = Ax - b \end{cases}$$

For greater m, we will have the same strategy and each time, we will combine two variable into a new variable and at the end we are going to have just one variable with lots of constrains. So the final result is like this:

$$\min t_{00}$$

$$\text{subject to : } \begin{cases} t_{K-1,j-1} = a_j^T x - b_j & j = 1, \ldots, m \\ t_{ik}^2 \leq t_{i+1,2^k} t_{i+1;2^k+1} & i = 0, \ldots, K-2, \ k = 0, \ldots, 2^i - 1 \\ Ax \succeq b \end{cases}$$

3 4.43 Eigenvalue optimization via SDP.

Suppose $A : R^n \to S^m$ is affine, i.e.,

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$$

where $A_i \in S^m$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$. Show how to pose the following problems as SDPs.

1. Minimize the maximum eigenvalue $\lambda_1(x)$.

2. Minimize the spread of the eigenvalues, $\lambda_1(x) - \lambda_m(x)$.

3. Minimize the condition number of $A(x)$, subject to $A(x) \succ 0$. The condition number is defined as $\kappa(A(x)) = \lambda_1(x)/\lambda_m(x)$, with domain $\{x | A(x) \succ 0\}$. You may assume that $A(x) \succ 0$ for at least one x.
   Hint. You need to minimize λ/γ, subject to

   $$0 \prec \gamma I \prec A(x) \prec \lambda I$$

   Change variables to $y = x/\gamma$, $t = \lambda/\gamma$, $s = 1/\gamma$.

4. Minimize the sum of the absolute values of the eigenvalues, $|\lambda_1(x)| + \cdots + |\lambda_m(x)|$.
   Hint. Express $A(x)$ as $A(x) = A_+ - A_-$, where $A_+ \succeq 0$, $A_- \succeq 0$.

Solution:

1.

$$\lambda_1(x) \leq t \Leftrightarrow A(x) \preceq tI$$

$\Rightarrow$ The SDP problem would be:

$$\min t$$

$$\text{subject to } A(x) \preceq tI$$

2.

$$\lambda_1(x) \leq t_1 \Leftrightarrow A(x) \preceq t_1 I$$

$$\lambda_2(x) \geq t_2 \Leftrightarrow A(x) \succeq t_2 I$$

$\Rightarrow$ The SDP problem would be:

$$\min(t_1 - t_2)$$

$$\text{subject to } t_2 I \preceq A(x) \preceq t_1 I$$

3. As the hint suggests, we need to solve this problem

$$\min \frac{\lambda}{\gamma}$$

$$\text{subject to } \lambda I \preceq A(x) \preceq \gamma I$$

This problem is quasiconvex, and can be solved by bisection.

$$\lambda \leq \gamma\alpha, \qquad \lambda I \preceq A(x) \preceq \gamma I, \gamma > 0$$

From the hint, by taking $y = x/\gamma$, $t = \lambda/\gamma$, $s = 1/\gamma$, we will have a SDP problem below

$$\min t$$
$$\text{subject to } I \preceq sA_0 + y_1A_1 + \cdots + y_nA_n \preceq tI, \qquad s \geq 0$$

We should show that these two problems are equivalent. We do this by showing that the optimal answer to first problem is feasible in the SDP problem and vise versa.

- Let (γ, λ, x) be the the optimal point for the first problem. By taking $s = 1/\gamma, y = x/\gamma, t = \lambda/\gamma$, we see that (s, y, t) is feasible in SDP problem. So the optimal value of first problem (p_1^*) is greater or equal with the optimal value of SDP problem(p_2^*).
- Let (s, y, t) be the the optimal point for the SDP problem. If $s > 0$, then by taking $\gamma = 1/s, x = y/s, \lambda = t/s$, we see that (γ, λ, x) is feasible in the first problem. If $s = 0$, we have $I \preceq y_1A_1 + \cdots + y_nA_n \preceq tI$. By taking $x = ky$, we will have

$$kI \preceq ky_1A_1 + \cdots + ky_nA_n = x_1A_1 + \cdots + x_nA_n = A(x) - A_0$$
$$\Rightarrow A(x) \succeq kI + A_0$$

  By tending k to ∞, we will have $A(x) \succeq kI + A_0 \succ 0$. So we will have these two bonds

$$\lambda_1(x) = \lambda_1(ky) \leq \lambda_1(A_0) + kt = \lambda_1(0) + kt$$
$$\lambda_m(x) = \lambda_m(ky) \geq \lambda_m(A_0) + k = \lambda_m(0) + k$$
$$\Rightarrow \frac{\lambda_1(x)}{\lambda_m(x)} = \frac{\lambda_1(0) + kt}{\lambda_m(0) + k} \to t$$

  Letting k go to infinity, we can construct feasible points in first problem. So the optimal value of SDP problem (p_2^*) is greater or equal with the optimal value of first problem(p_1^*).

As a result, $p_1^* = p_2^*$ and the two problems are equivalent.

4. By writing $A(x) = A_+ - A_-$, we will have SDP problem

$$\min tr(A_+) + tr(A_-)$$
$$\text{subject to } \begin{cases} A(x) = A_+ - A_- \\ A_+ \succeq 0A \succeq 0 \end{cases}$$

The reason for this is that we can divide the matrix A to two matrix A_+ and A_-. A_+ contains the positive eigenvalue of matrix A and A_- contains the negative eigenvalue of matrix A with opposite sign. So the sum of positive eigenvalues is the trace of A_+ and the minus sum of negative eigenvalues is the trace of A_-. So basically, if we have the eigenvalue decomposition of $A(x) = Q\Lambda Q^T$. Then $A_+ = Q\Lambda_+Q^T$ and $A_- = Q\Lambda_-Q^T$. Λ_+ keeps all the positive eigenvalue of $A(x)$ and replace all negative eigenvalues of $A(x)$ with zero. Λ_- keeps all the negative eigenvalue of $A(x)$ with positive sign and replace all positive eigenvalues of $A(x)$ with zero. So $A(x) = A_+ - A_-$. This is the idea. So for a fixed x, we calculate the A_+ and A_-. By minimizing over $x, A_+, andA$ we will minimize $\sum_{i=1}^m |\Lambda_i(A(x))|$.

4 3.6 Two problems involving two norms.

Show that $f(X;t) = nt\log t - t\log\det X$, with dom $f = S^n_{++} \times R_{++}$, is convex in $(X;t)$. Use this to show that

$$g(X) = n(trX)\log(trX) - (trX)(\log\det X) = n(\sum_{i=1}^{n}\lambda_i)(\log\sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n}\log\lambda_i)$$

where λ_i are the eigenvalues of X, is convex on S^n_{++}.

Solution:

This is the perspective function applied on $-\log\det X$.

$$f(X,t) = t[-\log\det(\frac{X}{t})] = nt\log t - \log\det X$$

This is because a scaled version of a matrix has an exponential scaled version of determinant.

$$\log\det(\frac{X}{t}) = \log t^{-n}\det(X) = -n + \log\det(X)$$

g is convexity because it is $g(X) = f(X, trX)$

$$g(X) = f(X, trX) = n\ trX\log trX - \log\det X$$

$$X = Q\Sigma Q^T \Rightarrow tr(X) = tr(Q\Sigma Q^T) = tr(Q^TQ\Sigma) = tr(\Sigma) = \sum_{i=1}^{n}\lambda_i$$

$$\det(X) = \prod_{i=1}^{n}\lambda_i$$

$$\Rightarrow g(X) = n(\sum_{i=1}^{n}\lambda_i)(\log\sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n}\log\lambda_i)$$

trX is both positive and a linear function of X. As a result, $g(X)$ is convex on S^n_{++}.

5 3.10 Weighted geometric mean.

The geometric mean $f(x) = (\prod_k x_k)^{1/n}$ with dom $f = R^n_{++}$ is concave, as shown on page 74. Extend the proof to show that

$$f(x) = \prod_{k=1}^{n} x_k^{\alpha_k} \qquad \text{dom } f = R^n_{++}$$

is concave, where α_k are nonnegative numbers with $\sum_k \alpha_k = 1$.

Solution:

We will prove this by showing the hessian of f is negative definite.

$$\begin{cases} \frac{\partial^2}{\partial^2 x_i} f(x) = \alpha_i(\alpha_i - 1)x_i^{\alpha_i - 2} \prod_{k \neq i} x_k^{\alpha_k} = \alpha_i(\alpha_i - 1)f(x)/x_i^2 \\ \frac{\partial^2}{\partial x_i \partial x_j} f(x) = \alpha_i x_i^{\alpha_i - 1} \alpha_j x_j^{\alpha_j - 1} \prod_{k \neq i,j} x_k^{\alpha_k} = \alpha_i \alpha_j f(x)/(x_i x_j) \\ \nabla^2 f(x) = f(x) \begin{bmatrix} \frac{\alpha_1(\alpha_1 - 1)}{x_1^2} & \frac{\alpha_1\alpha_2}{x_1x_2} & \dots & \frac{\alpha_1\alpha_n}{x_1x_n} \\ \frac{\alpha_1\alpha_2}{x_1x_2} & \frac{\alpha_2(\alpha_2-1)}{x_2^2} & \dots & \frac{\alpha_2\alpha_n}{x_1x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_1\alpha_n}{x_1x_n} & \frac{\alpha_2\alpha_n}{x_2x_n} & \dots & \frac{\alpha_n(\alpha_n-1)}{x_n^2} \end{bmatrix} \end{cases}$$

If the diagonal elements of $\nabla^2 f(x)$ were $\nabla^2_{i,i} f(x) = f(x)\frac{\alpha_1\alpha_1}{x_1x_1}$, we could write as

$$q = (\frac{\alpha_1}{x_1}, \frac{\alpha_2}{x_2}, \dots, \frac{\alpha_n}{x_n})$$

$$qq^T = \begin{bmatrix} \frac{\alpha_1^2}{x_1^2} & \frac{\alpha_1\alpha_2}{x_1x_2} & \dots & \frac{\alpha_1\alpha_n}{x_1x_n} \\ \frac{\alpha_1\alpha_2}{x_1x_2} & \frac{\alpha_2^2}{x_2^2} & \dots & \frac{\alpha_2\alpha_n}{x_1x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_1\alpha_n}{x_1x_n} & \frac{\alpha_2\alpha_n}{x_2x_n} & \dots & \frac{\alpha_n^2}{x_n^2} \end{bmatrix}$$

$$\nabla^2 f(x) = f(x)qq^T$$

We should change the diagonal elements of hessian. So we will use diagonal matrix:

$$\nabla^2 f(x) = f(x)(qq^T - diag(\alpha)^{-1}diag(q)^2)$$

$$y^T\nabla^2 f(x)y = f(x)\Big(y^Tqq^Ty - y^Tdiag(\alpha)^{-1}diag(q)^2y\Big)$$

$$= f(x)\Big(\sum_{k=1}^{n}(\frac{\alpha_k y_k}{x_k})^2 - \sum_{k=1}^{n}\frac{\alpha_k y_k^2}{x_k^2}\Big)$$

By taking $a = (\sqrt{\alpha_1}\frac{y_1}{x_1}, \sqrt{\alpha_2}\frac{y_2}{x_2}, \dots, \sqrt{\alpha_n}\frac{y_n}{x_n})$ and $b = (\sqrt{\alpha_1}, \sqrt{\alpha_2}, \dots, \sqrt{\alpha_n})$ and Cauchy-Schwarz inequality, we will have

$$||a^Tb||_2^2 \leq ||a||_2^2||b||_2^2 = ||a||_2^2 \qquad (||b||_2^2 = \sum_{i=1}^{n}\alpha_i = 1)$$

$$\sum_{k=1}^{n}(\frac{\alpha_k y_k}{x_k})^2 \leq \sum_{k=1}^{n}\frac{\alpha_k y_k^2}{x_k^2}$$

$$\Rightarrow y^T\nabla^2 f(x)y \leq 0 \Rightarrow \nabla^2 f(x) \preceq 0$$

So f in concave.

۶

مسئله به صورت زیر است.

$$Pr(X_1 = 1) = p_1$$
$$Pr(X_2 = 1) = p_2$$
$$Pr(X_3 = 1) = p_3$$
$$Pr(X_1 = 1 \wedge X_4 = 0 | X_3 = 1) = p_4$$
$$Pr(X_4 = 1 | X_2 = 1 \wedge X_3 = 0) = p_5$$

برای حل این مسئله 16 متغیر تعریف میکنیم. این متغیرها فضای حالت $[x_4, x_3, x_2, x_1]$ مشخص میکند.
مثلا متغیر x_1011 برابر $Pr(x_4 = 1, x_3 = 0, x_2 = 1, x_1 = 1)$ است. حال ۵ شرط را به زبان این متغیرها مینویسیم و مسئله بهینه‌سازی را حل می‌کنیم.

minimize and maximize $(x\_1000 + x\_1001 + x\_1010 + x\_1011 + x\_1100 + x\_1101 + x\_1110 + x\_1111) = Pr(X_4 = 1)$

$$\begin{cases} x\_0001 + x\_0011 + x\_0101 + x\_0111 + x\_1001 + x\_1011 + x\_1101 + x\_1111 = Pr(X_1 = 1) = p_1 \\ x\_0010 + x\_0011 + x\_0110 + x\_0111 + x\_1010 + x\_1011 + x\_1110 + x\_1111 = Pr(X_2 = 1) = p_2 \\ x\_0100 + x\_0101 + x\_0110 + x\_0111 + x\_1100 + x\_1101 + x\_1110 + x\_1111 = Pr(X_3 = 1) = p_3 \\ x\_0101 + x\_0111 = Pr(X_1 = 1 \wedge X_4 = 0 \wedge X_3 = 1) = p4p3 \\ x\_1010 + x\_1011 = Pr(X_4 = 1 \wedge X_2 = 1 \wedge X_3 = 0) = p5(x\_0010 + x\_0011 + x\_1010 + x\_1011) \end{cases}$$

با کمک کتاب‌خانه‌های بهینه سازی مسئله را تعریف کردم و جواب نهایی را با حداکثر سه رقم اعشار نوشتم.

۷ طراحی ماسک صورت

هزینه‌ی هر سانتی‌متر مربع از فیلتر را دو واحد پول و هر سانتی‌متر از نوار الاستیک را یک واحد پول در نظر میگیریم. پس هدف مینیمم کردن تابع هدف زیر است.

$$\text{minimum } [2(lw) + (\pi w + 2l)]$$

ترم اول مساحت فیلتر است که در دو واحد پول ضرب شده است. ترم دوم طول نوار است که در یک واحد پول ضرب شده است. مساحت فیلتر باید حداقل ۳۰۰ سانتی‌متر مربع باشد. همچنین روی نسبت ابعاد محدودیت داریم. آخرین محدودیت هم برابر است با محدودیت روی اندازه طول و عرض. با اعمال این محدودیت‌ها به مسئله زیر میرسیم.

$$\text{minimum } [2(lw) + (\pi w + 2l)]$$
$$\text{subject to} \begin{cases} lw \geq 300 \\ 1 \leq \frac{l}{w} \leq 2 \\ 10 \leq w \leq 20 \\ 20 \leq l \leq 30 \end{cases}$$

برای حل این مسئله از Geometric programming استفاده کردم. به این صورت که تعریف کردم

$$y_1 = \log l \qquad y_2 = \log w$$

پس مسئله بهینه‌سازی به شکل زیر تبدیل شد.

$$\text{minimum}[2(\exp(y_1 + y_2)) + (\pi \exp(y_2) + 2\exp(y_1))]$$
$$\text{to subject} \begin{cases} y_1 + y_2 \geq \log(300) \\ \log(20) \leq y_1 \leq \log(30) \\ \log(10) \leq y_2 \leq \log(20) \\ y_2 \leq y_1 \leq \log(2) + y_2 \end{cases}$$

مسئله به حالت خطی در آمد و نتیجه نهایی به شکل زیر شد

$$\begin{cases} l = 21.92686359686865 \\ w = 14.069807153270846 \\ \text{Optimal cost} : 686.1695947067898 \approx 687 \end{cases}$$