به نام خدا

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دانشکده مهندسی برق

1 3.18

Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.

1. $f(X) = tr(X^{-1})$ is convex on dom $f = S^n_{++}$.
2. $f(X) = (\det X)^{1/n}$ is concave on dom $f = S^n_{++}$.

Solution:

1. Let A be symmetric positive definite matrix hence there is a diagonal matrix D whose diagonal entries are nonzero and $A = PDP^{-1}$ so $A^{-1} = PD^{-1}P^{-1}$ and $Tr(A^{-1}) = Tr(D^{-1})$. Now D being diagonal matrix with non zero diagonal entries D^{-1} has diagonal entries reciprocal of the diagonal entries of D so $Tr(D^{-1})$ is sum of the inverses of the diagonal entries of D. We also know that $Tr(AB) = Tr(BA)$.
   By taking $g(t) = f(x + tv)$, where $x \in S^n_{++}$ and $v \in S^n$:
   $$\begin{aligned} g(t) = f(x+tv) = tr((x+tv)^{-1}) &= tr\big[x^{-1}(I + tx^{-1/2}vx^{-1/2})^{-1}\big] \\ &= tr\big[x^{-1}(BB^{-1} + tBWB^{-1})^{-1}\big] = tr\big[x^{-1}B(I+tW)^{-1}B^{-1}\big] \\ &= tr\big[B^{-1}x^{-1}B(I+tW)^{-1}\big] = \sum_{i=1}^{n}(B^{-1}x^{-1}B)_{ii}(1+t\lambda_i)^{-1} \end{aligned}$$
   which $x^{-1/2}vx^{-1/2} = BWB^{-1}$. Since sum of convex functions $1/(1+t\lambda_i)$ is convex, $g(t)$ is convex on domain $\{t | x + tv \in S^n_{++}\}$. So $f(x)$ is convex.
2. By taking $g(t) = f(x + tv)$, where $x \in S^n_{++}$ and $v \in S^n$:
   $$\begin{aligned} g(t) = f(x+tv) = (\det(x+tv))^{1/n} &= (\det(x^{1/2}(I + tx^{-1/2}vx^{-1/2})x^{1/2}))^{1/n} \\ = [\det(x^{1/2})\det(I + tx^{-1/2}vx^{-1/2})\det(x^{1/2})]^{1/n} &= \det(x^{1/2})^{2/n}\det(I + tx^{-1/2}vx^{-1/2})^{1/n} \\ &= \det(x^{1/2})^{2/n}\Big[\prod_{i=1}^{n}(1+t\lambda_i)\Big]^{1/n} \end{aligned}$$
   which λ_i is ith eigenvalue of $x^{-1/2}vx^{-1/2}$. Since geometric mean is concave on R^n_{++}. So $g(t)$ is concave on domain $\{t | x + tv \in S^n_{++}\}$. So $f(x)$ is convex.

2 3.25 Maximum probability distance between distributions

Let $p, q \in R^n$ represent two probability distributions on $\{1, \ldots, n\}$ (so $p, q \succeq 0, 1^T p = 1^T q = 1$). We define the *maximum probability distance* $d_{mp}(p, q)$ between p and q as the maximum difference in probability assigned by p and q, over all events:

$$d_{mp}(p, q) = \max\{|\text{prob}(p, C) - \text{prob}(q, C)||C \subseteq \{1, ..., n\}\}.$$

Here $\text{prob}(p, C)$ is the probability of C, under the distribution p, i.e., $\text{prob}(p, C) = \sum_{i \in C} p_i$.
Find a simple expression for d_{mp}, involving $|p-q|_1 = \sum_{i=1}^{n} |p_i - q_i|$, and show that d_{mp} is a convex function on $R^n \times R^n$. (Its domain is $\{(p, q)|p, q \succeq 0, 1^T p = 1^T q = 1\}$, but it has a natural extension to all of $R^n \times R^n$.)

Solution:

Imagine that C^* is the argmax of $\{|\text{prob}(p, C) - \text{prob}(q, C)\}$ over all sets. So C^* is a set like $\{n_1, n_2, \ldots, n_k\}$. So

$$d_{mp}(p, q) = |\sum_{i=1}^{k} \text{prob}(p, n_i) - \sum_{i=1}^{k} \text{prob}(q, n_i)| = |\sum_{i=1}^{k} \big(\text{prob}(p, n_i) - \text{prob}(q, n_i)\big)|$$

I claim that $\text{prob}(p, n_i) - \text{prob}(q, n_i)$ is either positive for all $i = 1 : k$ or negative for all $i = 1 : k$. Otherwise by omitting some n_i (with opposite sign from $\sum_{i=1}^{k} \text{prob}(p, n_i) - \sum_{i=1}^{k} \text{prob}(q, n_i)$) from C^*, we could get score better than $d_{mp}(p, q)$. I also clam that every n_i with same sign value with $\sum_{i=1}^{k} \text{prob}(p, n_i) - \sum_{i=1}^{k} \text{prob}(q, n_i)$ is in C^*. Because if it is not, by adding this number to our C^* , we could get score better than $d_{mp}(p, q)$. So C^* is the set of all $1 \leq n_i \leq n$ that $\text{prob}(p, n_i) - \sum_{i=1}^{k} \text{prob}(q, n_i)$ has the same sign for all members.

So we can divide the set $\{1, \ldots, n\}$ into two subsets S_1 and S_2 that for every item in S_1 like n_i, we have $\text{prob}(p, n_i) - \text{prob}(q, n_i) > 0$ and with opposite inequality for S_2. As described before,

$$d_{mp}(p, q) = \sum_{n \in S_1} \big(\text{prob}(p, n) - \text{prob}(q, n)\big) = -\sum_{n \in S_2} \big(\text{prob}(p, n) - \text{prob}(q, n)\big)$$

This equation is true because $\sum_{n \in S_1 \cup S_2} \big(\text{prob}(p, n) - \text{prob}(q, n)\big) = \sum_{n \in S_1 \cup S_2} \text{prob}(p, n) - \sum_{n \in S_1 \cup S_2} \text{prob}(q, n) = 1 - 1 = 0$. We can see that

$$|p-q|_1 = \sum_{i=1}^{n} |p_i - q_i| = \sum_{n \in S_1} \big(\text{prob}(p, n) - \text{prob}(q, n)\big) - \sum_{n \in S_2} \big(\text{prob}(p, n) - \text{prob}(q, n)\big) = d_{mp}(p, q) + d_{mp}(p, q) = 2d_{mp}(p, q)$$

So $d_{mp}(p, q) = |p - q|_1/2$

3 3.27 Diagonal elements of Cholesky factor.

Each $X \in S^n_{++}$ has a unique Cholesky factorization $X = LL^T$, where L is lower triangular, with $L_{ii} > 0$. Show that L_{ii} is a concave function of X (with domain S^n_{++}).

Hint. L_{ii} can be expressed as $L_{ii} = (w - z^T Y^{-1} z)^{1/2}$, where

$$\begin{bmatrix} Y & z \\ z^T & w \end{bmatrix}$$

is the leading $i \times i$ submatrix of X.

Solution:

Because $-()^{1/2}$ is a convex, non increasing, we only should prove that $w - z^T Y^{-1} z$ is concave.

$f(z, Y) = z^T Y^{-1} z$ with domain = $\{(z, Y) | Y \succeq 0\}$ is convex in both z and Y. To prove that we will use epigraph of f. (Helping form Example 3.4 of book)

$$epi_f = \{(z, Y, t) | (z, Y) \in \text{ domain}_f, f(z, Y) \leq t\} = \left\{(z, Y, t) | \begin{bmatrix} Y & z \\ z^T & t \end{bmatrix} \succeq 0, Y \succ 0\right\}$$

using the Schur complement condition for positive semi definiteness of a block matrix. The last condition is a linear matrix inequality in (z, Y, t), and therefore epi f is convex. So is f. Therefore, $w - z^T Y^{-1} z$ is a concave function of X. As a result of composition rule, L_{ii} will be concave.

4 3.34 The Minkowski function.

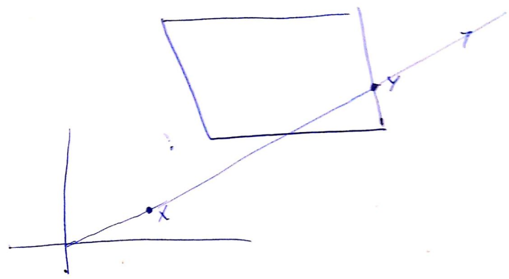
The *Minkowski function* of a convex set C is defined as

$$M_C(x) = \inf\{t > 0 | t^{-1}x \in C\}. \quad (1)$$

1. Draw a picture giving a geometric interpretation of how to find $M_C(x)$.
2. Show that M_C is homogeneous, i.e., $M_C(\alpha x) = \alpha M_C(x)$ for $\alpha \geq 0$.
3. What is dom M_C?
4. Show that M_C is a convex function.
5. Suppose C is also closed, bounded, symmetric (if $x \in C$ then $-x \in C$), and has nonempty interior. Show that M_C is a norm. What is the corresponding unit ball?

Solution:

1. Finding t is easy. You only need to draw an a arrow from 0 toward x. Then find the point (y) that this array exits from set C. Then $t = x/y$. As a result if this arrow does not even enters the set C, we will have $M_C(x) = \infty$ and if this arrow does not exit form the the set C, we will have $M_C(x) = 0$.



2. If $\alpha > 0$:
$$M_C(\alpha x) = \inf\{t > 0 | t^{-1}\alpha x \in C\} = \alpha \inf\{t/\alpha > 0 | t^{-1}\alpha x \in C\} = \alpha M_C(x)$$
If $\alpha = 0$:
$$M_C(\alpha x) = M_C(0) = \begin{cases} 0 & 0 \in C \\ \infty & 0 \notin C \end{cases}$$

3. Dom $M_C = \{x | x/t \in C \text{ for some } t > 0\}$

4. Dom M_C is a convex set because it is conic hull of C. Suppose $x_1, x_2 \in$ dom M_C so there exist t_1 and t_2 that $x_1/t_1 \in C$ and $x_2/t_2 \in C$. We should show that $z = \theta x_1 + (1-\theta)x_2 \in M_C$ for every $0 \leq \theta \leq 1$.
Because C is convex, any convex combination of x_1/t_1 and x_2/t_2 should be in C. So
$$v = \frac{\theta t_1}{\theta t_1 + (1-\theta)t_2}(x_1/t_1) + \frac{(1-\theta)t_2}{\theta t_1 + (1-\theta)t_2}(x_2/t_2) \in C \Rightarrow \frac{\theta x_1 + (1-\theta)x_2}{\theta t_1 + (1-\theta)t_2} \in C$$
So $M_C(\theta x_1 + (1-\theta)x_2) \leq \theta t_1 + (1-\theta)t_2$ because $\theta t_1 + (1-\theta)t_2$ was a candidate that could transfer $\theta x_1 + (1-\theta)x_2$ to C but maybe it is not the infimum value. So
$$M_C(\theta x_1 + (1-\theta)x_2) \leq \theta t_1 + (1-\theta)t_2 \leq \theta \inf\{t > 0 | x_1/t \in C\} + (1-\theta)\inf\{t > 0 | x_2/t \in C\} = \theta M_C(x_1) + (1-\theta)M_C(x_2)$$
So it is convex.

5. For being norm, we need some homogeneity, triangle inequality, and being zero only at the origin.

   - Homogeneity: We should prove that $M_C(\lambda x) = \lambda M_C(x)$. For $\lambda \geq 0$, we proved that $M_C(\lambda x) = \lambda M_C(x)$ in part 2. By symmetry of C, we also have $M_C(-x) = -MC(x)$. So $M_C(\lambda x) = \lambda M_C(x)$ for $\lambda < 0$ too.
   - Triangle inequality:

$$M_C(x+y) = 2M_C(1/2x + 1/2y) \geq 2M_C(1/2x) + 2M_C(1/2y) = M_C(x) + M_C(y)$$

   - Being zero only at the origin: By definition $M_C(0) = 0$. For all $x \neq 0$, we have $M_C > 0$ because we are taking infimum on $t > 0$.

5 3.37

Show that the conjugate of $f(X) = tr(X^{-1})$ with dom $f = S^n_{++}$ is given by

$$f^*(Y) = -2\,tr(-Y)^{1/2},\ dom f^* = -S^n_+.$$

Hint: The gradient of f is $\nabla f(X) = -X^2$

Solution:

The conjugate function is defined as

$$f^*(Y) = \sup_{X \in \text{ dom } f} (tr(YX) - f(X)) = \sup_{X \in \text{ dom } f} (tr(YX) - tr(X^{-1}))$$

since $tr(YX)$ is the standard inner product on S^n. We first show that $tr(YX) - tr(X^{-1})$ is unbounded above unless $Y \preceq 0$. If not, then Y has an eigenvector v_1, with $|v|_2 = 1$, and eigenvalue $\lambda_1 > 0$. Suppose Y has eigenvalue decomposition $Y = AWA^T$ with $W_{1,1} = \lambda_1 > 0$. By taking the same eigenvectors for $X = A\,\text{diag}(t, 1, \ldots, 1)A^T$ we will have:

$$tr(YX) - tr(X^{-1}) = tr(AW\,\text{diag}(t, 1, \ldots, 1)A^T) - tr(X^{-1}) = [t\lambda_1 + \sum_{i=2}^{n} \lambda_i] - [\frac{1}{t} + 1 + 1 + \cdots + 1]$$
$$= t\lambda_1 + \sum_{i=2}^{n} \lambda_i - \frac{1}{t} - n + 1$$

which is unbounded as $t \to \infty$. Now consider the case $Y \preceq 0$. By taking gradient of $tr(YX) - f(X)$ and making it equal to zero, we will have

$$tr(YX) = \sum_{i=1}^{n}\sum_{j=1}^{n} Y_{ij}X_{ij} \Rightarrow \frac{\partial tr(YX)}{\partial X_{ij}} = Y_{ij} \Rightarrow \nabla_X tr(YX) = Y \Rightarrow Y = \nabla f(X) = -X^2 \Rightarrow X = (-Y)^{-1/2}$$
$$f^*(Y) = tr(YX) - tr(X^{-1}) = tr(Y(-Y)^{-1/2}) - tr([(-Y)^{-1/2}]^{-1}) = -tr(-Y)^{1/2} - tr(-Y)^{1/2} = -2\,tr(-Y)^{1/2}$$

I assumed that Y is invertible. For singular Y, we know that conjugate functions are always closed.

6 3.49 Log-Concave

Show that the following functions are log-concave.

1. Logistic function: $f(x) = e^x/(1+e^x)$ with dom $f = R$.
2. Harmonic mean:
$$f(x) = \frac{1}{1/x_1 + \cdots + 1/x_n} \quad \text{dom} f = R^n_{++}.$$
3. Product over sum:
$$f(x) = \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i} \quad \text{dom} f = R^n_{++}.$$
4. Determinant over trace:
$$f(X) = \frac{\det(X)}{Tr(X)} \quad \text{dom} f = R^n_{++}.$$

Solution:

1.
$$f(x) = \frac{e^x}{1+e^x} \Rightarrow f'(x) = \frac{e^x}{(1+e^x)^2} \Rightarrow f''(x) = \frac{e^x - e^{2x}}{(1+e^x)^3}$$
$$f(x) \times f''(x) = \frac{e^{2x} - e^{3x}}{(1+e^x)^4} \qquad f'^2(x) = \frac{e^{2x}}{(1+e^x)^4}$$
$$f'^2(x) - f(x) \times f''(x) = \frac{e^{3x}}{(1+e^x)^4} > 0$$
So f is log-concave

2. Consider $g(x) = \log(f(x)) = -\log \sum_{i=1}^n \frac{1}{x_i}$. Now I want to show that $g(x)$ is convex by showing $\nabla^2 g(x) \preceq 0$
$$\frac{\partial g(x)}{\partial x_i} = \frac{1/x_i^2}{\sum_{i=1}^n \frac{1}{x_i}} \Rightarrow \begin{cases} \frac{\partial^2 g(x)}{\partial^2 x_i} = \frac{-2/x_i^3}{\sum_{i=1}^n \frac{1}{x_i}} + \frac{1/x_i^4}{\left[\sum_{i=1}^n \frac{1}{x_i}\right]^2} \\ \frac{\partial^2 g(x)}{\partial x_i \partial x_j} = \frac{1/(x_i x_j)^2}{\left[\sum_{i=1}^n \frac{1}{x_i}\right]^2} \end{cases}$$
$$y^T \nabla^2 g(x) y = \sum_{i=1}^n \sum_{j=1}^n y_i \nabla^2_{ij} g(x) y_j$$
$$= \sum_{i=1}^n \frac{-2y_i^2/x_i^3}{\sum_{i=1}^n \frac{1}{x_i}} + \frac{y_i^2/x_i^4}{\left[\sum_{i=1}^n \frac{1}{x_i}\right]^2} + \sum_{i=1}^n \sum_{j \neq i} \frac{y_i y_j/(x_i x_j)^2}{\left[\sum_{i=1}^n \frac{1}{x_i}\right]^2}$$
$$= \frac{1}{\left[\sum_{i=1}^n \frac{1}{x_i}\right]^2} \left[\Big(\sum_{i=1}^n y_i/x_i^2\Big)^2 - \Big(\sum_{i=1}^n 2y_i^2/x_i^3\Big)\Big(\sum_{i=1}^n 1/x_i\Big) \right] \leq 0$$
The last term is true because of Cauchy-Schwarz inequality $(x^T y)^2 \leq |x|_2^2 |y|_2^2$. So f is log-concave.

3. Consider $g(x) = \log(f(x)) = \log \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i} = \sum_{i=1}^n \log x_i - \log \sum_{i=1}^n x_i$. Now consider $h(t) = g(x + tv)$

$$h(t) = \sum_{i=1}^n \log(x_i + tv_i) - \log \sum_{i=1}^n (x_i + tv_i)$$
$$\Rightarrow h'(t) = \sum_{i=1}^n \frac{v_i}{x_i + tv_i} - \frac{\sum_{i=1}^n v_i}{\sum_{i=1}^n x_i + t\sum_{i=1}^n v_i}$$
$$\Rightarrow h''(t) = -\sum_{i=1}^n \frac{v_i^2}{(x_i + tv_i)^2} + \frac{(\sum_{i=1}^n v_i)^2}{(\sum_{i=1}^n x_i + t\sum_{i=1}^n v_i)^2}$$

We do not have to show it for other values of t except 0, because we have already shown this for arbitrary x. If you want to know whether this holds true for $x + t_0 v$, for some fixed $t_0 \in R \setminus \{0\}$, then simply redefine $x_0 = x + t_0 v$, consider the function $t \to f(x_0 + tv)$, and do all the below steps.

$$h''(0) = -\sum_{i=1}^n \frac{v_i^2}{x_i^2} + \frac{(\sum_{i=1}^n v_i)^2}{(\sum_{i=1}^n x_i)^2} \leq 0$$

If $\sum_{i=1}^n v_i = 0$, then the inequality will surely works because the second term will be zero and the first term is always non-positive.
Otherwise ($1^T v \neq 0$), we assume that $1^T v = 1^T x$. This assumption will not destroy the generality of solution because scaling x will not change the sign of the left hand of equation and scale both part of left hand of equation by same factor. So by our latest assumption, the problem reduces to

$$\sum_{i=1}^n \frac{v_i^2}{x_i^2} \geq 1$$

subject to $x \succ 0$ and $1^T v = 1^T x$. If we can show that the minimum of left hand side of equation is greater or equal as zero, the proof will be completed.

$$h(v) = \sum_{i=1}^n \frac{v_i^2}{x_i^2}$$
$$\text{subject to } x \succ 0, 1^T v = 1^T x$$

This is a least squares problem with linear constrain. (The constrain of $x \succ 0$ is not important in optimization) The result of this least squares problem is

$$v_i = \frac{\sum_{j=1}^n x_j}{\sum_{j=1}^n x_j^2} x_i^2$$

Therefore the minimum value is

$$v_i = \left(\frac{\sum_{j=1}^n x_j}{\sum_{j=1}^n x_j^2}\right)^2 \sum_{i=1}^n x_i^2 = \frac{(\sum_{j=1}^n x_j)^2}{\sum_{i=1}^n x_i^2} = \frac{||x||_1^2}{||x||_2} \geq 1$$

The last equation is true because $||x||_2 \leq ||x||_1^2$. So f is log-concave.

4. Consider $g(X) = \log(f(x)) = \log(\det(X)) - \log(Tr(X))$. We should prove that $g(X)$ is concave. By taking

$h(t) = g(X + tV)$ where $X \in S^n_{++}$ and $V \in S^n$:

$$
\begin{aligned}
h(t) = g(X+tV) &= \log(\det(X+tV)) - \log(Tr(X+tV)) \\
&= \log(\det(X)) + \log(\det(I + tX^{-1/2}VX^{-1/2})) - \log(Tr[X^{1/2}(I + tX^{-1/2}VX^{-1/2})X^{1/2}]) \\
&= \log(\det(X)) + \log(\det(I + tX^{-1/2}VX^{-1/2})) - \log(Tr[X(I + tX^{-1/2}VX^{-1/2})]) \\
&= \log(\det(X)) + \log(\det(I + tBWB^T)) - \log(Tr[X(I + tBWB^T)]) \\
&= \log(\det(X)) + \log(\det(B(tW + I)B^T)) - \log(Tr[X(B(tW + I)B^T)]) \\
&= \log(\det(X)) + \log(\det(B(tW + I)B^T)) - \log(Tr[B^T XB(tW + I)]) \\
&= \log(\det(X)) + \log\Big[\prod_{i=1}^{n}(1 + t\lambda_i)\Big] - \log\Big[\sum_{i=1}^{n}(b_i^T Xb_i)(1 + t\lambda_i)\Big] \\
&= \log(\det(X)) + \sum_{i=1}^{n}\log\big[1 + t\lambda_i\big] - \log\Big[\sum_{i=1}^{n}(b_i^T Xb_i)(1 + t\lambda_i)\Big] \\
&= \log(\det(X)) - \sum_{i=1}^{n}\log(b_i^T Xb_i) + \sum_{i=1}^{n}\log\big[(b_i^T Xb_i)(1 + t\lambda_i)\big] - \log\Big[\sum_{i=1}^{n}(b_i^T Xb_i)(1 + t\lambda_i)\Big] \\
&= C + \sum_{i=1}^{n}\log\big[(b_i^T Xb_i)(1 + t\lambda_i)\big] - \log\Big[\sum_{i=1}^{n}(b_i^T Xb_i)(1 + t\lambda_i)\Big]
\end{aligned}
$$

which $X^{-1/2}VX^{-1/2} = BWB^T = \sum_{i=1}^{n}\lambda_i b_i b_i^T$. The last term is proved to be concave in part 3. As a result, f is log-concave.