Lecture 5: Traffic Model

From "Broadband Integrated Networks" by M. Schwartz.

- Voice Traffic Model and Multiplexing
- Video Traffic Multiplexing Model

Traffic Characterization

- Future communication networks are expected to carry a variety of traffic types in an integrated fashion
- With traffic characterization, the statistical resource requirements, in terms of link and buffer capacities, can be obtained
- With traffic characterization, the queue waiting time, blocking probabilities, etc. can be determined.

Types of Traffic

Why do we need to study the "Types of Traffic"?

Question: we have learned M/M/1. How about D/D/1 with the same average arrival rate and service rate?

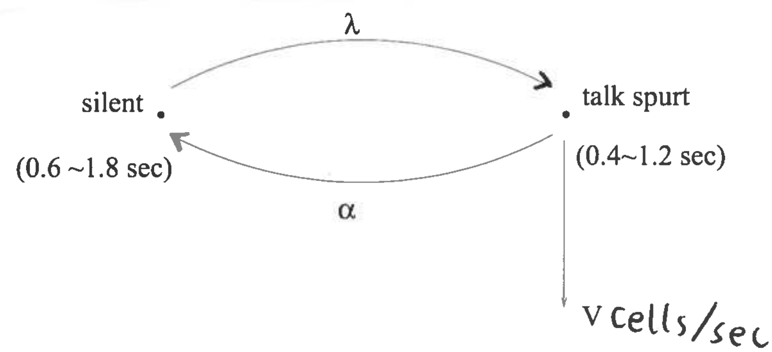
Traffic characterization is not only important when designing buffers, but also in studying admission, access and flow control.

Three types of traffic:

- Voice
- Video
- Data

Packet Voice Modeling

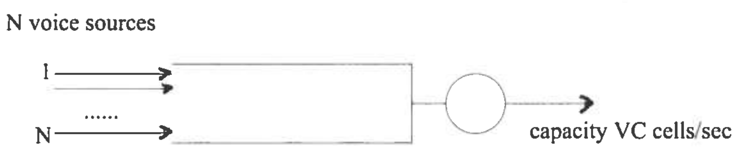
Human speech consists of an alternating sequence of active and inactive intervals. So a single voice source can be well represented by a two-state process (on-off model)



λ —— the rate of transition out of the slient state;

α —— the rate of transition out of the talk spurt.

The speaker activity factor $= \frac{1/\alpha}{1/\alpha + 1/\lambda} = \frac{\lambda}{\lambda+\alpha}$.



Suppose there are N voice sources, and their initiations of calls are independent of each other. The capacity of the server is VC cells/sec. It is clear that the parameter C must satisfy the lower bound inequality

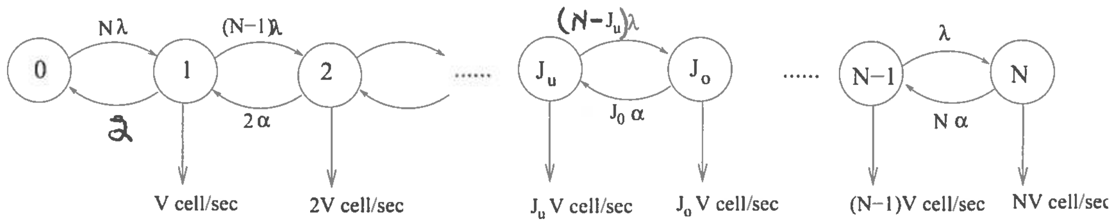
$$(\frac{\lambda}{\lambda+\alpha})N < C$$

i.e., the utilization factor of the queue $\rho = (\frac{\lambda}{\lambda+\alpha})\frac{N}{C} < 1$.

The probability of i active voices (out of N) is binomial, with probability

$$\pi_i = P(i) = \binom{N}{i}\left(\frac{\lambda}{\lambda+\alpha}\right)^i\left(\frac{\alpha}{\lambda+\alpha}\right)^{N-i}.$$

The N multiplexed independent voice sources give rise to the following $(N+1)$-state birth-death model. State i means there are i "on" voice sources.



$J_u = \lfloor C \rfloor$ is called the underload state

$J_o = \lceil C \rceil$ is called the overload state

e.g., $C = 17.6 \implies J_u = 17,\ J_o = 18.$

When $i > C$, the queue is filling and the system is said to be in overload; when $i < C$, with the queue emptying, the system is said to be in underload.

By setting up the balance equation at each state:

$i = 0$: $\qquad N\lambda\pi_0 = \alpha\pi_1$

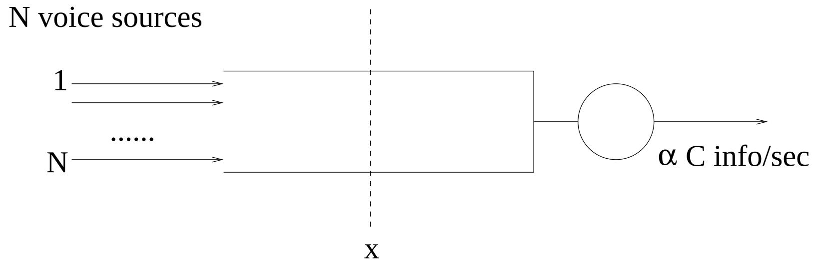
$1 \le i \le N-1$: $\pi_i[i\alpha + (N-i)\lambda] = [N-(i-1)]\lambda\pi_{i-1} + (i+1)\alpha\pi_{i+1}$

$i = N$: $\qquad N\alpha\pi_N = \lambda\pi_{N-1}$

Let $\mathbf{\Pi} = [\pi_0, \pi_1, \pi_2, ..., \pi_N]$, then $\mathbf{\Pi}\mathbb{M} = 0$ (you write down the matrix $\mathbb{M}$ by yourself). The solution for this continuous-time Markov Chain is exactly the binomial distribution.

Recall that the queue is stable as long as $\rho < 1$ and we have infinite queue size. But what if we have requirements on queueing delay and loss rate with a finite-size queue?

Fluid Source Modeling of Packet Voice



Assume N sources each generating V cells/sec during a talk spurt. The service capacity is VC cells/sec. When N and VC are very large relative to the small cell size, the discreteness of the buffer (due to cells arriving and leaving) may be neglected. So the cell arriving process appears like a *continuous* flow of fluid, and the buffer occupancy thus becomes a *continuous* random variable $\mathbf{x}$.

Let $1/\alpha$ be the average length of a talk spurt and define the number of cells arriving during an average talk spurt V/α as a "unit of information". The service capacity VC is normalized into αC units of information/sec by $\frac{VC}{V/\alpha} = \alpha C$.

Let l be the state of the buffer in units of cells. $l = \mathbf{x}V/\alpha$. Then

$$P(l > i) = P(\mathbf{x} > \alpha i/V)$$

i.e., the probability of $\mathbf{x}$ can be converted to buffer cell distribution.

Assume an exponentially distributed on-off process for each source, and there are i $(0 \leq i \leq N)$ sources in talk spurt.

P$\{$one source moves from "off" to "on" during $(t, t + \Delta t)|$ system is in state $i - 1$ at $t\}$ = $[N - (i - 1)]\lambda\Delta t$.

P$\{$one source moves from "on" to "off" during $(t, t + \Delta t)|$ system is in state $i + 1$ at $t\}$ = $(i + 1)\alpha\Delta t$.

P$\{$no source changes its state during $(t, t + \Delta t)|$ system is in state i at $t\}$ = $1 - [(N - i)\lambda + i\alpha]\Delta t$.

Define $F_i(t, x)$=P$\{i$ sources are "on" at t, $\mathbf{x} \leq x\}$ as the cumulative probability distribution at time t, when the system is at state i.

$$\begin{aligned} F_i(t + \Delta t, x) = {} & [N - (i - 1)]\lambda\Delta t F_{i-1}(t, x) + (i + 1)\alpha\Delta t F_{i+1}(t, x) \\ & + \{1 - [(N - i)\lambda + i\alpha]\Delta t\}F_i(t, x - (i - C)\alpha\Delta t) + O(\Delta t) \end{aligned}$$

where $\Delta x = (i - C)\alpha\Delta t$ is the net gain of the buffer in a small period from t to $t + \Delta t$.

Subtract $F_i(t, x)$ from both sides of the equation and divide by Δt

$$\frac{F_i(t + \Delta t, x) - F_i(t, x)}{\Delta t} = [N - (i-1)]\lambda F_{i-1}(t, x) + (i+1)\alpha F_{i+1}(t, x) \\ - [(N-i)\lambda + i\alpha]F_i(t, x - \Delta x) + \frac{F_i(t, x - \Delta x) - F_i(t, x)}{\Delta x/(i - C)\alpha}$$

Letting $\Delta t \to 0$, $\Delta x \to 0$, we get

$$\frac{\partial F_i(t, x)}{\partial t} = [N - (i-1)]\lambda F_{i-1}(t, x) + (i+1)\alpha F_{i+1}(t, x) \\ - [(N-i)\lambda + i\alpha]F_i(t, x) - (i - C)\alpha \frac{\partial F_i(t, x)}{\partial x}$$

Under the steady-state condition $\frac{\partial F_i(t,x)}{\partial t} = 0$, $F_i(t, x) \to F_i(x)$, we have

$$(i - C)\alpha \frac{dF_i(x)}{dx} = [N - (i-1)]\lambda F_{i-1}(x) - [(N-i)\lambda + i\alpha]F_i(x) + (i+1)\alpha F_{i+1}(x)$$

Here $0 \leq i \leq N$, with $F_{-1}(x) = F_{N+1}(x) = 0$.

The solution of this set of equations is readily obtained by first writing them out in their complete form:

$$\begin{aligned}
-C\alpha \frac{dF_0(x)}{dx} &= -N\lambda F_0(x) + \alpha F_1(x) \\
(1 - C)\alpha \frac{dF_1(x)}{dx} &= N\lambda F_0(x) - [(N-1)\lambda + \alpha]F_1(x) + 2\alpha F_2(x) \\
(2 - C)\alpha \frac{dF_2(x)}{dx} &= (N-1)\lambda F_1(x) - [(N-2)\lambda + 2\alpha]F_2(x) + 3\alpha F_3(x) \\
. &\quad . \\
. &\quad . \\
. &\quad . \\
(N - C)\alpha \frac{dF_N(x)}{dx} &= \lambda F_{N-1}(x) - N\alpha F_N(x)
\end{aligned}$$

Define the $(N+1)$-element row vector $\mathbf{F}(x) \equiv [F_0(x), F_1(x), ..., F_N(x)]$, then we have

$$\frac{d\mathbf{F}(x)}{dx}\mathbb{D} = \mathbf{F}(x)\mathbb{M}$$

with $\mathbb{D}$ an $(N+1)\times(N+1)$ diagonal matrix defined as

$$\mathbb{D} = \text{diag}[-C\alpha, (1-C)\alpha, ..., (N-C)\alpha]$$

and $\mathbb{M}$ an $(N+1)\times(N+1)$ matrix defined as

$$\begin{pmatrix} -N\lambda & N\lambda & 0 & \dots \\ \alpha & -[\alpha+(N-1)\lambda] & (N-1)\lambda & \dots \\ 0 & 2\alpha & -(N-2)\lambda-2\alpha & \dots \\ 0 & 0 & 3\alpha & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Multiplying by $\mathbb{D}^{-1}$ on both sides, we get

$$\frac{d\mathbf{F}(x)}{dx} = \mathbf{F}(x)\mathbb{M}'$$

where $\mathbb{M}' = \mathbb{M}\mathbb{D}^{-1}$.

The solution is the weighted sum of exponentials in the eigenvalues of the matrix $\mathbb{M}\mathbb{D}^{-1}$. Since the matrices $\mathbb{M}$ and $\mathbb{D}$ are of order $(N+1)$, there are $(N+1)$ eigenvectors. So the general solution is given by

$$\mathbf{F}(x) = \sum_{i=0}^{N} a_i \mathbf{\Phi}_i e^{z_i x}$$

where z_i the ith eigenvalue, $\mathbf{\Phi}_i$ the corresponding eigenvector given as the solution to the eigenvector equation

$$z_i \mathbf{\Phi}_i \mathbb{D} = \mathbf{\Phi}_i \mathbb{M}$$

and the $\{a_i\}$ a set of undetermined coefficients.

$F_j(x)$ = Prob[j sources on, buffer occupancy$\leq x$] = $\sum_{i=0}^{N} a_i \phi_{ij} e^{z_i x} \quad 0 \leq j \leq N$

with ϕ_{ij} the jth component of eigenvector $\mathbf{\Phi}_i$:

$$\mathbf{\Phi}_i = [\phi_{i0}, \phi_{i1}, ..., \phi_{ij}, ..., \phi_{iN}].$$

<u>Procedure of Finding $a_i, \mathbf{\Phi}_i, z_i$</u>

z_i and $\mathbf{\Phi}_i$ are the eigenvalue and eigenvector of the matrix $\mathbb{M}\mathbb{D}^{-1}$. They can be calculated by using Matlab.

For the infinite-buffer case, since $0 \leq F_j(x) \leq 1$, a_i should be 0 for any positive z_i. So we have

$$\mathbf{F}(x) = \sum_{i:Re[z_i \leq 0]} a_i \mathbf{\Phi}_i e^{z_i x}$$

Recall that $\mathbf{\Pi}\mathbb{M} = 0$, with $\mathbf{\Pi}$ the vector representing the probabilities of states of the underlying Markov chain. So we have

$$0\mathbf{\Pi}\mathbb{D} = \mathbf{\Pi}\mathbb{M}.$$

This means $z_0 = 0$ is one of the eigenvalues, and $\mathbf{\Phi}_0 = \mathbf{\Pi}$ is the corresponding eigenvector. Then

$$\mathbf{F}(x) = a_0\mathbf{\Pi} + \sum_{i:Re[z_i < 0]} a_i \mathbf{\Phi}_i e^{z_i x}$$

and we further have $\mathbf{F}(\infty) = a_0\mathbf{\Pi}$. On the other hand, $F_j(\infty)$ is just the probability that j sources are in talk spurt, which is precisely π_j. Therefore, we have $a_0 = 1$, and thus

$$\mathbf{F}(x) = \mathbf{\Pi} + \sum_{i:Re[z_i < 0]} a_i \mathbf{\Phi}_i e^{z_i x}.$$

It has been proved that the number of negative eigenvalues for this problem is $N - \lfloor C \rfloor$, which is also the number of values of a_i that have to be found.

When the system is in overload state, the probability that the buffer is empty is 0. So $F_i(0) = 0$, for $i > C$. So we have

$$F_j(0) = 0 = \pi_j + \sum_{i=J_o}^{N} a_i \phi_{ij}, \qquad J_o \leq j \leq N$$

For example, $N = 10$, $C = 5.5$, $J_o = \lceil C \rceil = 6$

$$F_j(0) = 0 = \pi_j + \sum_{i=6}^{10} a_i \phi_{ij}, \qquad 6 \leq j \leq 10$$

Given π_j and ϕ_{ij}, we can solve these equations to find the $N - \lfloor C \rfloor$ values of a_i.

The probability $F_j(x)$ and $F(x)$ (the buffer occupancy is less than or equal to x) are

$$F_j(x) = \pi_j + \sum_{i:Re[z_i < 0]} a_i \phi_{ij} e^{z_i x}; \quad F(x) = \sum_{j=0}^{N} F_j(x) = 1 + \sum_{i:Re[z_i < 0]} a_i \sum_{j=0}^{N} \phi_{ij} e^{z_i x}$$

The complementary probability cumulative distribution function

$$G(x) = 1 - F(x) = -\sum_{i:Re[z_i<0]} a_i \sum_{j=0}^{N} \phi_{ij} e^{z_i x}.$$

$G(x)$ is called the *survivor function*. It can be used to find the buffer size required for a given loss probability.

The buffer occupancy will exceed i cells with probability

$$P(l > i) = G(\alpha i/V)$$

since $l = \mathbf{x}V/\alpha$.

Approximation Approach

D. Anick *et al.*, "Stochastic theory of a data handling system with multiple sources," *Bell System Tech. J.*, vol. 61, no. 8, pp. 1871–1894, 1982.

Since the probability distribution is given by the sum of negative exponentials, the exponential with the smallest-magnitude negative eigenvalue will dominate. Let $-r$ to represent this eigenvalue, it can be shown that

$$r = \frac{(1-\rho)(1+\gamma)}{1 - C/N}$$

where $\rho = (\frac{\gamma}{1+\gamma})\frac{N}{C} < 1$, $\gamma = \frac{\lambda}{\alpha}$.

Then $G(x) \sim A_N \rho^N e^{-rx}$, where $A_N = \mathbf{\Pi}_{i=1}^{N-\lfloor C \rfloor -1} \frac{z_i}{z_i + r}$, and $z_1, z_2, ...z_{N-\lfloor C \rfloor -1}$ are the negative eigenvalues except $-r$.

Observation: The buffer size i required to keep the buffer overflow probability to some desired probability is independent of the number N of sources multiplexed, providing the capacity C is scaled accordingly. This says that the number of buffers per source, i/N, decreases as N increases. Actually, due to the factor ρ^N, the buffer size requirement reduces even more. This also means that the queueing delay decreases as N increases.

Example

Consider the special case of $N = 2$ voice sources. $1/\alpha = 1$ sec, and $1/\lambda = 1.5$ sec.

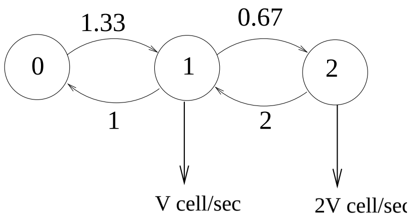
a) Find C, J_o, and J_u for the two cases $\rho = 0.75$ and $\rho = 0.85$.

Solution: using $\rho = (\frac{\lambda}{\lambda+\alpha})\frac{N}{C}$, we have

$$\rho = 0.75 : C = 1.07, J_o = 2, J_u = 1$$

$$\rho = 0.85 : C = 0.94, J_o = 1, J_u = 0$$

3-state diagram



b) Find and compare the eigenvalues of $\mathbb{M}'$ for the two cases.

Solution:

$$\mathbb{M} = \begin{pmatrix} -N\lambda & N\lambda & 0 \\ \alpha & -[\alpha + (N-1)\lambda] & (N-1)\lambda \\ 0 & N\alpha & -N\alpha \end{pmatrix} = \begin{pmatrix} -1.33 & 1.33 & 0 \\ 1 & -1.67 & 0.67 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\mathbb{D} = \text{diag}[-C\alpha, (1-C)\alpha, (2-C)\alpha]$$

so when $\rho = 0.75$:

$$\mathbb{D} = \text{diag}[-1.07, -0.07, 0.93]$$

$\mathbb{M}' = \mathbb{M}\mathbb{D}^{-1}$. Solve $\det(z\mathbb{I} - \mathbb{M}') = 0$ for eigenvalues:

$$z_0 = 0, z_1 = 24, z_2 = -0.89$$

Only one negative eigenvalue, so $A_N = 1$, and $G(x) \sim \rho^2 e^{-0.89x} = 0.5625e^{-0.89x}$.

When $\rho = 0.85$:

$$\mathbb{D} = \text{diag}[-0.94, 0.06, 1.06]$$

$$z_0 = 0, z_1 = -28, z_2 = -0.47$$

$$A_N = \frac{-28}{-28 + 0.47} = 1.02$$

$$G(x) \sim A_N \rho^2 e^{-0.47x} = 0.7348e^{-0.47x}$$

c) Find the eigenvectors for the two cases

Solution:

When $\rho = 0.75$:

$$\mathbf{\Phi}_0 = [-0.58, -0.77, -0.26]$$
$$\mathbf{\Phi}_1 = [-0.039, 1.0, 0.026]$$
$$\mathbf{\Phi}_2 = [0.36, 0.81, 0.46]$$

When $\rho = 0.85$:

$$\mathbf{\Phi}_0 = [0.58, 0.77, 0.26]$$
$$\mathbf{\Phi}_1 = [-0.036, -1.0, 0.024]$$
$$\mathbf{\Phi}_2 = [0.46, 0.81, 0.36]$$

d) Calculate and plot $P(l > i) = G(\alpha i/V)$, where $V = 170$ cells/sec.

Solution:

When $\rho = 0.75$

$$\mathbf{F}(x) = \mathbf{\Pi} + a_2\mathbf{\Phi}_2 e^{-0.89x}$$

Then $0 = \pi_2 + a_2\phi_{22}$. Since $\pi_2 = 0.16$ and $\phi_{22} = 0.46$, we have $a_2 = -0.35$. And thus

$$G(x) = -a_2 \sum_{j=0}^{2} \phi_{2j} e^{z_2 x} = 0.57e^{-0.89x}$$

When $\rho = 0.85$, it can be calculated that

$$G(x) = 0.12e^{-28x} + 0.73e^{-0.47x}$$

e) How large a buffer size is needed to have the buffer overflow probability equal to 10^{-2} for each of the two cases? Repeat for an overflow probability of 10^{-3}

Solution:

From the plot of $G(x)$, we get the buffer size in cells for overflow probability of 10^{-2} and 10^{-3}.

$P_L = 10^{-2}$: 750 for $\rho = 0.75$, 1550 for $\rho = 0.85$

$P_L = 10^{-3}$: 1200 for $\rho = 0.75$, 2350 for $\rho = 0.85$

Video Traffic Multiplexing Model

Video frames to be transmitted at the common rate of 30 frames/sec are sampled and quantized, picture element by picture element, providing a constant number of bits/frame.

North American Standard:

1 frame = 250,000 pixels

1 pixel = 8 bits (gray scale/monochrome video)

Transmitted source rate

= $30 \times 250,000 \times 8 = 60$ Mbits/sec (monochrome video)

or 3×60 Mbits/sec (color video)

Compression (using a variety of coding technique) can obtain 0.5 bit/pixel on average, i.e., average source rate = $30 \times 250,000 \times 0.5 = 3.75$ Mbits/sec.

The compressed video traffic is with variable bit rate (frame by frame).

Consider N independent video sources are multiplexed together. Source i has an average rate $\overline{\lambda_i}$ per second, and an autocovariance function

$$\begin{aligned} C_i(\tau) &= E[\lambda_i(t)\lambda_i(t+\tau)] - E^2[\lambda_i] \qquad \text{(continuous time)} \\ C_i(n) &= E[\lambda_i(m)\lambda_i(m+n)] - E^2[\lambda_m] \quad \text{(discrete time)} \\ \tau &= n/30 \end{aligned} \tag{1}$$

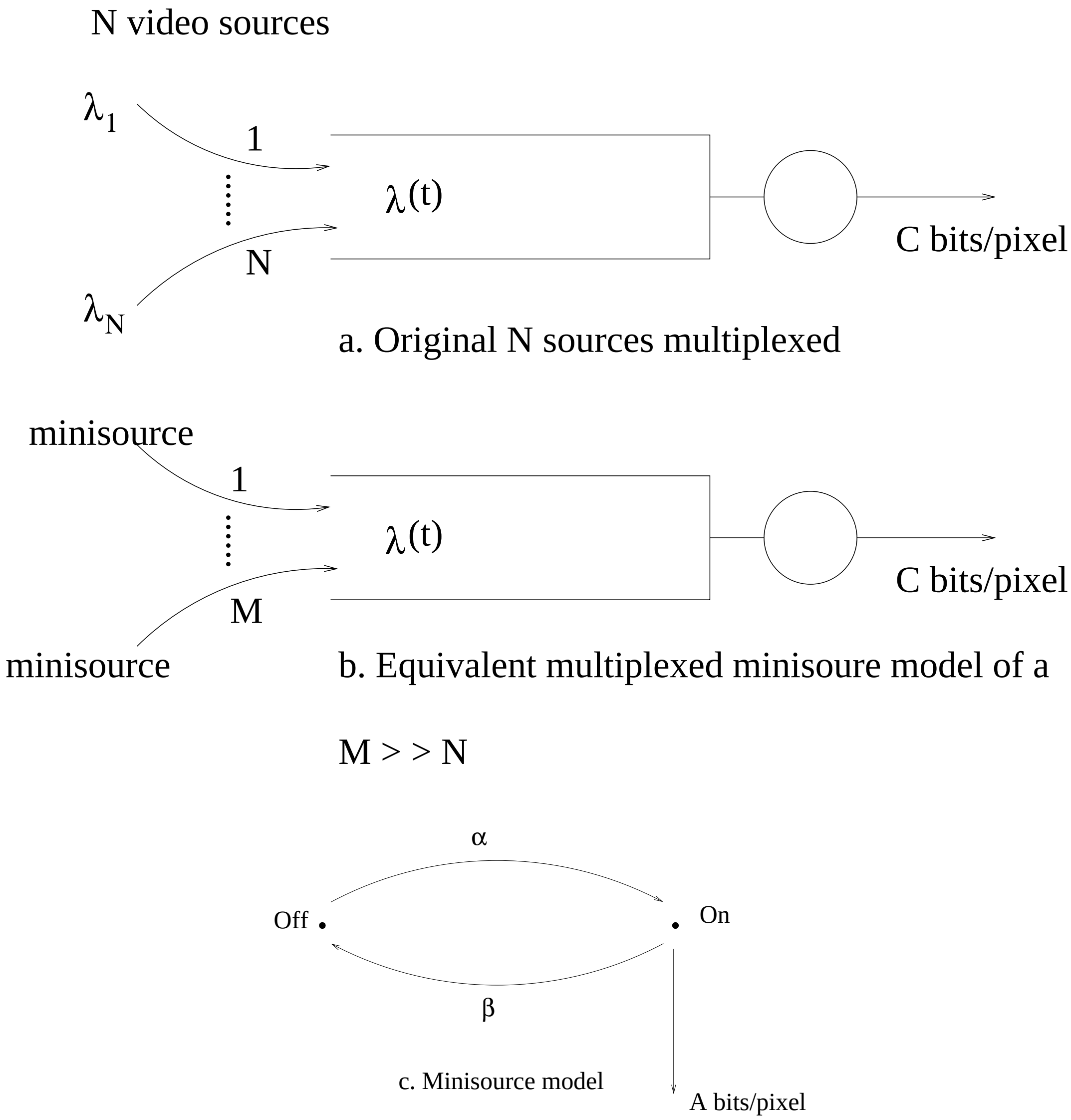
m is the frame number, and $m+n$ represents a frame n units (frames) away.

Let $\lambda(t)$ be the multiplexed (time varying) rate. Actually $\lambda(t)$ changes at frame intervals (30 frames/sec) rather than continuously.

The multiplexed process can be represented by an *equivalent process*: sum of $M(\gg N)$ identical two-state "minisources".

The time-varying bit rate $\lambda(t)$ is quantized to have the values 0, A, $2A$,..., MA bits/pixel. The original N sources multiplexed model can be represented by an $(M+1)$-state Markov chain with state-dependent transition rates.

Question: how to choose the three minisource model parameters α, β, A?



a. Original N sources multiplexed

b. Equivalent multiplexed minisoure model of a

M > > N

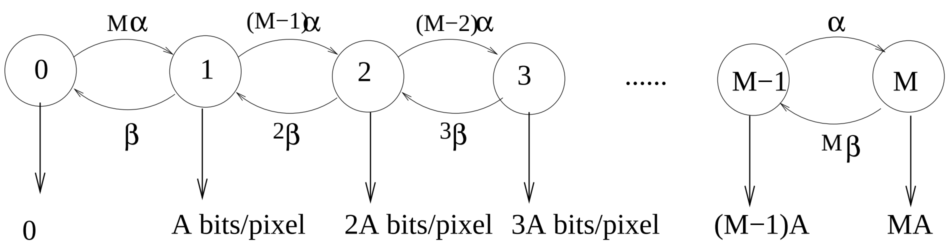
c. Minisource model

Approach: Match the first and second moments of the two models.

For N video sources:

$\lambda(t) = \sum_{i=1}^{N} \lambda_i(t)$

$C(t) = \sum_{i=1}^{N} C_i(t)$



Markov chain representation, equivalent process

Proof:

$$\begin{aligned}
C(\tau) &= E[\lambda(t)\lambda(t+\tau)] - E^2[\lambda(t)] \\
&= E\Big[\sum_{i=1}^{N}\lambda_i(t)\sum_{j=1}^{N}\lambda_j(t+\tau)\Big] - E^2\Big[\sum_{i=1}^{N}\lambda_i(t)\Big] \\
&= E\Big[\sum_{i=1}^{N}\lambda_i(t)\lambda_i(t+\tau) + \sum_{1\le i,j\le N,\ i\ne j}\lambda_i(t)\lambda_j(t+\tau)\Big] - \Big\{\sum_{i=1}^{N}E[\lambda_i(t)]\Big\}^2 \\
&= \sum_{i=1}^{N}E[\lambda_i(t)\lambda_i(t+\tau)] + \sum_{1\le i,j\le N,\ i\ne j}E[\lambda_i(t)]E[\lambda_j(t+\tau)] - \Big\{\sum_{i=1}^{N}E[\lambda_i(t)]\Big\}^2 \\
&= \sum_{i=1}^{N}E[\lambda_i(t)\lambda_i(t+\tau)] - \sum_{i=1}^{N}E^2[\lambda_i(t)] = \sum_{i=1}^{N}C_i(\tau)
\end{aligned}$$

Note that for wide-sense stationary process:

$$E[\lambda(t)] = E[\lambda(t+\tau)] = E(\lambda);$$

$$E[\lambda(t)\lambda(t+\tau)] = R_\lambda(\tau).$$

This completes the proof. □

$\sigma^2 = C(0) = \sum_{i=1}^{N} C_i(0) = \sum_{i=1}^{N}\sigma_i^2$

The parameter σ_i^2 represents the variance of video source i, while σ^2 is used to represent the variance of composite signal.

Assume all video sources have identical moments, and $E(\lambda_i) = \overline{\lambda_i} = 0.52$ bit/pixel, $\sigma_i^2 = 0.0536$, $C_i(\tau) = \sigma_i^2 e^{-3.9\tau}$.

So:

$E(\lambda) = NE(\lambda_i) = 0.52N$

$C(\tau) = NC(\tau_i) = N \cdot 0.0536e^{-3.9\tau}$.

For minisources:

It is shown that the autocovariance function of the two-state minisource model is given by

$$C_i(\tau) = A^2 \frac{\alpha\beta}{(\alpha+\beta)^2} e^{-(\alpha+\beta)\tau}$$

The probability that the minisource is in "on" state is $p = \frac{\alpha}{\alpha+\beta}$. So the minisource is in "off" state with probability $1-p$. Then we can rewrite $C_i(\tau)$:

$$C_i(\tau) = A^2 p(1-p) e^{-(\alpha+\beta)\tau}.$$

The M multiplexed minisources have the composite autocovariance function:

$$C(\tau) = MC_i(\tau) = MA^2 p(1-p) e^{-(\alpha+\beta)\tau}.$$

Since the average number of "on" minisources is $M\frac{\alpha}{\alpha+\beta} = Mp$, the average bit rate of the equivalent minisource model is

$$E(\lambda) = MpA.$$

The two models should be equivalent. So:

$$E(\lambda) = MpA = 0.52N$$

$$\sigma^2 = C(0) = MA^2 p(1-p) = 0.0536N$$

$$\alpha + \beta = 3.9.$$

Then we can get

$$\beta = \frac{3.9}{1+5.05N/M}$$

$$\alpha = 3.9 - \beta = \frac{1.97N}{M(1+5.05N/M)}$$

$$A = 0.4/\beta = 0.1 + 0.52N/M.$$

If $M/N = 20$, then $\beta = 3.12$, $\alpha = 0.78$, $A \doteq 0.13$ bit/pixel.

Fluid-flow queueing analysis for the minisources

Define:

$$F_i(x) \;=\; \text{P[queue} \leq x, i \text{ minisources on]}.$$

Then

$$\begin{aligned} F_i(\infty) &= \text{P[}i \text{ minisources on]} \\ &= \pi_i = \binom{M}{i} p^i (1-p)^{M-i} \end{aligned}$$

where $p = \frac{\alpha}{\alpha+\beta}$.

Following the same analysis procedure as voice model, we get the following equation:

$$\begin{aligned} F_i(t+\Delta t, x) &= [M-(i-1)]\alpha \Delta t F_{i-1}(t,x) + (i+1)\beta \Delta t F_{i+1}(t,x) \\ &\quad + \{1 - [(M-i)\alpha + i\beta]\Delta t\} F_i(t, x-(iA-C)K\Delta t) + o(\Delta t). \end{aligned}$$

$\Delta x = (iA - C)K\Delta t$ is the net gain of the buffer in a small time period t to $t + \Delta t$, and K is the number of pixels (picture elements) per second for a video source. For example, if a video frame contains 250,000 pixels, and the system transmits 30 frames/sec, then $K = 7,500,000$ pixels/sec.

Let $\Delta t \to 0$, $\Delta x \to 0$ and use the steady-state condition $\frac{\partial F_i}{\partial t} = 0$, we get

$$(iA-C)K\frac{dF_i(x)}{dx} = [M-(i-1)]\alpha F_{i-1}(x) - [i\beta + (M-i)\alpha]F_i(x) + (i+1)\beta F_{i+1}(x).$$

Rewrite the above equation

$$\frac{KA}{\beta}\left(i - \frac{C}{A}\right)\frac{dF_i(x)}{dx} = [M-(i-1)]\frac{\alpha}{\beta}F_{i-1}(x) - \left[i + (M-i)\frac{\alpha}{\beta}\right]F_i(x) + (i+1)F_{i+1}(x).$$

Define

$$\begin{aligned} \mathbb{D} &= \text{diag}(i - C/A) \quad 0 \leq i \leq M \\ \mathbf{F}(x) &= [F_0(x), F_1(x), ..., F_M(x)]. \end{aligned}$$

Then we have

$$\frac{KA}{\beta}\frac{d\mathbf{F}(x)}{dx}\mathbb{D} = \mathbf{F}(x)\mathbb{B}$$

where $(M+1)\times(M+1)$ matrix $\mathbb{B}$ is

$$\mathbb{B} = \begin{pmatrix} -M\alpha/\beta & M\alpha/\beta & 0 & 0 & \dots \\ 1 & -(M-1)\alpha/\beta - 1 & (M-1)\alpha/\beta & 0 & \dots \\ 0 & 2 & -(M-2)\alpha/\beta - 2 & (M-2)\alpha/\beta & \dots \\ 0 & 0 & 3 & -(M-3)\alpha/\beta - 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The solution of above differential equation:

$$\mathbf{F}(x) = \sum_{i=0}^{M} a_i \mathbf{\Phi_i} e^{\beta z_i x/KA}$$

with z_i the ith eigenvalue and $\mathbf{\Phi}_i$ the corresponding eigenvector satisfying the eigenvalue equation

$$z_i \mathbf{\Phi}_i \mathbb{D} = \mathbf{\Phi}_i \mathbb{B}.$$

Since $\mathbf{F}(\infty) = \mathbf{\Pi}$:

$$\mathbf{F}(x) = \mathbf{\Pi} + \sum_{i:Re[z_i<0]} a_i \mathbf{\Phi_i} e^{\beta z_i x/KA}.$$

Let $F(x)$ be the (marginal) probability distribution function that the queue occupancy is less than or equal to x, we have:

$$F(x) = \sum_{j=0}^{M} F_j(x) = 1 + \sum_{i:Re[z_i<0]} a_i \left[\sum_{j=0}^{M} \phi_{ij}\right] e^{\beta z_i x/KA}$$

where$\mathbf{\Phi}_i = [\phi_{i0}, \phi_{i1}, ..., \phi_{iM}]$.

The complementary distribution function or survivor function $G(x)$ (the probability the queue occupancy is greater than x) is given by

$$G(x) = 1 - F(x) = -\sum_{i:Re[z_i<0]} a_i \left[\sum_{j=0}^{M} \phi_{ij}\right] e^{\beta z_i x/KA}.$$

The a_i coefficients can be found in the same manner in the voice case.

Using the dominant eigenvalue to approximate $G(x)$, we have

$$G(x) \sim A_M \rho^M e^{-\beta r x/KA}$$

where the utilization ρ is $\rho \equiv MpA/C < 1$ for the minisource model or $\rho = N\bar{\lambda}/C < 1$ for the N multiplexed video sources model, and

$$r = (1-\rho)(1+\alpha/\beta)/[1-(C/MA)].$$

Example (from: B. Maglaris *et al.*, "Performance models of statistical multiplexing in packet video communications," *IEEE Transactions on Communications*, vol. 36, no. 7, pp. 834–844, 1988)

Let $N = 1$ video source (with 30 frames/sec and 250,000 pixels/frame) be approximated by $M = 20$ minisources. The average source bit rate is

$$\bar{\lambda} = 0.52\text{bit/pixel} = 30 \times 250,000 \times 0.52 = 3.9\text{Mbps}.$$

The transmission capacity = 4.875 Mbps = 0.65 bit/pixel.

The utilization ρ is then $\rho = 3.9/4.875 = 0.8$.

we get

$\alpha = 0.78, \beta = 3.12, A \doteq 0.13$ bit/pixel

$p = \frac{\alpha}{\alpha+\beta} = 0.2, \;\; K = 30 \times 250,000 = 7.5 \times 10^6$ pixel/sec

$r = (1-\rho)(1+\alpha/\beta)/[1-(C/MA)] = 0.333.$

Approximating the survivor function by the dominant eigenvalue term only:

$$\begin{aligned} G(x) &\sim A_M \rho^M e^{-\beta r x/KA} \\ &= A_M (0.8)^{20} e^{-1.066\times 10^{-6} x}. \end{aligned}$$