

# A Model-Free/Model-Aware Nonlinear Control of an Underactuated System

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## Abstract

In this proposal, a model-free nonlinear controller is designed for an underactuated nonlinear system. In this report, I focused on two papers [1, 2] among the nine papers. I proved the boundedness of the model-free control law under strict parameter selection in [1] on an underactuated and non-minimum phase system. In the end, I developed two ideas based on [1] and a new method called physics-informed AI [2]. In the latter, one can use reinforcement learning to construct a control law (In the original paper, the author used supervised learning). The second one creates a nonlinear model-aware approach (which is model-agnostic in control but still uses partial model information for optimization) for controlling an underactuated system.

## 1 Dynamical Model

In [1], a 3-DoF robotic arm was chosen as the benchmark. The position of each arm ( $q_i$ , where  $i \in \{1, 2, 3\}$ ) was controlled by a control signal  $u_i$ , where  $i \in \{1, 2, 3\}$ . Thus, the system was fully actuated. In addition, in [2], two examples were addressed, both of which are fully actuated. To investigate the effectiveness of the proposed method in [1], I selected a nonlinear model of an inverted pendulum as an example. The free body diagram (FBD) of the system is depicted in Figure 1.

From the FBD in Figure 1, the following system of equations are extracted:

$$\begin{cases} I\ddot{\theta} = Vl \sin \theta - Hl \cos \theta \\ m(\ddot{x} + l\ddot{\theta} \cos \theta - l\dot{\theta}^2 \sin \theta) = H \\ m(-l\ddot{\theta} \sin \theta - l\dot{\theta}^2 \cos \theta) = V - mg \\ M\ddot{x} = u - H \end{cases} \quad (1)$$

By eliminating  $H$  and  $V$ , the nonlinear system of equations will be as follows,

$$\begin{cases} (M + m)\ddot{x} + ml \cos \theta \ddot{\theta} - ml \sin \theta \dot{\theta}^2 = u \\ (I + ml^2)\ddot{\theta} + ml \cos \theta \ddot{x} - mgl \sin \theta = 0 \end{cases} \quad (2)$$

By solving for  $\ddot{x}$  and  $\ddot{\theta}$ , we will have

$$\begin{cases} \ddot{x} = \frac{1}{D}(4u + 4ml\dot{\theta}^2 \sin \theta - 3mg \sin \theta \cos \theta) \\ \ddot{\theta} = \frac{3}{Dl}((M + m)g \sin \theta - ml\dot{\theta}^2 \sin \theta \cos \theta - u \cos \theta) \end{cases}, \quad (3)$$

where,

$$D = 4M + m(1 + 3 \sin^2 \theta). \quad (4)$$

**Note:** In the case of a linear system of equations, we may consider the following substitutions:

$$\begin{cases} \sin \theta \approx \theta \\ \cos \theta \approx 1 \\ \theta \dot{\theta} \approx 0 \\ \dot{\theta}^2 \approx 0 \end{cases} \quad (5)$$

Linearization leads us to,

$$\begin{cases} (M + m)\ddot{x} + ml\ddot{\theta} = u \\ (I + ml^2)\ddot{\theta} + ml\ddot{x} - mgl\theta = 0 \end{cases} \quad (6)$$

Figure 2 shows the linear and nonlinear responses of the system for  $x(0) = 0$  m,  $\dot{x}(0) = 0$  m/s,  $\dot{\theta}(0) = 0$  rad/s,  $u = 0$  N, and  $\theta(0) = 0.1$  rad.

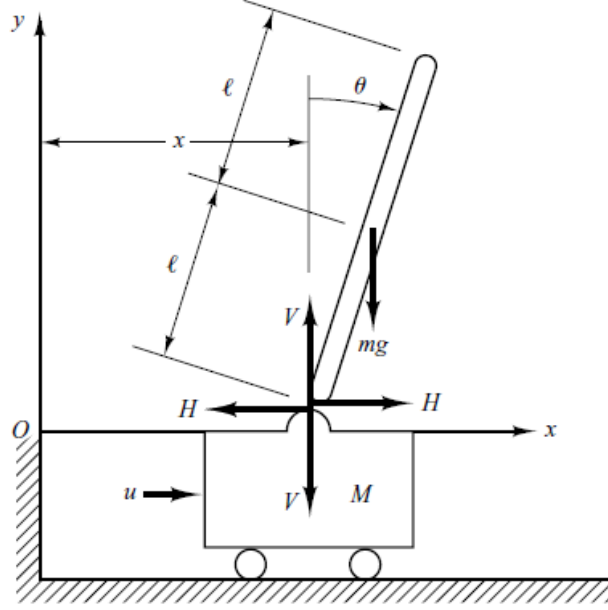


Figure 1: The FBD of an inverted pendulum [3]

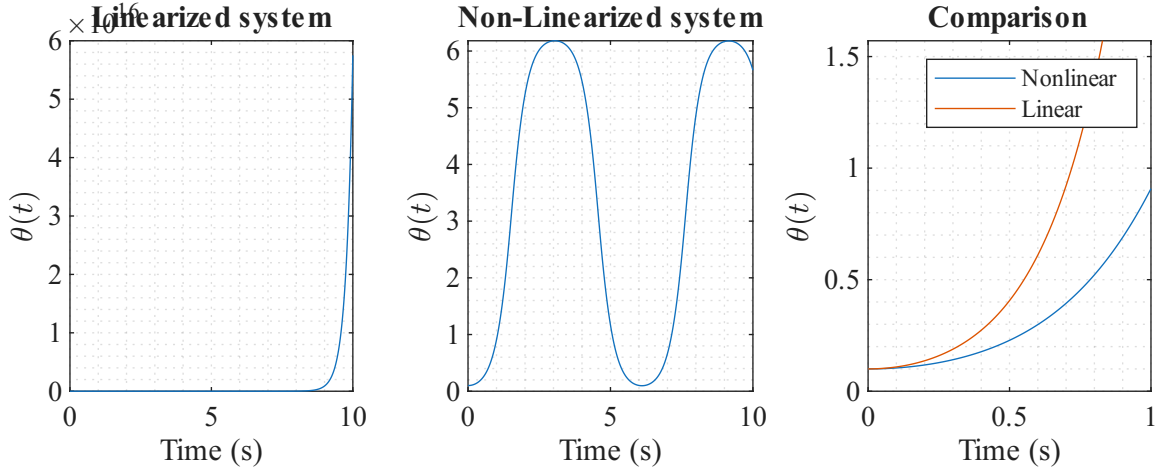


Figure 2: The system response to  $\theta(0) = 0.1$ ,  $x(0) = \dot{x}(0) = \dot{\theta}(0) = 0$ , while  $u = 0$ .

The dynamical model can be written as follows, which looks like equation (20) of [1]:

$$\begin{cases} \ddot{x} = f_1(\theta, \dot{\theta}) + g_1(\theta)u \\ \ddot{\theta} = f_2(\theta, \dot{\theta}) + g_2(\theta)u. \end{cases} \quad (7)$$

It should be noted that in (7), the sign of  $g_1(\theta)$  and  $g_2(\theta)$  must be addressed, since they may change the sign of  $u$ .

## 2 Controller Design

### 2.1 Objectives

The goal is to regulate  $\theta$  to be zero ( $\theta_d = 0$ ), while indirectly keeping  $\dot{\theta}$  to be zero ( $\dot{\theta}_d = 0$ ). In addition, the second goal is to control  $x$ . The first objective is more important; thus, we focus on that first. For  $\theta$ , our control law is model-free. After proving the boundedness of the proposed controller in [1] in controlling the  $\theta$ , we design a PD controller to control  $x$ . This gives us a cascade controller where  $\theta_d$  will be used to control the  $x$ .

We define the following tracking errors:

$$\begin{aligned} e_\theta &:= \theta - \theta_d, \\ y &:= \dot{\theta} + \lambda(t)\theta \quad \longrightarrow \quad \text{From [1]}, \\ e_y &:= y - y_d = \dot{\theta} + \lambda(t)\theta - \dot{\theta}_d - \lambda(t)\theta_d = \dot{\theta} + \lambda(t)e_\theta, \end{aligned} \tag{8}$$

where  $\lambda(t)$  is a positive function. Ref. [1] defines performance function  $\rho_j(t)$ , where  $j \in \{\theta, y\}$ , such that,

$$\begin{aligned} \rho_j(0) &> |e_j(0)|, \quad \text{And} \\ -M_j \rho_j(t) &< e_j(t) < \rho_j(t), \end{aligned} \tag{9}$$

where  $M_j \in \mathbb{R}$  is an overshoot index. To avoid distinguishing between positive/negative error cases in [1], I set  $M_j = 1$  in this report, ensuring that the allowable negative and positive errors are the same. We define the performance functions as follows,

$$\begin{aligned} \rho_\theta(t) &= (\rho_\theta(0) - \rho_\theta(\infty))e^{-l_\theta t} + \rho_\theta(\infty), \\ \rho_y(t) &= (\rho_y(0) - \rho_y(\infty))e^{-l_y t} + \rho_y(\infty) \end{aligned} \tag{10}$$

In (10),  $l_\theta$  and  $l_y$  are time constants that control the contraction of the error funnel.

We define the following normalized errors:

$$\begin{aligned} \hat{e}_\theta &= \rho_\theta^{-1}(t)e_\theta \\ \hat{y} &= \rho_y^{-1}(t)y \end{aligned} \tag{11}$$

Map them via a monotone log transform:

$$\begin{aligned} \varepsilon_\theta &= T(\hat{e}_\theta) = \ln \frac{1 + \hat{e}_\theta}{1 - \hat{e}_\theta} \\ \varepsilon_y &= T(\hat{y}) = \ln \frac{1 + \hat{y}}{1 - \hat{y}} \end{aligned} \tag{12}$$

By taking derivative from (12), we will have:

$$\begin{aligned} \dot{\varepsilon}_\theta &= J_\theta(\hat{e}_\theta)(\dot{e}_\theta + \alpha_\theta(t)e_\theta) \\ \dot{\varepsilon}_y &= J_y(\hat{y})(\dot{y} + \alpha_y(t)y) \end{aligned} \tag{13}$$

where

$$\begin{aligned} J_\theta &= \frac{1}{\rho_\theta} \frac{\partial T}{\partial \hat{e}_\theta} > 0, \quad \alpha_\theta(t) = -\frac{\dot{\rho}_\theta}{\rho_\theta} > 0, \\ J_y &= \frac{1}{\rho_y} \frac{\partial T}{\partial \hat{y}} > 0, \quad \alpha_y(t) = -\frac{\dot{\rho}_y}{\rho_y} > 0. \end{aligned} \tag{14}$$

**Goal:** If we prove  $\varepsilon_\theta$  and  $\varepsilon_y$  stay bounded, we conclude  $e_\theta(t) \in (\underline{b}_\theta(t), \bar{b}_\theta(t))$  and  $y(t) \in (\underline{b}_y(t), \bar{b}_y(t))$  for all  $t$ . Ref. [1] defines  $\underline{b}_j(t) = -\rho_j(t)$  and  $\bar{b}_j(t) = \rho_j(t)$ , where  $j \in \{\theta, y\}$ . Since  $\rho_j(t)$  is a strictly decreasing and positive function, the bound of error will decrease over time.

## 2.2 Control Law for $\theta$

We propose the following control law to control the system in section 1:

$$u(t) = -\frac{\text{sgn}(g_2(\theta))}{g_{min}}(K_\theta J_\theta \varepsilon_\theta + K_y J_y \varepsilon_y). \quad (15)$$

In (15),  $g_{min}$  is the lower bound of  $|g_2(\theta)|$ ,  $K_\theta > 0$  and  $K_y > 0$  are the gains. There are two differences from the original control law described in [1]. The first one is  $\text{sgn}(g_2(\theta))$ . In the original control law,  $u$  had nothing behind it; thus,  $u$  could be applied to the system without any sign change. The other difference is that  $\text{sgn}(g_2(\theta))$  is negative, which leaves the control law with a positive sign. The reason for this sign flip is that the inverted pendulum is a non-minimum phase system: an initial positive control force will lead to a negative change in  $\theta$ .

**Theorem 1:** Consider performance functions in (10) satisfying (9), and  $M_\theta = M_y = 1$ . The control law in (15) applied to the model (7) guarantees 1) the prescribed performance of tracking  $\theta$  and  $y$  in (8) for  $\forall t \geq 0$ , and 2) the boundedness of all closed-loop signals.

In order to prove the theorem, we do a little bookkeeping, we differentiate  $y$ :

$$\dot{y} = \ddot{\theta} + \lambda \dot{\varepsilon}_\theta = f_2(\theta, \dot{\theta}) + g_2(\theta)u + \lambda(\dot{\theta} - \dot{\theta}_d) \xrightarrow{\dot{\theta}_d=0} f_2(\theta, \dot{\theta}) + g_2(\theta)u + \lambda\dot{\theta}. \quad (16)$$

In order to make (16) simpler, we define:

$$w := f_2(\theta, \dot{\theta}) + \lambda\dot{\theta} \quad (17)$$

Thus,

$$\dot{y} = g_2(\theta)u + w. \quad (18)$$

In (18), we assume  $0 < g_{min} \leq |g_2(\theta)| \leq g_{max}$ . The condition  $g_{min}$  happens when  $\cos \theta$  is too small ( $\theta \approx \frac{\pi}{2}$ ), which makes the system uncontrollable no matter what happens (since the gain of  $u$  will be zero). This situation needs a swing-up mechanism, which is not too difficult to build, but I think it is extremely difficult to include in the proofs. The swing-up mechanism applies several sharp forces to bring  $|\theta| < \frac{\pi}{2}$ . In addition,  $w$  must be bounded so  $|w| \leq W_{max}$ . In addition,  $\lambda$  is a function of  $t$ . We embed this uncertainty in  $W_{max}$ .

Rewriting (13) leads us to:

$$\dot{\varepsilon}_y = J_y(\dot{y} + \alpha_y y) = J_y(g_2(\theta)u + w + \alpha_y y) \quad (19)$$

Note that  $g_2(\theta) = -\frac{3\cos\theta}{Dl}$ .

## 2.3 Proof of Theorem 1

We choose the Lyapunov function the same as [1].

$$V = \frac{1}{2}\varepsilon_y^2 + \frac{1}{2}\varepsilon_\theta^2. \quad (20)$$

### 2.3.1 Taking derivative from $\varepsilon_y$ term

By plugging the control law in (15) into (19) we will have:

$$\dot{\varepsilon}_y = J_y(g_2 u + w + \alpha_y y) = J_y\left(\frac{|g_2|}{g_{min}}(-K_\theta J_\theta \varepsilon_\theta - K_y J_y \varepsilon_y) + w + \alpha_y y\right). \quad (21)$$

Note that  $\text{sgn}(g_2)g_2 = |g_2|$ . In addition,  $\frac{|g_2|}{g_{min}} \geq 1$ . The latter expression will increase the controllable terms  $K_\theta$  and  $K_y$  over the uncertain terms. Thus,

$$\dot{V}_y = \varepsilon_y \dot{\varepsilon}_y = -J_y K_\theta J_\theta \varepsilon_y \varepsilon_\theta - K_y (J_y)^2 \varepsilon_y^2 + J_y w \varepsilon_y + J_y \alpha_y y \varepsilon_y. \quad (22)$$

Note that,

$$\left. \begin{array}{l} -\varepsilon_y \leq |\varepsilon_y| \\ -\varepsilon_\theta \leq |\varepsilon_\theta| \end{array} \right\} \rightarrow -|\varepsilon_y||\varepsilon_\theta| \leq \varepsilon_y \varepsilon_\theta \leq |\varepsilon_y||\varepsilon_\theta| \quad (23)$$

The upper bound of  $\dot{V}_y$  will be as follows,

$$\dot{V}_y \leq J_y^{max} K_\theta J_\theta^{max} |\varepsilon_y| |\varepsilon_\theta| - K_y (J_y^{min})^2 \varepsilon_y^2 + J_y^{max} W_{max} |\varepsilon_y| + J_y^{max} \alpha_y^{max} y |\varepsilon_y|. \quad (24)$$

Since  $T(\cdot)$  has the Lipschitz property, we can write  $|y| \leq c_y |\varepsilon_y|$ . We make the above equation neater:

$$\dot{V}_y \leq J_y^{max} K_\theta J_\theta^{max} |\varepsilon_y| |\varepsilon_\theta| - K_y (J_y^{min})^2 \varepsilon_y^2 + J_y^{max} W_{max} |\varepsilon_y| + J_y^{max} \alpha_y^{max} c_y \varepsilon_y^2. \quad (25)$$

We define:

$$\begin{aligned} a &:= J_y^{max} K_\theta J_\theta^{max} > 0 \\ b &:= K_y (J_y^{min})^2 > 0 \\ c &:= J_y^{max} W_{max} > 0 \\ d &:= J_y^{max} \alpha_y^{max} c_y > 0. \end{aligned} \quad (26)$$

### 2.3.2 Taking derivative from $\varepsilon_\theta$ term

From (13) we had,

$$\dot{\varepsilon}_\theta = J_\theta(\dot{e}_\theta + \alpha_\theta e_\theta) = J_\theta((\dot{\theta} - \dot{\theta}_d) + \alpha_\theta e_\theta) \xrightarrow{\dot{\theta}_d=0} J_\theta(y + (\alpha_\theta - \lambda)e_\theta). \quad (27)$$

Thus, we can write (Note  $|e_\theta| \leq c_\theta |\varepsilon_\theta|$ ):

$$\dot{V}_\theta = \varepsilon_\theta \dot{\varepsilon}_\theta \leq J_\theta^{max} c_y |\varepsilon_\theta| |\varepsilon_y| + J_\theta^{max} (\alpha_\theta^{max} - \lambda) c_\theta \varepsilon_\theta^2. \quad (28)$$

In (28), if  $\lambda > \alpha_\theta^{max}$ , we can make the RHS of the inequality negative.

We define  $h := \lambda - \alpha_\theta^{max} > 0$ .

### 2.3.3 Combining two derivative terms $\dot{V} = \dot{V}_\theta + \dot{V}_y$

We will have:

$$\dot{V} = \dot{V}_y + \dot{V}_\theta \leq (a + J_\theta^{max} c_y) |\varepsilon_y| |\varepsilon_\theta| - (b - d) \varepsilon_y^2 - J_\theta^{max} h c_\theta \varepsilon_\theta^2 + c |\varepsilon_y|. \quad (29)$$

We want to eliminate  $|\varepsilon_y|$  and  $|\varepsilon_\theta|$  from the inequality. We use Young's inequality:

$$\begin{aligned} |\varepsilon_y| |\varepsilon_\theta| &\leq \frac{\mu}{2} \varepsilon_y^2 + \frac{1}{2\mu} \varepsilon_\theta^2 \\ |\varepsilon_y| &\leq \frac{\mu}{2} \varepsilon_y^2 + \frac{1}{2\mu} \end{aligned} \quad (30)$$

We simplify the abovementioned inequality to:

$$\dot{V} \leq -[(b - d) - \frac{\mu}{2} (a + J_\theta^{max} c_y) - \frac{\mu c}{2}] \varepsilon_y^2 - [J_\theta^{max} h c_\theta - \frac{a}{2\mu} + \frac{J_\theta^{max} c_y}{2\mu}] \varepsilon_\theta^2 + \frac{c}{2\mu}. \quad (31)$$

With a proper choice of  $K_y$  and  $K_\theta$  we can make the RHS of Eq. (31) negative. Therefore:

$$|y(t)| < \rho_y(t), \quad |\theta(t)| < \rho_\theta(t) \quad \forall t \geq 0. \quad (32)$$

## 2.4 Controlling $x$

For controlling the second control objective  $x$ , we have two options. We can use the same scheme that was proposed in [1]. The objectives for  $x$  controller will be  $e_x = x - x_d$  and  $y = \dot{e}_x + \lambda(t)e_x$ . However, since the controller for  $x$  has a lower speed, we can use  $\theta_d$  as the control signal. Thus, I designed a PD controller to tackle the  $x$  part.

$$\theta_d = -K_d \dot{e}_x - K_p e_x. \quad (33)$$

## 2.5 Simulation

For the inverted pendulum of section 1, I assumed that:  $M = 2$  Kg,  $m = 0.5$  Kg,  $l = 0.5$  m, and  $g = 9.8$  m/s<sup>2</sup>. I selected the PD controllers for the  $x$  as  $K_d = K_p = 0.05$ . I selected  $K_\theta = 100$  and  $K_y = 10$ . In addition, the performance functions are as follows:

$$\begin{aligned} \rho_\theta(t) &= (0.5 - 0.01)e^{-5t} + 0.01 \\ \rho_y(t) &= (0.8 - 0.01)e^{-5t} + 0.01 \end{aligned} \quad (34)$$

Figure. 3 shows the simulation. The codes are uploaded on:  
[https://github.com/AmirhosseinAsgharnia/model\\_free\\_control](https://github.com/AmirhosseinAsgharnia/model_free_control)

## 3 Proposal

If we take a look at Figure 3 ( $\dot{x}$  and  $\dot{\theta}$ ), it is clear that we see sharp oscillations around  $t = 0.7$  seconds. During this period, the control signal has sharp jumps. Still, the performance remains within the defined boundary (there are instances that the performance jumps out of the boundary; however, I think it is caused by the numerical instabilities. I have to investigate more.). To address the control signal, using an optimal controller is a good choice. In [2], the authors defined a safe region within the Lyapunov function. We may say it has the same effect as the prescribed performance region in the section 2.1; however, they follow different principles. Then the authors derived a Hamilton-Jacobi-Bellman (HJB) equation, where the value function acts also as the Lyapunov function. Their Lyapunov function ensures safety (the boundary of each state is explicitly injected into the function), and the entire HJB considers optimality. Since they could not solve HJB analytically, they trained a neural network that satisfies the equations and constraints. They used a gradient-based supervised learning to train the neural network.

I propose two approaches as the framework:

- I can focus on [1] and try to evolve the control law in a way that it addresses both prescribed performance of  $\theta$  and  $x$  at the same time. It will be the complete version of the controller for an underactuated system. Unlike the current report, in which I designed a PD controller in a cascade scheme to control  $x$ , in the complete version, a single control law  $u(t)$  will take care of both states. This approach is truly model-free.
- I can focus on [2] approach since we can control the performance while also ensuring the optimality of the control signal. This approach is not completely model-free as [1]. It uses an interaction with the model to find a solution for the control law. In general, physics-informed AI is considered a model-aware approach, which can be viewed as a hybrid of model-free and model-based. In physics-informed AI, we use the model to shape the cost function, while data is used to construct a solution for the optimization problem. In this method, we will have a formal proof of the stability. Again, in this method, the goal will be to use a single control law to control all states.

## Acknowledgment

In this report, I used Maple 2025 to speed up the calculations, GPT 5 for searching and studying the topic, Grammarly for improving the language, and MATLAB 2025a for graphs and programs.

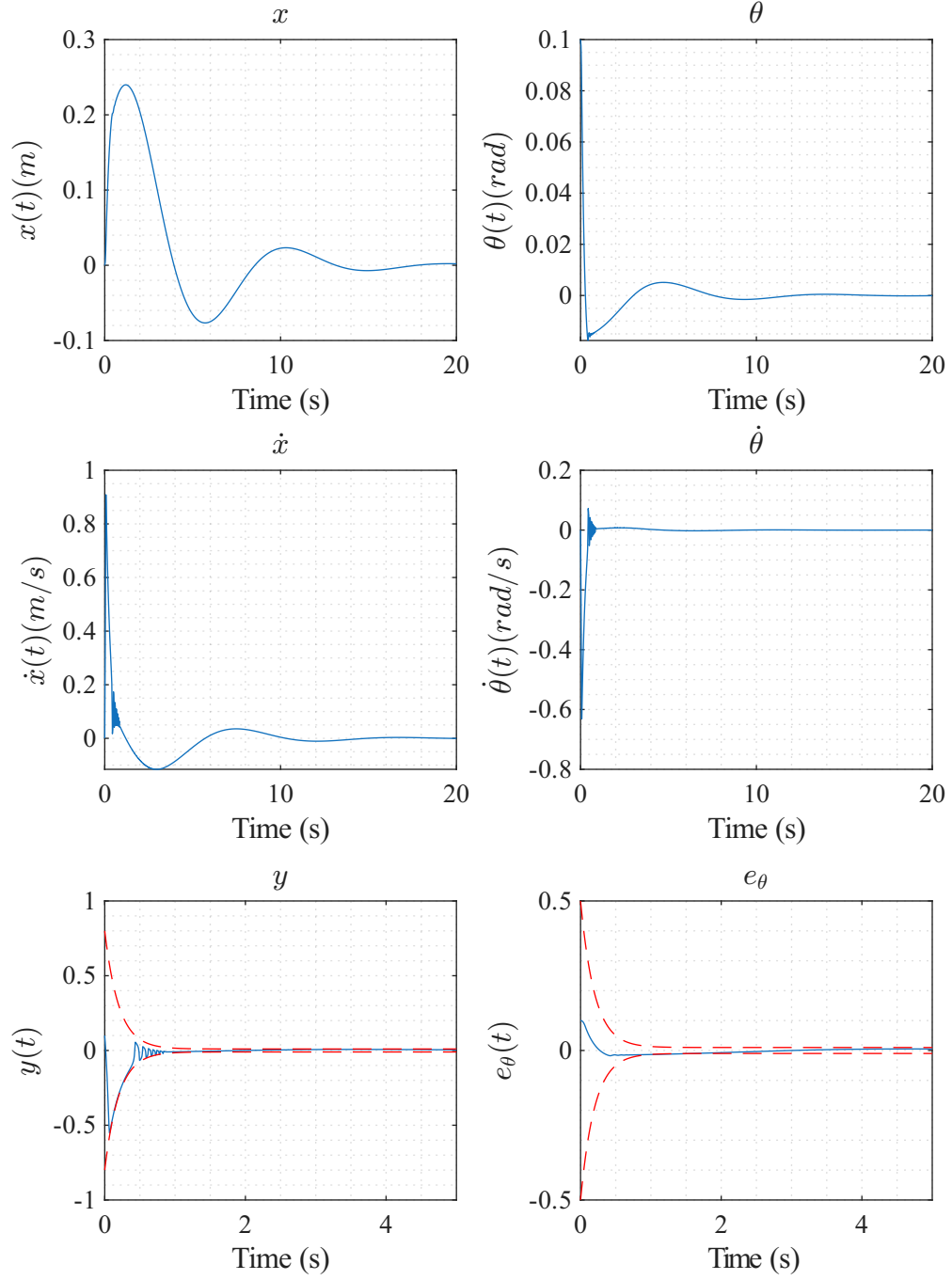


Figure 3: The effectiveness for the proposed controller by [1] on the system in section 1

## References

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