Fourier Series of $f(x) = x^2$

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We start by using the general formula for the Fourier series of a periodic function $f(x)$ with period $2L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

where a_0, a_n, and b_n are the Fourier coefficients, which can be calculated as follows:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L}) dx$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L}) dx$$

For our function $f(x) = x^2$ on the interval $-\pi < x < \pi$, we have $L = \pi$. Therefore, the Fourier coefficients become:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$
$$b_n = 0$$

To find a_0, we integrate x^2 from $-\pi$ to π:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

To find a_n, we integrate $x^2 cos(nx)$ from $-\pi$ to π:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)dx = \frac{4(-1)^n}{n^2\pi}$$

Therefore, the Fourier series for $f(x) = x^2$ on the interval $-\pi < x < \pi$ is given by:

$$f(x) = \frac{\pi^2}{3} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

This series converges pointwise to $f(x) = x^2$ for all x in the interval $-\pi < x < \pi$.