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1 Physical Setup

A ball is placed on a beam, where it is allowed to roll with one degree of freedom along the length of the beam. A lever arm is attached to the beam at one end and a servo gear at the other. As the servo gear turns by an angle θ, the lever changes the angle of the beam by α. When the angle is changed from the horizontal position, gravity causes the ball to roll along the beam. A controller will be designed for this system so that the ball's position can be manipulated.

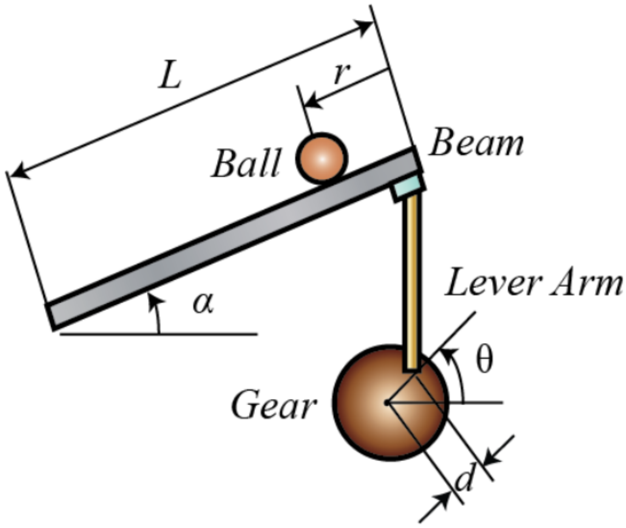

Figure 1: Ball And Beam: System Modeling

2 System Parameters

We assume the ball rolls without slipping, and the friction between the beam and ball is negligible. The system parameters are:

- mass of the ball: $m = 0.11$ kg
- radius of the ball: $R = 0.015$ m
- lever arm offset: $d = 0.03$ m
- gravitational acceleration: $g = 9.8$ m/s^2
- length of the beam: $L = 1.0$ m
- moment of inertia of the ball: $J = 9.99 \times 10^{-6}$ kg·m^2

3 State-Space Representation

This system has 4 state variables: $r, \dot{r}, \alpha, \dot{\alpha}$.

Numerical values chosen with respect to the figures are:

$$m = 0.111, \quad R = 0.015, \quad g = -9.8, \quad J = 9.99 \times 10^{-6}$$

$$H = \frac{-mg}{\frac{J}{R^2} + m}$$

Using the above values, the state-space model is defined as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{\frac{J}{R^2}+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

```
m = 0.111;
R = 0.015;
g = -9.8;
L = 1.0;
d = 0.03;
J = 9.99e-6;

H = -m*g/(J/(R^2)+m);
A = [0 1 0 0
0 0 H 0
0 0 0 1
0 0 0 0];
B = [0 0 0 1]';
C = [1 0 0 0];
D = [0];
ball_ss = ss(A,B,C,D)
```

```
ball_ss =

  A =
        x1  x2  x3  x4
   x1    0   1   0   0
   x2    0   0   7   0
   x3    0   0   0   1
   x4    0   0   0   0

  B =
        u1
   x1    0
   x2    0
   x3    0
   x4    1

  C =
        x1  x2  x3  x4
   y1    1   0   0   0

  D =
        u1
   y1    0

Continuous-time state-space model.
```

Figure 2: The Represented State Space of System

4 Linear Model Extraction in MATLAB

The Simulink model of the Ball & Beam system was first implemented based on the Lagrangian dynamics of the system. The block diagram of the model is shown in Figure 3.

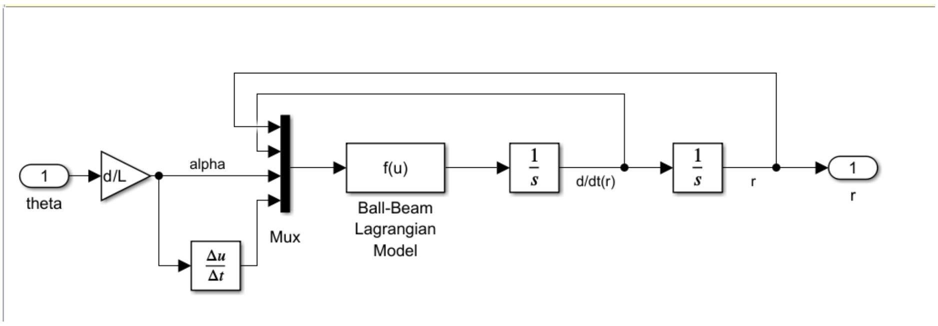

Figure 3: Simulink model of the Ball & Beam system

A step input was applied to the system and the corresponding response of the system was simulated. The step response of this nonlinear Simulink model is shown in Figure 4.

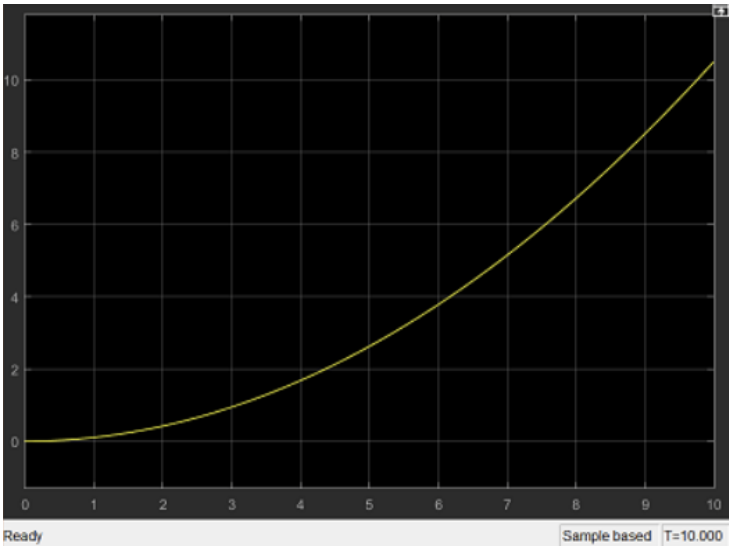

Figure 4: Step response of the nonlinear Ball & Beam Simulink model

After verifying the behavior of the nonlinear model, the system was linearized using the following MATLAB commands:

```
[a,b,c,d] = linmod('ball');
[num,den] = ss2tf(a,b,c,d);
```

Listing 1: Extracting state-space model using linmod

The process of linearization is shown in Figure 5.

```
a =

     0     1
     0     0


b =

         0
    0.2100


c =

     1     0


d =

     0


num =

         0         0    0.2100


den =

     1     0     0
```

Figure 5: Linearization of the Ball & Beam Simulink model using linmod

Finally, the step response of the linearized model was obtained using the transfer function generated from the linearization process. The step response of the linearized model is shown in Figure 6.

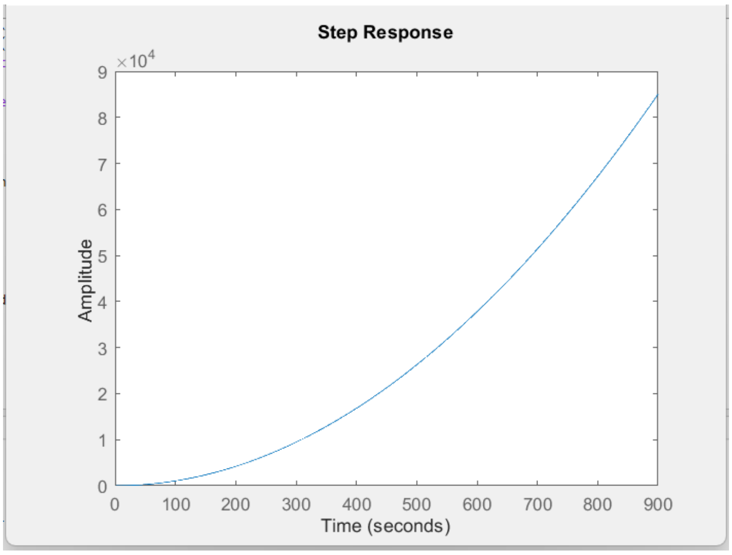

Figure 6: Step response of the linearized Ball & Beam model

5 Stability Check

Theoretical stability is checked by examining the eigenvalues of matrix A. If all eigenvalues have negative real parts, the system is stable.

In MATLAB, use:

```
eig(A)
```

Listing 2: Stability Check

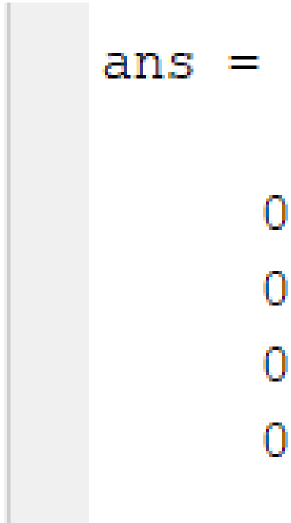

Figure 7: The Eign Value of System

In this system, all eigenvalues of matrix A are equal to zero. Since these eigenvalues do not lie in the open left half-plane (OLHP), the system is not stable.

Based on Lyapunov's stability theory, for a system to be asymptotically stable, all eigenvalues must have strictly negative real parts. In this case, because the eigenvalues are located at the origin, the system is not asymptotically stable.

6 Controllability and Observability

Controllability is checked using the controllability matrix:

$$\mathcal{C} = [B \ AB \ A^2B \ A^3B]$$

Observability is checked using the observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

In MATLAB:

```
Co = ctrb(A,B);
Ob = obsv(A,C);
rank(Co)
rank(Ob)
```

Listing 3: Controllability and Observability Check

The result of this code is shown in the following figure:

```
Controllability matrix rank

ans =

     4

Observability matrix rank

ans =

     4
```

Figure 8: Rank of controllability and observability matrices

As seen in the figure, both the controllability matrix and observability matrix have rank 4, which is equal to the number of state variables. This confirms that the system is fully controllable and fully observable.

7 Step Response of Open-loop System

To analyze system response:

```
[num,den] = ss2tf(A,B,C,D);
sys = tf(num,den);
step(sys);
```

Listing 4: Step Response Plot

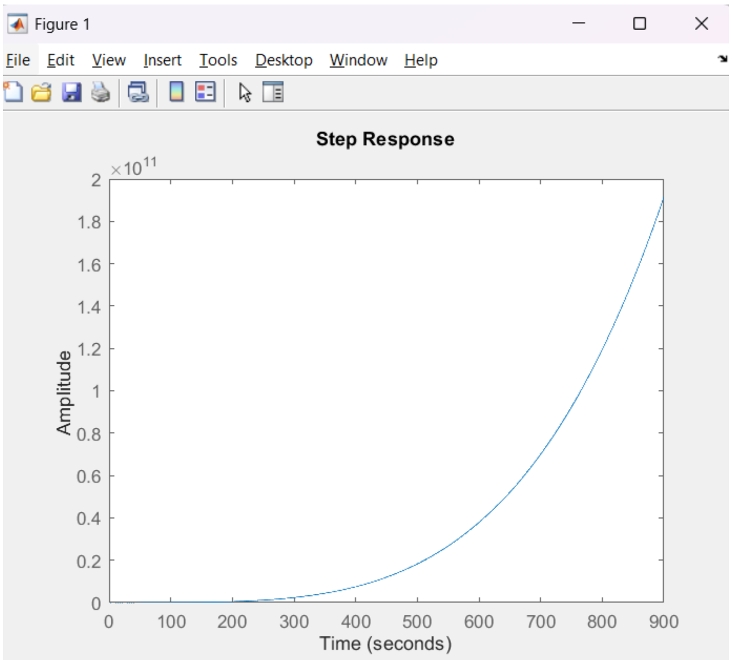

Figure 9: Step response of the open-loop system

The step response of the open-loop system is shown above. The output exhibits an unbounded growth, confirming that the system is not asymptotically stable.

8 Resources

This system is based on the University of Michigan's Control Tutorials for MATLAB and Simulink:

`https://ctms.engin.umich.edu/CTMS/index.php?example=BallBeam&section=SimulinkControl`

Contributing institutions include: University of Michigan, Carnegie Mellon University, and University of Illinois.