Multiple Objective Decision Making

1. Solve the following model using GDF and STEM methods by using customized starting point and assumptions in more than two steps.

Max $f_1(x)$ = $10x_1$ + $30x_2$ + $50x_3$ + $100x_4$

Max $f_2(x)$ = x_1 + x_2

Max $f_3(x)$ = x_1 + $4x_2$ + $6x_3$ + $2x_4$

S.t:

$5x_1 + 3x_2 + 2x_3 \leq 240$

$3x_3 + 8x_4 \leq 320$

$2x_1 + 3x_2 + 4x_3 + 6x_4 \leq 180$

$x_1, x_2, x_3, x_4 \geq 0$

GDF Solution

Step (1):

Determine the initial point:

x_{11}	x_{12}	x_{13}	x_{14}
10	10	10	10

Find the amount of Objective Functions:

f_{11}	f_{12}	f_{13}
1900	20	130

In interaction with the decision maker to determine weights, he considers the increase of the f2 function to 10 units by reducing the f1 function to 100 and the f3 function to the equivalent size of 40. So $w_1^2 = 0.1$, $w_3^2 = 0.25$

After simplifying the model will be like this:

```
1 Objective Declarations
    z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : maximize : 2.25*x[1] + 5.0*x[2] + 6.5*x[3] + 10.5*x[4]

1 Constraint Declarations
    st : Size=3, Index=st_index, Active=True
        Key : Lower : Body                                : Upper : Active
          1 :  -Inf :          5*x[1] + 3*x[2] + 2*x[3] : 240.0 :   True
          2 :  -Inf :                   3*x[3] + 8*x[4] : 320.0 :   True
          3 :  -Inf : 2*x[1] + 3*x[2] + 4*x[3] + 6*x[4] : 180.0 :   True
```

By solving the model, optimal solution is:

```
Variables are shown below:
x[ 1 ] = 0.0
x[ 2 ] = 0.0
x[ 3 ] = 0.0
x[ 4 ] = 30.0
Optimal value is shown Bellow:
315.0
```

So the improve direction is:

$z^1 = (-10, -10, -10, 20)$

To determine the optimal t1, we form the following table:

t	0	0.2	0.4	0.6	0.8	1
f1	1900	2120	2340	2560	2780	3000
f2	20	16	12	8	4	0
f3	130	116	102	88	74	60

The decision maker accepts values t1 = 0.4 and it means:

$x^2 = (6, 6, 6, 18)$

We also find that by increasing the t1 value, both f2 and f3 values decrease. So We recommend to the decision maker to change the reference function to f1.

Step (2):

Find the improvement direction and weights:

$f^1 = (2340, 12, 102)$

In interaction with the decision maker to determine weights, he claim that 100 units reduction in the amount of f1 function is equivalent to a 10 and 20 units increase in f2 and f3 relatively. The normalized weights are:

$w^2 = (0.1, 1, 0.5)$

After simplifying, the model will be like this:

```
1 Objective Declarations
    z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : maximize : 2.5*x[1] + 6.0*x[2] + 8.0*x[3] + 11.0*x[4]

1 Constraint Declarations
    st : Size=3, Index=st_index, Active=True
        Key : Lower : Body                                  : Upper : Active
          1 :  -Inf :          5*x[1] + 3*x[2] + 2*x[3] : 240.0 :   True
          2 :  -Inf :                   3*x[3] + 8*x[4] : 320.0 :   True
          3 :  -Inf : 2*x[1] + 3*x[2] + 4*x[3] + 6*x[4] : 180.0 :   True
```

By solving the model, optimal solution is:

```
Variables are shown below:
x[ 1 ] = 0.0
x[ 2 ] = 0.0
x[ 3 ] = 45.0
x[ 4 ] = 0.0
Optimal value is shown Bellow:
360.0
```

Find the improvement direction and weights:

$z^2 = (-6, -6, 39, -18)$

To determine the optimal t2, we form the following table:

t	0	0.2	0.4	0.6	0.8	1
f1	2340	2322	2304	2286	2268	2250
f2	12	9.6	7.2	4.8	2.4	0
f3	102	135.6	169.2	202.8	236.4	270

The decision maker accepts values t2 = 0.4.

It is also observed that by reducing the f1 value, the f2 value also decreases. So suggested to the decision maker, it is possible to change the reference function to f3.

The ratio of improvement in the second stage compared to the first is:

$$\frac{\Delta^2}{\Delta^1} = \frac{25.2}{29} = 0.86$$

If $\alpha = 0.35$ satisfy the decision maker, then the stoppage conditions have not yet been satisfied.

$x^3 = (3.6, 3.6, 21.6, 10.8)$

Step (3):

Find the improvement direction and weights:

$f^2 = (2304, 7.2, 169.2)$

In interaction with the decision maker to determine weights, he claim that 30 units reduction in the amount of f3 function is equivalent to a 100 and 15 units increase in f1 and f2 relatively. The normalized weights are:

$W^3 = (0.15, 1. 0.5)$

After simplifying, the model will be like this:

```
1 Objective Declarations
    z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : maximize : 3.0*x[1] + 7.5*x[2] + 10.5*x[3] + 16.0*x[4]

1 Constraint Declarations
    st : Size=3, Index=st_index, Active=True
        Key : Lower : Body                                   : Upper : Active
          1 :  -Inf :          5*x[1] + 3*x[2] + 2*x[3]      : 240.0 :   True
          2 :  -Inf :                   3*x[3] + 8*x[4]      : 320.0 :   True
          3 :  -Inf : 2*x[1] + 3*x[2] + 4*x[3] + 6*x[4]      : 180.0 :   True
```

By solving the model, optimal solution is:

```
Variables are shown below:
x[ 1 ] = 0.0
x[ 2 ] = 0.0
x[ 3 ] = 0.0
x[ 4 ] = 30.0
Optimal value is shown Bellow:
480.0
```

Find the improvement direction and weights:

$z^3 = (-3.6, -3.6, -21, 19.2)$

To determine the optimal t2, we form the following table:

t	0	0.2	0.4	0.6	0.8	1
f1	2304	2449.2	2594.4	2739.6	2884.8	3030
f2	7.2	5.76	4.32	2.88	1.44	0
f3	169.2	148.08	126.96	105.84	84.72	63.6

The decision maker accepts values t3 = 0.2

$$\frac{\Delta^3}{\Delta^1} = \frac{9.78}{29} = 0.337$$

That is less than 35.0, so the problem stops. The final answer is:

$x^4 = (2.88, 2.88, 17.4, 14.64)$

$f^3 = (2449.2, 5.76, 148.08)$

STEM Solution

Step (1):

$$f1^*$$

```
Variables:
  x : Size=4, Index=i
      Key : Lower : Value : Upper : Fixed : Stale : Domain
        1 :     0 :   0.0 :  None : False : False : NonNegativeReals
        2 :     0 :   0.0 :  None : False : False : NonNegativeReals
        3 :     0 :   0.0 :  None : False : False : NonNegativeReals
        4 :     0 :  30.0 :  None : False : False : NonNegativeReals

Objectives:
  z1 : Size=1, Index=None, Active=True
      Key  : Active : Value
      None :   True : 3000.0
```

$$f2^*$$

```
  Variables:
    x : Size=4, Index=i
        Key : Lower : Value              : Upper : Fixed : Stale : Domain
          1 :     0 :               20.0 :  None : False : False :
NonNegativeReals
          2 :     0 : 46.66666666666667 :  None : False : False :
NonNegativeReals
          3 :     0 :                0.0 :  None : False : False :
NonNegativeReals
          4 :     0 :                0.0 :  None : False : False :
NonNegativeReals

  Objectives:
    z2 : Size=1, Index=None, Active=True
        Key  : Active : Value
        None :   True : 66.66666666666667
```

$$f3^*$$

```
Variables:
  x : Size=4, Index=i
      Key : Lower : Value : Upper : Fixed : Stale : Domain
        1 :     0 :   0.0 :  None : False : False : NonNegativeReals
        2 :     0 :   0.0 :  None : False : False : NonNegativeReals
        3 :     0 :  45.0 :  None : False : False : NonNegativeReals
        4 :     0 :   0.0 :  None : False : False : NonNegativeReals

Objectives:
  z3 : Size=1, Index=None, Active=True
      Key  : Active : Value
      None :   True : 270.0
```

Step (2):

Calculate Objective Functions

Objective Functions	f1	f2	f3
f1	3000	0	60
f2	1600.1	66.7	206.7
f3	2250	0	270

Determine Alpha

a1	a2	a3
0.0086	0.6776	0.1325

Determine Weights

Π1	Π2	Π3
0.0105	0.8277	0.1618

Solve the Model

```
  Variables:
    x : Size=4, Index=i
        Key : Lower : Value              : Upper : Fixed : Stale : Domain
          1 :     0 :                0.0 :  None : False : False :
NonNegativeReals
          2 :     0 : 53.177950274645845 :  None : False : False :
NonNegativeReals
          3 :     0 :                0.0 :  None : False : False :
NonNegativeReals
          4 :     0 : 3.4110248626770776 :  None : False : False :
NonNegativeReals
    y : Size=1, Index=j
        Key : Lower : Value              : Upper : Fixed : Stale : Domain
          1 :     0 : 11.167369557675642 :  None : False : False :
NonNegativeReals

  Objectives:
    z : Size=1, Index=None, Active=True
        Key  : Active : Value
        None :   True : 11.167369557675642
```

Decision Variables

$x1^1$	$x2^1$	$x3^1$	$x4^1$
0	53.178	0	3.411

Objective Functions

$f1^1$	$f2^1$	$f3^1$
1936.44	53.178	219.534
f1*	**f2***	**f3***
3000	66.67	270

Step (3):

Decision making: We are not satisfied with the amount of the first objective function. By adjusting f2, we will improve the first objective function.

Determine Adjustment

Δf2
6.678

Determine Weights

Π1	Π2	Π3
1	0	0

Solve the Model

```
  Variables:
    x : Size=4, Index=i
        Key : Lower : Value              : Upper : Fixed : Stale : Domain
          1 :     0 :                0.0 :  None : False : False :
NonNegativeReals
          2 :     0 :               46.5 :  None : False : False :
NonNegativeReals
          3 :     0 :  4.292999999999998 :  None : False : False :
NonNegativeReals
          4 :     0 : 3.8880000000000012 :  None : False : False :
NonNegativeReals
    y : Size=1, Index=j
        Key : Lower : Value              : Upper : Fixed : Stale : Domain
          1 :     0 : 1001.5499999999997 :  None : False : False :
NonNegativeReals

  Objectives:
    z : Size=1, Index=None, Active=True
        Key  : Active : Value
        None :   True : 1001.5499999999997
```

Decision Variables

$x1^{2}$	$x2^{2}$	$x3^{2}$	$x4^{2}$
0	46.5	4.293	3.888

Objective Functions

$f1^{2}$	$f2^{2}$	$f3^{2}$
1998.45	46.5	219.534
f1*	**f2***	**f3***
3000	66.67	270

Step (4):

Decision making: The value of the first objective function is not desirable yet. This time, We will use the adjustment of the third objective function to improve the first objective function.

Determine Adjustment

Δf3
21.534

Determine Weights

Π1	Π2	Π3
1	0	0

Solve the Model

```
  Variables:
    x : Size=4, Index=i
        Key : Lower : Value : Upper : Fixed : Stale : Domain
          1 :     0 :   0.0 :  None : False : False : NonNegativeReals
          2 :     0 :  46.5 :  None : False : False : NonNegativeReals
          3 :     0 :   0.0 :  None : False : False : NonNegativeReals
          4 :     0 :  6.75 :  None : False : False : NonNegativeReals
    y : Size=1, Index=j
        Key : Lower : Value : Upper : Fixed : Stale : Domain
          1 :     0 : 930.0 :  None : False : False : NonNegativeReals

  Objectives:
    z : Size=1, Index=None, Active=True
        Key  : Active : Value
        None :   True : 930.0
```

Decision Variables

$x1^3$	$x2^3$	$x3^3$	$x4^3$
0	46.5	0	6.75

Objective Functions

$f1^3$	$f2^3$	$f3^3$
2070	46.5	199.5
f1*	**f2***	**f3***
3000	66.67	270

Decision making: At this stage, the decision maker is satisfied with the amount of all objective functions therefore the solution ends.

2. Consider the following issue. We're going to solve this problem with different MODM methods.

$Max\ f_1(x) = 5x_1 + 4x_2$

$Max\ f_2(x) = 10x_1 + 4x_2$

s.t:

$x_1 + x_3 \leq 15$

$x_1 + x_4 \leq 10$

$x_1 + x_2 \leq 20$

$x_2 + x_4 \leq 14$

$x_2 + x_3 + 2x_4 \leq 35$

a) Augmented Epsilon Constraint 2

First: The f1 Objective Function is chosen as the main objective function.

Second: Using the balance table, the range of the best and worst sub-functions are specified.

balance table

fi	f1	f2
f1	90	0
f2	0	350

According to the table, the value of r2 will be equal to:

r2 = 350 – 0 =350

Now, assuming q2= 10 and k= 6, we calculate the epsilon value.

$\varepsilon_2 = 0 + \frac{350}{10} \times 6 = 210$

So the model of the problem will be as bellow:

```
1 Objective Declarations
    Z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : maximize : 5*x[1] + 4*x[2] + 2.8571428571428575e-07*s2

1 Constraint Declarations
    ST : Size=6, Index=ST_index, Active=True
        Key : Lower : Body                   : Upper : Active
          1 : 210.0 : 10*x[3] + 20*x[4] - s2 : 210.0 :   True
          2 :  -Inf :            x[1] + x[3] :  15.0 :   True
          3 :  -Inf :            x[1] + x[4] :  10.0 :   True
          4 :  -Inf :            x[1] + x[2] :  20.0 :   True
          5 :  -Inf :            x[2] + x[4] :  14.0 :   True
          6 :  -Inf :   x[2] + x[3] + 2*x[4] :  35.0 :   True
```

By solving the model the optimal value is:

```
1 Objective Declarations
    Z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : maximize : 5*x[1] + 4*x[2] + 2.8571428571428575e-07*s2

1 Constraint Declarations
    ST : Size=6, Index=ST_index, Active=True
        Key : Lower : Body                   : Upper : Active
          1 : 210.0 : 10*x[3] + 20*x[4] - s2 : 210.0 :   True
          2 :  -Inf :            x[1] + x[3] :  15.0 :   True
          3 :  -Inf :            x[1] + x[4] :  10.0 :   True
          4 :  -Inf :            x[1] + x[2] :  20.0 :   True
          5 :  -Inf :            x[2] + x[4] :  14.0 :   True
          6 :  -Inf :   x[2] + x[3] + 2*x[4] :  35.0 :   True
```

Therefore, using the Augmented Epsilon Constraint 2, the optimal answer is:

X = (4.67, 8.67, 10.34, 5.34)

b) compromise programming

First mode: p=1

In this case, the model of the problem will be like this.

```
1 Objective Declarations
    Z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : maximize : (5*x[1] + 4*x[2] - 90)/90 + (10*x[3] + 20*x[4] - 350)/350

1 Constraint Declarations
    ST : Size=5, Index=ST_index, Active=True
        Key : Lower : Body                 : Upper : Active
          1 :  -Inf :          x[1] + x[3] :  15.0 :   True
          2 :  -Inf :          x[1] + x[4] :  10.0 :   True
          3 :  -Inf :          x[1] + x[2] :  20.0 :   True
          4 :  -Inf :          x[2] + x[4] :  14.0 :   True
          5 :  -Inf : x[2] + x[3] + 2*x[4] :  35.0 :   True
```

The optimal answer will be:

```
Variables are shown below:
x[ 1 ] = 8.0
x[ 2 ] = 12.0
x[ 3 ] = 7.0
x[ 4 ] = 2.0
Optimal value is shown Bellow:
-0.707936507936508
```

Second mode: p=2

In this case, the problem model is nonlinear:

```
1 Objective Declarations
    Z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : minimize : (((5*x[1] + 4*x[2] - 90)/90)**2 + ((10*x[3] + 20*x[4] - 350)/350)**2)**0.5

1 Constraint Declarations
    ST : Size=5, Index=ST_index, Active=True
        Key : Lower : Body                 : Upper : Active
          1 :  -Inf :          x[1] + x[3] :  15.0 :   True
          2 :  -Inf :          x[1] + x[4] :  10.0 :   True
          3 :  -Inf :          x[1] + x[2] :  20.0 :   True
          4 :  -Inf :          x[2] + x[4] :  14.0 :   True
          5 :  -Inf : x[2] + x[3] + 2*x[4] :  35.0 :   True
```

It should be noted that because of the type of Objective Function in this mode(p=2), which is inherently non-negative, the type of problem will change from maximization to minimization.

The optimal answer to this model is:

```
Variables are shown below:
x[ 1 ] = 4.739869239791856
x[ 2 ] = 8.73986936933747
x[ 3 ] = 10.260130795549859
x[ 4 ] = 5.260130685250208
Optimal value is shown Bellow:
0.5350951280244958
```

Third mode: p=∞

In this case, after linearizing the Objective Function, the model will be as follows:

```
1 Objective Declarations
    Z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : maximize :          y

1 Constraint Declarations
    ST : Size=7, Index=ST_index, Active=True
        Key : Lower : Body                                 : Upper : Active
          1 :  -Inf :     y - (5*x[1] + 4*x[2] - 90)/90 :   0.0 :   True
          2 :  -Inf : y - (10*x[3] + 20*x[4] - 350)/350 :   0.0 :   True
          3 :  -Inf :                       x[1] + x[3] :  15.0 :   True
          4 :  -Inf :                       x[1] + x[4] :  10.0 :   True
          5 :  -Inf :                       x[1] + x[2] :  20.0 :   True
          6 :  -Inf :                       x[2] + x[4] :  14.0 :   True
          7 :  -Inf :              x[2] + x[3] + 2*x[4] :  35.0 :   True
```

The optimal answer to this model is:

```
Variables are shown below:
x[ 1 ] = 8.00000007992563
x[ 2 ] = 12.00000011994711
x[ 3 ] = 7.000000070042604
x[ 4 ] = 2.0000000200285863
Optimal value is shown Bellow:
0.0
```

As it can be seen, the optimal answer to the problem will be the same in the first(p=1) and third(p=∞) cases.

c) Lexicographic

Part 1:

$$f1^*$$

```
  Variables:
    x : Size=4, Index=i
        Key : Lower : Value : Upper : Fixed : Stale :
Domain
          1 :     0 :  10.0 :  None : False : False :
NonNegativeReals
          2 :     0 :  10.0 :  None : False : False :
NonNegativeReals
          3 :     0 :   0.0 :  None : False : False :
NonNegativeReals
          4 :     0 :   0.0 :  None : False : False :
NonNegativeReals

  Objectives:
    z1 : Size=1, Index=None, Active=True
        Key  : Active : Value
        None :   True :  90.0
```

$f2^*$

```
  Variables:
    x : Size=4, Index=i
        Key : Lower : Value : Upper : Fixed : Stale :
Domain
          1 :     0 :   0.0 :  None : False : False :
NonNegativeReals
          2 :     0 :   0.0 :  None : False : False :
NonNegativeReals
          3 :     0 :  15.0 :  None : False : False :
NonNegativeReals
          4 :     0 :  10.0 :  None : False : False :
NonNegativeReals

  Objectives:
    z2 : Size=1, Index=None, Active=True
        Key  : Active : Value
        None :   True : 350.0
```

Part 2:

In this section, we solve the problem using Lexicographic method. In terms of decision making, the importance of the first Objective Function (f1) is much greater than the second Objective Function (f2). Therefore, the problem model and its optimal answer are as follows:

```
1 Objective Declarations
    z : Size=1, Index=None, Active=True
        Key  : Active : Sense    : Expression
        None :   True : minimize : 1000000*sm[1] + 10*sm[2]

1 Constraint Declarations
    ST : Size=7, Index=ST_index, Active=True
        Key : Lower : Body                                : Upper : Active
          1 :  -Inf :                         x[1] + x[3] :  15.0 :   True
          2 :  -Inf :                         x[1] + x[4] :  10.0 :   True
          3 :  -Inf :                         x[1] + x[2] :  20.0 :   True
          4 :  -Inf :                         x[2] + x[4] :  14.0 :   True
          5 :  -Inf :                x[2] + x[3] + 2*x[4] :  35.0 :   True
          6 :  90.0 :    5*x[1] + 4*x[2] + sm[1] - sp[1] :  90.0 :   True
          7 : 350.0 : 10*x[3] + 20*x[4] + sm[2] - sp[2] : 350.0 :   True
```

```
Variables are shown below:
x[ 1 ] = 10.0
x[ 2 ] = 10.0
x[ 3 ] = 5.0
x[ 4 ] = 0.0
Deviations from the goals are shown below:
d(-) 1  = 0.0     ,    d(+) 1 = 0.0
d(-) 2  = 300.0     ,    d(+) 2 = 0.0
Optimal value is shown Bellow:
3000.0
```

Results and Comparison with Ideal Values

x1	x2	x3	x4	f1	f1*	f2	f2*
10	10	5	0	90	90	50	350