حل مسیله به روش پرتابی خطی:

$$y'' = x^3y' + xy + xln(x) \qquad y(1) = 2 \ , \ y(2) = 1$$

در این مرحله مسیله ی مقدار مرزی ما به دو مسیله ی مقدار اولیه ی زیر تبدیل میشود:

$$y'' = x^3y' + xy + xln(x) \qquad y(1) = 2 \ , \ y'(1) = 0$$

$$y'' = x^3y' + xy \qquad y(1) = 0 \ , \ y'(1) = 1$$

با توجه به این که $h = \frac{b-a}{N} = \frac{1}{2}$ پس $N = 2$ و همچنین $p(x) = x^3$ و $q(x) = x$ و $r(x) = xln(x)$ پس در ادامه با توجه به مقادیر اولیه ی زیر داریم:

$$v_0 = 2 \qquad w_0 = 0 \qquad v_0' = 0 \qquad w_0' = 1$$

میدانیم فرمول ها برای گام $i + 1$ ام به صورت زیر اند:

$$k_1 = hw_i$$

$$l_1 = h\big(p(x_i)w_i + q(x_i)v_i + r(x_i)\big)$$

$$k_2 = h(w_i + \frac{l_1}{2})$$

$$l_2 = h(p\left(x_i + \frac{h}{2}\right)(w_i + \frac{l_1}{2}) + q\left(x_i + \frac{h}{2}\right)(v_i + \frac{k_1}{2}) + r\left(x_i + \frac{h}{2}\right))$$

$$k_3 = h(w_i + \frac{l_2}{2})$$

$$l_3 = h(p\left(x_i + \frac{h}{2}\right)(w_i + \frac{l_2}{2}) + q\left(x_i + \frac{h}{2}\right)(v_i + \frac{k_2}{2}) + r\left(x_i + \frac{h}{2}\right))$$

$$k_4 = h(w_i + l_3)$$

$$l_4 = h(p(x_i + h)(w_i + l_3) + q(x_i + h)(v_i + k_3) + r(x_i + h))$$

$$v_{i+1} = v_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$w_{i+1} = w_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$k_1' = hw_i'$$

$$l_1' = h(p(x_i)w_i' + q(x_i)v_i')$$

$$k_2' = h(w_i' + \frac{l_1}{2})$$

$$l_2' = h(p\left(x_i + \frac{h}{2}\right)(w_i' + \frac{l_1'}{2}) + q\left(x_i + \frac{h}{2}\right)(v_i' + \frac{k_1'}{2}))$$

$$k_3' = h(w_0' + \frac{l_2}{2})$$

$$l_3' = h(p\left(x_i + \frac{h}{2}\right)(w_i' + \frac{l_2'}{2}) + q\left(x_i + \frac{h}{2}\right)(v_i' + \frac{k_2'}{2}))$$

$$k_4' = h(w_0' + l_3)$$

$$l_4' = h(p(x_i + h)(w_i' + l_3') + q(x_i + h)(v_i' + k_3'))$$

$$v_i' = v_i' + \frac{1}{6}(k_1' + 2k_2' + 2k_3' + k_4')$$

$$w_i' = w_i' + \frac{1}{6}(l_1' + 2l_2' + 2l_3' + l_4')$$

اکنون با جایگذاری داده هایمان به محاسبات میپردازیم:

گام اول: $x_0 = 1$

$$k_1 = 0.5(0.0) = 0.0$$

$$l_1 = 0.5(1.0(0.0) + 1.0(2.0) + 0.0) = 1.0$$

$$k_2 = 0.5(0.5) = 0.25$$

$$l_2 = 0.5(1.953(0.5) + 1.25(2.0) + 0.279) = 1.87775$$

$$k_3 = 0.5(0.939) = 0.46944$$

$$l_3 = 0.5(1.953(0.939) + 1.25(2.125) + 0.279) = 2.38446$$

$$k_4 = 0.5(2.384) = 1.19223$$

$$l_4 = 0.5(3.375(2.384) + 1.5(2.469) + 0.608) = 6.17995$$

$$v_1 = 2.43852$$

$$w_1 = 2.61739$$

$$k_1' = 0.5(1.0) = 0.5$$

$$l_1' = 0.5(1.0(1.0) + 1.0(0.0) = 1.0$$

$$k_2' = 0.5(1.25) = 0.625$$

$$l_2' = 0.5(1.953(1.25) + 1.25(0.25) = 1.87775$$

$k_3' = 0.5(1.688) = 0.84424$

$l_3' = 0.5(1.953(1.688) + 1.25(0.313) = 2.38446$

$k_4' = 0.5(2.844) = 1.42211$

$l_4' = 0.5(3.375(2.844) + 1.5(0.844) = 6.17995$

$v_1' = 0.81010$

$w_1' = 3.06252$

گام دوم: $x_1 = 1.5$

$k_1 = 0.5(2.617) = 1.3087$

$l_1 = 0.5(3.375(2.617) + 1.5(2.439) + 0.608) = 6.54984$

$k_2 = 0.5(5.892) = 2.94616$

$l_2 = 0.5(5.359(5.892) + 1.75(3.093) + 0.979) = 18.98547$

$k_3 = 0.5(12.11) = 6.05506$

$l_3 = 0.5(5.359(12.11) + 1.75(3.912) + 0.979) = 36.36367$

$k_4 = 0.5(38.981) = 19.49053$

$l_4 = 0.5(8.0(38.981) + 2.0(8.494) + 1.386) = 165.11099$

$v_2 = 8.90546$

$w_2 = 49.67725$

$k_1' = 0.5(3.063) = 1.53126$

$l_1' = 0.5(3.375(3.063) + 1.5(0.81) = 6.54984$

$k_2' = 0.5(5.95) = 2.97516$

$l_2' = 0.5(5.359(5.95) + 1.75(1.576) = 18.98547$

$k_3' = 0.5(11.724) = 5.86219$

$l_3' = 0.5(5.359(11.724) + 1.75(2.298) = 36.36367$

$k_4' = 0.5(36.491) = 18.24534$

$l_4' = 0.5(8.0(36.491) + 2.0(6.672) = 165.11099$

$v_2' = 7.05198$

$w_2' = 46.38159$

اکنون با استفاده از اینکه $\hat{y}_0' = \frac{\beta - v_2}{v_2'} = \frac{1-8.90546}{7.05198} = -1.12103$ و $\hat{y}_0 = \alpha = 2$ همچنین

$\hat{y}_\mathrm{i} = \mathrm{v}_\mathrm{i} + \hat{y}_0' v_i'$ که $\hat{y}_\mathrm{i}$ ها تقریبی از مقدار تابع در نقطه ی گره i ام هستند و $\hat{y}_\mathrm{i}'$ ها تقریبی از مشتق تابع در نقطه ی گره i ام هستند پس داریم:

$x_0 = 1.0 \qquad \hat{y}_0 = 2.00000$

$x_1 = 1.5 \qquad \hat{y}_1 = 1.53038$

$x_2 = 2.0 \qquad \hat{y}_2 = 1.00000$

اکنون تمام این مراحل را برای $h = \frac{1}{4}$ نیز انجام میدهیم:

گام اول: $\mathrm{x} = 1.0$

$\mathrm{k}_1 = 0.25(0.0) = 0.0$

$\mathrm{l}_1 = 0.25(1.0(0.0) + 1.0(2.0) + 0.0) = 0.5$

$\mathrm{k}_2 = 0.25(0.25) = 0.0625$

$\mathrm{l}_2 = 0.25(1.424(0.25) + 1.125(2.0) + 0.133) = 0.68462$

$\mathrm{k}_3 = 0.25(0.342) = 0.08558$

$\mathrm{l}_3 = 0.25(1.424(0.342) + 1.125(2.031) + 0.133) = 0.72626$

$\mathrm{k}_4 = 0.25(0.726) = 0.18157$

$\mathrm{l}_4 = 0.25(1.953(0.726) + 1.25(2.086) + 0.279) = 1.0761$

$\mathrm{v}_1 = 2.07962$

$\mathrm{w}_1 = 0.73298$

$\mathrm{k}_1' = 0.25(1.0) = 0.25$

$\mathrm{l}_1' = 0.25(1.0(1.0) + 1.0(0.0) = 0.5$

$\mathrm{k}_2' = 0.25(1.125) = 0.28125$

$\mathrm{l}_2' = 0.25(1.424(1.125) + 1.125(0.125) = 0.68462$

$k_3' = 0.25(1.218) = 0.30445$

$l_3' = 0.25(1.424(1.218) + 1.125(0.141) = 0.72626$

$k_4' = 0.25(1.473) = 0.36826$

$l_4' = 0.25(1.953(1.473) + 1.25(0.304) = 1.0761$

$v_1' = 0.29828$

$w_1' = 1.48028$

گام دوم: $x = 1.25$

$k_1 = 0.25(0.733) = 0.18324$

$l_1 = 0.25(1.953(0.733) + 1.25(2.08) + 0.279) = 1.07751$

$k_2 = 0.25(1.272) = 0.31793$

$l_2 = 0.25(2.6(1.272) + 1.375(2.171) + 0.438) = 1.68233$

$k_3 = 0.25(1.574) = 0.39354$

$l_3 = 0.25(2.6(1.574) + 1.375(2.239) + 0.438) = 1.90202$

$k_4 = 0.25(2.635) = 0.65875$

$l_4 = 0.25(3.375(2.635) + 1.5(2.473) + 0.608) = 3.30276$

$v_2 = 2.45711$

$w_2 = 2.65781$

$k_1' = 0.25(1.48) = 0.37007$

$l_1' = 0.25(1.953(1.48) + 1.25(0.298) = 1.07751$

$k_2' = 0.25(1.888) = 0.47207$

$l_2' = 0.25(2.6(1.888) + 1.375(0.483) = 1.68233$

$k_3' = 0.25(2.177) = 0.54424$

$l_3' = 0.25(2.6(2.177) + 1.375(0.534) = 1.90202$

$k_4' = 0.25(3.079) = 0.76969$

$l_4' = 0.25(3.375(3.079) + 1.5(0.843) = 3.30276$

$v_2' = 0.82701$

$w_2' = 3.09916$

گام سوم: $x = 1.5$

$k_1 = 0.25(2.658) = 0.66445$

$l_1 = 0.25(3.375(2.658) + 1.5(2.457) + 0.608) = 3.31599$

$k_2 = 0.25(4.316) = 1.07895$

$l_2 = 0.25(4.291(4.316) + 1.625(2.789) + 0.789) = 5.9602$

$k_3 = 0.25(5.638) = 1.40948$

$l_3 = 0.25(4.291(5.638) + 1.625(2.997) + 0.789) = 7.46268$

$k_4 = 0.25(10.12) = 2.53012$

$l_4 = 0.25(5.359(10.12) + 1.75(3.867) + 0.979) = 15.49634$

$v_3 = 3.81901$

$w_3 = 10.26749$

$k_1' = 0.25(3.099) = 0.77479$

$l_1' = 0.25(3.375(3.099) + 1.5(0.827) = 3.31599$

$k_2' = 0.25(4.562) = 1.14042$

$l_2' = 0.25(4.291(4.562) + 1.625(1.214) = 5.9602$

$k_3' = 0.25(5.793) = 1.44815$

$l_3' = 0.25(4.291(5.793) + 1.625(1.397) = 7.46268$

$k_4' = 0.25(9.881) = 2.47021$

$l_4' = 0.25(5.359(9.881) + 1.75(2.275) = 15.49634$

$v_3' = 2.2307$

$w_3' = 10.01522$

گام چهارم: $x = 1.75$

$k_1 = 0.25(10.267) = 2.56687$

$l_1 = 0.25(5.359(10.267) + 1.75(3.819) + 0.979) = 15.67248$

$k_2 = 0.25(18.104) = 4.52593$

$l_2 = 0.25(6.592(18.104) + 1.875(5.102) + 1.179) = 32.52045$

$k_3 = 0.25(26.528) = 6.63193$

$l_3 = 0.25(6.592(26.528) + 1.875(6.082) + 1.179) = 46.86191$

$k_4 = 0.25(57.129) = 14.28235$

$l_4 = 0.25(8.0(57.129) + 2.0(10.451) + 1.386) = 119.83084$

$v_4 = 10.3465$

$w_4 = 59.31216$

$k'_1 = 0.25(10.015) = 2.50381$

$l'_1 = 0.25(5.359(10.015) + 1.75(2.231) = 15.67248$

$k'_2 = 0.25(17.213) = 4.30315$

$l'_2 = 0.25(6.592(17.213) + 1.875(3.483) = 32.52045$

$k'_3 = 0.25(25.014) = 6.25355$

$l'_3 = 0.25(6.592(25.014) + 1.875(4.382) = 46.86191$

$k'_4 = 0.25(53.292) = 13.32289$

$l'_4 = 0.25(8.0(53.292) + 2.0(8.484) = 119.83084$

$v'_4 = 8.38738$

$w'_4 = 55.30998$

اکنون با استفاده از اینکه $\hat{y}'_0 = \frac{\beta - v_4}{v'_4} = -1.11435$ و $\hat{y}_0 = \alpha = 2$ همچنین

$\hat{y}_i = v_i + \hat{y}'_0 v'_i$ که $\hat{y}_i$ ها تقریبی از مقدار تابع در نقطه ی گره i ام هستند و $\hat{y}'_i$ ها تقریبی از مشتق تابع در نقطه ی گره i ام هستند پس داریم:

$x_0 = 1.00 \qquad \hat{y}_0 = 2.00000$

$x_1 = 1.25 \qquad \hat{y}_1 = 1.74723$

$x_2 = 1.5 \qquad \hat{y}_2 = 1.53553$

$x_3 = 1.75 \qquad \hat{y}_3 = 1.33323$

$x_4 = 2.00 \qquad \hat{y}_4 = 1.00000$

اکنون با استفاده از روش برونیابی ریچاردسون داریم:

$$\boxed{\begin{matrix} N_1(h) \searrow \\ N_1(\frac{h}{2}) \rightarrow N_2(h) \end{matrix}} \Longrightarrow N_2(0.5) = N_1(0.25) + \frac{N_1(0.25) - N_1(0.5)}{4-1}$$

که با توجه به مقادیر بدست آمده در دو مرحله ی $h = \frac{1}{2}$ و $h = \frac{1}{4}$ داریم:

$$1.53553 + \frac{1.53553 - 1.53038}{3} = 1.53725$$

در نهایت مقدار دقیق تر تابع در نقطه ی $x = 1.5$ که با دقت $O(h^4)$ است برابر است با 1.53725

حل مسیله به روش ریلی ریتز:

ابتدا با تغییر متغیر $w = \frac{x-a}{b-a}$ داریم:

$$-\frac{d}{dw}\big(p\big((b-a)w+a\big)y'\big) + (b-a)^2 q\big((b-a)w+a\big)y = (b-a)^2 f\big((b-a)w+a\big)$$

$$0 \le w \le 1 \,.\, y(0) = \alpha \,.\, y(1) = \beta$$

پس داریم:

$$y'' = x^3 y' + xy + xln(x) \qquad y(1) = 2 \;\;,\;\; y(2) = 1 \;\;,\;\; h = 0/5$$

$$w = \frac{x-1}{2-1} \Longrightarrow x = w + 1$$

$$y'' = (w+1)^3 y' + (w+1)y + (w+1)ln(w+1) \qquad y(0) = 2 \;\;,\;\; y(1) = 1$$

با تغییر متغیر $z = y - \beta w - (1-w)\alpha$ داریم:

$$z = y - (w) - 2(1-w) \Longrightarrow y = z - w + 2$$

$$y' = z' - 1$$

$$y'' = z''$$

پس داریم:

$$z'' - (w+1)^3 z' - (w+1)z = (w+1)(\ln(w+1) + (2-w) - (w+1)^2)$$

$$0 \le w \le 1 \qquad z(0) = z(1) = 0$$

اکنون برای آنکه معادله به شکل استاندارد در بیاید طرفین معادله ی بالا را در $e^{\frac{-(w+1)^4}{4}}$ ضرب میکنیم و پس از فاکتور گیری معادله به صورت زیر درمیاید:

$$-\left(\left(-e^{\frac{-(w+1)^4}{4}}\right)z'\right)' + \left(-(w+1)e^{\frac{-(w+1)^4}{4}}\right)z = \left(e^{\frac{-(w+1)^4}{4}}(w+1)(\ln(w+1) + (2-w) - (w+1)^2)\right)$$

که یعنی ضرایب به صورت زیر تعریف میشوند:

$$p(w) = -e^{\frac{-(w+1)^4}{4}}$$

$$q(w) = -(w+1)e^{\frac{-(w+1)^4}{4}}$$

$$f(w) = -e^{\frac{-(w+1)^4}{4}}(w+1)(\ln(w+1) - w^2 - 3w + 1)$$

$$h = 0.5 \qquad x_i = 0.5i \qquad i = 0,1,2 \qquad n = 1$$

$$\phi(x) = \sum_{i=1}^{n} c_i \phi_i(x)$$

$$\phi_i(x) = \begin{cases} 0 & 0 \le x \le x_{i-1} \\ \frac{1}{h_{i-1}}(x - x_{i-1}) & x_{i-1} \le x \le x_i \\ \frac{1}{h_i}(x_{i+1} - x) & x_i \le x \le x_{i+1} \\ 0 & x_{i+1} \le x \le 1 \end{cases} \qquad i = 1. \dots . n$$

$$Q_{1.i} \approx \frac{h_i}{12}[q(x_i) + q(x_{i+1})] \qquad i = 1, \dots, n-1$$

$$Q_{2.i} \approx \frac{h_{i-1}}{12}[3\, q(x_i) + q(x_{i-1})] \quad i = 1, \dots, n$$

$$Q_{3.i} \approx \frac{h_i}{12}[3\, q(x_i) + q(x_{i+1})] \qquad i = 1, \dots, n$$

$$Q_{4.i} \approx \frac{h_{i-1}}{2}[p(x_i) + p(x_{i-1})] \quad i = 1, \dots, n+1$$

$$Q_{5.i} \approx \frac{h_{i-1}}{6}[2\,f(x_i) + f(x_{i-1})] \quad i = 1, \dots, n$$

$$Q_{6.i} \approx \frac{h_i}{6}[2\,f(x_i) + f(x_{i+1})] \quad i = 1, \dots, n$$

$$a_{i.i} = Q_{4.i} + Q_{4.i+1} + Q_{2.i} + Q_{3.i} \quad i = 1, \dots, n$$

$$a_{i.i+1} = -Q_{4.i+1} + Q_{1.i} \quad i = 1, \dots, n-1$$

$$a_{i.i-1} = -Q_{4.i} + Q_{1.i-1} \quad i = 2, \dots, n$$

$$b_i = Q_{5.i} + Q_{6.i} \quad i = 1, \dots, n$$

حال داریم:

$$\phi(w) = c_1\phi_1(w)$$

$$\phi_1(w) = \begin{cases} 0 & 0 \le w \le w_0 \\ \frac{1}{h_0}(w - w_0) & w_0 \le w \le w_1 \\ \frac{1}{h_1}(w_2 - w) & w_1 \le w \le w_2 \\ 0 & w_2 \le w \le 1 \end{cases}$$

$$\phi_1(w) = \begin{cases} 0 & 0 \le w \le 0 \\ 2w & 0 \le w \le 0.5 \\ 2\,(1-w) & 0.5 \le w \le 1 \\ 0 & 1 \le w \le 1 \end{cases}$$

$$Q_{2,1} \approx \frac{0.5}{12}\left[-3(1.5)e^{\frac{-1.5^4}{4}} - e^{\frac{-1^4}{4}}\right] = -0.085336$$

$$Q_{3,1} \approx \frac{0.5}{12}\left[-3(1.5)e^{\frac{-1.5^4}{4}} - 2e^{\frac{-2^4}{4}}\right] = -0.054413$$

$$Q_{4,1} \approx \frac{0.5}{2}\left[-e^{\frac{-1.5^4}{4}} - e^{\frac{-1^4}{4}}\right] = -0.265215$$

$$Q_{4,2} \approx \frac{0.5}{2}\left[-e^{\frac{-2^4}{4}} - e^{\frac{-1.5^4}{4}}\right] = -0.075095$$

$$Q_{5,1} \approx \frac{0.5}{6}\left[-2e^{\frac{-1.5^4}{4}}(1.5)(\ln(1.5)-0.75) - e^{\frac{-1^4}{4}}\right] = -0.040604$$

$$Q_{6,1} \approx \frac{0.5}{6}\left[-2e^{\frac{-1.5^4}{4}}(1.5)(\ln(1.5)-0.75) - 2e^{\frac{-2^4}{4}}(\ln(2)-3)\right] = 0.031337$$

$$a_{1.1} = -0.075095 - 0.265215 - 0.085336 - 0.054413 = -0.48006$$

$$b_1 = -0.040604 + 0.031337 = -0.009267$$

$$Ac = b \implies -0.329869\, c_1 = -0.009267 \implies c_1 = 0.01931$$

$$\phi(w) = 0.01931\, \phi_1$$

$$\phi_1(w) = \begin{cases} 0 & 0 \le w \le 0 \\ 2w & 0 \le w \le 0.5 \\ 2\,(1-w) & 0.5 \le w \le 1 \\ 0 & 1 \le w \le 1 \end{cases}$$

اکنون متغیر ها را به حالت اولیه بازمیگردانیم:

$$z = \mathrm{y} + w - 2$$

$$\mathrm{y} = 0.01931\, \phi_1 + 2 - w$$

$$w = x - 1$$

$$\mathrm{y} = 0.01931\phi_1 + 3 - x$$

$$\phi_1(x) = \begin{cases} 0 & 1 \le x \le 1 \\ 2x-2 & 1 \le x \le 1.5 \\ 4-2x & 1.5 \le x \le 2 \\ 0 & 2 \le x \le 2 \end{cases}$$

که جواب در نهایت به صورت بالا است که میتوان با جایگذاری نقاط مقادیر تابع را بدست آورد.

حل مسیله به روش تفاضلات متناهی:

$$y'' = x^3 y' + xy + xln(x) \qquad y(1) = 2 \ , \ y(2) = 1$$

با توجه به این که $h = \frac{b-a}{N+1} = \frac{1}{2}$ پس $N = 1$ و همچنین $p(x) = x^3$ و $q(x) = x$ و $r(x) = xln(x)$ در نتیجه ماتریس ضرایب یک ماتریس 1×1 خواهد بود و داریم:

$$A = [2 + h^2 q(x_1)] \qquad w = [w_1] \qquad b = [-h^2 r(x_1) + (1 - \frac{h}{2} p(x_1))w_2]$$

اکنون به توجه به این که $Aw = b$ و $x_1 = a + h = 1 + \frac{1}{2} = \frac{3}{2}$ پس با جاگذاری عبارت های معلوم، مقدار مجهول را میابیم.

$$\left[2 + \left(\frac{1}{2}\right)^2 \left(\frac{3}{2}\right)\right][w_1] = \left[-\left(\frac{1}{2}\right)^2 \left(\frac{3}{2}\right) \ln\left(\frac{3}{2}\right) + \left(1 - \frac{1}{4}\left(\frac{3}{2}\right)^3\right)\right]$$

$$2.375w_1 = 0.0042005845 \implies w_1 = 0.0017686672$$

صورت معادله دیفرانسیل به صورت زیر است.

$$y'' = \frac{1}{x}y' + 8xy^3 \qquad y(1) = 1 \ , \ y(2) = \frac{1}{2}$$

ابتدا به بررسی وجود و یکتایی جواب مسیله میپردازیم:

$$D = \{(x, y, y') |\ \ 1 \le x \le 2 \ , \ -\infty < y < \infty \ , \ -\infty < y' < \infty\}$$

که میبینیم برای هر $(x, y, y') \in D$ داریم:

$$f_y(x, y, y') = 24xy^2 > 0$$

$$\left|f_{y'}(x, y, y')\right| = \left|\frac{1}{x}\right| \le M = 1$$

اکنون به حل مسیله میپردازیم با توجه به اینکه $h = \frac{b-a}{N} = \frac{1}{2}$ پس $N = 2$ و همچنین

در کتاب گفته شده که $t_0 = \frac{\beta-\alpha}{b-a} = -\frac{1}{2}$ را نقطه ی شروع بگیریم ولی برای بهبود بخشیدن به همگرایی ما $t_0 = -2$ در نظر میگیریم و با توجه به اینکه تابع $f(x, v, w) = \frac{1}{x}w + 8xv^3$ و تابع $f_y(x, v, w) = 24xv^2$ و $f_{y'}(x, v, w) = \frac{1}{x}$ این شکل است پس داریم:

$$v_0 = 1 \qquad w_0 = -\frac{1}{2} \qquad u_1 = 0 \qquad u_2 = 1$$

میدانیم فرمول ها برای گام $i + 1$ ام به صورت زیر اند:

$$k_1 = h(w_i)$$

$$l_1 = hf(x_i, v_i, w_i)$$

$$k_2 = h(w_i + \frac{l_1}{2})$$

$$l_2 = hf\left(x_i + \frac{h}{2}, v_i + \frac{k_1}{2}, w_i + \frac{l_1}{2}\right)$$

$$k_3 = h(w_i + \frac{l_2}{2})$$

$$l_3 = hf\left(x_i + \frac{h}{2}, v_i + \frac{k_2}{2}, w_i + \frac{l_2}{2}\right)$$

$$k_4 = h(w_i + l_3)$$

$$l_4 = hf(x_i + h, v_i + k_3, w_i + l_3)$$

$$v_{i+1} = v_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$w_{i+1} = w_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$k_1' = h(u_2)$$

$$l_1' = h[f_y(x_i, v_i, w_i)u_1 + f_{y'}(x_i, v_i, w_i)u_2]$$

$$k_2' = h\left(u_2 + \frac{l_1'}{2}\right)$$

$$l_2' = h[f_y\left(x_i + \frac{h}{2}, v_i, w_i\right)\left(u_1 + \frac{k_1'}{2}\right) + f_{y'}\left(x_i + \frac{h}{2}, v_i, w_i\right)\left(u_2 + \frac{l_1'}{2}\right)]$$

$$k_3' = h\left(u_2 + \frac{l_2'}{2}\right)$$

$$l_3' = h[f_y\left(x_i + \frac{h}{2}, v_i, w_i\right)\left(u_1 + \frac{k_2'}{2}\right) + f_{y'}\left(x_i + \frac{h}{2}, v_i, w_i\right)\left(u_2 + \frac{l_2'}{2}\right)]$$

$$k_4' = h(u_2 + l_3')$$

$$l_4' = h[f_y(x_i + h, v_i, w_i)(u_1 + k_3') + f_{y'}(x_i + h, v_i, w_i)(u_2 + l_3')]$$

$$u_1 = u_1 + \frac{1}{6}(k_1' + 2k_2' + 2k_3' + k_4')$$

$$u_2 = u_2 + \frac{1}{6}(l_1' + 2l_2' + 2l_3' + l_4')$$

$$t_{i+1} = t_i - \frac{v_2 - \beta}{u_1}$$

گام اول نیوتون:

$$t_0 = -2.0$$

$$k_1 = 0.5(-2.0) = -1.0$$

$$l_1 = 0.5f(1.0,1.0,-2.0) = 3.0$$

$$k_2 = 0.5(-0.5) = -0.25$$

$$l_2 = 0.5f(1.25,0.5,-0.5) = 0.425$$

$$k_3 = 0.5(-1.787) = -0.89375$$

$$l_3 = 0.5f(1.25,0.875,-1.787) = 2.63461$$

$$k_4 = 0.5(0.635) = 0.3173$$

$l_4 = 0.5f(1.5,0.106,0.635) = 0.21873$

$v_1 = 0.50497$

$w_1 = -0.44367$

$k_1' = 0.5(1.0) = 0.5$

$l_1' = 0.5[f_y(1.0,1.0,-2.0)(0.0) + f_{y'}(1.0,1.0,-2.0)(1.0)] = 0.5$

$k_2' = 0.5(1.25) = 0.625$

$l_2' = 0.5[f_y(1.25,1.0,-2.0)(0.25) + f_{y'}(1.25,1.0,-2.0)(1.25)] = 4.25$

$k_3' = 0.5(3.125) = 1.5625$

$l_3' = 0.5[f_y(1.25,1.0,-2.0)(0.313) + f_{y'}(1.25,1.0,-2.0)(3.125)] = 5.9375$

$k_4' = 0.5(6.938) = 3.46875$

$l_4' = 0.5[f_y(1.5,1.0,-2.0)(1.563) + f_{y'}(1.5,1.0,-2.0)(6.938)] = 30.4375$

$u_1 = 1.39063$

$u_2 = 9.55208$

$k_1 = 0.5(-0.444) = -0.22184$

$l_1 = 0.5f(1.5,0.505,-0.444) = 0.62468$

$k_2 = 0.5(-0.131) = -0.06567$

$l_2 = 0.5f(1.75,0.394,-0.131) = 0.39078$

$k_3 = 0.5(-0.248) = -0.12414$

$l_3 = 0.5f(1.75,0.472,-0.248) = 0.66577$

$k_4 = 0.5(0.222) = 0.11105$

$l_4 = 0.5f(2.0,0.381,0.222) = 0.49736$

$v_2 = 0.42323$

$w_2 = 0.09551$

$k_1' = 0.5(9.552) = 4.77604$

$$l_1' = 0.5[f_y(1.5,0.505,-0.444)(1.391) + f_{y'}(1.5,0.505,-0.444)(9.552)] = 9.5668$$

$$k_2' = 0.5(14.335) = 7.16774$$

$$l_2' = 0.5[f_y(1.75,0.505,-0.444)(3.779) + f_{y'}(1.75,0.505,-0.444)(14.335)] = 24.32988$$

$$k_3' = 0.5(21.717) = 10.85851$$

$$l_3' = 0.5[f_y(1.75,0.505,-0.444)(4.974) + f_{y'}(1.75,0.505,-0.444)(21.717)] = 32.84247$$

$$k_4' = 0.5(42.395) = 21.19727$$

$$l_4' = 0.5[f_y(2.0,0.505,-0.444)(12.249) + f_{y'}(2.0,0.505,-0.444)(42.395)] = 85.56103$$

$$u_1 = 11.72826$$

$$u_2 = 44.46417$$

گام دوم نیوتون:

$$t_1 = -1.9934544915165546$$

$$k_1 = 0.5(-1.993) = -0.99673$$

$$l_1 = 0.5f(1.0,1.0,-1.993) = 3.00327$$

$$k_2 = 0.5(-0.492) = -0.24591$$

$$l_2 = 0.5f(1.25,0.502,-0.492) = 0.43443$$

$$k_3 = 0.5(-1.776) = -0.88812$$

$$l_3 = 0.5f(1.25,0.877,-1.776) = 2.66266$$

$$k_4 = 0.5(0.669) = 0.3346$$

$$l_4 = 0.5f(1.5,0.112,0.669) = 0.23147$$

$$v_1 = 0.51164$$

$$w_1 = -0.42197$$

$k_1' = 0.5(1.0) = 0.5$

$l_1' = 0.5[f_y(1.0,1.0,-1.993)(0.0) + f_{y'}(1.0,1.0,-1.993)(1.0)] = 0.5$

$k_2' = 0.5(1.25) = 0.625$

$l_2' = 0.5[f_y(1.25,1.0,-1.993)(0.25) + f_{y'}(1.25,1.0,-1.993)(1.25)] = 4.25$

$k_3' = 0.5(3.125) = 1.5625$

$l_3' = 0.5[f_y(1.25,1.0,-1.993)(0.313) + f_{y'}(1.25,1.0,-1.993)(3.125)] = 5.9375$

$k_4' = 0.5(6.938) = 3.46875$

$l_4' = 0.5[f\_y(1.5,1.0,-1.993)(1.563) + f_{y'}(1.5,1.0,-1.993)(6.938)] = 30.4375$

$u1 = 1.39063$

$u2 = 9.55208$

$k_1 = 0.5(-0.422) = -0.21098$

$l_1 = 0.5f(1.5,0.512,-0.422) = 0.66294$

$k_2 = 0.5(-0.091) = -0.04525$

$l_2 = 0.5f(1.75,0.406,-0.091) = 0.44311$

$k_3 = 0.5(-0.2) = -0.10021$

$l_3 = 0.5f(1.75,0.489,-0.2) = 0.76131$

$k_4 = 0.5(0.339) = 0.16967$

$l_4 = 0.5f(2.0,0.411,0.339) = 0.64199$

$v_2 = 0.45626$

$w_2 = 0.19699$

$k_1' = 0.5(9.552) = 4.77604$

$l_1' = 0.5[f_y(1.5,0.512,-0.422)(1.391) + f_{y'}(1.5,0.512,-0.422)(9.552)] = 9.7365$

$k_2' = 0.5(14.42) = 7.21017$

$$l_2' = 0.5[f_y(1.75,0.512,-0.422)(3.779) + f_{y'}(1.75,0.512,-0.422)(14.42)] = 24.89208$$

$$k_3' = 0.5(21.998) = 10.99906$$

$$l_3' = 0.5[f_y(1.75,0.512,-0.422)(4.996) + f_{y'}(1.75,0.512,-0.422)(21.998)] = 33.7476$$

$$k_4' = 0.5(43.3) = 21.64984$$

$$l_4' = 0.5[f_y(2.0,0.512,-0.422)(12.39) + f_{y'}(2.0,0.512,-0.422)(43.3)] = 88.66336$$

$$u_1 = 11.86468$$

$$u_2 = 45.49862$$

گام سوم نیوتون:

$$t_2 = 1.9897683107707487$$

$$k_1 = 0.5(-1.99) = -0.99488$$

$$l_1 = 0.5f(1.0,1.0,-1.99) = 3.00512$$

$$k_2 = 0.5(-0.487) = -0.24361$$

$$l_2 = 0.5f(1.25,0.503,-0.487) = 0.43976$$

$$k_3 = 0.5(-1.77) = -0.88494$$

$$l_3 = 0.5f(1.25,0.878,-1.77) = 2.67851$$

$$k_4 = 0.5(0.689) = 0.34437$$

$$l_4 = 0.5f(1.5,0.115,0.689) = 0.23872$$

$$v_1 = 0.51540$$

$$w_1 = -0.40971$$

$$k_1' = 0.5(1.0) = 0.5$$

$$l_1' = 0.5[f_y(1.0,1.0,-1.99)(0.0) + f_{y'}(1.0,1.0,-1.99)(1.0)] = 0.5$$

$k_2' = 0.5(1.25) = 0.625$

$l_2' = 0.5[f_y(1.25,1.0,-1.99)(0.25) + f_{y'}(1.25,1.0,-1.99)(1.25)] = 4.25$

$k_3' = 0.5(3.125) = 1.5625$

$l_3' = 0.5[f_y(1.25,1.0,-1.99)(0.313) + f_{y'}(1.25,1.0,-1.99)(3.125)] = 5\ .9375$

$k_4' = 0.5(6.938) = 3.46875$

$l_4' = 0.5[f_y(1.5,1.0,-1.99)(1.563) + f_{y'}(1.5,1.0,-1.99)(6.938)] = 30.4375$

$u_1 = 1.39063$

$u_2 = 9.55208$

$k_1 = 0.5(-0.41) = -0.20485$

$l_1 = 0.5f(1.5,0.515,-0.41) = 0.68488$

$k_2 = 0.5(-0.067) = -0.03363$

$l_2 = 0.5f(1.75,0.413,-0.067) = 0.47379$

$k_3 = 0.5(-0.173) = -0.08641$

$l_3 = 0.5f(1.75,0.499,-0.173) = 0.81819$

$k_4 = 0.5(0.408) = 0.20424$

$l_4 = 0.5f(2.0,0.429,0.408) = 0.73371$

$v_2 = 0.47528$

$w_2 = 0.25738$

$k_1' = 0.5(9.552) = 4.77604$

$l_1' = 0.5[f\_y(1.5,0.515,-0.41)(1.391) + f\_(y')(1.5,0.515,-0.41)(9.552)] =$
9.8332

$k_2' = 0.5(14.469) = 7.23434$

$l_2' = 0.5[f\_y(1.75,0.515,-0.41)(3.779) +$
$f\_(y')(1.75,0.515,-0.41)(14.469)] = 25.21244$

$k_3' = 0.5(22.158) = 11.07915$

$$l_3' = 0.5[f\_y(1.75,0.515,-0.41)(5.008) + f\_(y')(1.75,0.515,-0.41)(22.158)] = 34.26607$$

$$k_4' = 0.5(43.818) = 21.90908$$

$$l_4' = 0.5[f\_y(2.0,0.515,-0.41)(12.47) + f\_(y')(2.0,0.515,-0.41)(43.818)] = 90.45226$$

$$u_1 = 11.94264$$

$$u_2 = 46.09249$$

و در نهایت پاسخ های مسیله به صورت زیر بدست آمده اند که v_i تقریبی از مقدار تابع y در نقطه ی گره i ام است.

$$x_0 = 1.0 \qquad v_0 = 1.0$$

$$x_1 = 1.5 \qquad v_1 = 0.5154$$

$$x_2 = 2.0 \qquad v_2 = 0.47528$$

صورت معادله ی دیفرانسیل پاره ای بیضوی به صورت زیر است.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=6x-2 \qquad 1\le x\le 2\,,0\le y\le 1$$

$$u(x,1)=x^3 \qquad u(x,2)=x^3-1$$

$$u(0,y)=1-y^2 \qquad u(1,y)=8-y^2$$

حال با توجه به فرمول زیر برای $m=n=3$ ابتدا مقادیر زیر را حساب میکنیم.

$$w_{0,j}=g(x_0,y_j) \qquad w_{n,j}=g(x_n,y_j) \qquad j=0,1,2,3$$

$$w_{i,0}=g(x_i,y_0) \qquad w_{i,m}=g(x_i,y_m) \qquad i=1,2$$

$$w_{0,0}=1 \qquad w_{0,1}=\frac{8}{9} \qquad w_{0,2}=\frac{5}{9} \qquad w_{0,3}=0$$

$$w_{1,0}=\frac{64}{27} \qquad ? \qquad w_{1,3}=\frac{37}{27}$$

$$w_{2,0}=\frac{125}{27} \qquad w_{2,3}=\frac{98}{27}$$

$$w_{3,0}=8 \qquad w_{3,1}=\frac{71}{9} \qquad w_{3,2}=\frac{68}{9} \qquad w_{3,3}=7$$

اکنون با توجه به $h=k=\frac{1}{3}$ با استفاده از فرمول زیر به محاسبه ی مقادیر مجهول میپردازیم.

$$(2\left(\frac{h}{k}\right)^2+2)w_{i,j}-\left(w_{i+1,j}+w_{i-1,j}\right)-\left(\frac{h}{k}\right)^2(w_{i,j+1}+w_{i,j-1})=-h^2f(x_i,y_j)$$

که $\frac{h}{k}=1$ پس به ازای $i,j=1,2$ چهار معادله ی زیر را داریم.

$$4w_{1,1}-\left(w_{2,1}+w_{0,1}\right)-\left(w_{1,2}+w_{1,0}\right)=-\frac{1}{9}\left(6\times\frac{4}{3}-2\right)=-\frac{6}{9}$$

$$4w_{1,2}-\left(w_{2,2}+w_{0,2}\right)-\left(w_{1,3}+w_{1,1}\right)=-\frac{1}{9}\left(6\times\frac{4}{3}-2\right)=-\frac{6}{9}$$

$$4w_{2,1} - (w_{3,1} + w_{1,1}) - (w_{2,2} + w_{2,0}) = -\frac{1}{9}\left(6 \times \frac{5}{3} - 2\right) = -\frac{8}{9}$$

$$4w_{2,2} - (w_{3,2} + w_{1,2}) - (w_{2,3} + w_{2,1}) = -\frac{1}{9}\left(6 \times \frac{5}{3} - 2\right) = -\frac{8}{9}$$

که با جایگذاری مقادیر معلوم به دستگاه معادله ی زیر میرسیم.

$$4w_{1,1} - \left(w_{2,1} + \frac{8}{9}\right) - \left(w_{1,2} + \frac{64}{27}\right) = -\frac{6}{9}$$

$$4w_{1,2} - \left(w_{2,2} + \frac{5}{9}\right) - \left(\frac{37}{27} + w_{1,1}\right) = -\frac{6}{9}$$

$$4w_{2,1} - \left(\frac{71}{9} + w_{1,1}\right) - \left(w_{2,2} + \frac{125}{27}\right) = -\frac{8}{9}$$

$$4w_{2,2} - \left(\frac{68}{9} + w_{1,2}\right) - \left(\frac{98}{27} + w_{2,1}\right) = -\frac{8}{9}$$

$$+4w_{1,1} - w_{1,2} - w_{2,1} = \frac{70}{27}$$

$$-w_{1,1} + 4w_{1,2} - w_{2,2} = \frac{40}{27}$$

$$-w_{1,1} + 4w_{2,1} - w_{2,2} = \frac{287}{27}$$

$$-w_{1,2} - w_{2,1} + 4w_{2,2} = \frac{278}{27}$$

که با استفاده از روش عددی حل دستگاه با دقت 5 رقم بعد اعشار جواب های زیر بدست میاید.

$w_{0,0} = 1.00000$ $w_{0,1} = 0.88889$ $w_{0,2} = 0.55556$ $w_{0,3} = 0.00000$

$w_{1,0} = 2.37037$ $w_{1,1} = 2.19444$ $w_{1,2} = 1.94907$ $w_{1,3} = 1.37037$

$w_{2,0} = 4.62963$ $w_{2,1} = 4.23611$ $w_{2,2} = 4.12037$ $w_{2,3} = 3.62963$

$w_{3,0} = 8.00000$ $w_{3,1} = 7.88889$ $w_{3,2} = 7.55556$ $w_{3,3} = 7.00000$