به نام خدا

پاسخنامه تمرین سری ششم

ریاضیات مهندسی – دکتر حقی

ترم پاییز 1403 – دانشکده فنی دانشگاه تهران

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الف)

$$\mathcal{F}\{e^{-4|t|}\}=\frac{8}{16+(2\pi f)^2}\rightarrow\mathcal{F}\{e^{-4|t-5|}\}=\frac{8e^{-i10\pi f}}{16+(2\pi f)^2}$$

$$\rightarrow\mathcal{F}\{(-i2\pi t)^2e^{-4|t-5|}\}=\frac{d^2}{df^2}\left[\frac{8e^{-i10\pi f}}{16+(2\pi f)^2}\right]$$

$$\rightarrow\mathcal{F}\{t^2e^{-4|t-5|}\}=\frac{-1}{(2\pi)^2}\frac{d^2}{df^2}\left[\frac{8e^{-i10\pi f}}{16+(2\pi f)^2}\right]$$

ب)

$$X(f)=\int_{-\infty}^{\infty}x(t)e^{-i2\pi ft}dt=\int_{-1}^{0}(t+1)e^{-i2\pi ft}dt+\int_{0}^{1}(t-1)e^{-i2\pi ft}dt$$

$$=\left[\left(\frac{i}{2\pi f}t+\frac{1}{4\pi^2f^2}\right)e^{-i2\pi ft}\right]_{-1}^{0}+\left[\frac{i}{2\pi f}e^{-i2\pi ft}\right]_{-1}^{0}$$

$$+\left[\left(\frac{i}{2\pi f}t+\frac{1}{4\pi^2f^2}\right)e^{-i2\pi ft}\right]_{0}^{1}-\left[\frac{i}{2\pi f}e^{-i2\pi ft}\right]_{0}^{1}=\frac{1}{\pi f}(1-\sin(\pi f))$$

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الف) با توجه به خاصیت دوگانی داریم:

$$\mathcal{F}\{\Lambda(t)\} = sinc^2(f) \rightarrow \mathcal{F}\{sinc^2(t)\} = \Lambda(f)$$

$$\cos^2(\pi t)\, sinc^2(t) = \left(\frac{1}{2} + \frac{\cos(2\pi t)}{2}\right) sinc^2(t)$$

$$= \frac{1}{2} sinc^2(t) + \frac{1}{2}\cos(2\pi t)\, sinc^2(t)$$

$$\rightarrow X(f) = \mathcal{F}\left\{\frac{1}{2} sinc^2(t)\right\} + \mathcal{F}\{ sinc^2(t)\} * \mathcal{F}\left\{\frac{1}{2}\cos(2\pi t)\right\}$$

$$\rightarrow X(f) = \frac{1}{2}\Lambda(f) + [\Lambda(f)] * \left[\frac{1}{4}\delta(f-1) + \frac{1}{4}\delta(f+1)\right]$$

$$\rightarrow X(f) = \frac{1}{2}\Lambda(f) + \frac{1}{4}[\Lambda(f-1) + \Lambda(f+1)]$$

ب)

$$\mathcal{F}\{sinc^2(t)\} = \Lambda(f) \rightarrow \mathcal{F}\left\{sinc^2\left(\frac{t}{3}\right)\right\} = 3\Lambda(3f)$$

$$\rightarrow \mathcal{F}\left\{sinc^2\left(\frac{t-1}{3}\right)\right\} = 3\Lambda(3f)e^{-i2\pi f} \rightarrow \mathcal{F}\left\{\frac{1}{2} sinc^2\left(\frac{t-1}{3}\right)\right\} = \frac{3}{2}\Lambda(3f)e^{-i2\pi f}$$

ج) با توجه به خاصیت دوگانی داریم:

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{1}{i\pi f} \rightarrow \mathcal{F}\left\{\frac{1}{i\pi t}\right\} = -\mathrm{sgn}(f)$$

$$\mathcal{F}\left\{\frac{1 \times i}{i\pi t}\right\} = -\mathrm{sgn}(f) \times i \rightarrow \mathcal{F}\left\{\frac{1}{\pi t}\right\} = -i\ \mathrm{sgn}(f)$$

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الف) می‌دانیم:

$$\mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2 + (2\pi f)^2}$$

$$x(t) = e^{-2|t|} \rightarrow X(f) = \frac{4}{4(1+\pi^2 f^2)} \ (1)$$

همچنین داریم:

$$\int_{-\infty}^{\infty} X(f)df = x(0)$$

$$\int_{-\infty}^{\infty} \frac{2}{4(1+\pi^2 f^2)} df = \frac{1}{2}\int_{-\infty}^{\infty} \frac{4}{4(1+\pi^2 f^2)} = \frac{1}{2}x(0) = \frac{1}{2}e^0 = 0.5$$

ب)

$$\begin{cases} x(t) = \dfrac{2}{1+(2\pi t)^2} \xrightarrow{duality} X(f) = e^{-|f|} \\ y(t) = sinc(2t) \rightarrow Y(f) = \dfrac{1}{2}\Pi\left(\dfrac{f}{2}\right) \end{cases}$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df = \int_{-\infty}^{\infty} \frac{1}{2}e^{-|f|}\Pi\left(\frac{f}{2}\right)df = \int_{0}^{\infty} e^{-f}\Pi\left(\frac{f}{2}\right)df$$

$$= \int_0^1 e^{-f}df = -e^{-f}\Big|_0^1 = -e^{-1} + 1 = 1 - e^{-1}$$

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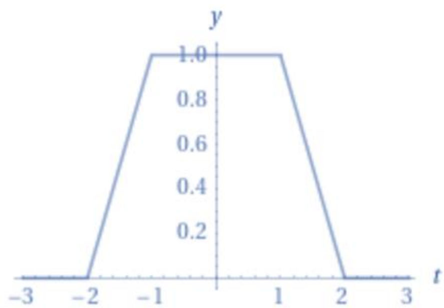
الف)

$$x(t)=\frac{\sin^2(2\pi t)-\sin^2(\pi t)}{t^2}= x(t)=\frac{4\pi^2\sin^2(2\pi t)}{4\pi^2t^2}-\frac{\pi^2\sin^2(\pi t)}{\pi^2t^2}$$
$$=4\pi^2 sinc^2(2t)-\pi^2 sinc^2 t$$

$$\rightarrow X(f)=4\times\pi^2\times\frac{1}{2}\Lambda\left(\frac{f}{2}\right)-\pi^2\times\Lambda(f)$$
$$=2\pi^2\Lambda\left(\frac{f}{2}\right)-\pi^2\Lambda(f)$$

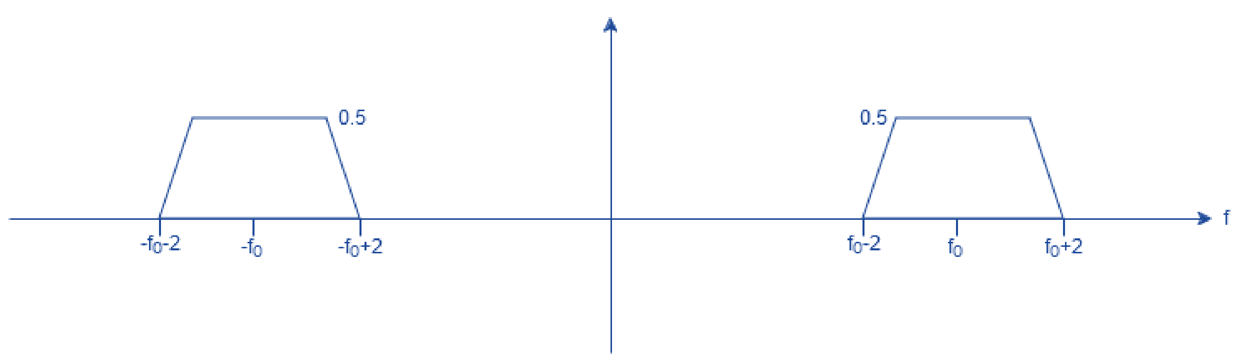


*** در بخش های بعدی، دامنه‌ی تمام سیگنال‌ها در π^2 ضرب گردد.***

ب)

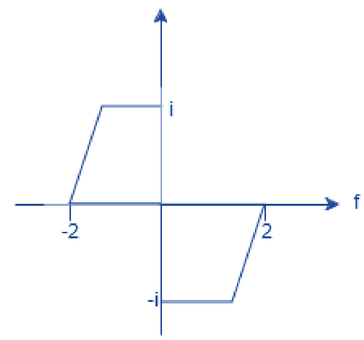
$$y(t)=x(t)\cos(2\pi f_0 t)\rightarrow Y(f)=X(f)*\frac{1}{2}\big(\delta(f-f_0)+\delta(f+f_0)\big)$$

$$\rightarrow Y(f)=\frac{1}{2}\big(X(f-f_0)+X(f+f_0)\big)$$



ج)

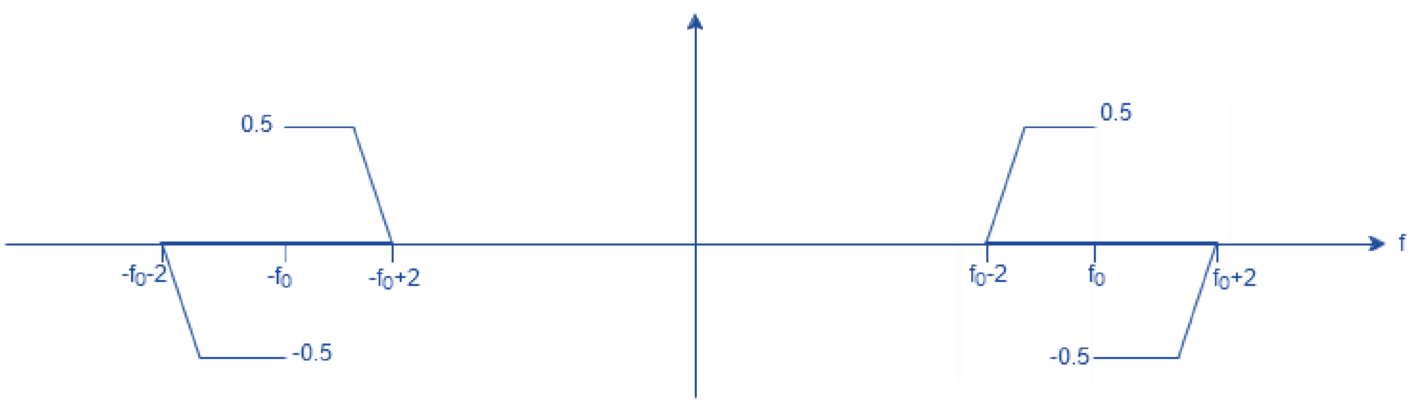
$$z(t) = \frac{1}{\pi t} * x(t) \xrightarrow{\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -i\,\mathrm{sgn}(f)} Z(f) = -i\,\mathrm{sgn}(f) X(f)$$



د)

$$w(t) = z(t)\sin(2\pi f_0 t) \rightarrow W(f) = Z(f) * \frac{1}{2i}\left(\delta(f - f_0) - \delta(f + f_0)\right)$$

$$\rightarrow W(f) = \frac{1}{2i}\left(Z(f - f_0) - Z(f + f_0)\right)$$



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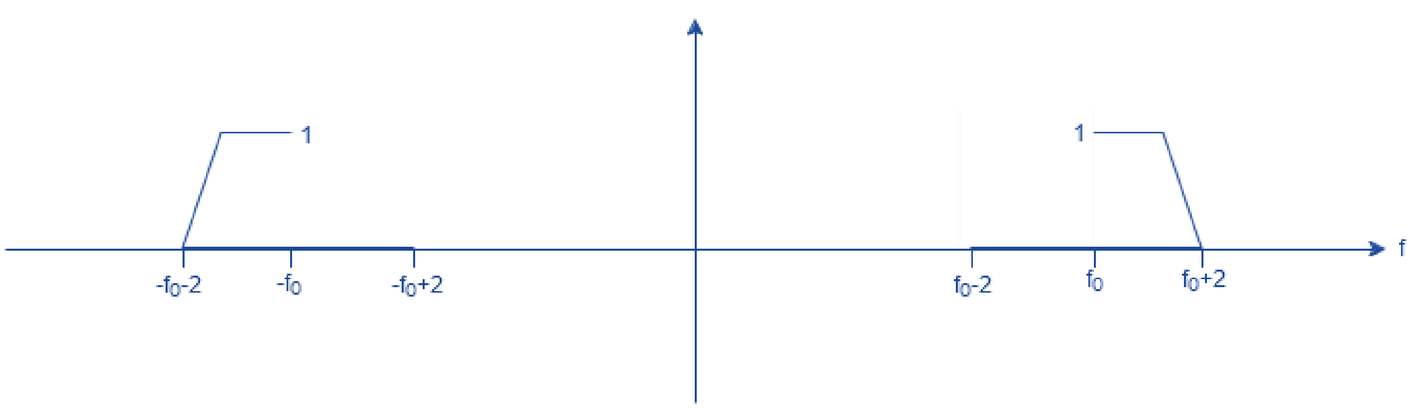

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الف)

$$X(f)|_{f=0} = \int_{-\infty}^{\infty} x(t)dt = 13$$

ب)

$$\int_{-\infty}^{\infty} X(f)df = x(0) = 2$$

ج)

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt = 8 + 9 + 8 + \int_{4}^{6} (6-t)^2 dt = \frac{83}{3}$$

تهیه و تنظیم: امیرمرتضی رضائی