

دانشگاه تهران

دانشکده مهندسی برق و کامپیوتر

ریاضیات مهندسی - بهار ۱۴۰۴

تمرین سری 5

دستیار تمرین: امیرمرتضی رضائی

سوال اول

(الف)

$$x(t) = \Pi\left(\frac{t}{2}\right) (1 + \cos(5\pi t)) = \Pi\left(\frac{t}{2}\right) + \Pi\left(\frac{t}{2}\right) \cos(5\pi t)$$

$$\begin{aligned}\mathcal{F}\{x(t)\} &= \mathcal{F}\left\{\Pi\left(\frac{t}{2}\right)\right\} + \mathcal{F}\left\{\Pi\left(\frac{t}{2}\right)\right\} * \mathcal{F}\{\cos(5\pi t)\} \\ &= 2\text{sinc}(2f) + 2\text{sinc}(2f) * \left\{\frac{1}{2}\delta(f - 2.5) + \frac{1}{2}\delta(f + 2.5)\right\} \\ &= 2\text{sinc}(2f) + \text{sinc}(2f - 5) + \text{sinc}(2f + 5)\end{aligned}$$

(ب)

$$y(t) = \sin^2(4\pi t) \text{sinc}^2\left(\frac{3t-2}{5}\right) = \left(\frac{1 - \cos(8\pi t)}{2}\right) \text{sinc}^2\left(\frac{3t-2}{5}\right)$$

$$= \frac{1}{2} \text{sinc}^2\left(\frac{3t-2}{5}\right) - \frac{1}{2} \cos(8\pi t) \text{sinc}^2\left(\frac{3t-2}{5}\right)$$

$$\rightarrow \mathcal{F}\{y(t)\} = \mathcal{F}\left\{\frac{1}{2} \text{sinc}^2\left(\frac{3t-2}{5}\right)\right\} - \mathcal{F}\left\{\text{sinc}^2\left(\frac{3t-2}{5}\right)\right\} * \mathcal{F}\left\{\frac{1}{2} \cos(8\pi t)\right\}$$

$$\begin{aligned}\mathcal{F}\{\text{sinc}^2(t)\} &= \Lambda(f) \rightarrow \mathcal{F}\left\{\text{sinc}^2\left(\frac{t}{5}\right)\right\} = 5\Lambda(5f) \rightarrow \mathcal{F}\left\{\text{sinc}^2\left(\frac{t-2}{5}\right)\right\} = 5\Lambda(5f) \cdot e^{-i4\pi f} \\ &\rightarrow \mathcal{F}\left\{\text{sinc}^2\left(\frac{3t-2}{5}\right)\right\} = \frac{5}{3}\Lambda\left(\frac{5f}{3}\right) \cdot e^{-i\frac{4}{3}\pi f} \rightarrow \mathcal{F}\left\{\frac{1}{2} \text{sinc}^2\left(\frac{3t-2}{5}\right)\right\} \\ &= \frac{5}{6}\Lambda\left(\frac{5f}{3}\right) \cdot e^{-i\frac{4}{3}\pi f}\end{aligned}$$

$$\rightarrow \mathcal{F}\{y(t)\} = \frac{5}{6}\Lambda\left(\frac{5f}{3}\right) \cdot e^{-i\frac{4}{3}\pi f} - \left(\frac{5}{6}\Lambda\left(\frac{5f}{3}\right) \cdot e^{-i\frac{4}{3}\pi f}\right) * \left\{\frac{1}{2}\delta(f-4) + \frac{1}{2}\delta(f+4)\right\}$$

$$= \frac{5}{6}\Lambda\left(\frac{5f}{3}\right) \cdot e^{-i\frac{4}{3}\pi f} - \frac{5}{12}\left(\Lambda\left(\frac{5(f-4)}{3}\right) \cdot e^{-i\frac{4}{3}\pi(f-4)} + \Lambda\left(\frac{5(f+4)}{3}\right) \cdot e^{-i\frac{4}{3}\pi(f+4)}\right)$$

(ج)

$$\begin{aligned} G(\omega) &= A \frac{1}{|\frac{1}{W}|} D\left(\frac{\omega}{\frac{1}{W}}\right) e^{-j\omega t_0} \\ &= AW D(W\omega) e^{-j\omega t_0} \\ &= \frac{AW}{2} \text{sinc}^2(W\omega/4) e^{-j\omega t_0} \end{aligned}$$

سوال دوم

(الف)

$$\left. \begin{aligned} \mathcal{F}\{\text{sinc}^2(t)\} &= \Lambda(f) \\ \mathcal{F}\{tx(t)\} &= \frac{i}{2\pi} \frac{d}{df} X(f) \end{aligned} \right\} \rightarrow \mathcal{F}\{t \cdot \text{sinc}^2(t)\} = \frac{i}{2\pi} \frac{d}{df} \Lambda(f) = \frac{i}{2\pi} \cdot \begin{cases} 1 & -1 \leq f \leq 0 \\ -1 & 0 \leq f \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

طبق پارسوال داریم:

$$\int_{-\infty}^{+\infty} x^2(t) dt = \int_{-\infty}^{+\infty} |X^2(f)| df \rightarrow \int_{-\infty}^{+\infty} (t \cdot \text{sinc}^2(t))^2 dt = \int_{-1}^{+1} \frac{1}{4\pi^2} df = \frac{1}{2\pi^2}$$

(ب) می دانیم:

$$\begin{aligned} \mathcal{F}\{e^{-a|t|}\} &= \frac{2a}{a^2 + (2\pi f)^2} \\ x(t) = e^{-2|t|} &\rightarrow X(f) = \frac{4}{4(1 + \pi^2 f^2)} \quad (1) \end{aligned}$$

همچنین داریم:

$$\begin{aligned} \int_{-\infty}^{\infty} X(f) df &= x(0) \\ \int_{-\infty}^{\infty} \frac{1}{2(1 + \pi^2 f^2)} df &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{4}{4(1 + \pi^2 f^2)} = \frac{1}{2} x(0) = \frac{1}{2} e^0 = 0.5 \end{aligned}$$

سوال سوم

می‌دانیم:

$$x^*(t) = \mathcal{F}^{-1}\{X^*(-f)\} \rightarrow \mathcal{F}\{x^*(t)\} = X^*(-f)$$

پس برای $x(t)$ های حقیقی:

$$x(t) = x^*(t) \rightarrow X(f) = X^*(-f)$$

و برای $x(t)$ های موهومی:

$$x(t) = -x^*(t) \rightarrow X(f) = -X^*(-f)$$

همچنین داریم:

$$x(-t) = \mathcal{F}^{-1}\{X(-f)\} \rightarrow \mathcal{F}\{x(-t)\} = X(-f)$$

پس برای $x(t)$ های زوج:

$$x(t) = x(-t) \rightarrow X(f) = X(-f)$$

و برای $x(t)$ های فرد:

$$x(t) = -x(-t) \rightarrow X(f) = -X(-f)$$

حال برای تابع $f(t)$ داریم:

$$F^*(-f) = (-4\pi^2 i f e^{-2\pi|f|})^* = 4\pi^2 i f e^{-2\pi|f|} = F(f)$$

$$F(-f) = -4\pi^2 i f e^{-2\pi|f|} = -4\pi^2 i f e^{-2\pi|f|} = -F(f)$$

بنابراین $f(t)$ تابعی حقیقی و فرد می‌باشد. همچنین برای هر تابع فرد داریم:

$$f(0) = -f(-0) = -f(0) = 0.$$

برای محاسبه تبدیل فوریه معکوس داریم:

$$\mathcal{F}\{e^{-\lambda|t|}\} = \frac{2\lambda}{\lambda^2 + (2\pi f)^2} \xrightarrow{\text{duality}} \mathcal{F}\left\{\frac{2\lambda}{\lambda^2 + (2\pi(-t))^2}\right\} = e^{-\lambda|f|}$$

$$\rightarrow \mathcal{F}^{-1}\{e^{-2\pi|f|}\} = \frac{2 \cdot 2\pi}{4\pi^2 + (2\pi t)^2} = \frac{1}{\pi + \pi t^2}$$

$$\rightarrow \mathcal{F}^{-1}\{2\pi e^{-2\pi|f|}\} = \frac{2}{1 + t^2}$$

$$\rightarrow \mathcal{F}^{-1}\{i2\pi f \cdot 2\pi e^{-2\pi|f|}\} = \frac{d}{dt} \frac{2}{1 + t^2} = \frac{-4t}{(1 + t^2)^2} = f(t)$$

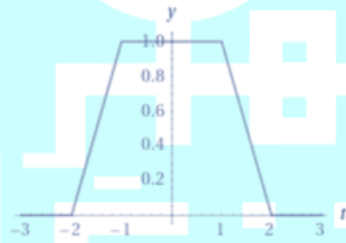
همانطور که مشخص است، $f(t)$ تابعی حقیقی و زوج است و در $t = 0$ مقدار 0 دارد.

سوال چهارم

(الف)

$$x(t) = \frac{\sin^2(2\pi t) - \sin^2(\pi t)}{\pi^2 t^2} = x(t) = \frac{4\sin^2(2\pi t)}{4\pi^2 t^2} - \frac{\sin^2(\pi t)}{\pi^2 t^2} = 4\text{sinc}^2(2t) - \text{sinc}^2 t$$

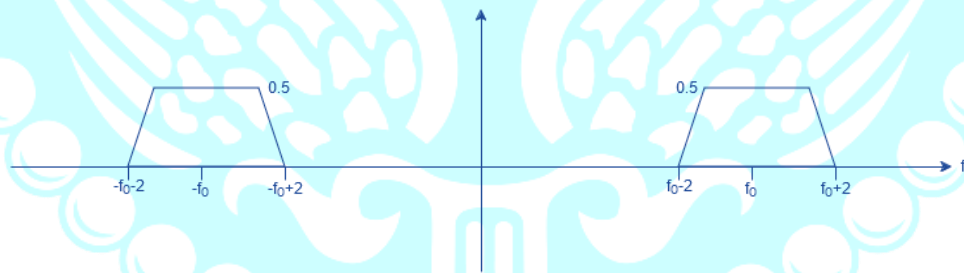
$$\rightarrow X(f) = 4 \times \frac{1}{2} \Lambda\left(\frac{f}{2}\right) - \Lambda(f) = 2\Lambda\left(\frac{f}{2}\right) - \Lambda(f)$$



(ب)

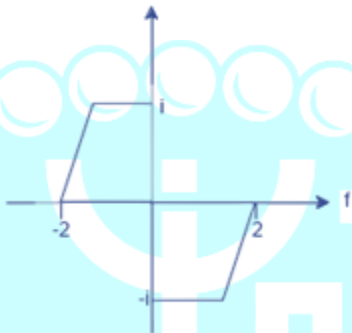
$$y(t) = x(t) \cos(2\pi f_0 t) \rightarrow Y(f) = X(f) * \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$$

$$\rightarrow Y(f) = \frac{1}{2}(X(f - f_0) + X(f + f_0))$$



(ج)

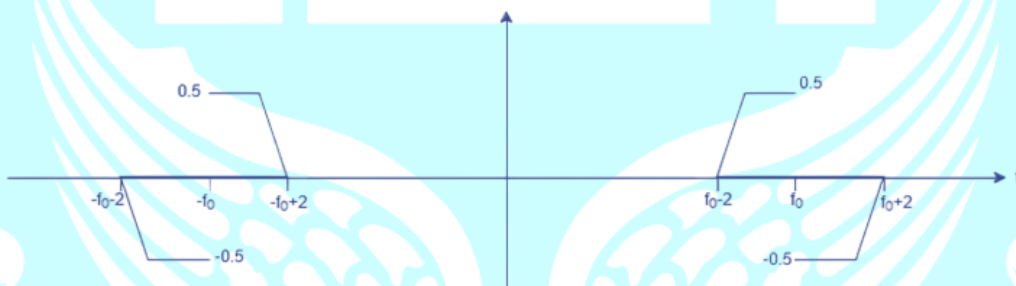
$$z(t) = \frac{1}{\pi t} * x(t) \xrightarrow{\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -i \operatorname{sgn}(f)} Z(f) = -i \operatorname{sgn}(f) X(f)$$



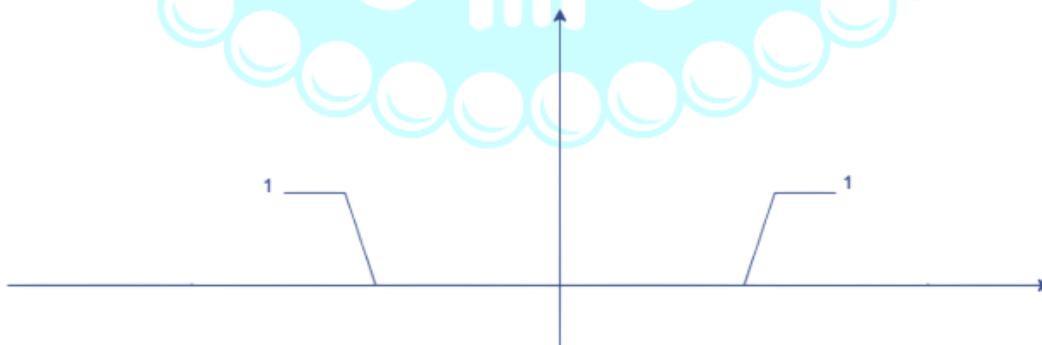
(د)

$$w(t) = z(t) \sin(2\pi f_0 t) \rightarrow W(f) = Z(f) * \frac{1}{2i} (\delta(f - f_0) - \delta(f + f_0))$$

$$\rightarrow W(f) = \frac{1}{2i} (Z(f - f_0) - Z(f + f_0))$$



(هـ)



سوال پنجم

متن سوال

(الف)

$$X(f)|_{f=0} = \int_{-\infty}^{\infty} x(t)dt = 5.5$$

(ب)

$$\int_{-\infty}^{\infty} X(f)df = x(0) = 2$$

(ج)

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-1}^0 (t+1)^2 dt + 8 + 1 = \frac{28}{3}$$