In The Name of God.
The Merciful, The Compassionate.

Basis, 4 Fundamental Subspaces

notes on Gilbert Strang videos, Lecture 09,10,11

1 Linear independence, Spanning a space, Basis, and Dimension

- independence is obvious!
- Vectors $\{v_1, \dots v_n\}$ span a space means: The space consists all combinations of those vectors.
- **Basis**: Basis for a space is a sequence of vectors $\{v_1, \dots, v_d\}$ that has two properties:
  1. They are independent
  2. They span a space
- Given a space, every basis for the space has the same number of vectors: This number is the dimension of the space. (proof, see Carlo Tomasi's note)
- $Rank(A) = \#$pivot columns = dimension of the column space
- $dim(C(A)) = r, dim(N(A)) = \#$free variables $= n - r$

2 The 4 fundamental subspaces

- 4 subspaces:
  - $A$is a$m \times n$ Matrix
  - $C(A), N(A), N(A^T)$ or left $N(A), R(A) = C(A^T)$
- $\mathbb{R}^n : R(A) \perp N(A)$
- $\mathbb{R}^m : C(A) \perp N(A^T)$
- $C(rref(A)) \neq C(A)$

- $R(rref(A)) = R(A)$, row operations $\Longrightarrow dim(R(A)) = r$
- $y^T A = \mathbf{0}^T$, y is in left nullspace of A,

  $rref(\begin{bmatrix} A_{m\times n} & \mathcal{I} \end{bmatrix}) \longrightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m} \end{bmatrix} := E \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} R & E \end{bmatrix}$
  $\Longrightarrow EA = R$,

  if A was square and invertible: $R = I \Rightarrow E = A^{-1}$, rows of E corresponding to zero rows of R are basis for $N(A.T)$, because they are row vectors producing zero when multiplied by A.

- New vector spaces: $M =$ all $m\, by\, m$ matrices.
- Subspaces of M:
  - All upper triangular matrices (U)
  - All symmetric matrices (S)
  - Diagonal matrices: $D = S \cap U$: dim(D) = m
  - $S + U$:any element in $S+$ any element in U: all $m\, by\, m$ matrices: dim=m^2
  - $dim(S) + dim(U) - dim(S \cap U) = dim(S \cup U)$
- Rank one matrices can be written as $A = \vec{u}.\vec{v}^T$

	$C(A)$	$N(A)$	$R(A)$	$N(A^T)$
Basis:	Pivot Cols	Special Sol'n	First r rows of $R := rref(A)$	rows of E corresponding to zero rows of R
DIM:	r	$n - r$	r	$m - r$