

In The Name of God.
The Merciful, The Compassionate.

Basis, 4 Fundamental Subspaces

notes on Gilbert Strang videos, Lecture 09,10,11

1 Linear independence, Spanning a space, Basis, and Dimension

- independence is obvious!
- Vectors $\{v_1, \dots, v_n\}$ span a space means: The space consists all combinations of those vectors.
- **Basis:** Basis for a space is a sequence of vectors $\{v_1, \dots, v_d\}$ that has two properties:
 1. They are independent
 2. They span a space
- Given a space, every basis for the space has the same number of vectors: This number is the dimension of the space. (proof, see Carlo Tomasi's note)
- $Rank(A) = \# \text{pivot columns} = \text{dimension of the column space}$
- $\dim(C(A)) = r, \dim(N(A)) = \# \text{free variables} = n - r$

2 The 4 fundamental subspaces

- 4 subspaces:
 - A is $m \times n$ Matrix
 - $C(A), N(A), N(A^T)$ or left $N(A), R(A) = C(A^T)$
- $\mathbb{R}^n : R(A) \perp N(A)$
- $\mathbb{R}^m : C(A) \perp N(A^T)$
- $C(rref(A)) \neq C(A)$

- $R(rref(A)) = R(A)$, row operations $\implies \dim(R(A)) = r$
- $y^T A = \mathbf{0}^T$, y is in left nullspace of A ,

$$rref([A_{m \times n} \quad I]) \longrightarrow [R_{m \times n} \quad E_{m \times m}] := E [A \quad I] = [R \quad E] \\ \implies EA = R,$$

if A was square and invertible: $R = I \Rightarrow E = A^{-1}$, rows of E corresponding to zero rows of R are basis for $N(A.T)$, because they are row vectors producing zero when multiplied by A .

- New vector spaces: M = all m by m matrices.
- Subspaces of M :
 - All upper triangular matrices (U)
 - All symmetric matrices (S)
 - Diagonal matrices: $D = S \cap U$: $\dim(D) = m$
 - $S + U$: any element in S + any element in U : all m by m matrices: $\dim = m^2$
 - $\dim(S) + \dim(U) - \dim(S \cap U) = \dim(S \cup U)$

	$C(A)$	$N(A)$	$R(A)$	$N(A^T)$
Basis:	Pivot Cols	Special Sol'n	First r rows of $R := rref(A)$	rows of E corresponding to zero rows of R
DIM:	r	$n - r$	r	$m - r$