

In The Name of God.  
The Merciful, The Compassionate.

# Symmetric Matrices, Positive Definiteness

notes on Gilbert Strang videos, Lecture 25

## 1 Symmetric Matrices: $A = A^T$

1. The eigenvalues are real:

$$\begin{aligned}Ax = \lambda x &\Rightarrow \bar{A}\bar{x} = \bar{\lambda}\bar{x} = A\bar{x} = \bar{\lambda}\bar{x} \\&\Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda} = \bar{x}^T A \\&\Rightarrow \bar{x}^T Ax = \bar{\lambda}\bar{x}^T x \\Ax = \lambda x &\Rightarrow \bar{x}^T Ax = \lambda\bar{x}^T x \\&\Rightarrow \lambda\bar{x}^T x = \bar{\lambda}\bar{x}^T x \\&\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \text{ is real}\end{aligned}$$

- Note that  $\bar{x}^T x$  is  $\geq 0$ .
- If  $A$  is complex: the proof is still working if  $\bar{A}^T = A$
- Good matrices are conjugate transpose.

2. The eigenvectors are (can be chosen) orthogonal  
 $A = S\Lambda S^{-1} \rightarrow Q\Lambda Q^T$  Assuming orthonormal eigenvectors

$$\begin{aligned}A = Q\Lambda Q^T &= \begin{bmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \\&= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots\end{aligned}$$