In The Name of God.
The Merciful, The Compassionate.

Positive Definite Matrices, Minima, Ellipsoids, and Similar Matrices

notes on Gilbert Strang videos, Lecture 27,28

1 Positive Definite Matrices

- Positive Definiteness Test: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$
  1. Eigen values must be greater than zero $\lambda_1 > 0, \lambda_2 > 0$
  2. Sub-determinants should be greater than zero: $a > 0, ac - b^2 > 0$
  3. Pivots should be greater than zero: $a > 0, \frac{ac-b^2}{a} > 0$
  4. Definition: $x^T A x > 0$ at all points except at $x = 0$
- Semi-definite: all $>$ becomes $\geq$.
- $x^T A x = ax^2 + 2bxy + cy^2, x = \begin{bmatrix} x \\ y \end{bmatrix}$
- Cutting this equation at level one: gives an ellipse
- MIN:
  - Calculus: $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} > 0$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 0$
  - Linear Algebra: min of $f(x_1, x_2, \ldots, x_n)$: Matrix of second derivatives should be positive definite.
- Matrix of second derivatives: $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$
- In higher dimensions, (e.g dim=3) : cutting at level one gives an ellipsoid.
- $A = Q\Lambda Q^T$. So, the ellipoid has n principal axes parallel to eigenvectors, and their scales are given by eigenvalues.
- Some notes about positive definite (PD) matrices:
  - if A is PD: A^{-1} is also PD. (eigenvalues are positive)

- if A and B are PD matrices: $A+B$ is also PD. ($x^T Ax > 0$ and $x^T Bx > 0$)
- if A is a m-by-n matrice: $A^T A$ is symmetric, square, and positive semi-definite!
$$x^T A^T Ax = \|Ax\|^2 \geq 0$$
- if A has rank n (n independent columns, or $N(A) = \{\}$), $A^T A$ becomes PD.

2 Similar Matrices

- Two n-by-n matrices A and B are **similar** means: for some invertible matrice M, $B = M^{-1}AM$.
- Similar matrices have the same λ's!!

$$\begin{aligned} Ax &= \lambda x \\ AMM^{-1}x &= \lambda x \\ M^{-1}AMM^{-1}x &= \lambda M^{-1}x \\ BM^{-1}x &= \lambda M^{-1}x \end{aligned}$$

- Eigenvecotrs of B is M^{-1}(eigenvectors of A)
- BAD CASE Example: $\lambda_1 = \lambda_2 = 4$:

1. one small family: $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
$$M^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$
It is only similar to itself!
2. big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \leftarrow$ Jordan From
3. More members of family: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$