In The Name of God.
The Merciful, The Compassionate.

Symmetric Matrices, Positive Definiteness

notes on Gilbert Strang videos, Lecture 25

1 Symmetric Matrices: $A = A^T$

1. The eigenvalues are real:

$$\begin{aligned} Ax = \lambda x &\Rightarrow \bar{A}\bar{x} = \bar{\lambda}\bar{x} = A\bar{x} = \bar{\lambda}\bar{x} \\ &\Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda} = \bar{x}^T A \\ &\Rightarrow \bar{x}^T A x = \bar{\lambda}\bar{x}^T x \\ Ax = \lambda x &\Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x \\ &\Rightarrow \lambda \bar{x}^T x = \bar{\lambda}\bar{x}^T x \\ &\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \text{ is real} \end{aligned}$$

   - Note that $\bar{x}^T x$ is ≥ 0.
   - If A is complex: the proof is still working if $\bar{A}^T = A$
   - Good matrices are conjugate transpose.

2. The eigenvectors are (can be chosen) orthogonal:

   - $A = S\Lambda S^{-1} \to Q\Lambda Q^T$ Assuming orthonormal eigenvectors

$$\begin{aligned} A = Q\Lambda Q^T &= \begin{bmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} {q_1}^T \\ \vdots \\ {q_n}^T \end{bmatrix} \\ &= \lambda_1 q_1 {q_1}^T + \lambda_2 q_2 {q_2}^T + \dots \end{aligned}$$

   - Each $q_1 {q_1}^T$ is a projection matrix.
   - Every symmetric matrix is a combination of (mutually) perpendicular projection matrices.
   - For symmetric matrices: the signs of the pivots are the same as the sign's of λ's

2 Positive definite (symmetric) matrix

- all the eigenvalues are positive
- all the pivots are positive
- all sub-determinant are positive