In The Name of God.
The Merciful, The Compassionate.

Linear Transformations, Change of basis

notes on Gilbert Strang videos, Lecture 30-32

1 Linear Transformations

- $T(cV + dW) = cT(V) + dT(w)$
- $T(0) = 0$
- $T(x) = Ax$ is a linear transformation
- Coordinates come from a basis: coordinates of $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ are $c_1, c_2, \cdots$
- Construct matrix A that represents linear transformation T:
  - $T : \mathbb{R}^n \to \mathbb{R}^m$
  - Choose bases $v_1, \cdots, v_n$ for inputs.
  - Choose bases $w_1, \cdots, w_m$ for outputs.
- Rule to find A given bases $v_1, \cdots, v_n$ and $w_1, \cdots, w_m$:
  - first column of A: write $T(v_1) = a_{11}w_1 + a_{21}w_2 + \cdots a_{m1}w_m$
  - second column of A: write $T(v_2) = a_{12}w_1 + \cdots a_{m2}w_m$
  - $A\,[\text{input coordinates}] = [\text{output coordinates}]$
- Example: $T = \frac{\mathrm{d}}{\mathrm{d}x}$
  - Input: $c_1 + c_2x + c_3x^2$, basis: $1, x, x^2$
  - Output: $c_2 + 2c_3x$, basis: $1, x$
  - $A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$
  - $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2 Change of basis

- Columns of W = new basis vectors
- $[x]_{\text{old basis}} \rightarrow [c]_{\text{new basis}} : x = Wc$

3 Transformation Matrices

- Transformation matrices with respect to different bases:
  - T with respect to $v_1, \cdots, v_8$ it has matrix A
  - with respect to $w_1, \cdots, w_8$ it has matrix B
  - A is similar to B, $A = M^{-1}BM$, M is the change of basis vector
- What is A? using $v_1, \ldots, v_8$.
  - know T completely from $T(v_1), T(v_2), \cdots, T(v_8)$.
  - Because every $x = c_1v_1 + c_2v_2 + \cdots, c_8v_8$, Then $T(x) = c_1T(v_1) + \cdots$.
    * Write $T(v_1) = a_{11}v_1 + a_{21}v_2 + \cdots + a_{81}v_8$
    * $T(v_2) = a_{12}v_1 + a_{22}v_2 + \cdots + a_{82}v_8$
- Eigenvector basis:
  - $T(v_i) = \lambda_i v_i$
  - $\Rightarrow A = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \lambda_n \end{bmatrix}$

A note: When we have orthogonal eigenvectors: $AA^T = A^TA$
symmetric, skew symmatric (i.e., $A^T = -A$), and orthogonal matrices pass this test!