In The Name of God.
The Merciful, The Compassionate.

Left and right inverse, pseudoinverse

notes on Gilbert Strang videos, Lecture 33

1 Left and right inverse

2-sided inverse: $AA^{-1} = I = A^{-1}A \longrightarrow r = m = n$ (full rank)

- Left inverse:
  - full column rank $r = n$
  - nullspace $= \{\}$
  - independent columns, zero or 1 solution to $Ax = b$
  - $A^T A$ is a n-by-n full rank matrix
  - $(A^T A)^{-1} A^T$ is A^{-1}_{left}
  - $A^{-1}_{\text{left}} A = I_{n \times n}$
- Right inverse:
  - full row rank $r = m < n$
  - $n(A^T) = \{\}$, independent rows
  - ∞ solutions to $Ax = b$, $n - r$ free variables
  - $A^T (AA^T)^{-1}$ is A^{-1}_{right}
  - $AA^{-1}_{\text{rigth}} = I_{m \times m}$
- $A(A^T A)^{-1} A^T = P$ is a projection onto columnspace, and $A^T (AA^T)^{-1} A = P'$ is a projection onto rowspace.

2 Pseudoinverse $(r < m, r < n)$

- If x, y are two different vectors in rowspace, then: $Ax \neq Ay$.(proof. by supposing they are the same. So, $x - y$ should be in nullspace, but it is a subtract of two vectors in rowspace! So, it is the zero vector)
- Find pseudoinverse A^+:

1. Start from SVD: $A = U\Sigma V^T$,
   - $\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}_{m\times n}$
   - $\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & 0 \\ & \ddots & & \\ & & 1/\sigma_r & \\ 0 & & & 0 \end{bmatrix}_{n\times m}$
   - $\Sigma\Sigma^+ = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}_{m\times m}$
   - $\Sigma^+\Sigma = \begin{bmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$
2. $A^+ = V\Sigma^+U^T$