In The Name of God.
The Merciful, The Compassionate.

Orthogonal Vectors and Subspaces, Projecting into subspaces

notes on Gilbert Strang videos, Lecture 14-16

1 Orthogonality

- Columnspace $\perp$ nullspace of A^T
- Orthogonal vectors: $x^T y = 0$
- Subspace $\mathcal{S}$ is orthogonal to subspace $\mathcal{T}$ means: every vector in $\mathcal{S}$ is perpendicular to every vector in $\mathcal{T}$
- Rowspace $\perp$ nullspace:
  $Ax = 0$, $\begin{bmatrix} \text{———} & row_1 & \text{———} \\ \text{———} & row_2 & \text{———} \\ & \vdots & \\ \text{———} & row_m & \text{———} \end{bmatrix} x = 0$, so, x is orthogonal to each row. Easy to see row combinations is also orthogonal.
- Rowspace and nullspace are orthogonal complement: nullspace contains all vectors $\perp$ to row space.

2 Projection

- Projection of a vector b onto a line a:
  - p is the projection of b
  - e is error, $e = b - p$, $p \parallel a$
  - $\Rightarrow p = xa$
  - $a^T(b - xa) = 0 => xa^T a = a^T b \Rightarrow x = \frac{a^T b}{a^T a}, p = a\frac{a^T b}{a^T a}$
  - Projection Matrix $= Pb, P = \frac{aa^T}{a^T a}$
  - $C(P)$: line through $a, rank(P) = 1$
  - Two properties of projection matrices:

1. $P^T = P$
2. $P^2 = P$

- Solve $Ax = b$ when there is no solution (e.g., $m > n$):
  - $A^T A$ is $n \times n$ and symmetric,
  - $A^T A\hat{x} = A^T b$, $\hat{x}$ shows estimation
  - $N(A^T A) = N(A)$
  - $rank(A^T A) = rank(A) \Rightarrow A^T A$ is invertible if A has independent columns
  - solve $A\hat{x} = p$, p is projection of b onto columnspace
  - again $e = b - p$
  - $b - A\hat{x}$ is $\perp$ to columnspace $\Rightarrow A^T(b - A\hat{x}) = 0$, (imagine columns of $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix}$ are its bases $\Rightarrow$ each of its columns must be $\perp$ to e. So, $a_1^T e = 0, \dots a_n^T e = 0 \Longrightarrow A^T e = 0$)
  - e is in nullspace of $A^T \Longrightarrow e \perp C(A)$, YES!
  - So, $A^T A\hat{x} = A^T b \Longrightarrow \hat{x} = (A^T A)^{-1} A^T b$
  - $p = A\hat{x} = A(A^T A)^{-1} A^T b, P = A(A^T A)^{-1} A^T$
  - If b is in columnspace: $Pb = b \Longleftarrow b$ has the form Ax, substitute by formulae of P
  - If $b \perp$ columnspace: $Pb = 0 \Longleftarrow b$ is in nullspace of A^T, substitute by formulae of P
  - $p + e = b, p = Pb \Rightarrow e = (I - P)b$,
  - $I - P$: proj to perpendicular space
- Least Squares: e.g., 2D:
  - $y = C + Dt$, and we have (a_i, b_i) for i in $[1 - 3]$.
  - Equations are: $C + Da_i = b_i \Longrightarrow Ax = b$, $\begin{bmatrix} \vdots & \vdots \\ 1 & a_i \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \vdots \\ b_i \\ \vdots \end{bmatrix}$: no solution
  - Minimize: $\|Ax - b\|^2 = \|e\|^2 = \sum_i e_i^2$

- p_i are points on the line: $p + e = b$ (if p was in the place of b, the equation could be solved. So, it is in the columnspace of A.
- to find p, $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$:
  $A^T A\hat{x} = A^T b \rightarrow$ find $\hat{x} \rightarrow$ best line $\rightarrow p_i \rightarrow e_i$
- Note: e is perpendicular to p and columns of A

- If A has independent columns, then $A^T A$ is invertible:

  1. Suppose $A^T Ax = 0$, So, x must be zero (nullspace is only the zero vector)
  2. IDEA: $x^T A^T Ax = 0 \Rightarrow (Ax)^T (Ax) = 0 \Rightarrow Ax = 0$ $\xRightarrow{\text{columns of } A \text{ are independent}} x = 0$