In The Name of God.
The Merciful, The Compassionate.

Singular Value Decomposition (SVD)

notes on Gilbert Strang videos, Lecture 29

1 Singular Value Decomposition

- $A = U\Sigma V^T$
- U, V are orthogonal matrices, Σ is diagonal.
- A can be any matrix. v_1 is a vector in rowspace. u_1 is a vector in columnspace. $u_1 = Av_1$. In SVD we need an orthogonal basis in rowspace get transformed to the orthogonal basis in columnspace. $\rightarrow$ $u_2 = Av_2, u_1 \perp u_2, v_1 \perp v_2, \ldots$. In order to have normalized bases we must have: $\sigma_1 u_1 = Av_1, \ldots$:

$$A \begin{bmatrix} | & | & & | & | & & | \\ v_1 & v_2 & \ldots & v_r & v_{r+1} & \ldots & v_n \\ | & | & & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | & & & | \\ u_1 & u_2 & \ldots & u_r & u_{r+1} & \ldots & u_m \\ | & | & & | & & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & \ldots & & & 0 \\ & \ddots & & & \\ & \ldots & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots \end{bmatrix}$$

$$\begin{aligned} AV &= U\Sigma \\ A &= U\Sigma V^T \\ A^T A &= V\Sigma^T U^T U \Sigma V^T \\ &= V \begin{bmatrix} \sigma_1^2 & \ldots & & 0 \\ 0 & \sigma_2^2 & \ldots & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & \end{bmatrix} \end{aligned}$$

- $A^T A = V\Sigma^2 V^T$, so, V's are eigenvectors of $A^T A$ and σ^2's are eigenvalues of that matrix.
- Similarly, U's are eigenvectors of AA^T.
- $v_1, \ldots, v_r$ are orthonormal basis for rowspace

- $u_1, \ldots, u_r$ are orthonormal basis for columnspace
- $v_{r+1}, \ldots, v_n$ are orthonormal basis for nullspace
- $u_{r+1}, \ldots, u_n$ are orthonormal basis for $N(A^T)$