In The Name of God.
The Merciful, The Compassionate.

Linear Equations

notes on Gilbert Strang videos, Lecture 07,08

1 Solutions to $Ax = b$

- Rank r is the number of pivots in elimination.
- Number of free variables equals to the number of columns without pivot.
- $R = [IF; 00]$, reduced echelon form.
- N is null space matrix (columns are special sol'n): $RN = 0 \Rightarrow N = [-F; I]$
- $Rx = 0 \Rightarrow [IF][x_{pivot}; x_{free}] = 0 \Rightarrow x_{pivot} = -Fx_{free}$
- $r \leq M, r \leq N$
- If A is an $M \times N$ matrix, use the following rules to know about the solutions:
  1. if $r = N < M$ (Full column rank matrix):
     - $N(A) = \{\text{zero vector}\}$, because we have zero free variables.
     - $x_{complete} = x_{particular}$ if a solution exists.
     - $R = [I; 0]$
     - zero or one solution
  2. if $r = M < N$ (Full row rank matrix):
     - $N - M$ free variables.
     - $R = [IF]$
     - has solution for every b!
     - infinite number of solutions
  3. if $r = M = N$:
     - $R = I$
     - invertible

- unique solution

4. if $r < M, r < N$:
   - $R = [IF; 00]$
   - if solution exists: $x_{complete} = x_{particular} + x_{nullspace}$
   - zero or infinite solutions

- In finding special solutions we have r pivot columns and $n - r$ free variables. we can set the free variables. e.g. to 1-hot encoding and find the other variables' values. Any combination of special solutions are also a special solution (they are solutions of $Ax = 0$)

- To find complete solution to $Ax = b$:

  1. $x_{particular}$: set all free variables to zero. Solve $Ax = b$ for other variables.
  2. Compute $x_{nullspace}$

     $\longrightarrow X = X_p + X_n$