In The Name of God.
The Merciful, The Compassionate.

Orthogonal Matrices and Gram Schmidt

notes on Gilbert Strang videos, Lecture 17

1 Orthonormality

- **Orthonormal basis**:
  bases $\{q_1, q_2, \ldots, q_n\}$ are orthonormal vectors:

$$\begin{cases} q_i^T.q_j = 0 & \text{if} i \neq j \\ q_i^T.q_j = 1 & \text{if} i = j \end{cases}$$

- **Orthonormal matrices**: columns are orthonormal $Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \ldots & q_n \\ | & | & & | \end{bmatrix}$:
  - $Q^T.Q = I$
  - if Q is squared, $Q^T = Q^{-1}$
  - Suppose Q has orthonormal columns, project onto its columns:
    * $P = Q(Q^TQ)^{-1}Q^T = QQ^T \rightarrow$ folows the properties. $P = I$ if Q is squared
    * $A^TA\hat{x} = A^Tb \Rightarrow Q^TQ\hat{x} = Q^Tb \rightarrow \hat{x}_i = q_i^Tb$

2 Gram Schmidt

- start with independent vectors $\{a, b, \ldots\}$, find orthogonal vectors $\{A, B, \ldots\}$ and orthonormal ones: $\{q_1 = \frac{A}{\|A\|}, \ldots\}$

- $A = a$, B must be orthogonal to a. $B = e$ (in projection). $\implies B = b - p = b - A^TbA/(A^TA)$
- $C = A^TcA/(A^TA) - B^TcB/(B^TB), \ldots$

- $A = QR:$ $\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \dots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_1^T q_1 & \dots \\ a_1^T q_2 & \dots \\ \vdots & \vdots \end{bmatrix}$

  $a_1^T q_2$ is 0, R is upper triangular. because later q's are set to be perpendicular to the earlier ones! (Look Book)