In The Name of God.
The Merciful, The Compassionate.

# Eigenvalues, Eigenvectors, Diagonalization, and Powers of $A$

notes on Gilbert Strang videos, Lecture 21,22

## 1 Eigenvalues and eigenvectors

- $Ax = \lambda x$, eigenvectors parallel to $x$
  - If $A$ is singular, then $\lambda = 0$ is an eigenvalue.
  - What are the $x$'s, and $\lambda$'s of projection matrix $P$? $\longrightarrow \lambda = 0, 1$
    - Any $x$ in the projection plane : $Px = x$, $\lambda = 1$
    - Any $x \perp$ to the plane: $Px = 0$, $\lambda = 0$
  - How about permutation matrix?
    - $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
    - swap rows!
    - $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = 1$, $Ax = x$
    - $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\lambda = -1$, $Ax = -x$
  - FACT: sum of $\lambda$'s $= a_{11} + a_{22} + \ldots + a_{nn}$
  - Solve $Ax = \lambda x$: $(A - \lambda I)x = 0$
    - $\Rightarrow det(A - \lambda I) = 0$. Find $\lambda$'s first.
    - For $2 \times 2$ matrices *trace* is the linear coefficient, and *det* is constant of degree 2 equation for eigenvalues.
    - For degenerate equations (e.g., when $A$ is upper triangular), the eigenvalues, and eigenvectors are repeated. A double or higher order degeneracy will cause a lost in one(or more) equations. We have infinite number of ways to choose eigenvectors. The most traditional way (almost in $2^{nd}$ order degeneracy) is to choose the first one to be the trivial normalized one. and the other vector to be orthogonal to the first one.

## 2 Diagonalization and Powers of $A$

- **Diagonalizing a matrix,** $S^{-1}AS = \Lambda$:
  - Suppose we have $n$ linearly independent eigenvectors of $A$.
  - Put them in columns of $S$:

$$
\begin{aligned}
AS &= A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \\ | & | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_n \end{bmatrix} \\
&= S\Lambda \\
&\Rightarrow AS = S\Lambda
\end{aligned}
$$

  - $A = S\Lambda S^{-1} \Rightarrow A^k = S\Lambda^k S^{-1}$

    **Theorem 1.** $A^k \to 0$ as $k \to \infty$ if all $|\lambda_i| < 1$.
  - $A$ is sure to be diagonalizable if all the $\lambda_i$'s are different.
  - If we have repeated eigenvalues: may or may not have independent eigenvectors (consider $I$).
  - Symmetric matrices: eigenvalues are real, eigenvectorss are orthogonal to each other
  - $e^{At}$:

$$
\begin{aligned}
e^{At} &= I + At + \frac{At^2}{2} + \dots + \frac{At^n}{n!} + \dots \\
&= SS^{-1} + S\Lambda S^{-1}t + S\Lambda^2 S^{-1}t/2 + \dots \\
&= Se^{\Lambda t}S^{-1}
\end{aligned}
$$

  - $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ & \ddots & \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix}$
  - Note: assumed $A$ is diagonalizable.
  - eigenvalues of $A$ and $A^T$ are the same. $det(A - \lambda I) = det(A^T - \lambda I)$