Université Grenoble Alpes
MSIAM 1st year

Numerical Optimization
Tuto 6: Linear and Quadratic Programs

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In this tutorial, we are going to investigate Linear and Quadratic problems, that is when minimizing linear or quadratic cost functions under linear inequalities constraints. Typical formulations of these problems is as such:

Linear program (LP):

$$\min_{x \in \mathbb{R}^n} \quad c^{\mathrm{T}} x$$
$$\text{subject to} \quad Gx \leq h$$

Quadratic program (QP):

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^{\mathrm{T}} P x + q^{\mathrm{T}} x$$
$$\text{subject to} \quad Gx \leq h$$

where $c, q \in \mathbb{R}^n$, $G \in \mathbb{R}^{m \times n}$, $h \in \mathbb{R}^m$, $P \in \mathbb{R}^{n \times n}$.

Altough these problems are quite specific, a number of (sub-)problems in signal and data processing can actually reformulate linearly or quadratically. The interest of these reformulation is that there exist a large number of standard libraries implementing computationally efficient LP and QP solvers[1].

Exercise 1 (Equivalent problems)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$; we consider the problem

$$\min_{x \in \mathbb{R}^n} \quad f(x)$$
$$\text{subject to} \quad x \in C$$

and we assume that a solution $\bar{x}$ exists. Show that this problem is *equivalent* to solving

$$\min_{(x,r) \in \mathbb{R}^{n+1}} \quad r$$
$$\text{subject to} \quad f(x) \leq r$$
$$(x, r) \in C \times \mathbb{R} \subset \mathbb{R}^{n+1}$$

in the sense that
  (i) if $\bar{x}$ is a solution of the first problem, then $(\bar{x}, f(\bar{x}))$ is a solution of the second one.
  (ii) if $(\bar{x}, \bar{r})$ is a solution of the second problem, then $\bar{x}$ is a solution of the first one.

Exercise 2 (Linear reformulation)**.** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Reformulate the problem

$$\min_{x \in \mathbb{R}^n} \quad \|Ax - b\|_\infty$$

as a linear problem. Notably, give the corresponding (c, G, h) from the LP formulation.

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[1]generally based on interior point, active sets, simplex, ... algorithms and variants.

Exercise 3 (Linear reformulation II)**.** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Reformulate the problem

$$\min_{x \in \mathbb{R}^n} \quad \|Ax - b\|_1$$

as a linear problem by extending the technique of Ex. 1 (without giving details). Notably, give the corresponding (c, G, h) from the LP formulation.

Do the same for the problem

$$\min_{x \in \mathbb{R}^n} \quad \|x\|_1$$
$$\text{subject to} \quad \|Ax - b\|_\infty \leq 1$$

Exercise 4 (Quadratic reformulation)**.** We consider the regression model

$$y = X\theta + \xi, \quad \xi \sim \mathcal{N}(0, \sigma I_m),$$

where $X \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$ are the observed values and $\theta \in \mathbb{R}^n$ is the unknown parameter we want to find. Show that maximizing the (log-)likelihood of θ amount to minimizing $\|X\theta - y\|_2^2$.

Reformulate the maximum likelihood problem under bounded output error as a Quadratic problem.

$$\max_{\theta \in \mathbb{R}^n} \quad \text{likelihood}(\theta) = p(y|\theta)$$
$$\text{subject to} \quad |y_i - X_i\theta| \leq \varepsilon$$

(X_i) is the row vector of the i-th line of X.

What would change if ξ followed a Laplace distribution?