A Brief Introduction to Yang-Mills Theory*

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Abstract

The main part of our report is our understandings on how to generalize physical theories from the basic principles and in a natural way, from the example of the Yang-Mills Theory. In this report, we will introduce the origin of gauge transformation and gauge invariance.Then we investigate the possible process of generalization, in which our principle is the invariance of Lagrangian under local gauge symmetry, with "the least" assumption on physical laws and mathematical forms. In the last section, we appeal to language of fiber bundle and its connection, during which progress we will find at last the quantities introduced in previous sections come out elegantly.

1 Introduction

1.1 Gauge Symmetry: Invariant under Transformation

In physics, gauge theories are a class of physical theories based on the idea that symmetry transformations can be performed locally as well as globally. Many powerful theories in physics are described by Lagrangians which are invariant under certain symmetry transformation groups. When they are invariant under a transformation identically performed at every space-time point they are said to have a **global symmetry**. Gauge theory extends this idea by requiring that the Lagrangians must possess **local symmetries** as well, it should be possible to perform these symmetry transformations in a particular region of space-time without affecting what happens in another region.

For example, in quantum physics, symmetry is a transformation between physical states that preserves the expectation values of all observables O (in particular the Hamiltonian)

$$S : |\varphi> \mapsto |\psi> = S|\varphi>$$

$$|<\psi|O|\psi>|^2 = |<\varphi|O|\varphi>|^2$$

The usual formulation of the physics theories uses fields, which sometimes are not physical quantities. Such are the gauge fields (fiber bundle connections for the mathematicians), which provide a redundant but convenient description of the physical degrees of freedom. The gauge (local) "symmetries" are a reflection of this redundancy. The physical quantities are certain equivalence classes of gauge fields. An analogy can be made with the construction of the real numbers. We can use sequences of rational numbers that have the same limit. Of course, each real number is represented by infinitely many such sequences. We can choose a particular well defined sequence to be a representative of the real number. This corresponds to the procedure of gauge fixing in gauge theories. The fact that gauge fields are not physical degrees of freedom becomes very clear when we try to quantize them. Then we are forced to work in one way or another with the physical quantities by removing the redundancy (the gauge symmetry).

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1.2 Motivation : From Global Symmetry to Local Symmetry

We use the scalar $O(n)$ gauge theory to illustrate how local gauge invariance can be "motivated" heuristically starting from global symmetry properties.

Consider a set of n non-interacting (free) scalar fields, with equal masses m. This system is described by an action which is the sum of the (usual) action for each scalar field φ_i

$$\mathcal{S} = \int \mathrm{d}^4x \sum_{i=1}^{n} \left[\frac{1}{2}(\partial_\mu \varphi_i)(\partial^\mu \varphi_i) - \frac{1}{2} m^2 \varphi_i^2 \right]$$

By introducing a vector of fields $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T$, the Lagrangian (density) can be compactly written as (use the Einstein notation)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi$$

It is now transparent that the Lagrangian is invariant under the transformation $\Phi \mapsto g\Phi$ whenever g is a constant matrix belonging to the n-by-n orthogonal group $O(n)$. This is the global symmetry of this particular Lagrangian, and the symmetry group is often called the **gauge group**.

Now, demanding that this Lagrangian should have local $O(n)$-invariance requires that the G matrices (which were earlier constant) should be allowed to become functions of the **space-time** coordinates x.

Unfortunately, the g matrices do not"pass through" the derivatives. When $g = g(x)$,

$$\partial_\mu (g\Phi)^T \partial^\mu (\Phi) \neq \partial_\mu \Phi^T \partial^\mu \Phi$$

This suggests defining a new Lagarangian, i.e., replace the functor ∂_μ by a new "derivative" D with the property

$$\mathcal{L}(g\Phi) = \mathcal{L}(\Phi)$$

that is,

$$\frac{1}{2}(D_\mu (g\Phi))^T D^\mu (g\Phi) - \frac{1}{2} m^2 (g\Phi)^T (g\Phi) = \frac{1}{2}(D_\mu \Phi)^T D^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi$$

Since $g(x) \in O(n)$ we have $(g\Phi)^T(g\Phi) = \Phi^T\Phi$, from the previous case it is tempting to define the new "derivative" (called covariant derivative) D as follows:

$$D_\mu = \partial_\mu + \gamma A_\mu(x)$$

such that

$$D_\mu (g\Phi) = g D_\mu \Phi$$

We now consider the form of $A_\mu(x)$ in order that the local symmetry holds. Since

$$D_\mu (g\Phi) = \partial_\mu (g\Phi) + \gamma \tilde{A}_\mu (g\Phi) = (\partial_\mu g)\Phi + g(\partial_\mu \Phi) + \gamma \tilde{A}_\mu (g\Phi)$$

it suffices to require that

$$(\partial_\mu g)\Phi + \gamma \tilde{A}_\mu g \Phi = g(\gamma A_\mu \Phi)$$

that is, the **gauge field** $A(x)$ is defined to have the transformation law

$$\tilde{A}_\mu(x) = g(x) A_\mu(x) g^{-1}(x) - \gamma^{-1}(\partial_\mu g(x)) g^{-1}(x)$$

Our picture of classical gauge theory is almost complete except for the fact that to define the covariant derivatives D,a.e. $A_\mu(x)$ which satisfies the local transformation mentioned above, one needs to know the value of the gauge field $A(x$ at all space-time points. Instead of manually specifying the values of this field, it can be given as the solution to a field equation from the principles of mechanics, besides satisfying the local invariance..

In fact, **Weil's Principle of Gauge Invariance** implies that if ψ satisfies Schrödinger's equations which involves the potential A, then $\exp(\frac{i}{e}\hbar f(x))\psi$ satisfies the Schrödinger's equations in which A has been replaced by $A + df$.

It will be indicated that the gauge field is an element of the Lie algebra, and can therefore be expanded as

$$A_\mu(x) = \sum_a A^a_\mu(x) T^a$$

There are therefore as many gauge fields as there are generators of the Lie algebra. We now have a **locally gauge invariant** Lagrangian

$$\mathcal{L}_{\text{loc}} = \frac{1}{2}(D_\mu \Phi)^T D^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi$$

The difference between this Lagrangian and the original globally gauge-invariant Lagrangian is seen to be the interaction Lagrangian

$$\mathcal{L}_{\text{local}} = \mathcal{L}_{\text{global}} + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = \frac{g}{2}\Phi^T A^T_\mu \partial^\mu \Phi + \frac{g}{2}(\partial_\mu \Phi)^T A^\mu \Phi + \frac{g^2}{2}(A_\mu \Phi)^T A^\mu \Phi$$

The term $\mathcal{L}_{\text{int}}$ introduces interactions between the n scalar fields just as a consequence of the demand for local gauge invariance. In the quantized version of this classical field theory, the quanta of the gauge field $A(x)$ are called gauge bosons. The interpretation of the interaction Lagrangian in quantum field theory is of scalar bosons interacting by the exchange of these gauge bosons.

2 Yang-Mills's Theory

2.1 Review of Electromagnetism

We will use the Hamilton theory to drive the equation s describing electromagnetic fields (i.e. Maxwell's equations) which turn out to be just the Euler-Lagrange equations corresponding to the particular action

$$S = S_f + S_m + S_{mf}$$

We begin with the term S_m. This is that part of the action due to the particles (i.e. charges) considered separately from the field through which they move, i.e. the action of the charges assuming the field absent. The appropriate action, as it is usually defined, is given by

$$S_m = -\sum_i m_i c \int_a^b dl,$$

where the sum is taken over all of the particles in the field, of masses m_i, c is the speed of light, and the integral $\int_a^b dl$(where l denotes arc length) is taken over the arc of the world -line of the particle in $\mathbb{R}^4_{1,3}$ between the two fixed events corresponding to the positions of the particle at an initial time t_1 and a later time t_2. The term S_{mf}, representing that part o f the action determined by the mutual interaction of the particles and the field, is usually defined by

$$S_{mf} = -\sum_j \frac{e_j}{c} \int A_k^{(j)} dx^k$$

where again the summation is over all particles indexed by j, where $e_j$is the charge on the jth particle, where, as for S_m, the integral is taken over an arc of the world-line of the the particle. Here (A_i) is a given 4-covector defined on $\mathbb{R}^4_{1,3}$(the so-called "4-potential") which characterizes the field Finally, the term S_f is that part of the action depending on the properties of the field alone, i.e. the action due to the field in the assumed absence of charges. If we are interested only in the motion of the particles in a given electromagnetic field, then the term S_f need not to be considered; on the other hand this term is crucial if our interest lies rather in finding equations characterizing the field. By way of preparing for the definition of S_f we introduce some already familiar concepts of electromagnetic-field theory, defining them in terms of the basic 4-potential (A_i). The three spatial components A^1, A^2, A^3 of the 4-vector (A^i) obtained by raising the index of the tensor (A_i) (for this purpose resorting , of course, to the Minkowski metric), define a 3-vector $\mathbf{A}$ called the *vector-potential* of the field. The remaining component A^0, perhaps more familiarly denoted by φ, is called the *scalar potential* of the field. The *electric field strength* is then the 3-vector

$$\mathbf{E} = \frac{1}{c}\frac{\mathbf{A}}{\partial t} - grad\varphi$$

while the *magnetic field strength* is by definition the 3-vector

$$\mathbf{H} = curl\mathbf{A}.$$

Finally the *electromagnetic field tensor* (F_{ik}) is defined by

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

We are now ready for the definition of S_f: we set

$$S_f = a \int 2(E^2 - H^2) d^4x,$$

where $H^2 =< \mathbf{H}, \mathbf{H} >, E^2 =< \mathbf{E}, \mathbf{E} >$ are the (Euclidean) scalar squares of the 3-vectors $\mathbf{H}$ and $\mathbf{E}$, where a is a constant (usually taken as $\frac{1}{16\pi c}$), and where with respect to the spatial co-ordinates x^1, x^2, x^3, the integral is taken over the whole of 3-space , while with respect to the variable x^0 (proportional to the time) it is taken over the interval between two fixed instants. Noting that $F_{ik}^2 \equiv F_{ik}F^{ik} = 2(H^2 - E^2)$, and substituting for a its customary value, we have

$$S_f = -\frac{1}{16\pi c} \int 2(H^2 - E^2) d^4x = \frac{1}{16\pi c} \int F_{ik}^2 d^4x.$$

Putting this together with the formulae for S_m and S_{mf} we obtain the formula for the total action S of an electromagnetic field containing charged particles:

$$S = -\sum \int mcdl - \sum \int \frac{e}{c} A_k dx^k - \frac{1}{16\pi c} \int F_{ik}^2 d^4x.$$

We can sometimes regard the total charge as being distributed continuously throughout space. In this case the amount of charge contained in the 3-dimensional volume element $dV = dx^1 \wedge dx^2 \wedge dx^3$ is given by ρdV, where ρ denotes the point-density of charge (thus ρ depends on

x^1, x^2, x^3 and the time t). We may also parameterize the world -line in $\mathbb{R}^4_{1,3}$ of a (variable) point -charge by the time:

$$x^0 = ct, x^i = x^i(t), i = 1, 2, 3.$$

Then (dx^i/dt) is the 4-dimensional velocity vector of the point-charge, and it is natural to call the 4-vector (j^i) defined as follow the *current 4-vector*

$$j^i = \rho \frac{dx^i}{dt}$$

. The three spatial components of this 4-vector define the usual current 3-vector

$$\mathbf{j} = \rho v$$

where v is the charge velocity at the given point, while the component j^0 is just $c\rho$. Direct calculation shows that in terms of the current vector (j^i) the total action takes the form

$$S = -\sum \int mcdl - \frac{1}{c^2} \int A_8 j^i d^4x - \frac{1}{16\pi c} \int F^2_{ik} d^4x.$$

Having defined the action S for an electromagnetic field we are now ready to show that it is an appropriate one, in the sense that Maxwell's equations for the field are just the Euler-Lagrange equations corresponding to S. Since we are interested only in the field , we may take the motions of the charges (i.e. the current) as predetermined, i.e. known in advance. Thus since we are, as it were, given the trajectories (i.e. world-lines) of the charges in advance, we can restrict our attention to the action $S = S_{mf} + S_f$. Our problem is therefore that of finding the conditions (in the form of the Euler-Lagrange equations) which the 4-potential (A_i) must satisfy for S to have an extreme value. Taking into account the assumption that in the term S_{mf} the current (j^i) is not to be regarded as subject to variation, we have that the corresponding Lagrangian is

$$L = L(A_i, \frac{\partial A_i}{\partial x^\alpha}) = -\frac{1}{c}(\frac{1}{c} j^i A_i + \frac{1}{16\pi} F^2_{ik}),$$

where $F_{ik} = \partial A_k/\partial x^i - \partial A_i/\partial x^k$. Looking into the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0$$

After some calculation we get

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i, i = 0, 1, 2, 3$$

2.2 From Electrodynamics to Yang-Mills Theory

In order to describe the two charge states of strong interaction holding together nucleons in atomic nuclei. Yang-Mills generalized the gauge invariance of electromagnetism to explore the possibility of requiring all interactions of the isotopic spin at all space-time points. In analogy to the case of electrodynamics, let ψ be a two-component wave function with the isotopic spin $\frac{1}{2}$. Under an isotopic gauge transformation

$$\psi = S\tilde{\psi}$$

As discussed above we know

$$D_\mu = \partial_\mu - i\epsilon B_\mu$$

with B_μ satisfies

$$\tilde{B}_\mu = S^{-1} B_\mu S + \frac{i}{\epsilon} S^{-1} \frac{\partial S}{\partial x_\mu}$$

Again in analogy to the procedure of obtaining gauge invariant field strengths in the electromagnetism case, Yang-Mills suggest that

$$F_{\mu\nu} = \frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu} + i\epsilon[B_\mu, B_\nu]$$

instead of

$$F_{\mu\nu} = \frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu}$$

in the former case, in order that $F_{\mu\nu}$ satisfies

$$\tilde{F}_{\mu\nu} = S^{-1} F_{\mu\nu} S$$

which also comes from the invariance of the Lagrangian. The form of $F_{\mu\nu}$ that Yang-Mills suggested was thought to be the most brilliant and powerful step in Yang-Mills theory.

So why $F_{\mu\nu}$ can be chosen in that way? Is it **natural**? We now consider what $F_{\mu\nu}$ should look like in order that $\tilde{F}_{\mu\nu} = S^{-1} F_{\mu\nu} S$ We suppose the form

$$F_{\mu\nu} = \frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu} + G_{\mu\nu} \equiv H_{\mu\nu} + G_{\mu\nu}$$

Then it requires that

$$\tilde{G}_{\mu\nu} - S^{-1} G_{\mu\nu} S = S^{-1} H_{\mu\nu} S - \tilde{H}_{\mu\nu}$$

$$\begin{aligned}
\partial_\nu \tilde{B}_\mu &= \partial_\nu (S^{-1} B_\mu S + \frac{i}{\epsilon} S^{-1} \frac{\partial S}{\partial x_\mu}) \\
&= (S^{-1})_\nu B_\mu S + S^{-1} \partial_\nu B_\mu S + S^{-1} B_\mu S_\nu + \frac{i}{\epsilon} (S^{-1})_\nu S_\mu + \frac{i}{\epsilon} S^{-1} S_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\partial_\mu \tilde{B}_\nu &= \partial_\mu (S^{-1} B_\nu S + \frac{i}{\epsilon} S^{-1} \frac{\partial S}{\partial x_\nu}) \\
&= (S^{-1})_\mu B_\nu S + S^{-1} \partial_\mu B_\nu S + S^{-1} B_\nu S_\mu + \frac{i}{\epsilon} (S^{-1})_\mu S_\nu + \frac{i}{\epsilon} S^{-1} S_{\nu\mu}
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{\mu\nu} &= (S^{-1})_\nu B_\mu S - (S^{-1})_\mu B_\nu S + S^{-1} H_{\mu\nu} S \\
&+ S^{-1} B_\mu S_\nu - S^{-1} B_\nu S_\mu + \frac{i}{\epsilon} (S^{-1})_\nu S_\mu - \frac{i}{\epsilon} (S^{-1})_\mu S_\nu
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{\mu\nu} - S^{-1} H_{\mu\nu} S &= (S^{-1})_\nu B_\mu S - (S^{-1})_\mu B_\nu S \\
&\quad + S^{-1} B_\mu S_\nu - S^{-1} B_\nu S_\mu + \frac{i}{\epsilon} (S^{-1})_\nu S_\mu - \frac{i}{\epsilon} (S^{-1})_\mu S_\nu \\
&= S^{-1} G_{\mu\nu} S - \tilde{G}_{\mu\nu}
\end{aligned}$$

It can be checked that when

$$G_{\mu\nu} = i\epsilon[B_\mu, B_\nu]$$

the transformation property holds, but it is apparent **there are still others which also work**, so in this sense that we choose $F_{\mu\nu}$ to be that form seems not so natural?

It can be seen that in the case of electrodynamics, in which $[B_\mu, B_\nu]$ vanishes, that is the difficulty generalizing the gauge invariance of electromagnetism to the particle physics since it is hard to imagine that there exist this special term which vanish in the case of electromagnetism but do not vanish in the latter case. Yang wrote in a commentary on his article as follows:

$\cdots$ Then I tried to define the field strengths $F_{\mu\nu}$ by $F_{\mu\nu} = \frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu}$, which was a "natural" generalization of electromagnetism. This led to a mess, and I had to give up. But the basic motivation remained attractive, and I came back to it several times in the next few years, always getting stuck at the same point $\cdots$

2.3 Yang-Mills equation

We now consider *Lagrangians* $L = L(A_\mu)$ *corresponding to a gauge field* A_μ*alone* (and to a prescribed matrix group G). Such a lagrangian must satisfy the following two conditions:
(i) $L(A_\mu)$ is a scalar.
(ii)$L(A_\mu)$ is invariant under gauge transformations.
The simplest functional satisfying these requirements has the form

$$L = -\frac{1}{4} g^{\mu\lambda} g^{\nu\chi} < F_{\mu\nu}, F_{\lambda\chi} >,$$

where $F_{\mu\nu}$ is the curvature form of the connection A_μ, $g_{\mu\nu}$ is an arbitrary metric on the region of interest of the underlying space, and $<,>$denotes the Killing form on the Lie algebra of the group G, defined by

$$< X, Y >= -tr(ad_X, ad_Y),$$

ad_X being the linear transformation of the Lie algebra defined by $ad_X(A) = [X, A]$.
Suppose that the metric $g_{\mu\nu}$ is Euclidean or pseudo-Euclidean, i.e. that (after choosing coordinates suitably) $g_{\mu\nu} = \epsilon_\mu \delta_{\mu\nu} = \pm 1$, then

$$S[A_\mu] = \int -\frac{1}{4} < F_{\mu\nu}, F_{\mu\nu} > d^n x,$$

where here the subscripts μ, ν are summed over and with the signs $\epsilon_\mu = \pm 1$taken into account.

Theorem 1. *The extremals of the previous functional satisfy the equations*

$$\nabla_\mu F_{\mu\nu} = 0$$

where μ is summed over.

Gauge fields corresponding to the group $SU(2)$ are generally know as *Yang-Mills fields* and the previous equations as *Yang-Mills equations.*

3 Natural from Mathematical Viewpoint?

Gauge theories are usually discussed in the language of differential geometry. Mathematically, a gauge is just a choice of a (local) section of some principal bundle. A gauge transformation is just a transformation between two such sections.

Physical Language	Mathematical Language
Gauge	Section
Gauge Potential	Connection Form
Field Strength	Curvature Form

To make things clear, we express the above ideas in mathematical terminology. that is, to use concepts from the theory of fiber bundle and connections.

Let M be a manifold of dimension n, oriented. Let P the bundle of its frames and $\pi : P \to M$ is the projection. Given a section $s : M \ni U \to P$ of the bundle, to a frame field $s = \{s_i\}_{1\leq i\leq n}$ the connection is given by an matrix of 1-forms

$$\omega = (\omega_i^j)$$

such that

$$\nabla s = sw$$

Suppose $\tilde{s} = \{\tilde{s}_i\}_{1\leq i\leq n}$ is another frame on the bundle. Let

$$\tilde{s} = sg$$

where $g(x) \in G$ and the elements in the matrix $g(x)$ are all smooth functions on U. We can easily get

$$g\tilde{w} = dg + wg$$

then take the exterior differential,

$$d(g\tilde{w}) = gd\tilde{w} + dg \wedge \tilde{w} = d(dg) + dwg - w \wedge dg$$

which implies that

$$g(d\tilde{w} + \tilde{w} \wedge \tilde{w}) = (dw + w \wedge w)g$$

Thus it is natural to define

$$\Omega = dw + w \wedge w$$

we obtain

$$\tilde{\Omega} = g^{-1}\Omega g$$

$\Omega = (\Omega_i^j)$ is called the curvature of the connection.

Now we take a look at what the above formulae stand for in physics. If we treat the a gauge as choice of a (local) section of some principal bundle, then we have the following corresponding expressions

$$w_\mu = -i\epsilon B_\mu$$

in which $w = w_\mu dx^\mu$. Then the transformation $g\tilde{w} = dg + wg$ can be describe as

$$\tilde{B_\mu} = g^{-1}B_\mu g + \frac{i}{\epsilon}g^{-1}dg$$

which is consistent with section 2.2. Furthermore,

$$\begin{aligned}
\Omega_{\mu\nu} &= \frac{\partial w_\nu}{\partial x_\mu} - \frac{\partial w_\mu}{\partial x_\nu} + [w_\mu, w_\nu] \\
&= -i\epsilon(\frac{\partial B_\nu}{\partial x_\mu} - \frac{\partial B_\mu}{\partial x_\nu}) - \epsilon^2[B_\mu, B_\nu] \\
&= i\epsilon(\frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu} + i\epsilon[B_\mu, B_\nu])
\end{aligned}$$

Thus the way that Yang-Mills defined the field strength by

$$F_{\mu\nu} = (\frac{\partial B_\mu}{\partial x_\nu} - \frac{\partial B_\nu}{\partial x_\mu}) + i\epsilon[B_\mu, B_\nu]$$

with the transformation law

$$\tilde{F} = g^{-1}Fg$$

now seems more natural from mathematics viewpoint.

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Motto

Read Euler, read Euler. He is the master of us all

—Pierre-Simon Laplace

Pauca sed matura. (Few, but ripe.)

—Carl Friedrich Gauss