A simpler proof of the Poincaré-Birkhoff theorem

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Introduction

In his search for periodic solutions in the restricted three body problem of celestial mechanics, H. Poincaré constructed an area-preserving section map of an annulus $\mathbf{A}$ on the energy surface. He asserted that an area-preserving homeomorphism of the closed annulus that satisfies some "twist condition" admits at least two fixed points. He proved it is true in some simple cases and conjectured it is also true in a general case [5]. Then he died. So we also call the theorem the last geometric theorem of Poincaré.

In 1913, Birkhoff [1] proved a result which was valid to find one fixed point but uncorrect to get the second one. A small modification of the argument was necessary and Birkhoff corrected this minor error in a paper [2] published in 1925. See also the well-detailed expository paper of Brown and Newman [3].

The goal of this short paper is to introduce the notion of *positive path* of a homeomorphism which seems to be a natural object to understand Birkhoff's arguments.

This short paper is a part of the paper [4]. Only for communion within the mathematics department of Tsinghua University.

Statement and proof of the Poincaré-Birkhoff theorem

In what follows a *path* on a topological space X is a continuous map $\gamma : I \to X$ defined on a segment $I = [a, b] \subset \mathbf{R}$. The *origin* and the *extremity* of γ are respectively $\gamma(a)$ and $\gamma(b)$. If X_1 and X_2 are two subsets of X, we will say that γ *joins* X_1 to X_2 if its origin belongs to X_1 and its extremity belongs to X_2. The restriction of γ to a compact interval $J \subset I$ is a *sub-path* of γ. If γ is one-to-one, γ is an *arc* ; if $\gamma(a) = \gamma(b)$, it is a *loop* ; if $\gamma(a) = \gamma(b)$ and γ is one-to-one on $[a, b)$, it is a *simple loop*. The concatenation of two paths (when it is defined) is denoted by $\gamma_1\gamma_2$. As it is usually done, we often will not make any distinction between a path and its image. In particular, if Y is a subset of X, we will write $\gamma \subset Y$ if the image of γ is included in Y.

We write $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ and we fix in this section and the next one a homeomorphism F of $\mathbf{A} = \mathbf{T}^1 \times [0,1]$ homotopic to the identity and a lift f of F to the universal cover $\widetilde{\mathbf{A}} = \mathbf{R} \times [0,1]$. We suppose that f satisfies the *boundary twist condition* :

$$\text{for every } x \in \mathbf{R}, \; p_1 \circ f(x,0) < x < p_1 \circ f(x,1),$$

where $p_1 : \widetilde{\mathbf{A}} \to \mathbf{R}$ is the first projection. We write $\mathrm{Fix}_*(F)$ for the set of fixed points of F that are lifted to fixed points of f. Let us state the Poincaré-Birkhoff theorem :

Theorem 1. *If F preserves the measure induced by $dx \wedge dy$, then $\sharp\mathrm{Fix}_*(F) \geq 2$.*

Let us recall the ideas of Birkhoff. The vector field $\widetilde{X} : z \mapsto f(z) - z$ is invariant by the covering automorphism $T : (x,y) \mapsto (x+1,y)$ and lifts a vector field X on $\mathbf{A}$ whose singular set is exactly $\mathrm{Fix}_*(F)$. If γ is a path in $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$, one may define the *variation of angle*

$$i_\gamma f = \int_{\widetilde{X}\circ\gamma} d\theta$$

where

$$d\theta = \frac{1}{2\pi}\frac{xdy - ydx}{x^2+y^2}$$

is the usual polar form on $\mathbf{R}^2 \setminus \{0\}$. The form $d\theta$ being closed can be integrated on any (even non-smooth) path in $\mathbf{R}^2 \setminus \{0\}$. The theorem will be proved if one finds a loop Γ in $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$ such that $i_\Gamma \neq 0$. Indeed if $\mathrm{Fix}_*(F)$ is finite (equivalently if $\mathrm{Fix}(f)$ is discrete) then

$$i_\Gamma f = \sum_{z\in \mathrm{Fix}(f)} i(\widetilde{X}, z) \int_{\xi_z \circ \Gamma} d\theta$$

where $i(\widetilde{X}, z)$ denotes the Poincaré index of $\widetilde{X}$ at z and ξ_z is the vector field $z' \mapsto z' - z$. This implies that $\sharp\mathrm{Fix}_*(F) \geq 2$ because $i(\widetilde{X}, z) = i(X, \pi(z))$ and because the Poincaré-Hopf formula asserts that

$$\sum_{z\in \mathrm{Fix}_*(F)} i(X,z) = \chi(\mathbf{A}) = 0.$$

If we can find two paths γ and γ' such that $i_\gamma f = i_{\gamma'} f \neq 0$, the first one that joins $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$, the second one that joins $\mathbf{R} \times \{1\}$ to $\mathbf{R} \times \{0\}$, the loop $\Gamma = \gamma\delta\gamma'\delta'$ obtained by adding horizontal segments on each boundary line will satisfy $i_\Gamma f = 2 i_\gamma f \neq 0$.

In [1] Birkhoff composes F with a small vertical translation to build such paths. Suppose that the displacement is positive, the iterates by the perturbed map G of $\mathbf{T}^1 \times \{0\}$ are pairwise disjoint, and they are not all included in the annulus (because G also preserves the area). Birkhoff chooses an arc α that joins a point $z \in \mathbf{T}^1 \times \{0\}$ to $G(z)$ and by concatenation of the iterates of α constructs an arc that joins $\mathbf{T}^1 \times \{0\}$ to $\mathbf{T}^1 \times \{1\}$ and that is lifted into an arc γ satisfying $i_\gamma f = -\dfrac{1}{2}$. We will give here a simple construction, in the spirit of Birkhoff's ideas, that does not need any perturbation and that is still valid under weaker hypothesis than the preservation of the area.

Definition 2. Let G be a homeomorphism of a topological space X. A *positive path* of G is a path $\gamma\ : I \to X$ such that for every t, t' in I

$$t' \geq t \Rightarrow G(\gamma(t')) \neq \gamma(t).$$

Observe that a positive path γ does not meet the fixed point set, that any sub-path of γ is positive and that the images $G^k \circ \gamma$, $k \in \mathbf{Z}$, are also positive.

Proposition 3. *If γ is a positive path of f that joins a boundary line of $\widetilde{\mathbf{A}}$ to the other one, then $i_\gamma f = -\dfrac{1}{2}$.*

Proof. We write the proof in the case where γ joins $\mathbf{R} \times \{0\}$ to $\mathbf{R} \times \{1\}$, the other case being similar. The boundary of the simplex

$$\Delta = \{(t, t') \in I^2 \,|\, t' \geq t\}$$

may be written $\partial\Delta = \delta_d \delta_h \delta_v$ where δ_d is the diagonal, δ_h a horizontal segment and δ_v a vertical one. The path γ being positive, the map

$$\Phi\ : (t, t') \mapsto f(\gamma(t')) - \gamma(t)$$

does not vanish on Δ and one has

$$\int_{\Phi\circ\delta_d} d\theta + \int_{\Phi\circ(\delta_h\delta_v)} d\theta = \int_{\Phi\circ\partial\Delta} d\theta = 0.$$

Observe now that the image by Φ of each segment δ_h and δ_v does not intersect the vertical half-line $\{0\} \times (-\infty, 0]$. This implies that

$$i_\gamma f = \int_{\Phi\circ\delta_d} d\theta = -\int_{\Phi\circ(\delta_h\delta_v)} d\theta = -\frac{1}{2}.$$

□

Recall that a *wandering point* of a homeomorphism f of a topological space X is a point that admits a *wandering neighborhood* U, that means a neighborhood U such that the $f^k(U)$, $k \geq 0$, are pairwise disjoint. Recall that a *Urysohn space* is a topological space such that two distinct points may be separated by closed neighborhoods.

Proposition 4. *Suppose that X is a connected and locally path-connected Urysohn space and that G is a fixed point free homeomorphism of X with no wandering point. If $Z \subset X$ satisfies $G(Z) \subset Z$, then for every $z \in X$ there exists a positive path of G that joins Z to z.*

Proof. One must prove the equality $Y = X$, where Y is the set of points that may be joined by a positive path of G whose origin belongs to Z. The space X being connected, it is sufficient to prove that $\overline{Y} \subset \mathrm{Int}(Y)$. Fix $z_0 \in \overline{Y}$. By hypothesis, one can find a path-connected neighborhood V of z_0 such that $\overline{V} \cap G(\overline{V}) = \emptyset$. We will prove that $V \subset Y$.

The fact that $z_0 \in \overline{Y}$ implies that there exists a positive path $\gamma_0 : I \to X$ from Z to V. The closures of the subsets $J = \gamma_0^{-1}(V)$ and $J' = \gamma_0^{-1}(G(V))$ do not intersect because $\overline{V} \cap G(\overline{V}) = \emptyset$. This implies that $\inf J \neq \inf J'$.

Suppose first that $\inf J < \inf J'$ (this includes the case where $J' = \emptyset$). In that case, there is a sub-path γ_1 of γ_0 from Z to V that does not meet $G(V)$. For every $z \in V$ one can find a path γ inside V that joins the extremity z_1 of γ_1 to z. The path $\gamma_2 = \gamma_1\gamma$ is positive because γ_1 is positive and $G(\gamma)$ is disjoint both from γ and γ_1. This implies that $z \in Y$.

Suppose now that $\inf J' < \inf J$. In that case, there is a sub-path γ_1 of γ_0 from Z to $G(V)$ that does not meet V. We denote by z_1 its extremity. The point $G(z_1)$ does not belong to γ_1 because this path is positive. The path being compact (X is Hausdorff), one can find a path-connected neighborhood $U \subset G(V)$ of z_1 such that $G(U)$ does not intersect γ_1. The set U being non wandering, one can find a point $z_2 \in U$ whose positive orbit meets $G^{-1}(U) \subset V$. Choose a path γ inside U that joins z_1 to z_2. The path $\gamma_2 = \gamma_1\gamma$ does not meet V and is positive because γ_1 is positive and $G(\gamma)$ is disjoint from γ_1 and γ. Let us consider the integer $k \geq 1$ such that $G^k(\gamma_2) \cap V \neq \emptyset$ and $G^{k'}(\gamma_2) \cap V = \emptyset$ if $0 \leq k' < k$. Since $G(Z) \subset Z$, the path $G^k(\gamma_2)$ is a positive path from Z to V that does not meet $G(V)$. We conclude like in the first case. □

Let us explain how to deduce Theorem 1 from Proposition 3 and 4.

Proof of Theorem 1. One may suppose that $\mathrm{Fix}_*(F)$ does not separate the two boundary circles (otherwise $\sharp\mathrm{Fix}_*(F) = +\infty$). If n is large enough, the homeomorphism F' of the annulus $\mathbf{A}' = \mathbf{R}/n\mathbf{Z} \times [0, 1]$ lifted by f has no fixed points but the ones that are lifted to fixed points of f. Therefore $\mathrm{Fix}(F') = \mathrm{Fix}_*(F')$ does not separate the boundary circles of $\mathbf{A}'$ and one may consider the connected component W of $\mathbf{A}' \setminus \mathrm{Fix}(F')$ that contains the boundary. Moreover F' has no wandering point because it preserves the area. Applying Proposition 4 to $X = W$, to $G = F'|_W$ and to $Z = \mathbf{R}/n\mathbf{Z} \times \{0\}$ or $Z = \mathbf{R}/n\mathbf{Z} \times \{1\}$, one constructs a positive path of F' from one of the boundary circle of $\mathbf{A}'$ to the other one. Such a path is lifted to a positive path of f from the corresponding boundary line to the other one. Theorem 1 follows from Proposition 3. $\square$

Remark 5. One can prove that F' has no wandering point if it is the case for F. Indeed, let T' be a generator of the (finite) group of automorphisms of the covering space $\mathbf{A}'$. The fact that F has no wandering point implies that for every non empty open set $U \subset \mathbf{A}'$, there exists $q \geq 1$ and $p \in \mathbf{Z}$ such that $F'^q(U) \cap T'^p(U) \neq \emptyset$. Let us fix a non empty open set $U_0 \subset \mathbf{A}'$ and define a sequence $(U_k)_{k\geq 0}$ of non empty open sets where U_{k+1} may be written $U_{k+1} = F'^{q_k}(U_k) \cap T'^{p_k}(U_k)$. One deduces that for every $k' > k$, one has $U_{k'} \subset F'^{q_k+\cdots q_{k'-1}}(U_k) \cap T'^{p_k+\cdots p_{k'-1}}(U_k)$. One can find $k' > k$ such that $p_k + \cdots + p_{k'-1} = 0 \pmod n$. This implies that U_k is non-wandering and therefore that U_0 itself is non-wandering. So Theorem 1 is valid if instead of supposing that F preserves the area, one supposes that F has no wandering point. Anyway, instead of working in a finite cover of $\mathbf{A}$, one can prove directly that the conclusion of Proposition 2 occurs if X is the connected component of $\widetilde{\mathbf{A}} \setminus \mathrm{Fix}(f)$ that contains the boundary, if $G = f|_X$ and if Z satisfies $f(Z) \subset Z$ and $T(Z) = Z$.

References

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数学趣闻——庞加莱最后几何定理

在1911年，庞加莱开始有了他可能不久于人世的预感，12月9日他写信给一个数学杂志的编辑，询问是否能接受一篇尚未完成的论文——与通常的习惯相反——关于一个庞加莱认为最最重要的问题的论文："……以我的年纪，我可能不能解决它了，所得到的结果，有可能把研究者们带到新的、意想不到的道路上去，尽管它们使我多次受骗，我认为它们太有前途了，我自愿献出它们……。"他已经把两年时间中较好的部分用来试着去克服他的困难，但没有收效。

他猜测的那个定理的证明，能够使他在三体问题上取得惊人的进展；特别是它将使他能够证明比以前考虑过的更一般的一些情形的无限多个周期解的存在。这个期望中的证明，在庞加莱的"未完成交响乐"发表以后不久，就由一个年轻的美国数学家乔治·戴维·伯克霍夫证明了。

1912年春天，庞加莱再次病倒，7月9日接受了第二次手术。手术是成功的，但是7月17日，他在穿衣服的时候，非常突然地死于血栓。他当时五十九岁，正处在他能力的顶峰——用潘勒韦的话说，"理性科学的活着的大脑。"

Mathematical Quotations

Poincaré, Jules Henri (1854-1912)

Mathematical discoveries, small or great are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labour, both conscious and subconscious.