Four Equivalent Statements Concerning Baire-1 Functions*

Shi Xiaojie†

编者按:本文假定读者熟悉点集拓扑中的基本概念并知道 Tietze 扩张定理、单位分解定理及度量空间都是仿紧的。

In this article, I will give four equivalent statements concerning Baire-1 functions. Let's start with some definitions. (X will be a metric space if not specified otherwise.)

Definition 1 *Let f be a function defined on X, we say f is a* Baire-1 function *if there exist continuous functions $\{f_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f_n(x) = f(x), \forall\ x \in X$.*[1] *The set of all Baire-1 functions defined on X is denoted by $\mathscr{B}_1(X)$.*

Definition 2 *A set V is called F_σ if $V = \bigcup_{n=1}^{\infty} V_n$ for some sequence of closed sets V_n.*

Definition 3 [2]*In set theory, two ordered sets X, Y are said to have the same* order type *when there exists a bijection $f : X \to Y$ such that both f and its inverse are monotone (order preserving).*

An ordinal number*, or just* ordinal*, is the order type of a well-ordered set. The finite ordinals are the natural numbers: 0,1,2,.... The least infinite ordinal is ω which is identified with the cardinal number $\aleph_0$. The set of all countable ordinals constitutes the first uncountable ordinal ω_1 which is identified with the cardinal $\aleph_1$.*

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†Department of Mathematics, National University of Singapore

[1]在通常的定义中 Baire-1 函数不包含连续函数，不过在本文中可以包含。

[2]编者加。参考 Wikipedia 中 Order type, Ordinal number, Successor ordinal, Limit ordinary 等词条。

The successor *of an ordinal number α is the smallest ordinal number greater than α. An ordinal number that is a successor is called a* successor ordinal. *A* limit ordinal *is an ordinal number which is neither zero nor a successor ordinal.*

Definition 4 *Let P be a closed subset of X, $f : X \to \mathbb{R}$, $\epsilon > 0$. $P(f, \epsilon) := \{x \in P :$ for all open neighborhood U of x in X, $\exists\, x_1, x_2 \in P \cap U$, such that $|f(x_1) - f(x_2)| \geq \epsilon\}$. Define $P^0(f, \epsilon) = P(f, \epsilon)$; $P^\alpha(f, \epsilon) = (P^{\alpha-1}(f, \epsilon))(f, \epsilon)$ if α is a successor ordinal; $P^\alpha(f, \epsilon) = \bigcap_{\gamma<\alpha} P^\gamma(f, \epsilon)$ if α is a limit ordinal.*

Define

$$\beta(f, \epsilon) = \begin{cases} \text{the smallest } \alpha < \omega_1 \text{ such that } P^\alpha(f, \epsilon) = \emptyset & \text{if such } \alpha \text{ exists} \\ \omega_1 & \text{otherwise.} \end{cases}$$

Define $\beta(f) = \sup_{\epsilon>0} \beta(f, \epsilon)$.

Note that $P(f, \epsilon)$ is a closed subset of P. If $0 < \epsilon_1 < \epsilon_2$, then $P(f, \epsilon_1) \supset P(f, \epsilon_2)$ and $\beta(f, \epsilon_1) \geq \beta(f, \epsilon_2)$. $\sup_{\epsilon>0} \beta(f, \epsilon) = \sup_{\epsilon_n} \beta(f, \epsilon_n)$ for any sequence ϵ_n goes to 0. If $\beta(f, \epsilon) < \omega_1, \forall\, \epsilon > 0$, then $\beta(f) < \omega_1$.

Definition 5 *A space X is said to be a* Baire space *if the following condition holds: Given any countable collection $\{A_n\}$ of closed subsets of X, if each of them has empty interior in X, then their union $\bigcup A_n$ also has empty interior in X.*

Theorem 6 (Baire Category Theorem) *If X is a complete metric space, then X is a Baire space.*

Proof: Let $\{A_n\}$ be as defined in Definition 5, we prove this theorem by showing that any nonempty open set U_0 in X contains a point x that does not lie in any set A_n, thus the interior of $\bigcup A_n$ must be empty, which is what we want.

Since $\text{Int}A_1 = \emptyset$, there exists $y_1 \in U_0$ such that $y_1 \notin A_1$. Then there exists $r > 0$ such that $B(y_1, r) \cap A_1 = \emptyset$ since A_1 is closed. Since U_0 is open, there exists $s > 0$ such that $B(y_1, s) \subset U_0$. Let $U_1 = B(y_1, \min\{s, t\}/4)$, then $\overline{U}_1 \subset U_0$ and $\overline{U}_1 \cap A_1 = \emptyset$.

Now U_1 is a nonempty open set in X, we can find $y_2 \in U_1$ such that $y_2 \notin A_2$ since $\text{Int}A_2 = \emptyset$. Do the same thing as above, there is an open ball U_2 such that $\overline{U}_2 \subset U_1$ and $\overline{U}_2 \cap A_2 = \emptyset$, we may shrink U_2 (if necessary) so that its radius is not more than $s/8$. Inductively, there is a sequence of open sets $\{U_n\}$ such that $\overline{U}_n \subset U_{n-1}$, $\overline{U}_n \cap A_n = \emptyset$ and the radius of U_n is not more than $s/2^{n+1}$.

Consider $\bigcap \overline{U}_n$, claim that $(\bigcap \overline{U}_n) \cap (\bigcup A_n) = \emptyset$; $\bigcap \overline{U}_n \subset U_0$; $\bigcap \overline{U}_n \neq \emptyset$. The first two statements are obvious since $\overline{U}_n \cap A_n = \emptyset$ and $\overline{U}_n \subset U_{n-1}$, respectively. For the third one, notice that if $z_1, z_2 \in U_n$, then $d(z_1, z_2) \leq s/2^n$. Therefore the sequence $\{y_n\}$ we construct is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} y_n = x$. For each n, since $\overline{U}_n$ is closed and $y_m \in U_n, \forall\, m \geq n+1$, we have $x \in \overline{U}_n$. Hence $x \in \bigcap \overline{U}_n$.

The point $x \in U_0$ does not lie in any A_n and thus is what we want. $\Box$

Theorem 7 *Let X be a complete separable metric space, $f : X \to \mathbb{R}$. The following four statements are equivalent (The first 3 are called Baire Characterization Theorem):*

1. *$f \in \mathscr{B}_1(X)$;*

2. *For all open set U of $\mathbb{R}$, $f^{-1}(U)$ is F_σ;*

3. *For all closed subset F of X, $f|_F : F \to \mathbb{R}$ has a point of continuity;*

4. *$\beta(f) < \omega_1$.*

Proof: (1)$\Rightarrow$(2)

For any open set U of $\mathbb{R}$, we can write $U = \bigcup_{i=1}^{\infty} V_i$, where $V_i = (a_i, b_i)$ are pairwise disjoint intervals. So we only need to show for any $a < b$, $f^{-1}(a,b)$ is F_σ.

Take arbitrary $c, d \in \mathbb{Q}$ with $a < c < d < b$, consider the set $\bigcap_{n=N}^{\infty} f_n^{-1}[c,d]$, where $\{f_n\}$ are as defined in Definition 1. Since f_n is continuous for each n, $f_n^{-1}[c,d]$ is closed, thus $\bigcap_{n=N}^{\infty} f_n^{-1}[c,d]$ is closed. Take

$$A = \bigcup_{\substack{a<c<d<b \\ c,d\in\mathbb{Q}}} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} f_n^{-1}[c,d].$$

A is F_σ since $\mathbb{Q}$ is countable. Now it suffices to prove $A = f^{-1}(a,b)$.

On one hand, for any $x \in f^{-1}(a,b)$, $a < f(x) < b$. Take $c, d \in \mathbb{Q}$ such that $a < c < f(x) < d < b$. Since $\lim_{n\to\infty} f_n(x) = f(x)$, there exists N such that for $n \geq N$, $f_n(x) \in [c,d]$, hence $x \in \bigcap_{n=N}^{\infty} f_n^{-1}[c,d] \subset A$.

On the other hand, for any $x \in A$, there exist $c, d \in \mathbb{Q}$, $N \in \mathbb{N}$ with $a < c < d < b$ such that $x \in \bigcap_{n=N}^{\infty} f_n^{-1}[c,d]$. Thus for $n \geq N$, $f_n(x) \in [c,d] \subset (a,b)$. Since $\lim_{n\to\infty} f_n(x) = f(x)$, we have $f(x) \in (a,b)$, hence $x \in f^{-1}(a,b)$.

(2)⇒(3)

Let F be a closed subset of X, define $F_0 = F$, then F_0 is a Baire space since F_0 is closed (hence complete) and metric. With F_{n-1} defined, we construct E_n, F_n and x_n as follows: (Note that we will use the information about F_n: F_n is closed in F_0 and $\mathrm{Int}_{F_0}(F_n)$ is not empty. For F_0 those two are clearly true, for other F_n we will prove them after construction.)

Use countable internals (a_i, b_i) with length $1/2^n$ to cover $\mathbb{R}$. For each (a_i, b_i), $f|_{F_0}^{-1}(a_i, b_i) = \bigcup_{j=1}^{\infty} V_{i,j}$, where each $V_{i,j}$ is closed in F_0. Let $T_{i,j} = V_{i,j} \cap F_{n-1}$, then $T_{i,j}$ is closed in F_0 since F_{n-1} does so. It is easy to show $F_{n-1} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} T_{i,j}$, hence $\mathrm{Int}_{F_0}(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} T_{i,j})$ $=\mathrm{Int}_{F_0}(F_{n-1}) \neq \emptyset$. By the Baire Category Theorem, there exists some $T_{i,j}$ such that $\mathrm{Int}_{F_0}(T_{i,j}) \neq \emptyset$. Let $E_n = T_{i,j}$, then there exists $x_n \in \mathrm{Int}_{F_0}(E_n)$. Since $\mathrm{Int}_{F_0}(E_n)$ is open in F_0, there exists $B(x_n, \delta_n) \subset \mathrm{Int}_{F_0}(E_n)$ where $\delta_n > 0$.

Let

$$F_n = \overline{B(x_n, \min\{\delta_n/2, 1/2^n\})} \cap F_0,$$

then we verify the two conditions we assume about F_n. For the first one, F_n is certainly closed in F_0. For the second one, since $x_n \in B(x_n, \min\{\delta_n/2, 1/2^n\})$, we have $x_n \in \mathrm{Int}_{F_0}(F_n)$, therefore $\mathrm{Int}_{F_0}(F_n)$ is not empty.

For each n, it is easy to check that $F_n \subset \mathrm{Int}_{F_0}(E_n) \subset E_n \subset F_{n-1}$ and for $x, y \in E_n$, $|f(x) - f(y)| < 1/2^n$.

Consider the sequence $\{x_n\}$. For $n, m > N$, $x_n, x_m \in F_N$, hence $d(x_n, x_m) \leq 1/2^{N-1}$, therefore $\{x_n\}$ is a Cauchy sequence. Since F_0 is complete, there exists $x \in F_0$ such that $x = \lim_{n\to\infty} x_n$. For each n, since F_n is closed, we have $x \in F_n$, thus $x \in \bigcap_{n=1}^{\infty} F_n \subset \mathrm{Int}_{F_0}(E_n)$ for any n.

Claim that for this $x \in F$, $(x, f(x))$ is a point of continuity. For any $\epsilon > 0$, let $1/2^n < \epsilon$, then for any $y \in \mathrm{Int}_{F_0}(E_n)$, $|f(x) - f(y)| < 1/2^n < \epsilon$. Therefore, $(x, f(x))$ is indeed a point of continuity.

(3)⇒(4)

First we state a theorem.

Theorem 8 (Cantor-Baire Stationary Principle) *Let X be a separable metric space. If there exist $\{F_\alpha\}, \alpha < \omega_1$, such that F_α is a closed subset of X for each α and $\alpha_1 < \alpha_2$ implies $F_{\alpha_2} \subset F_{\alpha_1}$, then there exists $\beta < \omega_1$ such that $F_\alpha = F_\beta$, $\forall\ \alpha \geq \beta$.*

Proof: Suppose there does not exist such β, then for any $\alpha_1 < \omega_1$, we can find α_2 such that $\alpha_2 > \alpha_1$ and $F_{\alpha_2} \subsetneqq F_{\alpha_1}$. So without loss of generality, we can assume that $\alpha_1 < \alpha_2$ actually implies $F_{\alpha_2} \subsetneqq F_{\alpha_1}$. If not, we can construct a subset of $\{F_\alpha : \alpha < \omega_1\}$ namely F'_α $(\alpha < \omega_1)$ with $\alpha_2 > \alpha_1$ implies $F'_{\alpha_2} \subsetneqq F'_{\alpha_1}$.

Since X is separable, let $X = \overline{\{x_i\}}$. Consider

$$\{B(x_n, q) : x_n \in \{x_i\}, q \in \mathbb{Q}^+\}$$

The set is countable, we denote it by $\{O_n\}$.

For each $\alpha < \omega_1$, we show that there exists $O_j \in \{O_n\}$ such that $O_j \cap F_\alpha \neq \emptyset$ and $O_j \cap F_{\alpha+1} = \emptyset$. Since $F_{\alpha+1} \subsetneqq F_\alpha$, there exists $x \in F_\alpha \setminus F_{\alpha+1}$. Since $F_{\alpha+1}$ is closed, there exists $B(x, \delta)$ such that $B(x, \delta) \cap F_{\alpha+1} = \emptyset$ and $\delta \in \mathbb{Q}^+$. Since $\{x_n\}$ is dense, there exists $x_i \in \{x_n\}$ such that $x_i \in B(x, \delta/4)$. Then consider $B(x_i, \delta/4)$, we have $x \in B(x_i, \delta/4) \cap F_\alpha$ and $B(x_i, \delta/4) \cap F_{\alpha+1} = \emptyset$. So $B(x_i, \delta/4)$ is just the O_j we want to find.

For each $\alpha < \omega_1$, we can find a corresponding O_α and $O_\alpha \neq O_\beta$ for $\alpha \neq \beta$. This contradicts with $\{O_n\}$ is countable. So we conclude that there exists $\beta < \omega_1$ such that $F_\alpha = F_\beta$, $\forall\ \alpha \geq \beta$. $\square$

Then we use Cantor-Baire Stationary Principle to complete the proof of (3)⇒(4).

Fix an $\epsilon > 0$, first show that $\beta(f, \epsilon) < \omega_1$. Suppose not, look at $P^\alpha(f, \epsilon)$ $(\alpha < \omega_1)$. It satisfies the assumption of Cantor-Baire Stationary Principle, so there exists $\beta < \omega_1$ such that $P^\alpha(f, \epsilon) = P^\beta(f, \epsilon)$, $\forall\ \alpha \geq \beta$.

Consider $P^\beta(f, \epsilon)$, it is a closed subset of X and is nonempty. So it has a point of continuity, namely x. By definition, $x \notin P^{\beta+1}(f, \epsilon)$. We get a contradiction.

So for any $\epsilon > 0$, $\beta(f, \epsilon) < \omega_1$. Thus, $\beta(f) = \sup_{\epsilon > 0} \beta(f, \epsilon) < \omega_1$.

(4)⇒(1)

First we state two lemmas.

Lemma 9 *Let X be a metric space. For a bounded function $f : X \to \mathbb{R}$, define the norm $||f|| = \sup_{x \in X} |f(x)|$. Then for a sequence of bounded functions $\{f_n\}$, if $\sum_{n=1}^\infty ||f_n(x)|| < \infty$ and each f_n is Baire-1, then $f(x) = \sum_{n=1}^\infty f_n(x)$ exists and f is also Baire-1.*

Proof: For any x, $\sum_{n=1}^\infty |f_n(x)| < \infty$ since $\sum_{n=1}^\infty ||f_n|| < \infty$. Then for any $\epsilon > 0$, there exists N such that for $m > N$, $\sum_{n=m}^\infty |f_n(x)| < \epsilon$. Therefore, $f(x) = \sum_{n=1}^\infty f_n(x)$ exists for each $x \in X$.

For each f_i, let $M_i = \sum_{x\in X}\{f_i(x)\}$ and $N_i = \inf_{x\in X}\{f_i(x)\}$. Since f_i is Baire-1, there exists a sequence of continuous functions $\{f_{ij} : j \in \mathbb{N}\}$ which converges to f_i. For each f_{ij}, define f'_{ij} by

$$f'_{ij}(x) = \max\{N_i, \min\{M_i, f_{ij}(x)\}\},$$

then f'_{ij} is continuous, $||f'_{ij}|| \leq ||f_i||$ and the sequence $\{f'_{ij} : j \in \mathbb{N}\}$ also converges to f_i.

Let $g_i(x) = \sum_{n=1}^{\infty} f'_{ni}(x)$, it is well-defined since

$$\sum_{n=1}^{\infty} ||f'_{ni}|| \leq \sum_{n=1}^{\infty} ||f_n|| < \infty.$$

We have $g_i(x) = \lim_{m\to\infty} \sum_{n=1}^{m} f'_{ni}(x)$ and $\sum_{n=1}^{m} f'_{ni}(x)$ uniformly converges to $g_i(x)$. Since $\sum_{n=1}^{m} f'_{ni}(x)$ is continuous for each m, $g_i(x)$ is continuous for each $i \in \mathbb{N}$.

Let $F_m(x) = \sum_{n=1}^{m} f_n(x)$, $F_{mi}(x) = \sum_{n=1}^{m} f'_{ni}(x)$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} ||f_n|| < \epsilon/3$. Therefore $||F_n - f|| < \epsilon/3$, for $n \geq N$. Hence $|F_n(x) - f(x)| < \epsilon/3$ for any $n \geq N$ and $x \in X$. Also we have for $n \geq N$, $x \in X$,

$$|F_{ni}(x) - g_i(x)| \leq \sum_{n=N}^{\infty} |f'_{ni}(x)| \leq \sum_{n=N}^{\infty} ||f_n|| < \epsilon/3.$$

Then for a fixed $x \in X$, since $F_N(x) = \lim_{i\to\infty} F_{Ni}(x)$, there exists an integer M such that for $i \leq M$, $|F_{Ni}(x) - F_N(x)| < \epsilon/3$. Therefore $|g_i(x) - f(x)| \leq |g_i(x) - F_{Ni}(x)| + |F_{Ni}(x) - F_N(x)| + |F_N(x) - f(x)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$, this implies that $\lim_{i\to\infty} g_i(x) = f(x)$, for any $x \in X$. Thus $\{g_i : i \in \mathbb{N}\}$ is a sequence of continuous functions which converges to f, hence f is Baire-1. $\Box$

Lemma 10 *If a sequence of Baire-1 functions $\{f_n\}$ uniformly converges to f, then f is also Baire-1.*

Proof: Since $\{f_n\}$ uniformly converges to f, then for any $\epsilon > 0$, there exists an integer N_ϵ such that for $n, m \geq N_\epsilon$, $||f_n - f_m|| < \epsilon$. Let $M_k = N_{1/2^k}$ for $k \in \mathbb{N}$. It is clear we can choose M_k such that $M_1 < M_2 < \cdots$. Take

$$g_n = \begin{cases} f_{M_1} & if n = 1 \\ f_{M_n} - f_{M_{n-1}} & if n \geq 2, \end{cases}$$

then g_n is bounder for $n \geq 2$. We have $\sum_{n=2}^{\infty} ||g_n|| < \infty$, thus by Lemma9, $\sum_{n=2}^{\infty} g_n$ is Baire-1. Therefore $f = f_{M_1} + \sum_{n=2}^{\infty} g_n$ is Baire-1 since f_{M_1} is Baire-1. $\square$

We now come back to prove (4)⇒(1).

If $\beta(f) < \omega_1$, then $\beta(f, \epsilon) < \omega_1$ for any $\epsilon > 0$. Fix arbitrary $\epsilon > 0$. Let X^0 be X; X^α be $X^{\alpha-1}(f, \epsilon)$ if α is a successor ordinal; X^α be $\bigcap_{\gamma<\alpha} X^\gamma(f, \epsilon)$ if α is a limit ordinal. ($P(f, \epsilon)$ is only defined when P is a closed subset of X. So to show each X^α is well-defined, we need to show X^α is closed in X for any α, we do this by induction. $X^0 = X$ is closed. Assume for any $\gamma < \alpha$, X^γ is closed in X. If α is a successor ordinal, $X^\alpha = X^{\alpha-1}(f, \epsilon)$ so a closed subset of $X^{\alpha-1}$, therefore X^α is closed in X since $X^{\alpha-1}$ is closed in X. If α is a limit ordinal, $X^\alpha = \bigcap_{\gamma<\alpha} X^\gamma(f, \epsilon)$, this is a intersection of closed sets in X, thus X^α is closed in X. So we can conclude for any α, X^α is closed in X.)

Since $\beta(f, \epsilon) < \omega_1$, we have

$$X = \bigcup_{\alpha<\beta(f,\epsilon)} X^\alpha \setminus X^{\alpha+1}.$$

Consider on $X^\alpha \setminus X^{\alpha+1}$ ($\alpha < \beta(f, \epsilon)$), for each $x \in X^\alpha \setminus X^{\alpha+1}$, there exists $\delta_x > 0$ such that for any $y, z \in B_{X^\alpha}(x, \delta_x)$, $|f(y) - f(z)| < \epsilon$. The balls form an open covering Λ of $X^\alpha \setminus X^{\alpha+1}$. Since $X^\alpha \setminus X^{\alpha+1}$ is a metric space hence paracompact, there exists a partition of unity Γ subordinated to the open covering Λ. For any $K_\beta \in \Gamma$, we can find a $U_\beta \in \Lambda$ such that $\text{supp}K_\beta \subset U_\beta$. Those U_β also form an open covering Λ_0 of $X^\alpha \setminus X^{\alpha+1}$. For each $K_\beta \in \Gamma$, fix a $x_\beta \in \text{supp}K_\beta$. Then for any $x \in X^\alpha \setminus X^{\alpha+1}$, there exists $r_x > 0$ such that $B_{X^\alpha \setminus X^{\alpha+1}}(x, r_x)$ only intersects finitely many $\text{supp}K_\beta$. Therefore x only belongs to finitely many $\text{supp}K_\beta$.

Let

$$g^\alpha(x) = \sum_\beta f(x_\beta) K_\beta(x),$$

$K_\beta(x) \neq 0$ for only finitely many β. Claim that for any $x \in X^\alpha \setminus X^{\alpha+1}$, $|f(x) - g^\alpha(x)| < \epsilon$ and also g^α is continuous on $X^\alpha \setminus X^{\alpha+1}$. In fact, by the definition of partition of unity, $\sum_\beta K_\beta(x) = 1$, therefore $f(x) = f(x) \sum_\beta K_\beta(x) = \sum_\beta f(x) K_\beta(x)$, $|f(x) - g^\alpha(x)| = |\sum_\beta (f(x) - f(x_\beta)) K_\beta(x)| \leq \sum_\beta |f(x) - f(x_\beta)| K_\beta(x)$. Since $x_\beta \in B_{X^\alpha}(x, \delta_x)$ for any β, $|f(x) - f(x_\beta)| < \epsilon$, thus $|f(x) - g^\alpha(x)| < \epsilon \sum_\beta K_\beta(x) = \epsilon$. For any $x \in X^\alpha \setminus X^{\alpha+1}$, on $B_{X^\alpha \setminus X^{\alpha+1}}(x, r_x)$, only finitely many $\text{supp}K_\beta$ intersect the ball. Thus, $\forall\ y \in B_{X^\alpha \setminus X^{\alpha+1}}(x, r_x)$, $g^\alpha(y) = \sum_\beta f(x_\beta) K_\beta(y)$ with only those finitely many K_β. Since each K_β is continuous on $X^\alpha \setminus X^{\alpha+1}$, we have g^α is continuous at x. So we conclude $g^\alpha(x)$ is continuous on $X^\alpha \setminus X^{\alpha+1}$.

Define $F_\epsilon(x) = g^\alpha(x)$ for $x \in X^\alpha \setminus X^{\alpha+1}$, $(0 \leq \alpha < \beta(f,\epsilon))$. For any $x \in X$, $x \in X^\alpha \setminus X^{\alpha+1}$ for some $0 \leq \alpha < \beta(f,\epsilon)$, thus $|F_\epsilon - f(x)| = |g^\alpha(x) - f(x)| < \epsilon$. On each $X^\alpha \setminus X^{\alpha+1}$, $F_\epsilon(x) = g^\alpha(x)$, therefore $F_\epsilon(x)$ is continuous on any $X^\alpha \setminus X^{\alpha+1}$. Define $f_n(x) = F_{1/2^n}(x)$, then f_n converges to f uniformly.

Fix $\epsilon > 0$, let X^α $(\alpha < \beta(f,\epsilon))$ be as defined before. Let $T_{r,\alpha} = X^\alpha \setminus N_X(X^{\alpha+1}, r)$ where $N_X(X^{\alpha+1}, r) = \bigcup_{x \in X^{\alpha+1}} B_X(x,r)$. Notice that $T_{r,\alpha}$ is closed in X since we have shown that X^α is closed in X for any α. We also define $V_r = \bigcup_{\alpha < \beta(f,\epsilon)} T_{r,\alpha}$. Claim that for any α,

$$N_X(T_{r,\alpha}, r) \cap (V_r \setminus T_{r,\alpha}) = \emptyset,$$

where $N_X(T_{r,\alpha}, r) = \bigcup_{x \in T_{r,\alpha}} B_X(x,r)$. In fact, it suffices to show $N_X(T_{r,\alpha}, r) \cap T_{r,\beta} = \emptyset$ for any $\alpha \neq \beta$. Without loss of generality, assume $\alpha > \beta$. For any $x \in N_X(T_{r,\alpha}, r)$, $x \in B_X(y,r)$ for some $y \in T_{r,\alpha} \subset X^\alpha$, thus $x \in N_X(X^\alpha, r)$. Since $\alpha > \beta$, $\alpha \geq \beta + 1$, we have $X^\alpha \subset X^{\beta+1}$. Therefore $x \in N_X(X^\alpha, r) \subset N_X(X^{\beta+1}, r)$, $x \notin T_{r,\beta}$. Hence $N_X(T_{r,\alpha}, r) \cap T_{r,\beta} = \emptyset$ for any $\alpha \neq \beta$.

Claim that V_r is closed in X for any r. In fact, for a sequence $\{x_n\}$ in V_r, if it is convergent in X, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$, $d(x_n, x_m) < r/2$. Since $x_N \in V_r$, we have $x_N \in T_{r,\alpha}$ for some α. For any $n > N$, $x_n \in B_X(x_N, r) \subset N_X(T_{r,\alpha}, r)$. Since $N_X(T_{r,\alpha}, r) \cap (V_r \setminus T_{r,\alpha}) = \emptyset$, we have $x_n \notin V_r \setminus T_{r,\alpha}$. Hence $x_n \in T_{r,\alpha}$ for $x_n \in V_r$. So we have for $n \geq N$, $x_n \in T_{r,\alpha}$ for a fixed α. Since $T_{r,\alpha}$ is closed in X, $\{x_n\}$ is convergent in X, we have $\lim_{n\to\infty} x_n \in T_{r,\alpha} \subset V_r$. So V_r is closed in X for any r.

We have shown that F_ϵ is continuous on each $X^\alpha \setminus X^{\alpha+1}$, so continuous on each $T_{r,\alpha}$. Then we show that F_ϵ is continuous on V_r. For any $x \in V_r$, $x \in T_{r,\alpha}$ for some α. Since F_ϵ is continuous on $T_{r,\alpha}$, we have for any $\Delta > 0$, there exists $\delta_1 > 0$ such that for $y \in B_{T_{r,\alpha}}(x, \delta_1)$, $|F_\epsilon(y) - F_\epsilon(x)| < \Delta$. Take $\delta_2 = \min\{r, \delta_1\}$. Since $N_X(T_{r,\alpha}, r) \cap (V_r \setminus T_{r,\alpha}) = \emptyset$, $B_{T_{r,\alpha}}(x, \delta_2)$ is actually $B_{V_r}(x, \delta_2)$. So we have for $y \in B_{V_r}(x, \delta_2)$, $|F_\epsilon(y) - F_\epsilon(x)| < \Delta$. Thus F_ϵ is continuous at x, F_ϵ is continuous on V_r.

Let $F_\epsilon|_{V_r}$ to be $F_{\epsilon,r}$. By Tietze Extension Theorem, $F_{\epsilon,r}$ can be extended from V_r to a continuous map of all of X, namely $\psi_{\epsilon,r}$. Let $\phi_{\epsilon,n}$ to be $\psi_{\epsilon,1/2^n}$. Claim that $\phi_{\epsilon,n}$ converges to F_ϵ. In fact, for any $x \in X$, $x \in X^\alpha \setminus X^{\alpha+1}$ for some α. Since $X^{\alpha+1}$ is closed in X, $x \notin X^{\alpha+1}$, then there exists $\delta > 0$ such that $B_X(x, \delta) \cap X^{\alpha+1} = \emptyset$. Take $1/2^N < \delta$, we have $x \notin N_X(X^{\alpha+1}, 1/2^N)$. Therefore $x \in X^\alpha \setminus N_X(X^{\alpha+1}, 1/2^N) \subset V_{1/2^N}$. For $n > N$, $\phi_{\epsilon,n}(x) = \psi_{\epsilon,1/2^n}(x)$. Since $x \in V_{1/2^N} \subset V_{1/2^n}$, we have $\psi_{\epsilon,1/2^n}(x) = F_{\epsilon,1/2^n}(x) = F_\epsilon(x)$. Thus $\phi_{\epsilon,n}(x)$ converges to $F_\epsilon(x)$ for any x. Since each $\phi_{\epsilon,n}$ is continuous, F_ϵ is Baire-1. Therefore each f_n we defined before is Baire-1. Since f_n converges to f uniformly, by Lemma10, f is Baire-1, which completes the proof. □

参考文献

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[3] Thomas Jech, *Set Theory*

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数学家逸事

有一天，波兰数学家Sierpinski要搬家，他的夫人把行李拿出来以后对他说："我去叫辆出租车，你在这儿看好行李，总共有10个箱子。" 过一会儿，他的夫人回来了，他对夫人说道："刚才你说有10个箱子，可是我数了只有9个箱子。"

"不对，肯定是10个。"

"说什么呢，我再数一遍，0，1，2，3……"