A Canonical Dual Approach for Solving Quadratic Programs with Linear and a Quadratic Constraint

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摘　要

In this article, we apply the canonical dual approach to solve the quadratic program with a quadratic constraint and linear constraints. we form the canonical dual problem under the dual Slater condition. The relationship between dual variables and primal solutions are investigated. Then we design a computational method to produce a sequence that converges to the dual optimal solutions. Strong duality will hold under some sufficient conditions. Examples are also presented.

Keywords: quadratic programming, canonical duality, global optimization

§1 Introduction

We study the quadratic program with a quadratic and linear constraints $(\mathcal{P})$ of the following form:

$$\begin{aligned} \nu = \quad \min \quad & P(x) = \tfrac{1}{2}x^TQx + f^Tx \\ s.t. \quad & \tfrac{1}{2}x^TBx \le \mu, \\ & Ax \le b, \end{aligned} \tag{1-1}$$

where x is an $n \times 1$ vector, Q an $n \times n$ symmetric matrix, f an $n \times 1$ vector, B an $n \times n$ symmetric matrix, μ a constant, A an $m \times n$ matrix, and b an $m \times 1$ vector. We assume the feasible domain of $(\mathcal{P})$ is not empty.

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In general, the problem $(\mathcal{P})$ is NP-hard. [1] In other words, it is impossible to be solved generally in polynomial time, unless P=NP. The problem $(\mathcal{P})$ is a subclass of the quadratically constrained quadratic programs $(\mathcal{QCQP})$:

$$\begin{array}{ll} \min & \frac{1}{2}x^TQx + f^Tx \\ s.t. & \frac{1}{2}x^TB_ix + q_i^Tx + r_i \leq 0, \quad i = 1, 2, \cdots, m, \\ & Ax \leq b. \end{array} \tag{1-2}$$

with m=1, i.e., there is only one quadratic constraint.

$(\mathcal{QCQP})$ is a well-studied problem in the global optimization literature with many applications, frequently arising from Euclidean distance geometry. [2] Many special subclasses of $(\mathcal{QCQP})$ have been studied, some of which are closely related to the problem $(\mathcal{P})$ here.

For instance, if there is no quadratic constraint, the problem $(\mathcal{P})$ becomes the linearly constrained quadratic program. It is generally NP-hard and a canonical dual iterate scheme has been developed to produce a sequence of point that converges to a KKT solution. [3] Another example is the quadratic program with one quadratic constraint but no linear constraints, i.e., $A = 0$ in $(\mathcal{P})$. This program can be solved by SDP relaxations. [4] Meanwhile, a canonical dual method has been proposed and can find the global minimizer. [5]

The problem $(\mathcal{P})$ can also be regarded as generalization of the trust region subproblem [6]:

$$\begin{array}{lll} \nu = & \min & P(x) = \frac{1}{2}x^TQx + f^Tx \\ & s.t. & \frac{1}{2}x^TBx \leq \mu, \\ & & Ax = b, \end{array} \tag{1-3}$$

where B is positive semidefinite. Here we make no assumptions about B except it is symmetric.

Recently, Gao [7][8] pioneered a canonical duality theory for nonconvex programming problems. The primal-dual relationship provides new insights to generating near-optimal solutions when the corresponding canonical dual problem is solved. Fang et al [9] [10] adopted this approach to tackle the 0-1 quadratic programming and sum-of-quadratic-ratios problems, Wang et al [11] further extended the approach for handling multi-integer quadratic programming problems. Their computational results showed that the approach could be computationally efficient, but much more theoretical analysis is needed. The details of the canonical duality theory can be referred to [7][8][5].

In this article, we try to apply the canonical dual approach to solve the problem $(\mathcal{P})$. The article is organized as follows. In chapter 2 we form the canonical dual problem and prove the convexity of it. The relationship between dual variables and primal solutions are revealed in chapter 3. A computational method is designed in chapter 4 to produce a sequence that converges to the dual optimal solutions. Then in the following chapter 5 we make further discussions about strong duality and sufficient conditions. Examples are presented in chapter 6 and conclusions can be found in chapter 7.

§2 The Canonical Dual Problem

In this chapter we form the canonical dual problem that can provide a lower bound for ν. We begin with the Lagrangian function associated with $(\mathcal{P})$:

$$L(x, \lambda, \sigma) = \frac{1}{2} x^T(\lambda B + Q)x + (f + A^T\sigma)^T x - \mu\lambda - b^T\sigma, \tag{2-4}$$

where $\lambda \geq 0$ and $\sigma \geq 0$.

The key idea of the canonical duality theory [7][8] is to find all stationery points $x(\lambda, \sigma)$ of $L(x, \lambda, \sigma)$ such that $\frac{\partial L(x, \lambda, \sigma)}{\partial x} = 0$, with the assumption $\lambda B + Q$ is invertible. Then the relationship between $x(\lambda, \sigma)$ and (λ, σ) is given by

$$x(\lambda, \sigma) = -(\lambda B + Q)^{-1}(f + A^T\sigma). \tag{2-5}$$

Substituting it to $L(x, \lambda, \sigma)$, we have the canonical dual function $P^d(\lambda, \sigma)$:

$$P^d(\lambda, \sigma) = -\frac{1}{2}(f + A^T\sigma)^T(\lambda B + Q)^{-1}(f + A^T\sigma) - \mu\lambda - b^T\sigma. \tag{2-6}$$

In general, $P^d(\lambda, \sigma)$ is an "over-estimator" of the Lagrangian dual $l(\lambda, \sigma) := \min_{x\in\mathbb{R}^n} L(x, \lambda, \sigma)$. However, the two dual functions $P^d(\lambda, \sigma)$ and $l(\lambda, \sigma)$ coincide on the set $\{\lambda \geq 0 | \lambda B + Q \succ 0\}$.

Before we move on, we need an assumption here. Define $S_\lambda^+ = \{\lambda \geq 0 | \lambda B + Q \succ 0\}$. We assume that S_λ^+ is not empty, i.e., $\exists \lambda'$ satisfying $\lambda' \geq 0$ and $\lambda' B + Q \succ 0$. Under this assumption, the set S_λ^+ is actually a convex interval with non-empty interior. [5] We denote $intS_\lambda^+$ as (λ_1, λ_2). Note this assumption is in fact dual Slater condition.

Denote $S_\sigma^+ = \{\sigma | \sigma \geq 0\}$, and $\mathcal{F} = S_\lambda^+ \times S_\sigma^+$. Now we form the canonical dual problem $(\mathcal{P}^d)$:

$$\begin{aligned} \nu^d = \quad & \sup \quad P^d(\lambda, \sigma) \\ & s.t. \quad \lambda \geq 0, \\ & \qquad \lambda B + Q \succ 0, \\ & \qquad \sigma \geq 0. \end{aligned} \tag{2-7}$$

Now we establish

Lemma 2.1. *(Weak Duality)* $\nu^d \leq \nu$.

Proof On $\mathcal{F}$,

$$\begin{aligned} P^d(\lambda, \sigma) &= L(x(\lambda, \sigma), \lambda, \sigma) \\ &= \min_{x \in \mathbb{R}^n} L(x, \lambda, \sigma) \\ &\leq \min_{\frac{1}{2}x^T Bx \leq \mu, Ax \leq b} L(x, \lambda, \sigma) \\ &\leq \min_{\frac{1}{2}x^T Bx \leq \mu, Ax \leq b} P(x) = \nu. \end{aligned}$$

Then $\nu^d \leq \nu$. $\square$

Lemma 2.1 implies that the optimal value of $(\mathcal{P}^d)$ serves as a lower bound for the primal optimal value. The following lemma shows that the canonical dual problem $(\mathcal{P}^d)$ is a convex optimization problem with a concave objective function and a convex domain.

Lemma 2.2. *$P^d(\lambda, \sigma)$ is a continuous concave function over $int\mathcal{F}$.*

Proof Obviously, $P^d(\lambda, \sigma)$ is continuous over $int\mathcal{F}$.

For any $(\lambda, \sigma) \in int\mathcal{F}$, simple computations show:

$$\frac{\partial P^d(\lambda, \sigma)}{\partial \lambda} = \frac{1}{2}(f + A^T\sigma)^T(\lambda B + Q)^{-1}B(\lambda B + Q)^{-1}(f + A^T\sigma) - \mu, \tag{2-8}$$

$$\frac{\partial P^d(\lambda, \sigma)}{\partial \sigma} = -A(\lambda B + Q)^{-1}(f + A^T\sigma) - b, \tag{2-9}$$

$$\frac{\partial^2 P^d(\lambda,\sigma)}{\partial\sigma\partial\lambda} = (\frac{\partial^2 P^d(\lambda,\sigma)}{\partial\lambda\partial\sigma})^T = A(\lambda B+Q)^{-1}B(\lambda B+Q)^{-1}(f+A^T\sigma), \tag{2-10}$$

$$\frac{\partial^2 P^d(\lambda,\sigma)}{\partial\lambda^2} = -(f+A^T\sigma)^T(\lambda B+Q)^{-1}B(\lambda B+Q)^{-1}B(\lambda B+Q)^{-1}(f+A^T\sigma), \tag{2-11}$$

$$\frac{\partial^2 P^d(\lambda,\sigma)}{\partial\sigma^2} = -A(\lambda B+Q)^{-1}A^T, \tag{2-12}$$

Then the Hessian matrix of $P^d(\lambda,\sigma)$ is

$$\begin{aligned} & \frac{\partial^2 P^d(\lambda,\sigma)}{\partial(\lambda,\sigma)^2} \\ = & -[B(\lambda B+Q)^{-1}(f+A^T\sigma),A^T]^T(\lambda B+Q)^{-1}[B(\lambda B+Q)^{-1}(f+A^T\sigma),A^T] \\ \preceq & \ 0. \end{aligned} \tag{2-13}$$

This completes the proof. □

So we may solve the dual problem $(\mathcal{P}^d)$ to obtain useful information about the lower bound of ν, and the primal solution x due to the relationship (2-5). We present the relationship in details in the next chapter.

§3 Primal-Dual Relationship

As we assume at the beginning of this paper, the primal problem has a non-empty feasible domain. Therefore the supremum of $(\mathcal{P}^d)$ is finite, i.e. $\nu^d < +\infty$.

We first consider a subproblem that will appear in the following lemma and chapter **??**. For any fixed $\lambda \in intS_\lambda^+$, the relaxed canonical dual problem $(\mathcal{RP}^d)$ is defined as follows.

$$\begin{aligned} \max \quad & RP^d(\sigma) = P^d(\lambda,\sigma) \\ s.t. \quad & \sigma \geq 0. \end{aligned} \tag{3-14}$$

Noting that the primal problem is feasible, we know $\nu < +\infty$. Hence $(\mathcal{RP}^d)$ is bounded above and has a feasible point $\sigma = 0$. This implies that $(\mathcal{RP}^d)$ has a finite maximizer. Denote the maximizer by $\sigma(\lambda)$.

Lemma 3.1. *Suppose ν^d occurs at $(\bar{\lambda}, \bar{\sigma})$ and $\bar{\sigma}$ is finite. For every sequence $\{\lambda^k\}$ such that $\lim_{k\to\infty} \lambda^k = \bar{\lambda}$, we assume that the corresponding sequence $\{\sigma^k\} = \{\sigma(\lambda^k)\}$ satisfies $\lim_{k\to\infty} \sigma^k = \bar{\sigma}$. Then we have three kinds of cases depending on where $\bar{\lambda}$ lies:*

(i) If $\bar{\lambda} \in intS_\lambda^+$, then

$$\bar{x} = -(\bar{\lambda}B + Q)^{-1}(f + A^T\bar{\sigma}),$$

satisfies

$$\frac{1}{2}\bar{x}^T B\bar{x} - \mu = 0,$$

$$\nu^d = \frac{1}{2}(f + A^T\bar{\sigma})^T\bar{x} - \mu\bar{\lambda} - b^T\bar{\sigma},$$

(ii) If $\bar{\lambda}$ is the left boundary λ_1 of S_λ^+, then

$$\bar{x}_1 := \lim_{\lambda\to\lambda_1^+}[-(\lambda B + Q)^{-1}(f + A^T\sigma(\lambda))]$$

and

$$\Delta_1 := \lim_{\lambda\to\lambda_1^+} \frac{\partial P^d(\lambda, \bar{\sigma})}{\partial\lambda} = \frac{1}{2}\bar{x}_1^T B\bar{x}_1 - \mu \le 0$$

exist.

$$\nu^d = \frac{1}{2}(f + A^T\bar{\sigma})^T\bar{x}_1 - \mu\lambda_1 - b^T\bar{\sigma},$$

(iii) If $\bar{\lambda}$ is the right boundary λ_2 of S_λ^+, then

$$\bar{x}_2 := \lim_{\lambda\to\lambda_2^-}[-(\lambda B + Q)^{-1}(f + A^T\sigma(\lambda))]$$

and

$$\Delta_2 := \lim_{\lambda\to\lambda_2^-} \frac{\partial P^d(\lambda, \bar{\sigma})}{\partial\lambda} = \frac{1}{2}\bar{x}_2^T B\bar{x}_2 - \mu \ge 0$$

exist.

$$\nu^d = \frac{1}{2}(f + A^T\bar{\sigma})^T \bar{x}_2 - \mu\lambda_2 - b^T\bar{\sigma},$$

Moreover, if $\lambda_2 = \infty$*, we have*

$$\mu = 0, \Delta_2 = 0.$$

Proof In case (i), if $\bar{\lambda} \in intS_\lambda^+$, $(\bar{\lambda}B + Q)$ is invertible and $\bar{\lambda}$ is a stationary point of $P^d(\lambda, \bar{\sigma})$. Then $\dfrac{\partial P^d(\bar{\lambda}, \bar{\sigma})}{\partial \lambda} = 0$, which implies $\frac{1}{2}\bar{x}^T B\bar{x} - \mu = 0$, where $\bar{x} = -(\bar{\lambda}B + Q)^{-1}(f + A^T\bar{\sigma})$.

Direct computation shows that

$$\nu^d = \frac{1}{2}(f + A^T\bar{\sigma})^T\bar{x} - \mu\bar{\lambda} - b^T\bar{\sigma}.$$

Now we turn to case (ii).

Recall that $(\lambda' B + Q) \succ 0$. Then there exists an orthogonal matrix M_1 s.t. $M_1^T(\lambda' B + Q)M_1 = I$. There is also an orthogonal matrix M_2 s.t. $M_2^T M_1^T B M_1 M_2 = H$, where $H = diag(h_1, h_2, \cdots, h_n)$ is diagonal. Let $M = M_1M_2$, then for any $\lambda \in S_\lambda^+$,

$$M^T(\lambda B + Q)M = M^T[(\lambda - \lambda')B + \lambda' B + Q]M = I + (\lambda - \lambda')H,$$

Denote $D = I + (\lambda - \lambda')H = diag(d_1(\lambda), d_2(\lambda), \cdots, d_n(\lambda))$, where $d_i(\lambda) = 1 + (\lambda - \lambda')h_i > 0, i = 1, 2, \cdots, n$. It follows that

$$(\lambda B + Q)^{-1} = MD^{-1}M^T,$$

where $D^{-1} = diag(d_1^{-1}(\lambda), d_2^{-1}(\lambda), \cdots, d_n^{-1}(\lambda))$, and $d_i^{-1}(\lambda) = \frac{1}{1+(\lambda-\lambda')h_i}, i = 1, 2, \cdots, n$.

Then,

$$P^d(\lambda, \sigma(\lambda)) = -\frac{1}{2}(f + A^T\sigma(\lambda))^T MD^{-1}M^T(f + A^T\sigma(\lambda)) - \mu\lambda - b^T\sigma(\lambda). \qquad (3\text{-}15)$$

The function $d_i^{-1}(\lambda)$ is monotone with a lower bound 0 on $intS_\lambda^+$. When $\lambda \to \lambda_1^+$, $d_i^{-1}(\lambda)$ must tend to a finite positive number or $+\infty$. If $\lim\limits_{\lambda\to\lambda_1^+} d_i^{-1}(\lambda) = +\infty$, the corresponding $[M^T(f + A^T\sigma(\lambda))]_i$ must tend to 0 since $\nu^d > -\infty$, where $[M^T(f + A^T\sigma(\lambda))]_i$ is the ith component of $[M^T(f + A^T\sigma(\lambda))]$.

Therefore, $\lim\limits_{\lambda\to\lambda_1^+} d_i^{-1}(\lambda)[M^T(f + A^T\sigma(\lambda))]_i$ exists for all $i = 1, 2, \cdots, n$. We define

$$\bar{x}_1 := M \lim_{\lambda\to\lambda_1^+} D^{-1}(\lambda)[M^T(f + A^T\sigma(\lambda))] = \lim_{\lambda\to\lambda_1^+}[-(\lambda B + Q)^{-1}(f + A^T\sigma(\lambda))].$$

Consequently, the following limit exists.

$$\Delta_1 := \lim_{\lambda\to\lambda_1^+} \frac{\partial P^d(\lambda,\bar{\sigma})}{\partial\lambda} = \frac{1}{2}\bar{x}_1^T B\bar{x}_1 - \mu \le 0.$$

Direct computation shows

$$\nu^d = \frac{1}{2}(f + A^T\bar{\sigma})^T\bar{x}_1 - \mu\lambda_1 - b^T\bar{\sigma}.$$

Finally, we come to the case (iii). Note that λ_2 may be $+\infty$. If λ_2 is finite, existence of the limits can be proved by similar argument for case (ii).

If $\lambda_2 = +\infty$, we note B is necessarily positive semidefinite. So $h_i \ge 0$ for all $i = 1, 2, \cdots, n$. Then

$$\lim_{\lambda\to+\infty} d_i^{-1}(\lambda)[M^T(f + A^T\sigma(\lambda))]_i = \begin{cases} 0, & \text{if } h_i > 0,\\ [M^T(f + A^T\sigma(\lambda))]_i, & \text{if } h_i = 0. \end{cases} \tag{3-16}$$

So it follows that

$$\bar{x}_2 := \lim_{\lambda\to+\infty}[-(\lambda B + Q)^{-1}(f + A^T\sigma(\lambda))]$$

and

$$\Delta_2 := \lim_{\lambda\to+\infty} \frac{\partial P^d(\lambda,\bar{\sigma})}{\partial\lambda} = \frac{1}{2}\bar{x}_2^T B\bar{x}_2 - \mu \ge 0$$

exist. By $\nu^d = \frac{1}{2}(f + A^T\bar{\sigma})^T\bar{x}_2 - \mu\lambda_2 - b^T\bar{\sigma}$,, only when $\mu = 0$ can $-\infty < \nu^d < +\infty$ occur at $\lambda_2 = +\infty$. And $\Delta_2 = 0$ can be shown by computation:

$$\begin{aligned}
& \lim_{\lambda\to+\infty} \tfrac{\partial P^d(\lambda,\sigma(\lambda))}{\partial\lambda} \\
= & \lim_{\lambda\to+\infty} \tfrac{1}{2}(f + A^T\sigma(\lambda))^T(\lambda B + Q)^{-1}B(\lambda B + Q)^{-1}(f + A^T\sigma(\lambda)) \\
= & \lim_{\lambda\to+\infty} \tfrac{1}{2}[M^T(f + A^T\sigma(\lambda))]^T[M^T(\lambda B + Q)M]^{-1}M^T BM[M^T(\lambda B + Q)M]^{-1}[M^T(f + A^T\sigma(\lambda))] \\
= & \lim_{\lambda\to+\infty} \tfrac{1}{2}[M^T(f + A^T\sigma(\lambda))]^T diag(\tfrac{h_1}{[1+(\lambda-\lambda')h_1]^2}, \tfrac{h_2}{[1+(\lambda-\lambda')h_2]^2}, \cdots, \tfrac{h_n}{[1+(\lambda-\lambda')h_n]^2})[M^T(f + A^T\sigma(\lambda))] \\
= & \ 0
\end{aligned}$$

The proof is now completed. $\square$

§4 A Computational Method

In this chapter we propose a numerical computational method to produce a sequence $\{(\lambda^k, \sigma^k)\}$ that will have a convergent subsequence if $\{\sigma^k\}$ is bounded. The supremum of the canonical dual problem $(\mathcal{P}^d)$ occurs at the limit point $(\bar{\lambda}, \bar{\sigma})$.

§4.1 Compute (λ_1, λ_2)

The first challenge we are facing is how to compute $S_\lambda^+ = (\lambda_1, \lambda_2)$.

λ_1 is easy to decide. Actually, λ_1 is the optimal value of the following semi-definite program.

$$\begin{array}{ll} \min & s \\ s.t. & s \geq 0 \\ & sB + Q \succeq 0 \end{array} \tag{4-17}$$

λ_2 is also not difficult to compute. If $B \succeq 0$, λ_2 is simply $+\infty$. Otherwise, λ_2 is the optimal value of the following SDP

$$\begin{array}{ll} \max & t \\ s.t. & t \geq 0 \\ & tB + Q \succeq 0 \end{array} \tag{4-18}$$

Hence, $S_\lambda^+ = (\lambda_1, \lambda_2)$.

§4.2 Compute $(\bar{\lambda}, \bar{\sigma})$

§4.2.1 The Subproblem $(\mathcal{RP}^d)$

Recall the subproblem $(\mathcal{RP}^d)$.

$$\begin{array}{ll} \max & RP^d(\sigma) = P^d(\lambda, \sigma) \\ s.t. & \sigma \geq 0. \end{array} \tag{4-19}$$

KKT conditions are satisfied by its finite maximizer $\sigma\lambda$:

$$\begin{aligned}
\sigma(\lambda) &\geq 0 && \text{(4-20)}\\
\tau(\lambda) &\geq 0 && \text{(4-21)}\\
\tau(\lambda)^T\sigma(\lambda) &= 0 && \text{(4-22)}\\
\tau(\lambda) + \triangledown RP^d(\sigma) &= 0. && \text{(4-23)}
\end{aligned}$$

Recall that $\triangledown RP^d(\sigma) = \dfrac{\partial P^d(\lambda,\sigma)}{\partial\sigma} = -A(\lambda B+Q)^{-1}(f+A^T\sigma) - b$. Denote $x(\lambda) = -(\lambda B+Q)^{-1}(f+A^T\sigma(\lambda))$. The KKT conditions can be reexpressed by

$$\begin{aligned}
\sigma(\lambda) &\geq 0 && \text{(4-24)}\\
b - Ax(\lambda) &\geq 0 && \text{(4-25)}\\
\sigma(\lambda)^T(b - Ax(\lambda)) &= 0 && \text{(4-26)}\\
x(\lambda) &= -(\lambda B+Q)^{-1}(f+A^T\sigma(\lambda)). && \text{(4-27)}
\end{aligned}$$

Lemma 4.1. *If $\bar{\lambda}$ is where ν^d occurs, and also suppose a sequence $\{\lambda^k\}$ satisfies $\lambda^k \to \bar{\lambda}$, when $\lambda^k \to \infty$. Denote $\sigma^k = \sigma(\lambda^k)$. If $\{\sigma^k\}$ is bounded, denote the limit point of any convergent subsequence as $\bar{\sigma}$, then $(\bar{\lambda},\bar{\sigma})$ is where ν^d lies.*

Proof We extend the domain where $P^d(\lambda,\sigma)$ is defined to its boundary. Since $P^d(\lambda,\sigma)$ is continuous on $int\mathcal{F}$, we let $P^d(\lambda_1,\sigma) = \lim\limits_{\lambda\to\lambda_1^+} P^d(\lambda,\sigma)$ and $P^d(\lambda_2,\sigma) = \lim\limits_{\lambda\to\lambda_2^-} P^d(\lambda,\sigma)$, if $\lambda_2 < +\infty$.

For convenience, We still use $\{(\lambda^k,\sigma^k)\}$ to denote the convergent subsequence. We just need to show that $\bar{\sigma}$ is the optimal solution of the relaxed problem $(\mathcal{RP}^d)$ with λ fixed as $\bar{\lambda}$. This is implied by the KKT conditions. Since (λ^k,σ^k) satisfy

$$\begin{aligned}
\sigma^k &\geq 0 && \text{(4-28)}\\
b - Ax(\lambda^k) &\geq 0 && \text{(4-29)}\\
(\sigma^k)^T(b - Ax(\lambda^k)) &= 0 && \text{(4-30)}\\
x(\lambda^k) &= -(\lambda B+Q)^{-1}(f+A^T\sigma^k). && \text{(4-31)}
\end{aligned}$$

Let $k\to\infty$. Noting $\lambda^k\to\bar{\lambda}$, $\sigma^k\to\bar{\sigma}$ and $\lim\limits_{k\to\infty} x(\lambda^k)$ exists due to Lemma 3.1. It holds that

$$\bar{\sigma} \geq 0 \quad (4\text{-}32)$$

$$b - Ax(\bar{\lambda}) \geq 0 \quad (4\text{-}33)$$

$$\bar{\sigma}^T(b - Ax(\bar{\lambda})) = 0 \quad (4\text{-}34)$$

$$x(\bar{\lambda}) = \lim_{k\to\infty} -(\lambda^k B + Q)^{-1}(f + A^T\bar{\sigma}). \quad (4\text{-}35)$$

These are the KKT conditions for the relaxed problem $(\mathcal{RP}^d)$ with $\bar{\lambda}$. The proof is completed. □

§4.2.2 An Iterate Scheme

We basically use dichotomy on λ to find $\bar{\lambda}$. At every step for λ, the corresponding $\bar{\sigma}$ is determined by solving $(\mathcal{RP}^d)$.

If $\lambda_2 < +\infty$, we just apply dichotomy to the finite interval $(a_1, a_2) = (\lambda_1, \lambda_2)$. If $\lambda_2 = +\infty$, however, we have to decide the interval (a_1, a_2) where $\bar{\lambda}$ lies, or to the contrast, make sure that $\bar{\lambda} = +\infty$ if the interval can not be found.

As for solving $(\mathcal{RP}^d)$, we use the disciplined convex programming solver **cvx**[12].

The numerical iterate scheme (NIS) is as follows.

STEP 0 Set $tol = 1 \times 10^{-4}, k = 0, flag = 0$. Let $a_1 = \lambda_1$.

STEP 1 If $\lambda_2 < +\infty$, let $a_2 = \lambda_2, flag = 1$. If $\lambda_2 = +\infty$, let $a_2 = 10a_1 + 10$.

STEP 2 Let $\lambda^k = (a_1 + a_2)/2$. Solve $(\mathcal{RP}^d(\lambda^k))$ to find the corresponding $\sigma^k = \sigma(\lambda^k)$. Compute $\delta^k = \dfrac{\partial P^d(\lambda^k, \sigma^k)}{\partial \lambda} = \frac{1}{2}(f + A^T\sigma^k)^T(\lambda^k B + Q)^{-1}B(\lambda^k B + Q)^{-1}(f + A^T\sigma^k) - \mu$.

STEP 3 If $|\delta^k| < tol$, go to STEP 5. If $\delta^k > tol$ and $flag = 0$, set $a_2 = 2a_2$ and $a_1 = \lambda^k$. If $\delta^k > tol$ and $flag = 1$, set $a_1 = \lambda^k$. If $\delta^k < tol$, set $a_2 = \lambda^k$, and let $flag = 1$.

STEP 4 If $|a_2 - a_1| < tol$, go to STEP 5. Otherwise, let $k = k + 1$ and go to STEP 2.

STEP 5 $\bar{\lambda} = \lambda^k$, $\bar{\sigma} = \sigma^k$. Stop.

Note if the procedures above do not stop, we know $\bar{\lambda} = +\infty$. $\bar{\sigma}$ can be chosen as the limit point of any convergent subsequence.

Numerically, we have the dual optimal solution $(\bar{\lambda}, \bar{\sigma})$ now. Further properties of it will bring results about strong duality.

§5 Strong Duality and Sufficient Conditions

We follow the results before to discuss on what occasions we can obtain a global minimizer of the primal problem $(\mathcal{P})$.

Theorem 5.1. *(Sufficient conditions for strong duality) Suppose ν^d is finite, and the supremum occurs at $(\bar{\lambda}, \bar{\sigma})$ such that $\bar{\sigma}$ is finite.*

(i) If $\bar{\lambda} \in intS_\lambda^+$, $\bar{x} = -(\bar{\lambda}B + Q)^{-1}(f + A^T\bar{\sigma})$ is a global minimizer.

(ii) If $\bar{\lambda}$ is the left boundary λ_1 of S_λ^+, $\bar{x}_1$ is a feasible solution of $(\mathcal{P})$,

$$P(\bar{x}_1) - \nu^d = -\lambda_1 \Delta_1.$$

Then either $\lambda_1 = 0$ or $\Delta_1 = 0$ assures $\bar{x}_1$ is a global minimizer.

(iii) If $\bar{\lambda}$ is the right boundary λ_2 of S_λ^+,

- *If $\lambda_2 = \infty$, $\bar{x}_2$ is a global minimizer of $(\mathcal{P})$.*
- *If $\lambda_2 < \infty$, $\bar{x}_2$ is a feasible solution only when $\Delta_2 = 0$. And when it holds, $\bar{x}$ is a global minimizer.*

(iv) (Boundarification Technique) In the remaining cases, ($\lambda_1 > 0$, $\Delta_1 < 0$ or $\lambda_2 < \infty$, $\Delta_2 > 0$), if there exists $\tilde{x}$ s.t. $(\lambda_i B + Q)\tilde{x} = 0$ and $A\tilde{x} = 0$, we can construct a global minimizer by the "boundarification technique". Specifically,

$$x^* = \bar{x}_i + \alpha\tilde{x},$$

where α satisfies

$$\alpha^2\tilde{x}^T B\tilde{x} + 2\alpha\bar{x}^T B\tilde{x} + \bar{x}^T B\bar{x} - 2\mu = 0.$$

Proof First, we consider the case (i). From Lemma 3.1 we know $\bar{x} = -(\bar{\lambda}B + Q)^{-1}(f + A^T\bar{\sigma})$ satisfies $\frac{1}{2}\bar{x}^T B\bar{x} = \mu$. The proof in Lemma 4.1 implies $A\bar{x} - b \leq 0$, then $\bar{x}$ is a feasible solution of $(\mathcal{P})$. Further computation shows $P(\bar{x}) = P^d(\bar{\lambda}, \bar{\sigma})$. This implies strong duality holds, and $\bar{x}$ is a global minimizer.

Then we work on the case (ii). Similar proof to that in the case (i) can perform here and $\bar{x}$ is then a feasible solution of the primal problem $(\mathcal{P})$. Furthermore, the duality gap

$P(\bar{x}_1) - \nu^d = -\lambda_1 \Delta_1$. If $\lambda_1 = 0$ or $\Delta_1 = 0$, the duality gap vanishes, and $\bar{x}_1$ is a global minimizer of the primal problem $(\mathcal{P})$.

As for the case (iii), similar argument reveals the strong duality when either $\lambda_2 = \infty, \mu = 0$ or $\Delta_2 = 0$ is satisfied.

Finally, we come to the remaining two cases.

When $\bar{\lambda} = \lambda_1 > 0$, we have $\lambda_1 B + Q \succeq 0$ and $\lambda B + Q \succ 0$ for $\lambda \in intS^+_\lambda$. Suppose that there is $\tilde{x}$ s.t. $(\lambda_i B + Q)\tilde{x} = 0$ and $A\tilde{x} = 0$. Then $\tilde{x}^T(\lambda B + Q)\tilde{x} > 0$ and $\tilde{x}^T(\lambda_1 B + Q)\tilde{x} = 0$. Hence $(\lambda - \lambda_1)\tilde{x}^T B \tilde{x} > 0$, and then $\tilde{x}^T B \tilde{x} > 0$. Now consider the following quadratic equation of α:

$$\alpha^2 \tilde{x}^T B \tilde{x} + 2\alpha \bar{x}_1^T B \tilde{x} + \bar{x}_1^T B \bar{x}_1 - 2\mu = 0. \tag{5-36}$$

Compute

$$\Delta = 4(\bar{x}_1^T B \tilde{x})^2 - 4\tilde{x}^T B \tilde{x}(\bar{x}_1^T B \bar{x}_1 - 2\mu) > 0.$$

So there is α_0 satisfying 5-36. Denote $x^* = \bar{x}_1 + \alpha_0 \tilde{x}$. Then

$$\frac{1}{2} x^{*T} B x^* - \mu = \frac{1}{2}[\bar{x}_1^T B \bar{x}_1 + 2\alpha_0 \bar{x}_1^T B \tilde{x} + \alpha_0^2 \tilde{x}^T B \tilde{x} - 2\mu] = 0 \tag{5-37}$$

and

$$Ax^* - b = A(\bar{x}_1 + \alpha_0 \tilde{x}) - b = A\bar{x}_1 - b \leq 0. \tag{5-38}$$

So x^* is a feasible solution of the primal problem $(\mathcal{P})$.

And

$$(f + A^T \bar{\sigma})^T \tilde{x} = -[(\lambda_1 B + Q)\bar{x}_1]^T \tilde{x} = \bar{x}_1^T (\lambda_1 B + Q)\tilde{x} = 0.$$

$$\bar{\sigma}^T (Ax^* - b) = \bar{\sigma}^T (A\bar{x}_1 - b) = 0.$$

Compute

$$\begin{aligned}
P(x^*) &= \frac{1}{2}x^{*T}Qx^* + f^Tx^* \\
&= \frac{1}{2}x^{*T}Qx^* + f^Tx^* + \lambda_1(\frac{1}{2}x^{*T}Bx^* - \mu) + \bar{\sigma}^T(Ax^* - b) \\
&= \frac{1}{2}x^{*T}(\lambda_1 B + Q)x^* + (f + A^T\bar{\sigma})^Tx^* - \mu\lambda_1 - b^T\bar{\sigma} \\
&= \frac{1}{2}(f + A^T\bar{\sigma})^T\bar{x} - \mu\lambda_1 - b^T\bar{\sigma} \\
&= \nu^d.
\end{aligned}$$

Therefore, x^* is a global minimizer.

When $\lambda_2 < \infty, \Delta_2 > 0$, the proof is similar. □

We see the strong duality can not always hold. This can be expected since the primal problem $(\mathcal{P})$ is in general NP-hard. Some examples will show various kinds of instances.

§6 Examples

Example 6.1. $Q = [1,0;0,1]; f = [1;1]; A = [-1,-1;-1,1;1,0]; b = [2;2;2]; B = [1,0;0,1]; \mu = 1.5$.

In this example, we see Q and B are both semi-definite. Thus the primal problem is a convex program. More specifically, it is a second order cone program. $S_\lambda^+ = (\lambda_1, \lambda_2) = [0, +\infty)$. The dual optimal solution is $(\bar{\lambda}, \bar{\sigma}) = (0,0,0,0)^T$. Then $\bar{x} = -(\bar{\lambda}B + Q)^{-1}(f + A^T\bar{\sigma}) = (-1,-1)^T$. Moreover, $P(\bar{x}) = -1$ and $\nu^d = -1$. Hence $\bar{x}$ is a global minimizer of the primal problem, which is also shown from Theorem 5.1 since $\lambda_1 = 0$.

Example 6.2. $Q = [1,0;0,-1]; A = [1,1;-1,1;0,-1]; b = [1;1;5]; f = [-1;-1]; B = [1,0;0,1]; \mu = 10$.

Now we see Q is indefinite, while B is still semi-definite. In fact, the quadratic constraint is redundant and the feasible domain is the polyhedral corresponding with the linear constraints. This also indicates that to some degree we may deal with a linearly constrained quadratic program by adding a redundant ball constraint.

$S_\lambda^+ = (\lambda_1, \lambda_2) = [1, +\infty)$. $(\bar{\lambda}, \bar{\sigma}) = (1,1,0,0)^T$. Then $\bar{x} = \lim_{\lambda \to 1^+}(\lambda B + Q)^{-1}(f + A^T\bar{\sigma}) = (0,1)^T$. Theorem 5.1 shows strong duality fails to hold since $\bar{\lambda} = \lambda_1 = 1 > 0$ and $\Delta_1 = -9.5 < 0$. In fact, simple computational result is $P(\bar{x}) - \nu^d = 9.5 > 0$. Unfortunately, the "boundarification technique" is not valid since the required nonzero $\tilde{x}$ does not exist.

Example 6.3. $Q = [1, 0; 0, -1]; A = [1, 1; -1, 1; 0, -1]; b = [1; 1; 5]; f = [-1; -1]; B = [1, 0; 0, 1]; \mu = 0.4$.

This example is almost the same as Example 6.2 except $\mu = 0.4$. Note the quadratic constraint is no longer redundant now. And the solution in Example 6.2 $x = (0, 1)^T$ is not feasible here.

We still have $S_\lambda^+ = (\lambda_1, \lambda_2) = [1, +\infty)$. But the dual optimal solution turns out that $(\bar{\lambda}, \bar{\sigma}) = (1.2905, 0.7417, 0, 0)^T$. Then the corresponding $\bar{x} = (0.1127, 0.8872)^T$. $\bar{x}$ is a global minimizer since $\bar{\lambda} \in intS_\lambda^+$.

Example 6.4. $Q = [1, 0; 0, -1]; A = [1, 1; -1, 1; 0, -1]; b = [1; 1; 5]; f = [-1; -1]; B = [-0.1, 0; 0, 1]; \mu = 0$.

Note that now Q and B are both indefinite. Follow the procedures in section 4.1 we compute $S_\lambda^+ = (\lambda_1, \lambda_2) = (1, 10)$. The dual optimal solution is $(\bar{\lambda}, \bar{\sigma}) = (3.1709, 0.4805, 0, 0)^T$. Since $\bar{\lambda} \in intS_\lambda^+$, the corresponding $\bar{x} = (0.7597, 0.2403)^T$ is the global minimizer of the primal problem.

§7 Conclusions

In this article, we form the canonical dual problem $(\mathcal{P}^d)$ of the quadratic program $(\mathcal{P})$ by applying the canonical duality theory. Under the dual Slater condition in S_λ^+, we propose an iterate scheme that may produce the dual optimal solution. Moreover, we show that the strong duality holds for some kinds of instances. As to the remaining ones, fortunately, if some extra conditions are satisfied, the strong duality can still hold by the "boundarification technique".

The approach provides a potential way to solve the subproblem $(CDP)_k$ in [3]. We may search for a better $\bar{\lambda}$ by directly solving $(CDP)_k$. Even the LCQP itself may be dealt with by adding a redundant quadratic constraint, which transforms LCQP to the problem in this article.

This is only the initial work on the program $(\mathcal{P})$. There are several further questions. A more efficient algorithm is worth developing. Precise analysis of the instances that strong duality does not hold is promising and valuable. Finally, we hope to extend the canonical dual approach to quadratic programs with multiple quadratic constraints and general $(\mathcal{QCQP})$.

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