An extension of the theorem of contraction mapping and fixed point[1]

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(Ⅰ) A specific problem

T is a mapping on Banach space and satisfies

$$\|Tx - Ty\| \le \alpha[p\|Tx - y\| + (1-p)\|x - Ty\|] \quad , \tag{*}$$

where $0 < \alpha < 1$, $0 \le p \le 1$, and x y are any two points in the space. One can claim that there is a unique point in the space that satisfies $Tx = x$.

Lemma 1. For any trial point x, the sequence $\{x, Tx, T^2x, T^3x \ldots\}$ is bounded.

Proof. Define index sets $A_k = \{1,2,3\ldots 2^k\}$ for $k \ge 0$. Define $M_k = \max_{i \in A_k}\{\|T^i x - x\|\}$. Then for $i \in A_{k+1} \setminus A_k$

$$\|T^i x - x\| \le \|T^i x - T^{2^k} x\| + \|T^{2^k} x - x\| \le \|T^i x - T^{2^k} x\| + M_k$$

For the first term on the right, apply (*) once and one can get

$$\|T^i x - T^{2^k} x\| \le \alpha[p\|T^i x - T^{2^k - 1}x\| + (1-p)\|T^{i-1}x - T^{2^k}x\|]$$

Apply (*) to every term on the right and one gets

$$\|T^i x - T^{2^k} x\| \le \alpha^2[p^2\|T^i x - T^{2^k-2}x\| + 2p(1-p)\|T^{i-1}x - T^{2^k-1}x\|$$
$$+ (1-p)^2\|T^{i-2}x - T^{2^k}x\|]$$

Carry out the process for a total of 2^k times and one gets

$$\|T^i x - T^{2^k} x\| \le \alpha^{2^k}[p^{2^k}\|T^i x - x\| + \binom{2^k}{1}p^{2^k-1}(1-p)\|T^{i-1}x - T^1 x\|$$
$$+ \binom{2^k}{2}p^{2^k-2}(1-p)^2\|T^{i-2}x - T^2 x\| + \ldots + (1-p)^{2^k}\|T^{i-2^k}x - T^{2^k}x\|]$$

For any term on the right which has the form of $\|T^{i-j}x - T^j x\|$, where $1 \le j \le 2^k$

$$\|T^{i-j}x - T^j x\| \le \|T^{i-j}x - x\| + \|x - T^j x\| \le 2M_{k+1}$$

Thus

[1] 本文是作者在 2008 年春季学期泛函分析课程中推导出的结果。
[2] 基数 52。

$$\left\|T^i x - T^{2^k} x\right\| \le 2\alpha^{2^k} M_{k+1}[p^{2^k} + \binom{2^k}{1} p^{2^k-1}(1-p) + \binom{2^k}{2} p^{2^k-2}(1-p)^2 + \ldots + (1-p)^{2^k}]$$
$$= 2\alpha^{2^k} M_{k+1}$$

Suppose i_0 is the index that $\left\|T^i x - x\right\|$ reaches M_{k+1}. If $i_0 \in A_{k+1} \setminus A_k$, then from result above one gets $M_{k+1} \le 2\alpha^{2^k} M_{k+1} + M_k$, or $M_{k+1} \le M_k /(1-2\alpha^{2^k})$ for large enough k such that $1-2\alpha^{2^k} > 0$. Otherwise, if $i_0 \in A_k$, then $M_{k+1} \le M_k$, and the relation $M_{k+1} \le M_k /(1-2\alpha^{2^k})$ also holds. Thus for any $N > k$

$$M_N \le M_k \frac{1}{1-2\alpha^{2^k}} \frac{1}{1-2\alpha^{2^{k+1}}} \frac{1}{1-2\alpha^{2^{k+2}}} \cdots \frac{1}{1-2\alpha^{2^{N-1}}}$$

The product $\frac{1}{1-2\alpha^{2^k}} \frac{1}{1-2\alpha^{2^{k+1}}} \frac{1}{1-2\alpha^{2^{k+2}}} \cdots \frac{1}{1-2\alpha^{2^{N-1}}} \cdots$ converges if and only if the summation $\sum_{i \ge k} \ln(1-2\alpha^{2^i})$ converges. And the convergence of latter is equivalent to the convergence of $\sum_{i \ge k} 2\alpha^{2^i}$, which is quite obvious. So there is an upper bound on M_k.

QED

Define $M = \sup\{\left\|T^i x - x\right\|\}$. According to the lemma, it is finite.

Corollary. For any point x, the sequence $\{x, Tx, T^2 x, T^3 x \ldots\}$ converges to some point x_0

Proof. Suppose $m \ge n$, then apply (*) n times. As in the proof of lemma 1, one gets

$$\left\|T^m x - T^n x\right\| \le \alpha^n [p^n \left\|T^m x - x\right\| + \binom{n}{1} p^{n-1}(1-p)\left\|T^{m-1} x - T^1 x\right\|$$
$$+ \binom{n}{2} p^{n-2}(1-p)^2 \left\|T^{m-2} x - T^2 x\right\| + \ldots + (1-p)^n \left\|T^{m-n} x - T^n x\right\|]$$
$$\le 2\alpha^n M$$

Obviously it goes to 0 when $m, n \to \infty$. $\{x, Tx, T^2 x, T^3 x \ldots\}$ is a Cauchy sequence and converges to some point x_0 in the space. **QED**

Lemma 2. For T that satisfies (*), it has a fixed point.

Proof. The only thing one has to do is to prove that for the convergent sequence $\{T^i x'\}$, its limit point x_0 is a fixed point. Let $x = T^n x'$, and $y = x_0$ in the inequality (*), then

$$\|T^{n+1}x - Tx_0\| \le \alpha[p\|T^{n+1}x - x_0\| + (1-p)\|T^n x - Tx_0\|]$$

Let $n \to \infty$, one gets

$$\|x_0 - Tx_0\| \le \alpha[p\|x_0 - x_0\| + (1-p)\|x_0 - Tx_0\|] = \alpha(1-p)\|x_0 - Tx_0\|$$

It holds only when $\|x_0 - Tx_0\| = 0$. This is what we want. **QED**

Lemma 3. T has at most one fixed point.

Proof. Suppose $Tx = x$ and $Ty = y$. Apply (*),

$$\|x-y\| = \|Tx - Ty\| \le \alpha[p\|Tx - y\| + (1-p)\|x - Ty\|] = \alpha\|x-y\|$$

It holds only when $\|x-y\| = 0$. **QED**

Thus the claim in the beginning is proved.

(Ⅱ)General cases

For T that satisfies(*), there is a much easier way to prove that $\{x, Tx, T^2x, T^3x \ldots\}$ converges. But the method used above can be applied to more general cases. Consider T that satisfies

$$\|T^m x - T^n y\| \le \alpha \sum_{i=1}^{t} p_i \|T^{m-a_i} x - T^{n-b_i} y\| \qquad (**)$$

Where $\sum_{i=1}^{t} p_i = 1, 0 \le p_i \le 1, 0 < \alpha < 1$, integers $a_i, b_j \ge 0$, $\max\{a_i\} \le m$, $\max\{b_i\} \le n$.

Lemma 4. For mapping that satisfies(**), the sequence$\{x, Tx, T^2x, T^3x \ldots\}$ is bounded for any point x.

Proof. Define A_k and M_k as one does in the proof of lemma 1. Use $\begin{pmatrix} & n & & \\ s_1 & s_2 & \cdots & s_t \end{pmatrix}$, $\sum_{i=1}^{t} s_i = n$ to denote the coefficient of the term $p_1^{s_1} p_2^{s_2} \cdots p_k^{s_t}$ when one expands $(p_1 + p_2 + \cdots + p_t)^n$. Define $l_k = [\min_{1\le i\le t}\{2^k / a_i, 2^k / b_i\}]$, where $[x]$ is the largest integer that is no larger than x. For large enough k such that $2\alpha^{l_k} < 1$, one has similar results as in lemma 1. For $i \in A_{k+1} \setminus A_k$

$$\left\|T^i x - x\right\| \le \left\|T^i x - T^{2^k} x\right\| + M_k .$$

Then apply (**) l_k times to the first term on the right

$$\left\|T^i x - T^{2^k} x\right\| \le \alpha^{l_k} \sum_{s_1+\cdots+s_t=l_k} \binom{l_k}{s_1 \quad s_2 \quad \cdots \quad s_t} p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t} \left\|T^{i-s_1a_1-\cdots-s_ta_t} x - T^{2^k-s_1b_1-\cdots s_tb_t} x\right\|$$

Notice that $i - s_1 a_1 - \cdots - s_t a_t$ and $2^k - s_1 b_1 - \cdots - s_t b_t$ in the terms on the right are always larger than or equals 0, according to the definition of l_k.

Then $\left\|T^{i-s_1a_1-\cdots-s_ta_t} x - T^{2^k-s_1b_1-\cdots s_tb_t} x\right\| \le 2M_{k+1}$ holds for all terms on the right, and

$$\left\|T^i x - T^{2^k} x\right\| \le 2\alpha^{l_k} M_{k+1} \sum_{s_1+\cdots+s_t=l_k} \binom{l_k}{s_1 \quad s_2 \quad \cdots \quad s_t} p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t} = 2\alpha^{l_k} M_{k+1}$$

$$M_{k+1} \le M_k / (1 - 2\alpha^{l_k})$$

Notice that as k adds up, l_k also increases. Then for large enough k, $\sum_{i\ge k} 2\alpha^{l_i}$ is smaller than $\sum_{j\ge 1} 2\alpha^j$ (the set $\{l_i\}_{i\ge k}$ being just a subset of all positive integers) and thus converges. So M_k have an upper bound M.

QED

Corollary. For T that satisfies (**) and any point x, the sequence $\{x, Tx, T^2x, T^3x \ldots\}$ converges to some point x_0.

Lemma 5. For T that satisfies (**), there is at most one fixed point.

Lemma 6. Suppose $m \ge 1$, $n = 1$, then the limit point x_0 for a convergent sequence $\{x', Tx', T^2x', T^3x' \ldots\}$ is the fixed point for T

Proof. Let $x = T^k x'$, and $y = x_0$ as in the proof of lemma 2. Apply (**)

$$\left\|T^{m+k} x' - Tx_0\right\| \le \alpha [\sum_{b_i=1} p_i \left\|T^{m+k-a_i} x' - x_0\right\| + \sum_{b_i=0} p_i \left\|T^{m+k-a_i} x' - Tx_0\right\|]$$

Let $k \to \infty$, one gets

$$\|x_0 - Tx_0\| \le \alpha \sum_{b_i=0} p_i \|x_0 - Tx_0\|$$

It holds only when $\|x_0 - Tx_0\| = 0$. **QED**

There is some difficulty to prove that the limit point x_0 is the fixed point for T when both m and n is larger than 1. The condition of continuity of T is required to deal with such cases easily. Then one can apply the limit process on both sides of the equation $T^{n+1}x = T \circ T^n x$ and claim that the limit point is the fixed point. Thus the following theorem is now valid.

Theorem. If a mapping T on Banach space satisfies

$$\|T^m x - T^n y\| \le \alpha \sum_{i=1}^{t} p_i \|T^{m-a_i} x - T^{n-b_i} y\|,$$

where $\sum_{i=1}^{t} p_i = 1$, $0 \le p_i \le 1$, $0 < \alpha < 1$, $a_i, b_j \ge 0$, $\max\{a_i\} \le m$, $\max\{b_i\} \le n$. And if m or n equals 1, or T is continuous, then there is a unique fixed point for T.

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Mathematical Quotations

Hardy, Godfrey H. (1877 - 1947)
The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.

Hilbert, David (1862-1943)
Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.