Maxwell Electromagnetic Theory
from a Viewpoint of Differential Forms[*]

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编者按：这篇关于 *Maxwell* 电磁场理论和下一篇关于 *Yang-Mills* 理论的论文均为"微分流形"课上小组合作完成。这两篇文章都体现了数学，更确切的说是几何学，与理论物理学的密切联系。这篇文章用微分形式的语言深入解读 *Maxwell* 电磁场理论，并涉及主丛的相关理论；后者成为在下一篇文章中详细描述的 *Yang-Mills* 理论的核心基础。

§1   Background

§1.1   Maxwell Equations

As we all know, the Maxwell equations are:

$$\nabla \cdot \mathbf{D} = \rho \quad (Coulomb's\ Law) \tag{1.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (Ampere's\ Law) \tag{1.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (Faraday's\ Law) \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0 \quad (Absence\ of\ Free\ Magnetic\ Poles) \tag{1.4}$$

The first two equations are inhomogeneous, while the other two are homogeneous. Here $\mathbf{E}$ is the electric field vector, $\mathbf{D}$ is the electric displacement, $\mathbf{H}$ is the magnetic field and $\mathbf{B}$ is the magnetic induction. We know that in vacuum (for simplicity we assume that we are always dealing with electromagnetic fields in vacuum) they are related by:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{1.5}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} \tag{1.6}$$

where ε_0 and μ_0 are constants satisfying $\varepsilon_0 \mu_0 = 1/c^2$, and c is the speed of light in vacuum. $\mathbf{J}$ is the current density, following the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{1.7}$$

where ρ is the charge density.

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[*]This article is a final report of the course *Differential Manifolds*.

§1.2   Vector and scalar potentials

It is convenient to introduce the scalar potential Φ and the vector potential $\mathbf{A}$, obtaining a smaller number of equations. According to (1.4), we can define $\mathbf{B}$ in terms of a vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1.8}$$

Then the equation (1.3) can be written as

$$\nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \tag{1.9}$$

So, in this equation, the quantity with vanishing curl can be written as the gradient of some scalar potential Φ. We have

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \tag{1.10}$$

That is

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{1.11}$$

From the above equations we find that $\mathbf{B}$ and $\mathbf{E}$ are determined by $\mathbf{A}$ and Φ, and the dynamic behavior of $\mathbf{A}$ and Φ are determined by the two inhomogeneous Maxwell equations, which can be written as follows:

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho/\varepsilon_0 \tag{1.12}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial^2 t} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) = -\mu_0 \mathbf{J} \tag{1.13}$$

Considering that $\nabla \times (\nabla \Lambda) = 0$ where Λ is any scalar function, $\mathbf{B}$ will be left unchanged by the transformation

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \Lambda \tag{1.14}$$

$\mathbf{E}$ is unchanged as well if

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \tag{1.15}$$

We can choose a set of potentials $(\mathbf{A}, \Phi)$ to satisfy the *Lorenz condition*

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{1.16}$$

Then the equations (1.12) and (1.13) have the form of wave equations, one for Φ and one for $\mathbf{A}$:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial^2 t} = -\rho/\varepsilon_0 \tag{1.17}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial^2 t} = -\mu_0 \mathbf{J} \tag{1.18}$$

These equations show us that the electromagnetic fields have similar characters as waves.

§1.3 Maxwell equations in special theory of relativity

The laws of mechanics should be the same in different coordinate systems moving uniformly relative to one another. For a long time, we believe that the laws of mechanics are invariant under Galilean transformation. However, the form of the wave equation is not invariant under classical Galilean transformations, which is in contrast with electromagnetic phenomena.

Based on the results of Michelson's and Morley's experiments, Lorentz got new transformations, under which Maxwell equations (1.17) and (1.18) were invariant. Einstein improved the theory and finally gave the well-known Lorentz transformations. The transformations from one frame moving with velocity parallel to x_1-axis is:

$$\left.\begin{aligned} x_0' &= \gamma(x_0 - \beta x_1) \\ x_1' &= \gamma(x_1 - \beta x_0) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \tag{1.19}$$

where $\beta = |\frac{\mathbf{v}}{c}|$, $\gamma = (1 - \beta^2)^{(-1/2)}$

Or we may have the matrix form of Lorentz transformations

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1.20}$$

The 4-dimensional space-time replaces the traditional space. We get an invariance of

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{1.21}$$

In four-dimensional vector space, we will employ the notations

$$\partial_\alpha \equiv (\frac{\partial}{\partial x^0}, \nabla) \tag{1.22}$$

$$\Box \equiv \frac{\partial^2}{\partial(x^0)^2} - \nabla^2 \tag{1.23}$$

where $x^0 = ct$. With these notations, we will see that if we put Φ and $\mathbf{A}$ together to form a 4-vector potential $A^\alpha = (\Phi, c\mathbf{A})$, and define $J^\alpha = (c\rho, \mathbf{J})$, then the wave equations (1.17)(1.18) have the form

$$\Box A^\alpha = \frac{1}{c\varepsilon_0} J^\alpha \tag{1.24}$$

and the Lorenz condition turns into

$$\partial_\alpha A^\alpha = 0 \tag{1.25}$$

If we define

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -E_{x^1} & -E_{x^2} & -E_{x^3} \\ E_{x^1} & 0 & -cB_{x^3} & cB_{x^2} \\ E_{x^2} & cB_{x^3} & 0 & -cB_{x^1} \\ E_{x^3} & -cB_{x^2} & cB_{x^1} & 0 \end{pmatrix} \tag{1.26}$$

and

$$
(\mathfrak{F}^{\alpha\beta}) = \begin{pmatrix} 0 & -cB_{x^1} & -cB_{x^2} & -cB_{x^3} \\ cB_{x^1} & 0 & E_{x^3} & -E_{x^2} \\ cB_{x^2} & -E_{x^3} & 0 & E_{x^1} \\ cB_{x^3} & E_{x^2} & -E_{x^1} & 0 \end{pmatrix}
\tag{1.27}
$$

then the inhomogeneous Maxwell equations (1.1) (1.2) can be written as

$$
\partial_\alpha F^{\alpha\beta} = \frac{1}{c\varepsilon_0} J^\beta
\tag{1.28}
$$

and the homogeneous Maxwell equations (1.3) (1.4) are

$$
\partial_\alpha \mathfrak{F}^{\alpha\beta} = 0
\tag{1.29}
$$

§2 Maxwell equations on 3-dimensional manifolds

In this section, we shall translate the classical Maxwell equations into the language of differential forms and discuss gauge invariance.

§2.1 Translation into the language of differential forms

Let M be a 3-dimensional manifold, which is contractible or, at least, has trivial cohomology groups $H^k(M)$ for $k \geq 1$. We express the physical quantities $\mathbf{E} = \mathbf{E}(t)$, $\mathbf{B} = \mathbf{B}(t)$ and so on as differential forms depending on the parameter $t \in \mathbb{R}$ smoothly.

More precisely, taking $\mathbf{E}$ for example, by saying that $E = E(t), t \in \mathbb{R}$ is a family of smooth differential 1-forms depending on t smoothly, we mean that E can be represented as $E(t, x^1, x^2, x^3) = E_i(t, x^1, x^2, x^3)dx^i$ in a certain local coordinate system, where E_i has continuous partial derivative with respect to t for $i = 1, 2, 3$. It is obvious that the smooth dependence on t defined above does not depend on the choice of local coordinate systems.

We consider $E = E(t), H = H(t)$ as 1-forms, $D = D(t), B = B(t), J = J(t)$ as 2-forms, and $\rho = \rho(t)$ as a 3-form. We define two operators $\frac{\partial}{\partial t} : \Omega^k(M) \times \mathbb{R} \to \Omega^k(M) \times \mathbb{R}$ and $* : \Omega^k(M) \to \Omega^{3-k}(M)$ as follows ($k = 0, 1, 2, 3$).

Suppose $\omega = \omega(t)$ is a family of k-forms depending on t smoothly. In a local coordinate system, it can be represented as $\omega(t, x^1, x^2, x^3) = \omega_{i_1, \ldots, i_k}(t, x^1, x^2, x^3)dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. [1]Define

$$
\frac{\partial}{\partial t}\omega(t, x^1, x^2, x^3) = \frac{\partial \omega_{i_1, \ldots, i_k}}{\partial t}(t, x^1, x^2, x^3)dx^{i_1} \wedge \cdots \wedge dx^{i_k}
$$

in the same coordinate system. Since $\omega_{i_1, \ldots, i_k}$ is continuously differentiable with respect to all its parameters, it is easy to check that the operator $\frac{\partial}{\partial t}$ is well defined. For example, $\frac{\partial E}{\partial t}(t, x^1, x^2, x^3) = \frac{\partial E_i}{\partial t}(t, x^1, x^2, x^3)dx^i$ in a local coordinate system, and $\frac{\partial E}{\partial t}(t, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial \tilde{E}_i}{\partial t}(t, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)d\tilde{x}^i$ in another. Furthermore, with the help of local representation, it can be seen that $\frac{\partial}{\partial t}$ is commutative with the exterior differential operator d.

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[1]More precisely, we should have written $\varphi^* \omega(t, x^1, x^2, x^3)$, where φ is a local chart, and φ^* is the corresponding pull-back map.

We define a linear mapping $* : \Omega^k(M) \to \Omega^{3-k}(M)$ for $k = 0, 1, 2, 3$ by:

$$*1 = dx^1 \wedge dx^2 \wedge dx^3$$

$$*dx^1 = dx^2 \wedge dx^3, *dx^2 = dx^3 \wedge dx^1, *dx^3 = dx^1 \wedge dx^2$$

$$*dx^2 \wedge dx^3 = dx^1, *dx^3 \wedge dx^1 = dx^2, *dx^1 \wedge dx^2 = dx^3$$

$$*dx^1 \wedge dx^2 \wedge dx^3 = 1$$

in a local coordinate system. It is not difficult to verify that the operator $*$ is well defined and commutative with $\frac{\partial}{\partial t}$. Obviously, $*^2 = * \circ *$ is an identity map.

Now we can transform the classical Maxwell equations as follows:

$$dD = \rho \tag{2.30}$$

$$dH = J + \frac{\partial D}{\partial t} \tag{2.31}$$

$$dB = 0 \tag{2.32}$$

$$dE + \frac{\partial B}{\partial t} = 0 \tag{2.33}$$

In our case, E and H are 1-forms but D and B are 2-forms. So we should use the operator $*$ to express their relations precisely:

$$D = \varepsilon_0 * E \tag{2.34}$$

$$H = \frac{1}{\mu_0} * B \tag{2.35}$$

§2.2 Potentials

In classical electrodynamics, it is convenient to introduce the scalar potential Φ and the vector potential $\mathbf{A}$, obtaining a smaller number of equations. Now we do the analogous things in the language of differential forms.

Since $dB = 0$ and the manifold M has trivial cohomology groups, there exists a 1-form A such that

$$B = dA \tag{2.36}$$

Then equation (2.33) can be written as

$$d\left(E + \frac{\partial A}{\partial t}\right) = 0 \tag{2.37}$$

This means that $E + \frac{\partial A}{\partial t}$ is the exterior differential of a 0-form, namely, a smooth function $-\Phi$:

$$E + \frac{\partial A}{\partial t} = -d\Phi, \quad \text{or} \quad E = -d\Phi - \frac{\partial A}{\partial t} \tag{2.38}$$

The definition of B and E in terms of the 1-form A and the 0-form Φ according to (2.36) and (2.38) satisfies identically the two homogeneous Maxwell equations (2.32) and (2.33). Now let's substitute these expressions into the other two Maxwell equations.

According to (2.34) and (2.38), $D = -\varepsilon_0 \left(*d\Phi + \frac{\partial}{\partial t}(*A) \right)$, so $dD = -\varepsilon_0 \left(d*d\Phi + \frac{\partial}{\partial t}(d*A) \right)$. It is not difficult to see[2] that $d*d\Phi = *\nabla^2\Phi$.

According to (2.35) and (2.36), $H = \frac{1}{\mu_0} *dA$, so $dH = \frac{1}{\mu_0} d*dA$. Using the above expression for D and the commutativity of $\frac{\partial}{\partial t}$ with $*$ and d, we can see that $\frac{\partial D}{\partial t} = -\varepsilon_0 * \left(d\left(\frac{\partial \Phi}{\partial t}\right) + \frac{\partial^2 A}{\partial t^2} \right)$. Thus we can transform equation (2.31) into

$$d*dA + \frac{1}{c^2} * \left(d\left(\frac{\partial \Phi}{\partial t}\right) + \frac{\partial^2 A}{\partial t^2} \right) = \mu_0 J \tag{2.39}$$

It is not difficult to verify that $d*d*A$ corresponds to $\nabla(\nabla\cdot\mathbf{A})$, and $*d*dA$ to $\nabla\times(\nabla\times\mathbf{A})$.[3] In fact, if we introduce an operator $\delta = *d* : \Omega^k(M) \to \Omega^{k-1}(M)$, then δA corresponds to $\nabla\cdot\mathbf{A}$. Recalling the formula $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$, we can define an operator $\nabla^2 : \Omega^k(M) \to \Omega^k(M)$ by $\nabla^2 = [d,\delta] = d\delta - \delta d = d*d* - *d*d$ for $k = 1,2,3$.[4] It will turn out to be useful for simplifying the Maxwell equations.

Now we can rewrite the four equations into two:

$$*\nabla^2\Phi + \frac{\partial}{\partial t}(d*A) = -\rho/\varepsilon_0 \tag{2.40}$$

$$* \left(\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - d\left(*d*A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} \right) \right) = -\mu_0 J \tag{2.41}$$

§2.3 Gauge invariance

To simplify the above equations, we must exploit the arbitrariness involved in the definition of the potentials and choose what is convenient. Since B is defined through (2.36) by $B = dA$ and $d^2 = 0$, it is left unchanged by the transformation,

$$A \to A' = A + d\Lambda \tag{2.42}$$

For E defined through (2.38) to be unchanged as well, Φ must be simultaneously transformed,[5]

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \tag{2.43}$$

The freedom implied by (2.42) and (2.43) means that we can choose a set of (A, Φ) to satisfy the *Lorenz condition* (1867),

$$*d*A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{2.44}$$

This will uncouple the equations (2.40) and (2.41), leaving two wave equations only:

$$* \left(\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} \right) = -\rho/\varepsilon_0 \tag{2.45}$$

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[2] In a local coordinate system, $*d\Phi = \frac{\partial \Phi}{\partial x^i}dx^1 \wedge \widehat{dx^i} \wedge dx^3$, and $d*d\Phi = \nabla^2\Phi dx^1 \wedge dx^2 \wedge x^3 = *\nabla^2\Phi$.

[3] For example, $d*d*A = d*\left(\frac{\partial A_i}{\partial x^i}dx^1 \wedge dx^2 \wedge dx^3 \right) = \frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial x^i} \right)dx^j$. The other one is left to the reader.

[4] For $k=0$, $d*d* - *d*d = -\sum_{i=1}^3 \frac{\partial^2}{\partial(x^i)^2}$, which differ from the classical Laplacian operator by a negative sign. So we use ∇^2 in the sense of the above definition only for $k \geq 1$, especially for $k = 1$, and as the classical Laplacian operator for $k = 0$.

[5] $E = -d\Phi - \frac{\partial A}{\partial t} = -d\Phi + d\left(\frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}(d\Lambda) - \frac{\partial A}{\partial t} = -d\Phi' - \frac{\partial A'}{\partial t}$

$$* \left(\nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \right) = -\mu_0 J \tag{2.46}$$

Equations (2.45) and (2.46), plus (2.44), form a set of equations equivalent to the Maxwell equations in vacuum. The transformation (2.42) and (2.43) is called a *gauge transformation*, and the invariance of the fields under such transformations is called *gauge invariance*.

In fact, suppose A and Φ do not satisfy (2.44), making a gauge transformation (2.42) and (2.43), we can demand that A', Φ' satisfy Lorenz condition:

$$*d * A' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0 = *d * A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} \tag{2.47}$$

provided a gauge function Λ can be found to satisfy

$$\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = - \left(*d * A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) \tag{2.48}$$

Suppose the Lorenz condition is satisfied by A, Φ, then it is preserved by the *restricted gauge transformation*.

$$\begin{aligned} A &\to A + d\Lambda \\ \Phi &\to \Phi - \frac{\partial \Lambda}{\partial t} \\ \nabla^2 \Lambda &- \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = 0 \end{aligned} \tag{2.49}$$

All potentials satisfying the Lorenz condition are said to belong to the *Lorenz gauge*. The Lorenz gauge is a concept independent of coordinate system chosen and so fits naturally into the considerations of special relativity.

§3   Maxwell equations in special relativity

§3.1   4-dimensional Maxwell Equations in differential forms

In special relativity, we take space and time as a whole into consideration and deal with the 4-dimensional space $\mathbb{R}^{1,3}$ called space-time with a constant Lorentz metric. Now we will study the 4-dimensional case.[6]

Recall that we expressed the physical quantities $\mathbf{E} = \mathbf{E}(t)$, $\mathbf{B} = \mathbf{B}(t)$ and so on as differential forms depending on $t \in \mathbb{R}$ smoothly in 3-dimensional case. Now it is convenient to, for example, express $\mathbf{E}$ as a single differential form $E(x^0, x^1, x^2, x^3) = E_i(x^0, x^1, x^2, x^3)dx^i$, where $\{x^0, x^1, x^2, x^3\}$ is a coordinate system and $x^0 = ct$.[7] We use E_i, B_i, J_i and so on to denote the components of corresponding physical quantities, which are smooth functions of (x^0, x^1, x^2, x^3).

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[6]We are unable to study general 4-manifolds in special relativity because the metric matrix is required to be unchanged under coordinate transformations. But what we do here in the language of differential forms may be easily generalized to general relativity.

[7]In $\mathbb{R}^{1,3}$ it can be even chosen as a global coordinate system.

We rewrite the classical Maxwell equations as follows:(where we have used the relations (1.5), (1.6) and $x^0 = ct$)

$$\nabla \cdot \mathbf{B} = 0 \tag{3.50}$$

$$\nabla \times \mathbf{E} + c\frac{\partial \mathbf{B}}{\partial x^0} = 0 \tag{3.51}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho \tag{3.52}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial x^0} \tag{3.53}$$

To simplify the homogeneous Maxwell equations (3.50) and (3.51), we define a 2-form F, called "Faraday":

$$\begin{aligned}
F = &- E_1 dx^0 \wedge dx^1 - E_2 dx^0 \wedge dx^2 - E_3 dx^0 \wedge dx^3 \\
&+ c(B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2)
\end{aligned} \tag{3.54}$$

We compute

$$\begin{aligned}
dF = &(c\nabla \cdot \mathbf{B})dx^1 \wedge dx^2 \wedge dx^3 \\
&+ (c\frac{\partial B_1}{\partial x^0} + \frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3})dx^0 \wedge dx^2 \wedge dx^3 \\
&+ (c\frac{\partial B_2}{\partial x^0} + \frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1})dx^0 \wedge dx^3 \wedge dx^1 \\
&+ (c\frac{\partial B_3}{\partial x^0} + \frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2})dx^0 \wedge dx^1 \wedge dx^2 \\
= &0
\end{aligned} \tag{3.55}$$

where we have used the first two equations. Conversely, $dF = 0$ also implies the validity of these two equations.

To deal with the last two equations, observe that by putting $c\mathbf{B} \to \mathbf{E}$ and $\mathbf{E} \to -c\mathbf{B}$ in the first two equations, we actually have something quite similar to the last two. More precisely, we may as well define a new 2-form M, called "Maxwell":

$$\begin{aligned}
M = &c(B_1 dx^0 \wedge dx^1 + B_2 dx^0 \wedge dx^2 + B_3 dx^0 \wedge dx^3) \\
&+ E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2
\end{aligned} \tag{3.56}$$

It is now convenient to introduce a new operator $*$ for 4-dimensional case. In general, on an n-manifold with a metric g, the *Hodge star operator* $* : \Omega^k \to \Omega^{n-k}$ is such a linear operator that

$$\alpha \wedge *\beta = g(\alpha, \beta)d\text{vol}_g \tag{3.57}$$

where $\alpha = \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \beta = \beta_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}, g(\alpha, \beta) = \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_k} g^{i_1 j_1} \cdots g^{i_k j_k}$, $d\text{vol}_g = \sqrt{|\det(g_{ij})|}dx^1 \wedge \cdots \wedge dx^n$ and (g^{ij}) is the inverse of the metric matrix (g_{ij}). In special

relativity the metric is given by

$$(g_{ij}) \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{3.58}$$

and the star is determined by the following rules:

$$*1 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

$$*dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = -1$$

$$*dx^0 = dx^1 \wedge dx^2 \wedge dx^3 \qquad * dx^1 \wedge dx^2 \wedge dx^3 = dx^0$$

$$*dx^1 = dx^0 \wedge dx^2 \wedge dx^3 \qquad * dx^0 \wedge dx^2 \wedge dx^3 = dx^1$$

$$*dx^2 = dx^0 \wedge dx^3 \wedge dx^1 \qquad * dx^0 \wedge dx^3 \wedge dx^1 = dx^2$$

$$*dx^3 = dx^0 \wedge dx^1 \wedge dx^2 \qquad * dx^0 \wedge dx^1 \wedge dx^2 = dx^3$$

$$*(dx^0 \wedge dx^1) = -dx^2 \wedge dx^3 \quad * (dx^0 \wedge dx^2) = -dx^3 \wedge dx^1 \quad *(dx^0 \wedge dx^3) = -dx^1 \wedge dx^2$$

$$*(dx^1 \wedge dx^2) = dx^0 \wedge dx^3 \quad * (dx^3 \wedge dx^1) = dx^0 \wedge dx^2 \quad *(dx^2 \wedge dx^3) = dx^0 \wedge dx^1$$

It can be concluded that $* * \omega = (-1)^{k+1}\omega$ if ω is a k-form, $k = 0, \ldots, 4$.

Then it is ready to see that

$$M = *F \tag{3.59}$$

and we compute (keep the last two equations in mind)[8]

$$
\begin{aligned}
dM =& (\nabla \cdot \mathbf{E})dx^1 \wedge dx^2 \wedge dx^3 \\
&+ \left(\frac{\partial E_1}{\partial x^0} - c \left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} \right) \right) dx^0 \wedge dx^2 \wedge dx^3 \\
&+ \left(\frac{\partial E_2}{\partial x^0} - c \left(\frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1} \right) \right) dx^0 \wedge dx^3 \wedge dx^1 \\
&+ \left(\frac{\partial E_3}{\partial x^0} - c \left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2} \right) \right) dx^0 \wedge dx^1 \wedge dx^2 \\
=& \frac{1}{\varepsilon_0} \rho dx^1 \wedge dx^2 \wedge dx^3 \\
& - c\mu_0 J_1 dx_0 \wedge dx^2 \wedge dx^3 - c\mu_0 J_2 dx_0 \wedge dx^3 \wedge dx^1 - c\mu_0 J_3 dx_0 \wedge dx^1 \wedge dx^2
\end{aligned} \tag{3.60}
$$

or

$$*d * F = \frac{1}{c\varepsilon_0} \tilde{J} \tag{3.61}$$

where $\tilde{J}$ is a 1-form, called the "4-current":[9]

$$\tilde{J} = c\rho dx^0 - (J_1 dx^1 + J_2 dx^2 + J_3 dx^3) \tag{3.62}$$

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[8]Here we regard ρ as a smooth function, not a 3-form.

[9]Don't forget that $c^2 \mu_0 \varepsilon_0 = 1$.

Introducing a new operator $\delta = *d* : \Omega^k \to \Omega^{k-1}$, we have

$$\delta F = \frac{1}{c\varepsilon_0}\tilde{J} \tag{3.63}$$

We remark that the relation above implies the last two equations.

To sum up, we have reduced Maxwell equations into two very simple identities involving the differential forms:

$$dF = 0$$

$$\delta F = \frac{1}{c\varepsilon_0}\tilde{J}$$

§3.2   Discussions

Since $dF = 0$ in $\mathbb{R}^4$, we must have

$$F = dI \tag{3.64}$$

where the "potential" 1-form I is

$$I = -\Phi dx^0 + c(A_1 dx^1 + A_2 dx^2 + A_3 dx^3) \tag{3.65}$$

From the relation $F = dI$ we deduce

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.66}$$

$$\mathbf{E} = -c\frac{\partial \mathbf{A}}{\partial x_0} - \nabla\Phi \tag{3.67}$$

where $\mathbf{A} = (A_1, A_2, A_3)$. Our choice of I is not unique. Indeed I is determined up to an "exact" 1-form, i.e.

$$I' = I + d\Lambda \tag{3.68}$$

serves our demand as well, thus we have the following *gauge transformation* leaving $\mathbf{E}$ and $\mathbf{B}$ unchanged:

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda \tag{3.69}$$

$$\Phi' = \Phi - c\frac{\partial \Lambda}{\partial x_0} \tag{3.70}$$

I also plays a fundamental role in our differential form of Maxwell equations, since now we have only one equation remained:

$$\delta dI = *d*dI = \frac{1}{c\varepsilon_0}\tilde{J} \tag{3.71}$$

Curiously, we might want to know about $*d*I$:

$$*I = -\Phi dx^1 \wedge dx^2 \wedge dx^3 + c(A_1 dx_0 \wedge dx^2 \wedge dx^3 + A_2 dx_0 \wedge dx^3 \wedge dx^1 + A_3 dx_0 \wedge dx^1 \wedge dx^2) \tag{3.72}$$

$$d*I = -(\frac{\partial \Phi}{\partial x^0} + c\nabla \cdot \mathbf{A})dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{3.73}$$

$$*d*I = \frac{\partial \Phi}{\partial x^0} + c\nabla \cdot \mathbf{A} \tag{3.74}$$

If $*d*I = 0$, we say that I satisfies the *Lorenz condition*:

$$\frac{\partial \Phi}{\partial x^0} + c\nabla \cdot \mathbf{A} = 0 \tag{3.75}$$

which is equivalent to (1.16). It can be proved that we can always choose I(more precisely, Λ) to satisfy the Lorenz condition.

From the identity

$$\delta F = \frac{1}{c\varepsilon_0}\tilde{J}$$

we deduce

$$\delta\tilde{J} = 0 \tag{3.76}$$

or more explicitly

$$c\frac{\partial \rho}{\partial x_0} + \nabla \cdot \mathbf{J} = 0 \tag{3.77}$$

i.e.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{3.78}$$

This is the same as equation (1.7) and also called *law of conservation* which states the total charge is invariant.

§4 Classical electrodynamics as the curvature of a line bundle

§4.1 About principal bundle

To interpret electromagnetism as a gauge theory, we need first to introduce some definitions.

A differentiable fiber bundle consists of following ingredients:

1. Three manifolds: the total space E, the base space X, and the fiber F.

2. A map $\pi : E \to X$, the projection.

3. A covering U of X by a family $(U_\alpha)_{\alpha \in A}$, of open sets.

4. For each U_α in U a diffeomorphism, the local trivialization, $h_\alpha : \pi^{-1}(U_\alpha) \times F$ such that for U_β in U with $U_\alpha \cap U_\beta \neq \emptyset$ the map

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times F$$

is given by

$$h_\alpha \circ h_\beta^{-1}(x, f) = (x, h_{\alpha\beta}(x)(f))$$

where

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to Aut(F)$$

and are called the transition maps.

Next, a principal G-bundle is a fibre bundle whose fibre is a Lie group G, and each of its transition maps $h_{\alpha\beta}$ corresponds to the left action of an element $g_{\alpha\beta}$ in G.

A differential-geometric G-connection on a principal G-bundle is a 1-form ω on E taking its values in the Lie algebra $\mathfrak{g}$ of G, with the following two properties:

(1)the "normalization" property: restriction of the form ω to the fiber G yields the above-defined form $\omega_0 = -(dg)g^{-1}$;

(2)invariance property: under the natural left action of the group G on E we have that $g^*\omega = Ad(g)\omega = g\omega g^{-1}$.

On each local trivialization $h_U : \pi^{-1}(U) \to U \times G$, the connection defined above can be represented by a $\mathfrak{g}$-valued 1-form A_U on U, which we shall call the *local principal gauge potential*. Note that in doing so we have passed from an object defined on the total space P to one defined on the base space M. As in physics M is generally space-time, this is in keeping with the notion that physical field theory deals with objects defined directly on space-time.

Conversely, suppose we have a set of trivializations $h_U : \pi^{-1}(U) \to U \times G$ such that the open sets U cover X. Suppose that for each trivialization we have a $\mathfrak{g}$-valued 1-form A_U such that if $U \cap V \neq \emptyset$, then in $U \cap V$, the equality

$$A_V = Ad_{g_{VU}}A_U - dg_{VU} \cdot g_{VU}^{-1} \qquad (*)$$

holds, in which g_{VU} is the fibre transition map. Then there is a unique invariant connection in P for which the representatives with respect to the given trivializations are the given A_U.

From gauge potential we can define a $\mathfrak{g}$-valued 2-form in U called the *curvature 2-form* F that satisfies $F(\eta, \xi) = dA(\eta, \xi) + [A(\eta), A(\xi)]$, or in short, $F = dA + [A, A]$.

With the definitions above, we can now further the understanding of electromagnetism. The 1-form A in electromagnetism takes its value in $\mathbb{R}$ which is the Lie algebra of Lie group $U(1)$. By checking the equation $(*)$ one can conclude that A is the local principal gauge potential of an invariant connection on the principal $U(1)$-bundle. What's more, the curvature 2-form $\tilde{F} = dA + [A, A] = dA$ because of the commutativity of $\mathbb{R}$, thus the electromagnetic 2-form is actually the curvature 2-form of the connection that A decides.

§4.2   Developments after Maxwell's equations

In the preceding subsection, we can attach the quantities (differential forms) produced by electromagnetic fields to our 4-dimensional space-time by description to electromagnetic fields in 4-dimensional space. Thus our universe can be explained as a structure of 4-dimensional manifold, with a structure of connections deduced from electromagnetic fields attached to it.

This means that our description to the universe is deeper than 4-dimensional space.

Of course, what we do here is actually a special case of Yang-Mills theory.

But an idea to construct models for the universe is given here: 4-dimensional manifold plus structure of principal bundle.

In fact, what we added is $U(1)$, for it is a simplest Lie group. A particularity of $U(1)$ is that it is Abelian, which is not a common property of Lie groups. When we replace $U(1)$ into a more complex Lie group, Yang-Mills theory comes out.

参考文献

[1] John David Jackson: *Classical Electrodynamics*, Third Edition

[2] George Svetlichny: *Preparation for Gauge Theory*

[3] Laurent Claessens: *Field theory from a bundle point of view*

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数学谜语

1. What is yellow and complete?

2. What is round, purple and commutative?

3. How is a set in a topological space different from a door? (James R. Munkres: TOPOLOGY, p93)

4.（原创）谁是抽象代数学的"外祖父"？

<div align="center">（答案见第 76 页）</div>

数学家趣闻

▲ Levy, Paul（1886-1971）这个人数学做的虽然不错，但是记忆力却很差。有一次 Errera, Alfred（1886-1960）（Landau 的一个学生）为 Levy 举办了一场晚宴。第二天，Errera 碰见 Levy，毕恭毕敬的说："我很高兴昨天度过一个美好的夜晚""恩？那么你昨晚在哪？"

▲一次拓扑课，Minkowski（闵可夫斯基）向学生们自负的宣称："这个定理没有证明的最主要的原因是至今只有一些三流的数学家在这上面花过时间。下面我就来证明它……"于是 Minkowski 开始拿起粉笔。这节课结束的时候，没有证完，到下一次课的时候，Minkowski 继续证明，一直几个星期过去了……一个阴霾的早上，Minkowski 跨入教室，那时候，恰好一道闪电划过长空，雷声震耳，Minkowski 很严肃的说："上天被我的骄傲激怒了，我的证明是不完全的……"

▲由于 Fermat 大定理的名声，在 New York 的地铁车站出现了乱涂在墙上的话：$x^n + y^n = z^n$ 没有解，对此我已经发现了一种真正美妙的证明，可惜我现在没时间写出来，因为我的火车正在开来。