Performance Analysis of Cell-Free Massive MIMO Systems: A Stochastic Geometry

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1 Abstract

This report aims to carry out a performance analysis on a Cell-Free Massive MIMO network whilst considering the randomness of access point (AP) locations across an area under analysis. This randomness is modelled using a Poisson Point Process (PPP) which is a more realistic model for the spatial randomness than a grid or uniform deployment. We model the system and show that the analytical and simulated results match in terms of the downlink coverage probability and throughput.

2 Introduction

Massive MIMO technology has been adopted in the 5G standard as it can leverage spatial multiplexing of many users on the same time-frequency resources using a large number of antennas. MIMO capitalizes on factors like channel hardening (turns the multi-antenna fading channel gain into nearly deterministic) and favourable propagation (makes the channel vectors nearly orthogonal) which are consequences of the law of large numbers. In addition to this, network MIMO has made strides in literature as it can help jointly serve all users by utilising local CSI at each AP. The channels are estimated via an uplink training phase under the assumption of channel reciprocity and Time-Division Duplex (TDD) design.

Further, cell-free massive MIMO abolishes cell boundaries demarcated in traditional cellular systems and the fact that a user is serviced by only a single AP. Here, one user is serviced by multiple APs in the vicinity which helps in increasing quality of service for users at the so called cell edge. Thus, cell-free massive MIMO reaps the benfits of both network and massive MIMO allowing many users to berved simulatneously with high quality of service due to macro-diversity, low path losses and increased expected coverage.

The PPP assumption for APs is a crucial one considering the fact that cell-free massive MIMO systems are designed based on heterogeneous and ad-hoc deployments.

3 System Model

3.1 AP Arrangement

We assume that an SDN-cum-CPU manages the control and data planes by means of a perfect backhaul for the CF massive MIMO network. We assume that a large number of APs each equipped with N antennas under a network MIMO concept serves jointly a set of K single-antenna users in the same time-frequency resources. All AP locations are generated randomly assuming they follow a two dimensional homogeneous PPP with density λ_{AP}. Moreover, we constrain our analysis to a 1km x 1km area. The number of APs M is a random variable obeying Poisson distribution with mean $\tilde{M} = \lambda_{AP}$. The total number of antennas is denoted by $W = MN$ and we assume $W >> K$ corresponding to a cell free massive MIMO scenario.

3.2 Channel Model

We consider both small scale fading and independent large scale fading in terms of path loss. Independence is assumed by virtue of the user staying static for multiple coherence intervals. We include the uplink training phase of τ_{tr} symbols as well as the uplink and downlink data transmission phases of τ_u and τ_d samples, respectively. The two data transmission phases assume identical channels based on the property of channel reciprocity being achievable under TDD operation and calibration of the hardware chains

We consider a specific realization of the PPP, where the number of the APs is M. Let $\mathbf{h}_{mk}$ be the $N \times 1$ channel vector between the mth AP and the typical user denoted henceforth by the arbitrary index k. In particular, the channel vector $\mathbf{h}_{mk}$ is expressed as

$$\mathbf{h}_{mk} = l_{mk}^{1/2} \mathbf{g}_{mk} \tag{1}$$

where l_{mk} and $\mathbf{g}_{mk}$ with m = 1, . . . , M and k = 1, . . . , K represent the independent large-scale and small-scale fadings between the mth AP and the typical user. Specifically, the large-scale fading considers geometric attenuation (path-loss) by means of $l_{mk}(r_{mk}) = \min(1, r_{mk}^{-\alpha})$ being a non-singular bounded pathloss model with $\alpha > 0$ being the path-loss exponent while r_{mk} expresses the distance between the mth AP and the typical user.

The distances from other users are independent and follow the uniform distribution. Furthermore, g_{mk},modeling Rayleigh fading, consists of small-scale fading elements, which are assumed to be independent and identically distributed (i.i.d.) $\mathcal{CN}(0, 1)$ random variables.

4 Uplink Channel Estimation

In each realization of the network, there is an uplink training phase in order to estimate g_{mk}, where all K users send simultaneously non-orthogonal pilot sequences with duration equal to $\tau_{tr} < K$ samples due to the limited length of the coherence interval.

By denoting $\psi_k \in \mathbb{C}^{\tau_{tr} \times 1}$ the normalized sequence of the k th user with $\|\psi_k\|^2 = 1$, the $N \times \tau_{\mathrm{tr}}$ received channel by the m th AP is given by

$$\tilde{\mathbf{y}}_{\mathrm{tr},m} = \sum_{i=1}^{K} \sqrt{\tau_{\mathrm{tr}} \rho_{\mathrm{tr}}} l_{mi}^{1/2} \mathbf{g}_{mi} \psi_i^{\mathrm{H}} + \mathbf{n}_{\mathrm{tr},m},$$

where $\mathbf{n}_{\mathrm{tr},m}$ is the $N \times \mathrm{tr}$ additive noise vector at the m th AP consisting of i.i.d. $\mathcal{CN}(0, 1)$ random elements, and ρ_{tr} is the normalized signal-to-noise ratio (SNR). By assuming orthogonality among the pilot sequences, the m th AP estimates the channel by projecting $\overline{\mathbf{y}}_{\mathrm{tr},mk}$ onto $\frac{1}{\sqrt{\tau_{\mathrm{tr}} \rho_{\mathrm{tr}}}} \psi_k$, i.e., we have

$$\begin{aligned}
\overline{\mathbf{y}}_{mk} &= \frac{1}{\sqrt{\tau_{\mathrm{tr}} \rho_{\mathrm{tr}}}} \overline{\mathbf{y}}_{\mathrm{tr},m} \psi_k \\
&= \mathbf{g}_{mk} l_{mk}^{1/2} + \sum_{i \neq k}^{K} l_{mi}^{1/2} \mathbf{g}_{mi} \psi_i^{\mathrm{H}} \psi_k + \frac{1}{\sqrt{\tau_{\mathrm{tr}} \rho_{\mathrm{tr}}}} \mathbf{n}_{\mathrm{tr},m} \psi_k
\end{aligned}$$

where the summation in the second term corresponds to the multi-user interference. Actually, this term is the source of pilot contamination.

Assuming, that the distance r_{mk} is known a priori, the m th AP obtains the linear minimum mean-squared error (MMSE) estimate as

$$\begin{aligned}\hat{\mathbf{h}}_{mk} &= \mathrm{E}\left[\mathbf{h}_{mk}^{\mathrm{H}}\overline{\mathbf{y}}_{\mathrm{tr},mk}\right]\left(\mathrm{E}\left[\overline{\mathbf{y}}_{\mathrm{tr},mk}\overline{\mathbf{y}}_{\mathrm{tr},mk}^{\mathrm{H}}\right]\right)^{-1}\overline{\mathbf{y}}_{mk}\\ &= \frac{l_{mk}}{\sum_{i=1}^{K}\left|\psi_i^{\mathrm{H}}\psi_k\right|^2 l_{mi}+\frac{1}{\tau_{\mathrm{tr}}\rho_{\mathrm{tr}}}}\overline{\mathbf{y}}_{mk}.\end{aligned}$$

Having obtained the estimated channel vector $\hat{\mathbf{h}}_{mk}$, the estimation error vector, based on the orthogonality property of MMSE estimation, is written $\tilde{\mathbf{e}}_{mk} = \mathbf{h}_{mk} - \hat{\mathbf{h}}_{mk}$. The estimated channel and estimation error vectors are uncorrelated and Gaussian distributed with N identical elements having zero mean and variances given by

$$\sigma_{mk}^2 = \frac{l_{mk}^2}{d_m}$$

and

$$\bar{\sigma}_{mk}^2 = l_{mk}\left(1-\frac{l_{mk}}{d_m}\right),$$

where $d_m = \left(\sum_{i=1}^{K}\left|\psi_i^{H}\psi_k\right|^2 l_{mi}+\frac{1}{\tau_{\mathrm{tr}}\ \rho_{tr}}\right)$. Hence, we have $\mathbf{h}_{mk}\in\mathbb{C}^{N\times 1}\sim\mathcal{CN}\left(\mathbf{0}, l_{mk}\mathbf{I}_N\right)$, $\quad \hat{\mathbf{h}}_{mk}\in\mathbb{C}^{N\times 1}\sim\mathcal{CN}\left(\mathbf{0},\sigma_{mk}^2\ \mathbf{I}_N\right)$ and $\bar{\mathrm{e}}_k\in\mathbb{C}^{N\times 1}\sim\mathcal{CN}\left(\mathbf{0},\bar{\sigma}_{mk}^2\mathbf{I}_N\right)$. At this point, it is better for the sake of following algebraic manipulations to denote the vectors $\mathbf{h}_k = \left[\mathbf{h}_1^T\cdots\mathbf{h}_{Mk}^T\right]^{\top}\in\mathbb{C}^{\mathcal{W}\times 1}\sim\mathcal{CN}\left(\mathbf{0},\mathbf{L}_k\right), \hat{\mathbf{h}}_k = \left[\hat{\mathbf{h}}_{1k}^T\cdots\hat{\mathbf{h}}_{Mk}^T\right]^{\top}\in\mathbb{C}^{\mathcal{W}\times 1}\sim\mathcal{CN}\left(\mathbf{0},\mathbf{\Phi}_k\right)$ and $\tilde{\mathbf{e}}_k\in\mathbb{C}^{\mathcal{W}\times 1}\sim\mathcal{CN}\left(\mathbf{0},\mathbf{L}_k-\mathbf{\Phi}_k\right)$, where the matrices $\mathbf{L}_k$, $\Phi_k = \mathbf{L}_k^2\mathbf{D}^{-1}$, and $\mathbf{D}$ are $\mathcal{W}\times\mathcal{W}$ are block diagonal, i.e., $\mathbf{L}_k = \mathrm{diag}\left(l_{1k}\mathbf{I}_N,\ldots,l_{Mk}\mathbf{I}_N\right)$

Also, $\mathbf{\Phi}_k = \mathrm{diag}\left(\sigma_{1k}^2\mathbf{I}_N,\ldots,\sigma_{Mk}^2\mathbf{I}_N\right)$, and $\mathbf{D} = \mathrm{diag}\left(d_1\mathbf{I}_N\ldots,d_M\mathbf{I}_N\right)$, respectively. In addition, we denote $\mathbf{C}_k = \mathbf{\Phi}_k^{-1}$ with $\mathbf{C}_k = \mathrm{diag}\left(c_{1k}\mathbf{I}_N,\ldots,c_{Mk}\mathbf{I}_N\right)$, where $c_{mk} = \sigma_{mk}^{-2}$.

5 Downlink Transmission

This section elaborates on the modeling and characterization of the downlink transmission in one realization of the network, and aims at presenting the downlink SINR, when the APs are PPP distributed and apply conjugate beamforming while the system is impaired by pilot contamination. Having in mind that the users are jointly served by the coordinated APs, we highlight that the received signal by the typical user is given by

$$y_{\mathrm{d},k} = \sqrt{\rho_{\mathrm{d}}}\sum_{i\in\Phi_{\mathrm{AP}}}\tilde{\mathbf{h}}_i^H\mathbf{s}_i + z_{\mathrm{d},k},$$

where ρ_{d} is the downlink transmit power, $\overline{\mathbf{h}}_i$ is the $N\times 1$ channel vector between the associated AP located at $x_i\in\mathbb{R}^2$ and the typical user including large and small-scale fadings, $z_{\mathrm{d},k}\sim\mathcal{CN}(0,1)$ is the additive Gaussian noise at the k th user, and s_i denotes the transmitted signal from the i th AP.

Given that the number of PPP distributed APs in the area $\mathcal{A}$ is M, we can rewrite this as

$$y_{\mathrm{d},k} = \sqrt{\rho_{\mathrm{d}}}\sum_{m=1}^{M}\mathbf{h}_{mk}^{\mathrm{H}}\mathbf{s}_m + z_{\mathrm{d},k},$$

where $\mathbf{h}_{mk}$ is the channel between the m th AP and user k while s_m denotes the transmitted signal from the m th associated AP. The transmit signal is written as

$$\mathbf{s}_m = \sqrt{\mu}\sum_{k=1}^{K} \mathbf{f}_{mk} q_k$$

with $q_k \in \mathcal{C}$ being the transmit data symbol for the typical user satisfying $\mathbb{E}\left[|q_k|^2\right] = 1$. Actually, the overall transmit signal to users can be written in a vector notation as $\mathbf{q} = [q_1, \ldots, q_K]^\top \in \mathbb{C}^K \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_K)$ for all users. Moreover, $\mathbf{f}_{mk}$ represents the (m, k) th element of a linear precoder. In order to avoid sharing channel state information between the APs, we assume scaled conjugate beamforming. We select conjugate beamforming precoding because of its computational efficiency and good performance in both massive MIMO and SCs designs. Thus, the expression of the precoder is $\mathbf{f}_{mk} = c_{mk}\hat{\mathbf{h}}_{mk}$. Regarding the scaling, it relies on a statistical channel inversion power-control policy that also eases the algebraic manipulations henceforth. Also, μ is a normalization parameter obtained by means of the constraint of the transmit power $\mathbb{E}\left[\rho_{\mathrm{d}} \mathrm{s}\mathrm{s}^{\mathrm{H}}\right] = \rho_{\mathrm{d}}$. Hence, we have

$$\mu = \frac{1}{\mathbb{E}\left[\operatorname{tr} \mathbf{F}_m \mathbf{F}_m^{\mathrm{H}}\right]},$$

where $\mathbf{F}_m = [\mathbf{f}_{m1} \cdots \mathbf{f}_{mK}] \in \mathbb{C}^{N \times K}$ is the precoding matrix. This can further be simplified to

$$\mu = \left(\frac{1}{W}\sum_{i=1}^{K} tr(\mathbf{C}_i^2 \mathbf{\Phi}_i)\right)^{-1} \tag{2}$$

Taking into account for the imperfect CSIT due to pilot contamination, the received signal by the typical user is written as

$$\begin{aligned}
y_{\mathrm{d},k} &= \sqrt{\mu\rho_{\mathrm{d}}}\sum_{m=1}^{M}\sum_{i=1}^{K} c_{mi}\mathbf{h}_{mk}^{\mathrm{H}}\hat{\mathbf{h}}_{mi} q_i + z_{\mathrm{d},k} \\
&= \sqrt{\mu\rho_{\mathrm{d}}}\mathbb{E}\left[\mathbf{h}_k^{\mathrm{H}}\mathbf{C}_k\hat{\mathbf{h}}_k\right] q_k + \sqrt{\mu\rho_{\mathrm{d}}}\mathbf{h}_k^{\mathrm{H}}\mathbf{C}_k\hat{\mathbf{h}}_k q_k \\
&- \sqrt{\mu\rho_{\mathrm{d}}}\mathbb{E}\left[\mathbf{h}_k^{\mathrm{H}}\mathbf{C}_k\hat{\mathbf{h}}_k\right] q_k + \sqrt{\mu\rho_{\mathrm{d}}}\sum_{i\neq k}^{K}\mathbf{h}_k^{\mathrm{H}}\mathbf{C}_i\hat{\mathbf{h}}_i q_i + z_{\mathrm{d},k},
\end{aligned}$$

where the second and fourth terms describe the desired signal and the multi-user interference. This transformation has been done since the users are not aware of the instantaneous CSI, but only of its statistics which can be easily acquired, especially, if they change over a long-time scale. Hence, user k has knowledge of only $\mathbb{E}\left[\mathbf{h}_k^H\mathbf{C}_k\hat{\mathbf{h}}_k\right]$. If we consider that this equation represents a single-input single-output (SISO) system, the effective SINR of the downlink transmission from all the APs to the typical user under imperfect CSIT, conditioned on the distances of APs l_{mk} for $m = 1, \ldots, M$, is given by

$$\gamma_k = \frac{\left|\mathbb{E}\left[\mathbf{h}_k^{\mathrm{H}}\mathbf{C}_k\hat{\mathbf{h}}_k\right]\right|^2}{\operatorname{var}\left[\mathbf{h}_k^{\mathrm{H}}\mathbf{C}_k\hat{\mathbf{h}}_k\right] + \sum_{i\neq k}^{K}\mathbb{E}\left[\left|\mathbf{h}_k^{\mathrm{H}}\mathbf{C}_i\hat{\mathbf{h}}_i\right|^2\right] + \frac{1}{\mu\rho_{\mathrm{d}}}},$$

where we assume that the APs treat the unknown terms as uncorrelated additive noise.

5.1 Deterministic Equivalent

The deterministic equivalent of a sequence of random complex values $(X_n)_{n\geq 1}$ is a deterministic sequence $\left(\bar{X}_n\right)_{n\geq 1}$, which approximates $\bar{X}_n$ such that

$$X_n - \bar{X}_n \xrightarrow[n\to\infty]{\text{a.s.}} 0,$$

where $\underset{n\to\infty}{\text{a.s.}}$ 0 is taken to mean almost sure convergence.

Now, conditioned on the distances of APs, the deterministic SINR $\bar{\gamma}_k$, obtained such that $\gamma_k - \bar{\gamma}_k \xrightarrow[M\to\infty]{\text{a.s.}} 0$, is provided below.

Given a realization of Φ_{AP} and conditioned on the APs distances, the deterministic SINR of the downlink transmission from the PPP distributed APs to the typical user in a CF massive MIMO system, accounting for pilot contamination and conjugate beamforming, is given by

$$\bar{\gamma}_k \asymp \frac{\mathcal{W}}{\frac{1}{\mathcal{W}}\sum_{i=1}^{K} \operatorname{tr}\mathbf{D}\mathbf{L}_i^{-2}\left(\mathbf{L}_k + \frac{\mathcal{W}}{\rho_{\text{d}}}\right) - 1}$$

6 Coverage Probability

The coverage probability is calculated both by simulated and DE analytical expression of SINR over many channels and AP configurations for each user. The SINR estimates over these epochs are stored and for various SINR targets, the probability of a user having a receive SINR greater than the target SINR is then estimated by finding the ratio of instances where obtained SINR is above the target SINR over the total number of instances. This can be expressed as

$$P_c^{cf} = P(\text{SINR} > T) \tag{3}$$

where T is the target SINR

7 Average Achievable Spectral Efficiency

The achievable rate is calculated both by simulated and DE analytical SINR expression of SINR over many channels and AP configurations for each user similar to the coverage probability case. The SINR estimates over these epochs are stored and the average achievable SE is found by calculating

$$R_k^{\text{cf}} = \left(1 - \frac{\tau_{\text{tr}}}{\tau_c}\right)\mathbb{E}\left[\log_2\left(1+\gamma_k\right)\right] \quad \text{b/s/Hz}, \tag{21}$$

where τ_c is the channel coherence interval in number of samples, τ_{tr} is the duration of the uplink training phase, and γ_k is the SINR and expectation is found as a time average over all SINR instances.

8 Numerical Results

We choose a finite window of area of 1 km $\times$ 1 km, where we distribute the APs, each having $N = 5$ antennas, according to a PPP realization with density $\lambda_{\text{AP}} = 40\text{APs/km}^2$ unless otherwise stated. Given that the analytical expressions rely on the assumption of an infinite plane while the simulation considers a finite square, we assume that this area is wrapped around at the edges to prevent any boundary effects. In addition, the structure of the system includes a number of APs serving simiitaneously $K = 10$ randomly distributed users. We use the default values in Table I unless otherwise stated. The normalized uplink training transmit power per pilot symbol ρ_{tr} and downlink transmit power ρ_{d} result by dividing $\bar{\rho}^{\mathrm{tr}}$ and $\bar{p}_{\mathrm{d}}$ by the noise power N_{P} given in W by $N_{\mathrm{P}} = W_{\mathrm{c}} \times \kappa_{\mathrm{B}} \times T_0 \times\ N_{\mathrm{F}}$, where the various parameter values can be found in Table I.

Table 1: Parameter Values for Numerical Results

Description	Values
Number users	$K = 10$
Number of Antennas/AP	$N = 5$
AP density	$\lambda_{\mathrm{AP}} = 40\mathrm{APs/km}^2$
Communication bandwidth, carrier frequency	$W_{\mathrm{c}} = 20\mathrm{MHz}$, $f_0 = 1.9\mathrm{GHz}$
Uplink training transmit power per pilot symbol	$\rho_{\mathrm{tr}} = 100$ mW
Downlink transmit power	$p_{\mathrm{d}} = 200$ mW
Path loss exponent	$\alpha = 3.5$
Coherence bandwidth and time	$B_{\mathrm{c}} = 200\mathrm{KHz}$ and $T_{\mathrm{c}} = 1$ ms
Duration of uplink training	$\tau_{\mathrm{tr}} = 10$ samples
Duration of uplink and downlink training is SCs	$\tau_{\mathrm{tr}} = \tau_{\mathrm{d}} = 10$ samples
Boltzmann constant	$\kappa_{\mathrm{B}} = 1.381 \times 10^{-23}$ J/K
Noise temperature	$T_0 = 290$ K
Noise figure	$N_{\mathrm{F}} = 9$ dB

In Fig. 1, we plot the coverage probability obtained by the simulated and analytically obtained SINR across AP densities of 40, 60, 100 and 200.

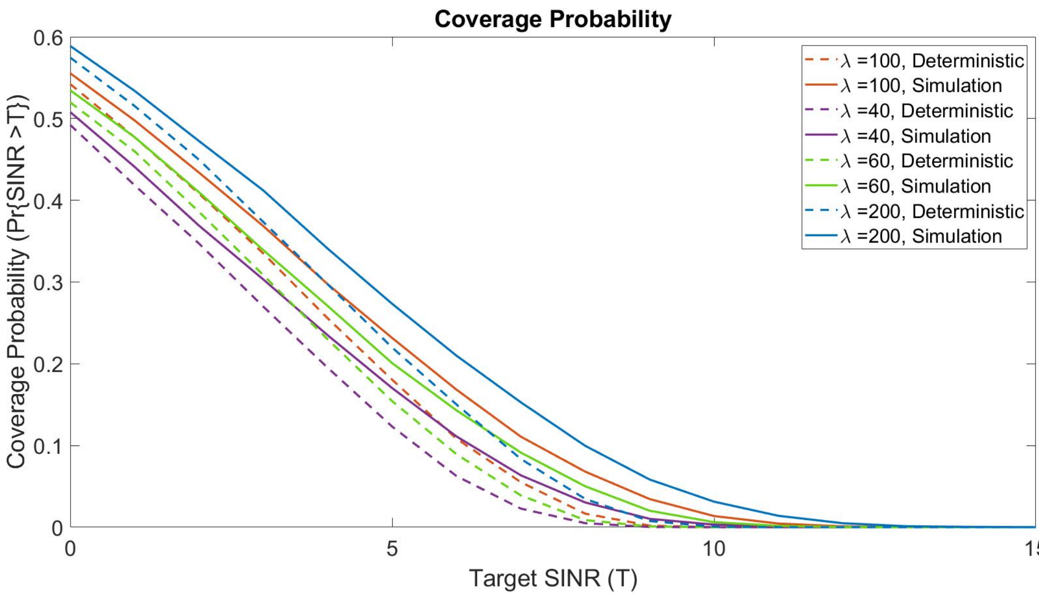

Figure 1: Coverage Probability vs. Target SINR

Next, in Fig. 2, we plot the average achievable downlink spectral efficiency (SE_k^{cf}) vs. number of users (K) for pilot sequence lengths (τ_{tr}) of 10 and 20 where $\lambda_{AP} = 100$.
Here we observe that when the number of users exceeds 20 for the $\tau_{tr} = 20$ case, the average achievable downlink SE drops sharply. This is because of the effect of pilot contamination on the

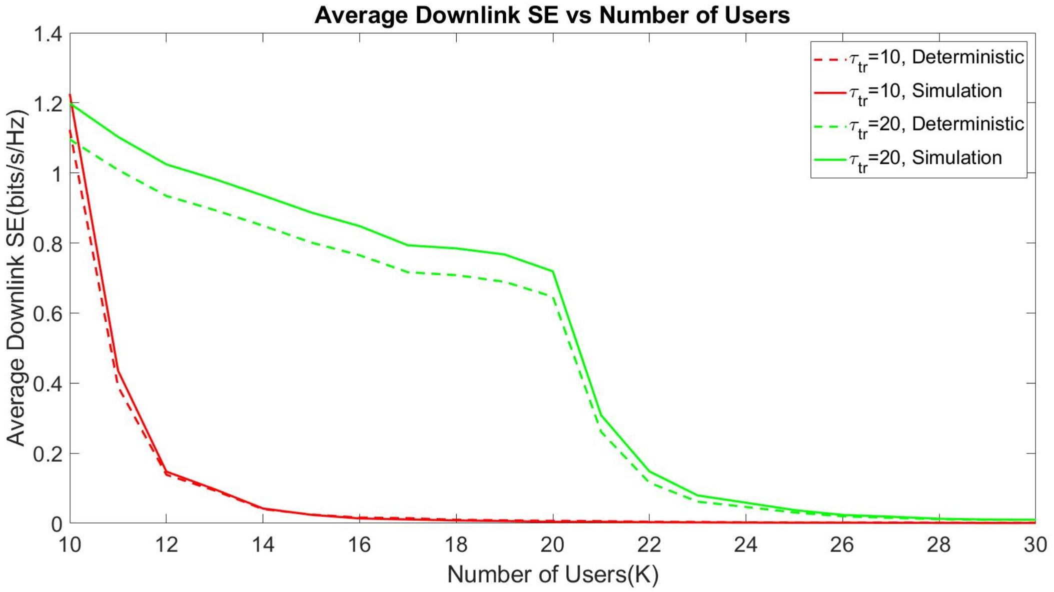

Figure 2: Average Achievable Downlink SE vs Number of Users

system wherein users share pilot sequences thereby increasing the interference in the SINR term at the receiver.

We also plot the average achievable downlink SE across various AP densities ranging from 40 to 110 in increments of 10 as shown in Fig. 3. Here we also increase the user count (K) along with τ_{tr}, meaning, when $\tau_{tr} = 10$, $K = 10$ and when $\tau_{tr} = 20$, $K = 20$. This is to avoid the effect of pilot contamination which as we have previously seen causes a sharp degradation in performance.

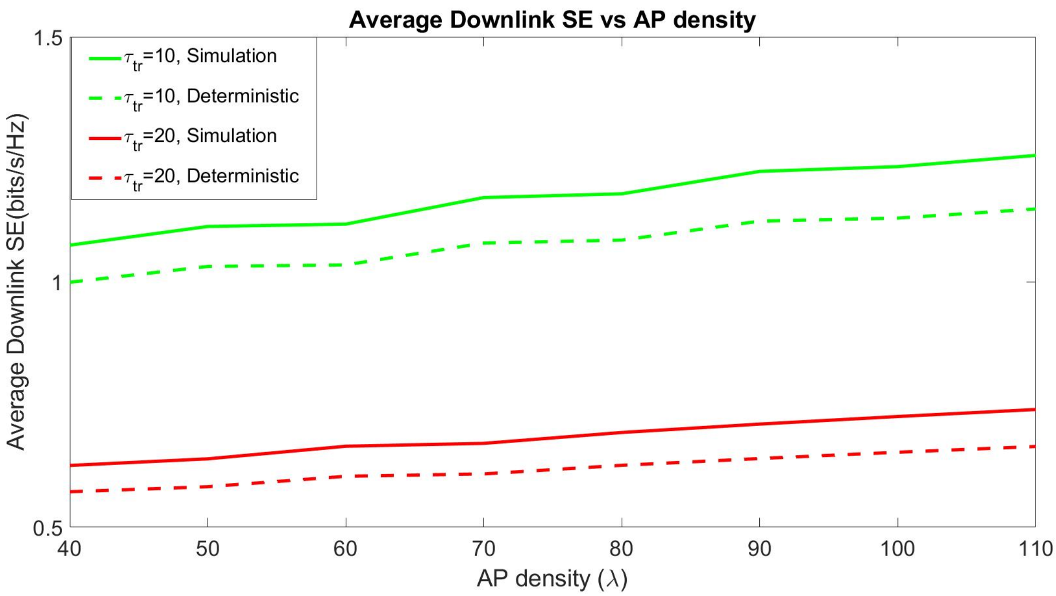

Figure 3: Average Achievable Downlink SE vs AP density

We then plot the average achievable downlink SE for various values of path loss exponent (α) which is described in Fig. 4.

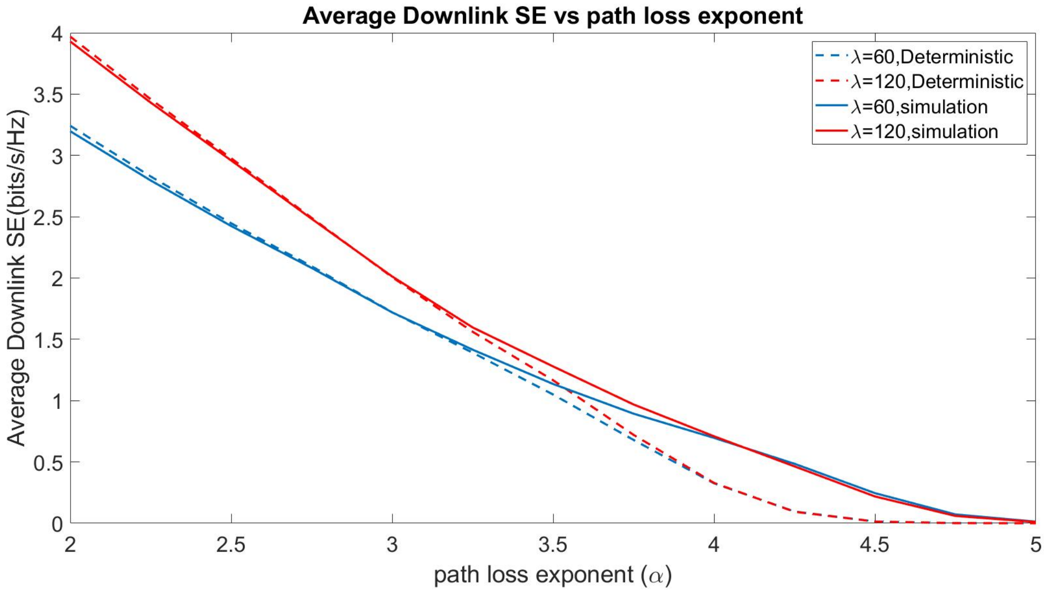

Figure 4: Average Achievable Downlink SE vs Path Loss Exponent

9 Conclusions

We have shown that the results obtained via simulation and by employing the deterministic equivalent closely match each other. Further, we carried out a performance analysis on a typical cell-free massive MIMO setup in terms of coverage probability and average achievable downlink SE upon varying various parameters like AP density, No. of Users, Target SINR and Path Loss Exponent.

References

[1] Papazafeiropoulos, Anastasios, et al. "Performance analysis of cell-free massive MIMO systems: A stochastic geometry approach." *IEEE Transactions on Vehicular Technology* 69.4 (2020): 3523-3537.