NEXT STOP: INFINITY

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ABSTRACT

This essay provides a philosophical-mathematical overview of the concept of infinity.
As we know, Infinity is a useful tool in mathematical calculations and it occurs in many shapes and forms in various mathematical areas, such as the infinitesimal quantities, the transfinite numbers in set theory, or limiting processes. But does "actual infinity" exist in our everyday life?
In this article we will lay out infinity's main properties and discuss two approaches to actual infinity:

1. The theory of actual infinity - finite and infinite quantities both fall under the same theory, meaning an actual infinite quantity is the same as a finite quantity, only bigger.
2. The idea of potential infinity - infinity is simply a figure of speech, it does not make an appearance in our physical world, and can only correspond to an iterative process of growing finite numbers.

In addition, we will address a few examples of how infinity applies to our daily lives, illustrate some of the paradoxes inherent in it, and demonstrate the magic of infinity and its behavior.
We will see that no matter how we try to look at this topic, we will always be driven back to the discipline where these roots lie – mathematics, which has also offered the most promising path towards their eventual resolution.
This article concludes that infinity's existence and meaning is still an important open question, which is relevant today more than ever. Even though infinity is widely accepted In the mathematical community, we argue its use should be more carefully queried with other alternatives addressed.

INTRODUCTION

"The infinite! No other question has ever moved so profoundly the spirit of man, no other idea has so fruitfully stimulated his intellect, yet no other concept stands in greater need of clarification than that of the infinite…" -David Hilbert

There are many ways in which infinity has been used throughout history.
A concept that has been passed down from generation to generation for 3000 years, must have undergone many changes throughout those years, and infinity is but a great example.
In fact its uses are so various, that the distinction between them is one of the most important steps towards understanding them.
Dowden (Dowden,2013) names three types of infinity: transcendental infinity, potential infinity, and actual infinity.

The first, is the vaguest and probably the one most referred to by the layman.
It is usually used as a figure of speech referring to the unimaginably large and indefinite, describing properties such as god's infinite compassion or the number of grains of sand and stars in the sky.
It corresponds to the eternal, everlasting and transcendent of the human mind.

The second and third were born under Aristotle's revolution against the first.
Aristotle was trying to ground logic and condemn the overuse of god in arguments, which was very common in his time (384–322 BC). One major step towards his goal was Aristotle's finitism. He claimed there were no transcendental nor actual infinities, the only infinity that exists in our world is the potential infinity. That which, given a certain size, can always be grown by one more. For example: The largest number imaginable can always be greater by adding 1. It is important to note the iterative process is not perceived to be carried out to completion, it is the potential to do so which Aristotle defines as a potential infinity.
On the other hand, actual infinity was defined by him as a complete mathematical object with an infinite number of elements, and so for Aristotle its existence in the real world, was strictly forbidden.
Aristotle's revolution is still heard today in the ontology of mathematics. Many have agreed with him that actual infinity can not exist. They claim our understanding can permit the potentially infinite, but the actually infinite is just a useful idea, an inherently incorrect model of reality, that has miraculously led us to magnificent breakthroughs. Others object to infinity's exclusion, they urge to persist in the direction of great progress that has been made and accept infinity's existence no matter its unintuitive nature, while some still require mathematics to retrace its steps, seeking ultimately to ground mathematics - the bedrock of science and logic itself - on solid foundations.

Much more will be discussed about Aristotle's revolution and his distinction between actual and potential infinity, but to start our journey towards a further understanding of the eternal and Infinite, we would like to step back and discuss infinity's importance.
Infinity is irreplaceable in modern physics, the infinitely small electron, the infinite singularity of a massive black hole, the infinity of time and space itself and much more.
Infinity has left its marks in many fields of mathematics as well: limits, integrals and derivatives, functions and probabilities. All of these and many more, were made rigorous using infinity.
Indeed, despite the difficulty comprehending infinity, it is truly remarkable how people - scientists and engineers, can use it in their calculations without appreciating the care and thought that went into it,

using a limit stretching to infinity or the result of some infinite sum.
Writing just three digits out of the infinite decimal expansion of an irrational number such as e or pi, then adding "…" for the reader to fill in. Starting their proofs casually claiming there exists a number in the Naturals or the Reals, no matter how large. We have gotten used to thinking about abstract objects, our brains have been trained well enough in the manipulation of numerals, to allow us not to feel discomfort with these symbols expressing objects that are beyond our intuition's reach. Therefore, to begin to understand infinity and its size we would like to construct a bridge between our intuition and infinity's realm:

Imagine a deck of 52 cards. Now shuffle them. This specific ordering resting in the palm of your hands has, most probably, never been ordered this way before. For the number of orderings possible is 52 factorial and this is an astronomically big number. If you were to take 52 factorial to the power of 52 factorial, repeating this 10 times to build a tower of unfathomably big numbers and write it down, the number you have written could only truly exist in our world - in the symbols you have written it down in. If you tried to write out its digits, giving each digit to an atom in our universe, you wouldn't be able to climb past the second factorial. Yet this number is still just finite, and even if you did this every single day for the rest of your life, it would still remain finite.
Infinity is bigger than our minds and intuition are able to grasp, and though we can only hope to one day comprehend its size, with the help of Dedekind, Cantor and many other great minds, we have been able to define, symbolize and prove statements about it, allowing us to use it in an almost careless and unworthy manner in our everyday calculations.

Our goal in this essay is to shake the mathematical foundations we have grown so accustomed to by visiting the paradoxes infinity brings with it, we wish to challenge the sense of calm we have when using mathematics and its efficient tools to describe our world.
The main questions we will discuss are:
Does infinity exist in nature, or is it just a "man-made" concept?
Paradoxes of infinity and its appliance in daily life - how is it possible?

In section 1 we will take a historical tour motivating our current understanding of infinity's size through the lens of different paradoxes.
In section 2 we will address the possibility of an infinite process ending and the question: does actual infinity exist in the real world. It is hopeless to cover all the paradoxes and all the relevant ideas in just one essay but we will try and tell the main storyline through the core concepts.
In the third section we will give our conclusions and additional remarks.

SECTION 1

GALILEO'S PARADOX

Hermann Weyl in his book: "levels of infinity" lays out 4 levels of arithmetic which will guide us throughout this essay. The first is based on a "part-whole intuition", which claims the set "||" is contained in the set "|||". (Weyl,1930, p. 19)
Such a simple example, the mind can hardly consider otherwise, for we see the amount of the first set is replicated in the amount on the second one, with one more "|" remaining, and so we are forced to accept this relation. But what if we were asked: which set is larger, two elephants or three dogs?

Weyl's second level of arithmetic answers how to measure sets which are composed of different elements: he symbolizes each set's size with a number, the larger number corresponds to the larger set. This method works well enough for finite sets, allowing us to see the surprising fact that we actually have more dogs than elephants even though the elephants can not physically be placed inside the 3 dogs.
What of infinite sets? Are there more odd numbers than even numbers?

Another rather simple question, but it opposes the first hard challenge for the part-whole intuition. For our comparison method fails, we can not assign a single number to these sets that will describe their size, there are an infinite number of both and infinity is larger than any finite number.
The Islamic mathematician Ibn Qurrah (9th century CE) faced this challenge. He claimed there were the same amount of each. Ibn Qurrah's argument is as follows: for every odd number there is a following even number and so they are "similarly frequent". By his logic all multiples of eight would be half the amount of all multiples of four, which would then be half the amount of all even numbers and so on.(see Rashed 2009)

700 years later Galileo Galilei envisions a paradox Ibn Qurrah can not resolve.
To understand his paradox, we will first describe an ancient counting method:
It is of Jewish tradition not to count people. Instead if one wishes to know the total number of people present somewhere, every person is told to bring a single coin and these coins are alternatively counted. We say there is a one to one correspondence between the coins we end up counting and the people who brought them.
To better generalize this method to counting infinite sets, we can observe that the above might be alternatively stated as - after all the coins are counted, no person will be left uncounted.

Galileo's paradox can now be stated as the following question: are there more natural numbers than square numbers?
At first, this seems an easier question than the last, obviously both the part-whole intuition and the frequentist intuition respond instantly: "there are more Natural numbers! For the squares are contained in the Natural numbers and they appear less and less frequent than the Naturals". Yet Galileo isn't easily convinced, he uses the counting method and the infinite collection of natural numbers as labels for the squares.
[Remark: The natural numbers are sometimes called the counting numbers because of such frequent use of this method.]

The counting process takes place by applying these labels to the squares in a unique and reversible fashion: Match each natural number with its square, 1 will be paired with 1, 2 with 4, 3 with 9, etc. After we have paired them all, every natural number will be grouped with one and only one square. If we were to close our eyes and imagine an outside observer labeling each square number with a Natural number, with no former knowledge of what he was counting, he would never believe there were more from one than the other! For which natural number would that be? Which number would be "left out" without a square corresponding to it?

Galileo can not make sense of the paradoxical nature of infinity's size and so he abolishes it:

> *"So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less," are not applicable to infinite, but only to finite, quantities." (Galilei, 1638,* pp. 31–33*)*

200 years later, Georg Cantor, a German mathematician, was the first to find a consistent measure for the infinite. In a sense he took the exact opposite approach from Galileo, supposing the size of infinite sets can only be measured through a one-to-one correspondence. Cantor's approach is motivated by a mathematically pleasing argument:

> *"In order for there to be a variable quantity in some mathematical study, the "domain" of its variability must strictly speaking be known beforehand through a definition. However, this domain cannot itself be something variable…. Thus this "domain" is a definite, actually infinite set of values. Thus each potential infinite…presupposes an actual infinite." (Cantor, 1887*, p.9)

Cantor realizes that if we were to treat potential infinity as we do normal varying quantities in mathematics, we must first fix the domain in which it is allowed to vary. But for this type of domain to exist, we then must accept an actual infinity. Though seemingly innocent, Cantor's argument isn't just a mathematical necessity, it is a revolutionary uprising. For almost two thousand years, potential infinity had the upper hand and actual infinity was considered an impossibility. Described as the theoretical infinite continuation of the potential infinity, its existence was disallowed and most paradoxes were overturned by blaming the challenger of the use of actual infinities in his arguments. However, Cantor realizes that the power relations should be reversed, for potential infinities to exist they need actual infinities to exist and not vice versa.

This great philosophical upending allows Cantor the peace of mind to define the measure of the infinite, by using the infinite collection of Natural numbers. Cantor generalizes the meaning of size - or cardinality - to include infinite sets. Naming the new unit size of the collection of Natural numbers - aleph null - $\aleph_0$.

Cantor thereby declares the existence of the first actual infinity and challenges the extremely "human" part-whole intuition. Bolzano famously argued in favor of Cantor's decision (before it was made), claiming an infinite quantity needed an infinite measure. (1950)

The second revolution came with Cantor's new method of proof – diagonalization.
A method fitted for infinite sets like no other before it. It proves by construction that a certain infinite

set isn't countable.
Cantor considered the set of all infinite sequences of binary digits. (which we will now prove is an uncountable set) Cantor shows that If $E_1, E_2, \ldots, E_n$, ... is any enumeration of elements from that set, [and so contra-positively supposing there is a one-to-one correspondence between the given set and the Natural numbers] we can then construct an element E of the set, such that it doesn't correspond to any E_i in the given enumeration. E is constructed by taking the *inverse* of the i^{th} digit of E_i to be the i^{th} digit of E. For example:

$$
\begin{aligned}
&E1 = (\mathbf{0}, 0,0,0,0,0,0,\ldots)\\
&E2 = (1, \mathbf{1}, 1,1,1,1,1,\ldots)\\
&E3 = (0,1, \mathbf{0}, 1,0,1,0,\ldots)\\
&E4 = (1,0,1, \mathbf{0}, 1,0,1,\ldots)\\
&E5 = (1,1,0,1, \mathbf{0}, 1,1,\ldots)\\
&E6 = (0,0,1,1,0, \mathbf{1}, 1,\ldots)\\
&E7 = (1,0,0,0,1,0, \mathbf{0},\ldots)\\
&\vdots\\
&=> E = (1,0,1,1,1,0,1,\ldots)
\end{aligned}
$$

Cantor uses this method to constructively show that certain sets are of the "same size", while others can have no one-to-one correspondence between them. Furthermore, he uses it to show a method of generating larger and larger infinite sets, by using a set's power set.
A power set of a certain set, is the set containing all possible subsets of the given set.
Cantor shows, using the diagonalization method, that the power set must have a larger cardinality than its corresponding set – this proof is now called Cantor's theorem. Thus, he generates his infinite series of aleph's. Infinity so it seemed could be measured and not all infinities were made the same.

Dedekind joined his side, not only approving of Cantor's assumption of an actual infinity, but he also tried to prove it. Claiming the set of all possible thoughts capable of the human mind was infinite.
Another step forward in the taming of the infinite was Dedekind's algebraic definition of infinite sets. Dedekind claimed the property of having a one-to-one correspondence to a proper subset of itself - in much the same way we have depicted above between the natural numbers and the squares - was the defining feature of an infinite set.

One of the most famous paradoxes on the subject is Hilbert's hotel. Though it is not a paradox in the usual sense, pure Cantorian's would not be bothered by it. Rather, it was made by Hilbert for pedagogical purposes, to better picture Cantor's results.

Hilbert imagines a hotel with infinite rooms, one day a guest arrives and asks for a room to stay in. The clerk searches through the list of rooms, but to his dismay the hotel is completely full - every room occupied. Hilbert then asks how can the clerk find a room for the waiting guest to stay?

Well, Hilbert says, we can apply Dedekind's defining feature of infinite sets, allowing for a one-to-one correspondence between the infinite set and a proper subset of itself.

We can use the infinite set of natural numbers as the room numbers, then we can move each guest to the room with number one larger than its own. 1 will move to 2, 2 to 3 and so on, the new set of rooms will correspond to the set of all natural numbers not including 1. Which of course, is a proper subset of the natural numbers. But now notice, there is an additional room for our guest in room 1!

Contrary to our initial intuition, any finite amount - x - of new guests can be roomed in a similar manner: Moving each guest to the room with number x larger than their current room, each would receive a corresponding room and there would be no guest left without a room. The clerk would then have the needed rooms at his disposal and could ready them for the new guests. While the part-whole intuition would never allow for such arithmetic, Dedekind approves, saying there is no error in the possibility of these one-to-one correspondences.

Hilbert then goes further and assigns rooms to an infinite number of new guests, thus illustrating some infinities are of the "same size" as the natural numbers. He finds rooms for the integers, by first moving all the guests to double their room number, in this way all the odd numbered rooms are vacated. Then, ordering the integers by {0,1,-1,2,-2…} and relabelling them using just the odd numbered rooms, he finds a room for all: 0 will occupy room 1, 1 will be in room 3 and so on. But Hilbert's appetite only grows, He manages to find room for the Rational's! Even the algebraic numbers can all be hosted. But then the Real numbers arrive, and Hilbert holds up his hands in defeat. For, he shows, there aren't enough rooms for them all. This time, using Cantor's diagonalization argument, Hilbert supposes contra positively they could all be placed in the infinitely many rooms, then by the construction he finds a guest from the Reals, that couldn't be in any one of these labeled rooms and so proves otherwise. This infinite hotel isn't big enough, thus the set of real numbers is a much larger infinity than the others we have described. Remarkably, Cantor has shown us the sets of numbers that could ever possibly be labeled by man, are but specks of dust in the ocean of the continuum!

Many had disapproved of Cantor's and Dedekind's constructions, indeed the sense of unease was put best by Weyl:

> *"The set counts as given if a determinate criterion decides which elements belong to it and which do not… one can scarcely rid oneself of the feeling that through the attachment of "accessible" sets and sequences to the law, a chaotic plenum of possibilities, of "arbitrarily tossed together," "lawless" sets is neglected…Set theory is untroubled by its use of such alternatives in the criteria it erects to determine the membership of a number in a set or in the laws by which it defines an infinite sequence. One can see that it thereby entangles itself in a pernicious logical circle." (Weyl,1930, p. 25)*

In the end, Weyl's critique led to set theory's demise, its strengths overturned to its weaknesses.

Cantor's vague all-encompassing definition of a set gave rise to Russel's self-referencing paradox. In which he defines a set that contains all sets that don't contain themselves. If this set were to contain itself, why then it couldn't. But if it didn't contain itself, then it must. Similarly, Cantor's power set method, which kept churning out bigger and bigger infinities came to a screeching halt by another paradox, defining the set that contains all sets. On the one hand it includes all its subsets, hence its power set is a subset of itself. Yet by Cantor's theorem, the power set has a larger cardinality, and so can not be contained in the original set. Thus, Cantor's dream to measure the infinite failed, as we have just shown there are infinite sets that cannot be measured by a cardinality, for if they had a cardinality, a paradox would ensue.

These were fatal blows, and the mathematical world was thrust back into turmoil. By now Dedekind and Cantor's triumphs were becoming more and more essential, defining the linear continuum and uniting mathematics in an unprecedented manner, forcing Hilbert's now infamous remark:

> *“Wherever there is any hope of salvage, we will carefully investigate fruitful definitions and deductive methods. We will nurse them, strengthen them, and make them useful. No one shall drive us out of the paradise which Cantor has created for us.” (Hilbert, 1926, p. 191)*

There were some theories that tried to save the day. The one that rose above the rest was Zermelo and Fraenklel’s system of axioms.” ZF” for short, takes the approach set down by Hilbert and is able to fix the problems of set theory by restricting it. For example, the self referencing paradox - solved by including the axiom of regularity and so opposing a set to be an element of itself. These axioms, are now considered to be the foundation of mathematics and are an equivalent accomplishment to the extremely sought-after physical “theory of everything”.

There are those that question the ZFC revision, (referring to the “ZF” axioms with the addition of the axiom of choice) naming the Banach Tarski paradox: In which a ball is taken apart into a finite number of parts, then without stretching or deforming the parts, is re-assembled to form two balls of the exact same size! This paradox sharpens the dagger for those that believe the part-whole intuition must stand and set theory has gone too far, for it exaggerates the part to twice the whole! But it still doesn’t pose a threat to the purists, for them our intuition doesn’t have precedence when it comes to infinity.

> *“The logical problems with the actual infinite are not problems of incoherence but arise from the features that are characteristic of infinite sets. When the intuitive notion of “smaller than” is replaced by a precise definition, finite sets and infinite sets just behave somewhat differently, that is all.” (Rundle,2004, p. 170)*

Furthermore, recent discoveries in the LHC have linked possible physical interactions in quark theory to this exact paradoxical nature of infinite sets. (Augenstein,1984)

Though it seems today the mathematical community has come to terms with the paradoxical behavior. A relatively new theory developed by Benci, Di Nasso and Mancosu, called the theory of numerosities. (cf. Benci, Di Nasso,2003 and Benci, Di Nasso, and Forti, 2006) Addresses the un-intuitiveness of Cantor's cardinals, while trying to retain its ability to capture the size of infinity.

It agrees with Cantorian cardinalities on finite sets, but then diverges on infinite sets. For instance, in this approach the set of squares has numerosity strictly less than the set of natural numbers. Furthermore, the defined numerosities pertain to additional “nice” axioms that cardinalities do not: such as, distributivity and commutativity.(see Easwaran, Kenny, Alan Hájek, Paolo Mancosu, and Graham Oppy, 2021) It is yet to be addressed in theory and in the philosophy of mathematics what the meaning of these numerosities are, but that they can face the challenge of describing the size of infinity while obtaining a part-whole intuition is a historic achievement.

SECTION 2

ACTUAL INFINITY vs POTENTIAL INFINITY

The second element that shines bright in the infinite, is the infinite process. Indeed, for much of history, infinity was only grasped by thinking of processes. Such as, the continuous division of a line segment or the ever-ticking clock of time - growing one second each second.

A canonical example is that of Tristram Shandy, a famous paradox by Bertrand Russel, based on a popular novel: “The Life and Opinions of Tristram Shandy, Gentleman” by Laurence Sterne. In Russel’s adapted paradox, Tristram writes his own diary. Every day in his life takes him a year to write, and the question Russel asks is: if Tristram was to live forever, would he ever “complete” his diary?

Russel answers with the now familiar counting method: there is a one-to-one correspondence between each entry and the corresponding year it will be written and so although the amount Tristram has yet to write each year grows by 364 days. There will be “no day left unwritten”. On the other hand, his life will never come to an end and so he could not possibly be able to write “all” of his diary. This paradox magnificently illustrates the discomfort one can have when dealing with infinite processes. For it seems that the ever-growing number of days Tristram has yet to write, suddenly evaporates when Russel calmly decides to count the days written instead of the days remaining.

Indeed, (Diamond,1964) resolves the paradox by questioning the counting process itself and its ability to complete infinite sets. He surfaces the importance of the mapping between language and mathematical use. When Cantor defines “size” as an equivalence relation, he redefines the meaning of the sentence: “the set of natural numbers and the set of square numbers have the same size”. While before, in a part-whole sense of size, this sentence might have been paradoxical or false. For Cantor it has been redefined to mean there is a mapping between them.

Diamonds argues that this definition orders a separation of ways. For finite sets “equally numbered” and “same size” coincide, but for the infinite they differ. In a sense, Diamond is demanding we take seriously the remainder growing to infinity. Saying, it is true that at every step the number of days remaining is only finite and will never overtake the infinity of time Tristram has yet to write his diary, but there is no number which can bound this ever-growing amount - and in the limit - it grows to infinity! For Diamond, Tristram can never write “all” the days in his life and Russel’s paradox arises by taking too literally the mathematical definitions.

This brings us to another class of infinite process paradoxes called - supertasks:

Which describe infinite processes that occur in a finite interval of time - unlike the former paradox they use a more demanding timestamp. For instead of the outstretching infinity of time Tristram had at his disposal, supertasks probe the “inward” infinity. Probably the most famous is Zeno’s dichotomy, which he conjured up almost 2500 years ago to refute the possibility of motion itself.

There are many versions of this paradox, but we will focus on the most intuitive and simple of the lot. The setup is as follows: a runner completes a 100-meter-long race, because he ran the whole segment, there must’ve been a time t1, in which he had crossed the 50-meter mark. Similarly, there must’ve been a time t2, in which he had crossed the 75-meter mark and so on. By continuously dividing the remaining segment in half, we can construct an infinite number of points in which the runner must've crossed before he could reach the finish line. But if it is so, then the runner has completed an infinite number of steps in a finite amount of time!

Zeno's paradox has two subtle elements to it:

The first is arithmetic, that of summing an infinite amount of smaller and smaller pieces but which results in a finite amount. It is with this paradox that Zeno himself uses to argue against the possibility of motion. Though an important argument in human history, it was also solved by calculus and made rigorous by Weierstrass and so will not be our main focus. The second point refers to the infinite amount of points the runner must cross in a finite amount of time.

It is this point which Aristotle uses to circumvent Zeno's paradox. Aristotle claims there are two possibilities for the result of the race, precisely because there are two views of the segment and the run itself. There is the potential infinity of the complete 100-meter line segment and the actual infinity of the totality of all its infinitely many divisible parts. Aristotle then argues the running of the whole race - we can experience because no division was needed, the line was only potentially infinite.

The problem is when we think this potential infinity can be realized by the division process of always generating the next half. Aristotle forbids it, arguing that to realize such a run the runner must change the run to a series of discontinuous runs and so this race will never be completed. The base of Aristotle's claim is that for the runner to run the race as Zeno depicts in his argument, a physical distinction must be made at each timestamp. Thus, Aristotle reverts the logical process to a physical process which the runner must apply.(cf. Huggett, 2019)

This argument that ties the impossibility of the run, to the impossibility of a series of such physical events, is a very intuitive argument taken many times throughout history, with progressively tighter and tighter physical bounds. But at each bound a corresponding counter example exists, some such examples include: if the runner were to stop at each timestamp, then this would lead to discontinuous speed and acceleration (Grunbaum, 1969) Or the boundedness of space by the Planck length. (Wisdom ,1951)

A certain challenge to those upholding these methods was put nicely by Earman and Norton:

> *"The deeper problem with Black and Wisdom's conclusion is that it preempts the use of continua in physical theories involving motion. If Wisdom's escape were correct, we would have a philosophical demonstration of the falsity of the major theories of modern physics all of which take for granted that spacetime is a continuum. of course it is conceivable that attempts to marry quantum physics and the general theory of relativity will force the abandonment of the continuum concept for space and time but the notion that armchair philosophizing - and not very good armchair philosophizing at that - can achieve the same aim gives philosophy a bad name" (Earman and Norton,1996, p. 233)*

Earman and Norton dislike using this physical connection to bind the logical debate. Even more so if the "physical argument" contradicts the current physical theories. [which all accept that space can be infinitely divided]

So to put this argument to bed, they adapt another infamous paradox called Thomson's lamp:
One minutes before midnight Thomson decides to turn his lamp on, then half a minute later regrets it and turns it back on, a quarter of a minute later he turns the lamp off and so on, he continues to turn the lamp on and off, each time after waiting exactly $\frac{1}{2}$ the time interval he waited the last time he turned the lamp on/off.
This infinite series of time intervals sums up to 1 minute.
Thomson's question was: what will be the state of the lamp at exactly midnight?

Earman and Norton show an analogous construction: a ball bounces on a rough surface. With each bounce its speed is reduced to half of its speed before the bounce. Assuming an ideal ball that rebounds without taking any time. It is impossible for the ball to rest after only a finite number of bounces, since no bounce can be the last.
They claim there is no paradox in this scenario, even though the ball completes an infinite amount of bounces in a finite amount of time.

For Earman and Norton, the question of the physical state at the end of the 1 minute, can be answered if the problem is stated correctly, with the initial conditions further elaborated.
They argue there are two main fallacies which many incorporate in their arguments to falsely claim there is a paradox:
The first is a misuse of the word "completeness". Infinity defies the orthodox meaning of "completion" for finite tasks, which includes a distinct "last step". After all, infinity is an unending process. Earman and Norton argue it is the search for the "last step" which incorporates an additional constraint of a discontinuous limit, which then leads to contradictions and paradoxes. Yet this constraint isn't necessary for the problem and if it were removed, the paradox could be resolved.
The second common fallacy, claims a property which is conserved throughout a series of finite steps, should then be preserved at the completion of the totality of all infinite steps. For example, in Thomson's lamp, though the state of the lamp at any finite step decides the state at the next step. It would be false to infer that the state after the infinite totality of steps occurred is dependent on the "last step" before it. Furthermore, a schedule of switching lamps before midnight could be fitted to leave the lamp at midnight in either state.

Earman and Norton challenge Aristotle's ancient coupling between the infinite number of logical steps to the infinite number of physical points, claiming the connection should finally be severed. They go on to show a possible realization of a supertask in general relativity, which is too complicated to describe here, but acts as the final nail in the coffin of the physical argument, thereby declaring there is no inherent inconsistency in the completion of an infinite number of distinct tasks in a finite amount of time.

After discussing both major properties of infinity it's time to move on to the main question of its existence - the verdict here is split: their arguments and opinions can be described by the connection they believe exists between the human world and the mathematical world:

Firstly, there are those who believe math is fundamentally a product of the human mind and has no existence independent of us. They are the "constructivist ", they believe what we perceive to be true in the world is tightly bound to our thoughts and experiences, and so there is no all-encompassing truth which we are to discover.

The constructivist wants to epistemically constrain truth, allowing a statement to be true only if a proof of it can be constructed. A further departure from the mathematical “realist” views, was made by Kronecker and Brouwer, the predominant figures in a strain of “constructivist” called “intuitionism”. They argued the “classical logic” used in proofs needed to be revised and only “intuitionist logic” could be used. They completely eradicated the use of actual infinity and were famously known for not allowing the use of irrational numbers in their proofs. Though they are free of possible inconsistencies, a certain challenge they face is rebuilding mathematics with just the “intuitionistic logic”. Many important logical steps cannot be used in this revision, most notably the law of excluded middle and the double negation rule, which are very commonly used in mathematical study.

> *"Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. To prohibit existence statements and the principle of excluded middle is tantamount to relinquishing the science of mathematics altogether.” (Hilbert, 1927, p. 476)*

Moreover, Michele Friend in her book “*Introducing Philosophy of Mathematics*“ (Friend, 2007, p. 69), brings up certain statements which may seem to us must have truth values, either true or false, no matter if we can perceive of them or not, such as:

- The last dinosaur was female.
- There are an infinite number of prime pairs.
- The world was created eight minutes ago, but we cannot detect this, since it was created complete with a history.

But by depending the truth value of a statement on our knowledge, the intuitionist disregards them. For intuitionist there is no place for statements which are impossible for us to know if they are true.

A more “productive” approach, taken by Weyl and Hilbert, allows mathematical objects a certain distance from our knowledge of them.

> *“when at the third level I locate the actually occurring number-signs in the sequence of all possible numbers arising through a process of generation…Here the existent is projected onto the background of the possible, of an ordered manifold of possibilities producible according to a fixed procedure and open to infinity…If one takes mathematics by itself, one should restrict oneself with Brouwer to the truths of insight, in which infinity only enters as an open field of possibilities; there is no motive discoverable that presses farther than that. But in natural science, we touch a sphere impervious to the demands of open-eyed certainty. Here, knowledge necessarily becomes symbolic shaping, which is why when mathematics is taken along by physics in the process of theoretical world-construction it is no longer necessary that the mathematical let itself be isolated as a particular region of the intuitively certain.” (Weyl,1930, p. 19)*

It is important to note that at Weyl and Hilbert’s time, the physical theories were being quantized. Each continuous quantity turned out to be made of discrete chunks (electrons, atoms,energy and light) and so their argument for the impossibility of actual infinity was based on this apparent absence of it from nature:

> *"The infinite divisibility of a continuum is an operation which exists only in thought. It is merely an idea which is in fact impugned by the results of our observations of nature and of our physical and chemical experiments." (Hilbert,1926, p. 186)*

It then follows that in our modern view of reality, we can use this same argument to say that actual infinity exists. The original claim is that infinity is not found in nature and therefore it does not exist. However, theories change and evolve over time, and some of today's modern theories do find actual infinity in nature and it is even essential to the description of it, so one can claim on the contrary, that infinity does exist. Thus, this argument is not impervious to time and depends on the current physical theories. A common continuation to this argument. [if the known physical laws of the time were to agree with it] Then asks contra positively, if actual infinity doesn't exist, how then could it describe reality so effectively?

> *"Can thought about things be so much different from things? Can thinking processes be so unlike the actual processes of things? In short, can thought be so far removed from reality?" (Hilbert,1926, p. 191)*

Another possible objection to this theory, made by the "intuitionist", would be to ask about the placement of the line whereby one would cross from the infinite realm into reality, this vagueness of what exists and what does not can be troublesome.

Lastly, are Plato's successor's, they put the line of existence between mathematics and reality but on the opposite side! Most notable amongst them were Godel and Cantor. To them the way we perceive the world is questionable and flawed. Where as, mathematical objects are consistent, there existence transcends nature as we know it. As Cantor expressed in a letter to Hermite:

> *"the (whole) numbers seem to me to be constituted as a world of realities which exist outside of us…the reality and absolute existence of the natural numbers was in fact much greater than that of Nature because their existence is not based on knowledge that we receive from our senses…In fact, the physical world, i.e. space and time, could contribute nothing to mathematical inquiry" (Dauben,1977, p. 228)*

Cantor is claiming there is no reason for mathematicians to ground mathematics in arbitrary physical objects, the rigor in which they are studied gives them confidence enough.

put exquisitely well by Godel:

> *"Classes and concepts may, however, also be conceived as real objects [...]existing independently of our definitions and constructions. It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions [...].*

> *But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception[...]" (Godel,1944, p. 120)*

An infamous objection by Benacerraf (Benacerraf, 1973) and Field after him (cf. Field, 1989), argues that abstract mathematical objects cannot exist, since us, humans, exist within spacetime, and if they exist, then they do not exist in spacetime.
So if there exist an abstract mathematical object, we would not be able to know about it.
But if that is correct, then how did we come to perceive the mathematical knowledge we already have? aren't our senses and experiences the only way for us to gain new knowledge?
Hence, human beings have attained mathematical knowledge, which proves that Mathematical platonism is not correct.

CONCLUSION

In this essay we have visited many paradoxes and possible resolutions of them, we have explored the possibility of the infinite and questioned its connection to our perception of reality.
It is not our goal to arrive at a verdict on the subject, but to act as the aperitif to the mind, for infinity's history and internal paradoxes are far removed from the black boxes young mathematicians are given for their use and study.
We hope that the reader will obtain a renewed curiosity of the brave decisions made throughout the century and other possible alternatives that remain to be discovered.

> *"The completed, actual infinite as a closed realm of absolute existence cannot be given to the mind. Nevertheless, mind is ineluctably compelled by the demand of totality and the metaphysical belief in reality to represent infinity as closed Being by means of a symbolic construction." (Weyl,1930, p. 30)*

Just like the symbol of the number 1 helps us represent all possible objects of size 1, encapsulating all infinitely many possible objects of amount one in a single symbol and forgetting about the objects it can refer to.So to, symbolizing infinity lets our brain forget the nature of the beast it has tamed.
The subject of infinity and its presence in our everyday life is still being investigated and studied in the fields of physics and mathematics and though infinity's existence is an all-out rebellion against what we can believe to be true, we must not limit reality with the boundaries of what the human mind can grasp.

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