IMAGE TRANSFORMS

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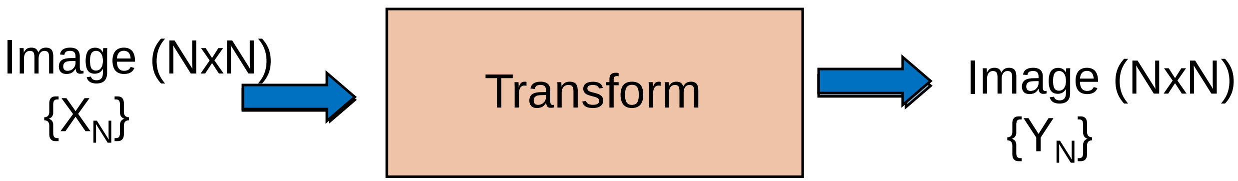
Image Transforms

- Image transforms can be simple arithmetic or complex mathematical operations on images which convert images from one representation to another.



$$Y=AX$$

$$X=A^{-1}Y$$

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Applications

- Preprocessing
  - Filtering
  - Enhancement

- Image Compression

- Feature Extraction
  - Edge detection
  - Corner detection

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Unitary Matrix

- Image transformation represents a given image into series sum of unitary matrices

- A matrix is called unitary if

$$A^{-1} = \underbrace{A^{*T} \equiv A^{H}}_{\text{Hermitian conjugate}}$$

- Unitary Matrices are basis images

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Properties of Unitary Transform y =Ax

- Energy Conservation
  - $\| \underline{y} \|^2 = \| \underline{x} \|^2$
    - $\| \underline{y} \|^2 = \| A\underline{x} \|^2 = (A\underline{x})^{*T} (A\underline{x}) = \underline{x}^{*T} A^{*T} A \, \underline{x} = \underline{x}^{*T} \underline{x} = \| \underline{x} \|^2$
- Rotation
  - A unitary transformation is a rotation of a vector in an N-dimension space, i.e., a rotation of basis coordinates

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Contd..

- Energy Compaction
  - Many common unitary transforms (like DCT, KLT etc.) tend to pack a large fraction of signal energy into just a few transform coefficients
- Decorrelation
  - Highly correlated input elements ➔ quite uncorrelated output coefficients
  - Covariance matrix $E[(\underline{y} - E(\underline{y}))(\underline{y} - E(\underline{y}))^{*T}]$

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What is orthogonal ?

- A set of real valued continuous functions $\{a_n(t)\} = \{a_o(t), a_1(t), \ldots\}$ is said to be orthogonal over (t_0, t_0+T) if

$$\int_T a_m(t).\, a_n(t)\, dt = \begin{cases} k \ \ if \ m = n \\ 0 \ \ if \ \ m \neq n \end{cases}$$

if $k = 1$ then $a_n(t)$ is orthonormal

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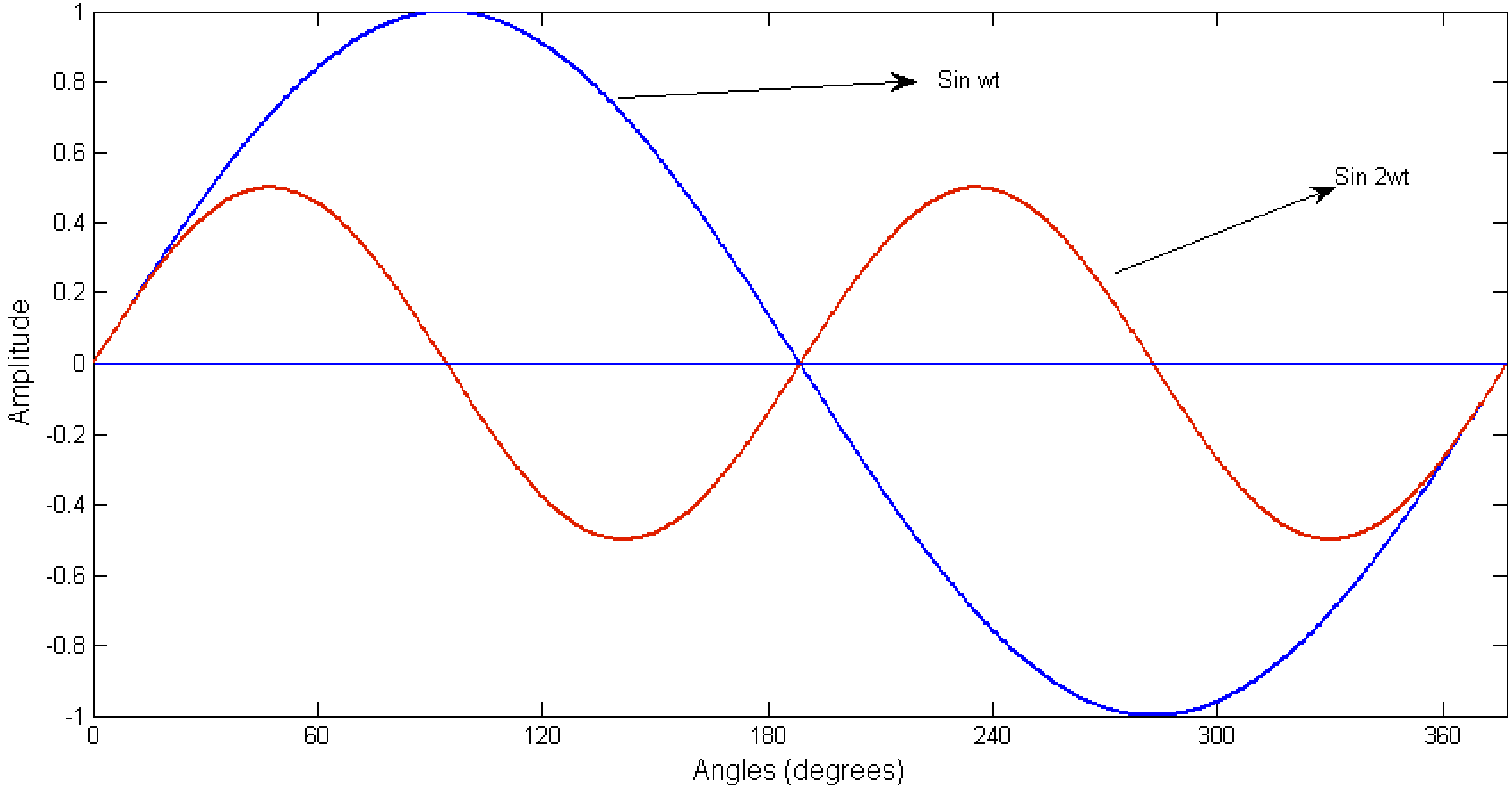
Example

- {sinωt, sin2ωt}



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Orthogonal Expansion

Any arbitrary signal *x(t); {t_0, t_0+T}*

Can be represented by series summation of a set of orthogonal basis functions.

$$x(t) = \sum_{n=0}^{\infty} c_n a_n(t)$$

$c_n \longrightarrow$ nth coefficient of expansion

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Contd..

$$x(t) = \sum_{n=0}^{\infty} c_n a_n(t), t_0 \le t < t_0 + T$$

$$\Rightarrow \int_T x(t) a_m(t) dt = \int_T \sum_{n=0}^{\infty} c_n a_n(t) a_m(t) dt$$

$$= c_0 \int_T a_0(t) a_m(t) dt + c_1 \int_T a_1(t) a_m(t) dt + \ldots + c_m \int_T a_m(t) a_m(t) dt + \ldots$$

$$\Rightarrow \int_T x(t) a_m(t) dt = k c_m \qquad \Rightarrow \int_T x(t) a_m(t) dt = c_m, \text{ if } k = 1$$

Orthonormal Set

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Complete, Closed(orthogonal functions)

- There is no signal x(t) with $\int_T x^2(t)dt < \infty$

such that

$$\int_T x(t)a_n(t)dt = 0, n = 0,1,..$$

- x(t) with $\int_T x^2(t)dt < \infty$

$$\hat{x}(t) = \sum_{n=0}^{N-1} c_n a_n(t)$$

Such that $\int |x(t) - \hat{x}(t)|^2 dt < \varepsilon \quad \varepsilon > 0$

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Discrete Formulation

- Let the set of samples represented by $\{u(n): 0 \leq n \leq N-1\}$ is a vector of dimension N

- Premultiply **u** by a unitary matrix A of dimension N*N, we get another vector

$$v = Au$$

Transformed vector

Transformation matrix

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Contd..

In the form of series summation

$$v(k) = \sum_{n=0}^{N-1} a(k,n)u(n) \qquad 0 \le k \le N-1$$

where $\quad A^{-1} = A^{*T}$

As A is a unitary matrix we can get back u as

$$u(n) = \sum_{k=0}^{N-1} a^{*}(k,n)v(k) \qquad 0 \le n \le N-1$$

The columns of A^{*T}

i.e., $(a_k^{*} \approx \{a^{*}(k,n), 0 \le n \le N-1\}^{T})$ are called the basis vectors of A

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Example

- Let

$$U=\begin{bmatrix}u_0\\u_1\\u_2\end{bmatrix}\quad V=\begin{bmatrix}v_0\\v_1\\v_2\end{bmatrix}\quad A=\begin{bmatrix}a_{00}&a_{01}&a_{02}\\a_{10}&a_{11}&a_{12}\\a_{20}&a_{21}&a_{22}\end{bmatrix}$$

$$A^{-1}=A^{*T}=\begin{bmatrix}a_{00}&a_{10}&a_{20}\\a_{01}&a_{11}&a_{21}\\a_{02}&a_{12}&a_{22}\end{bmatrix}^*$$

$$U=A^{-1}V=A^{*T}V$$

$$U=\begin{bmatrix}u_0\\u_1\\u_2\end{bmatrix}=A^{-1}\begin{bmatrix}v_0\\v_1\\v_2\end{bmatrix}=\underbrace{\begin{bmatrix}a_{00}\\a_{01}\\a_{02}\end{bmatrix}^*}_{b_1}v_0+\underbrace{\begin{bmatrix}a_{10}\\a_{11}\\a_{12}\end{bmatrix}^*}_{b_2}v_1+\underbrace{\begin{bmatrix}a_{20}\\a_{21}\\a_{22}\end{bmatrix}^*}_{b_3}v_2$$

basis vectors $\longrightarrow$ b_1 b_2 b_3

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Contd..

In case of an image, an image can be represented by a set of basis images

Image $\longrightarrow$ $u(m,n) \qquad 0 \le m,n \le N-1$

$$v(k,l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{k,l}(m,n)u(m,n) \quad 0 \le k,l \le N-1$$

$a_{k,l}(m,n) \longrightarrow$ N × N matrix/ N^2 numbers of such matrices

Inverse Transform

$$u(m,n) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_{k,l}^{*}(m,n)v(k,l) \quad 0 \le m,n \le N-1$$

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Contd..

- Set of complete orthonormal discrete basis function satisfies the following properties

$a_{k,l}(m,n) \longrightarrow$ Orthonormality

$$\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{k,l}(m,n)a^{*}_{k',l'}(m,n) = \delta(k-k', l-l')$$

Completeness

$$\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_{k,l}(m,n)a^{*}_{k,l}(m', n') = \delta(m-m', n-n')$$

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Computational Complexity

- To Compute transform coefficient $v(k,l)$

$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{k,l}(m,n)u(m,n) \quad 0\le k,l\le N-1$$

- For each $v(k,l)$ no. of complex multiplications and additions ~ O(N^2), so for all $v(k,l)$~O(N^4) which is quite expensive for practical size Images.
- How to reduce computation time??
- Using separable unitary transforms

$$a_{k,l}(m,n)=a_k(m).b_l(n)\approx a(k,m)b(l,n)$$

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Separable Transforms

where $\{a_k(m), k = 0,1,..N-1\}$ are 1D orthonormal sets of basis vectors
$\{b_l(n), l = 0,1,..N-1\}$

should be unitary matrices themselves

In most cases we choose A & B to be same. $A \approx \{a(k,m)\}$ *and* $B = \{a(l,n)\}$

$$v(k,l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a(k,m)\,u(m,n)\,a(l,n);\ V = AUA^T$$

$$u(m,n) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a^*(k,m)\,v(k,l)\ a^*(l,n);\ U = A^{*T}VA^*$$

$$V = AU\,A^T$$

$$V^T = A[AU]^T$$

- These are called 2D separable transformation
- The 2D transform can be performed first by transforming each column of U and then transforming each row of the result to obtain rows of V complexity is $2N^3$

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Concept of Basis Images

- Let a_k^* denotes the k[th] column of A^{*T}
- Define the matrices

$$A_{k,l}^* = a_k^* a_l^{*T}$$

- Define the inner product of two NxN matrices F and G

$$\langle F, G \rangle = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) g^*(m,n)$$

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Concept of Basis Images

- Then the transformation equation can be written as

$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{k,l}(m,n)u(m,n) \approx \left\langle U, A_{k,l}^{*} \right\rangle$$

$$u(m,n)=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_{k,l}^{*}(m,n)v(k,l)$$

$$\Rightarrow U = u(m,n)=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l)A_{k,l}^{*}$$

- U is represented by linear combination of N^2 Matrices.
- These matrices are called basis images

$$A_{k,l}^{*} \quad k,\ l = 0,1,2..N-1$$

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Example

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Transformed Image $V = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 4 & 6 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -1 \\ -2 & 0 \end{pmatrix}$$

To get basis images, take outer product of the columns of A^{*T}

$$A^*_{0,0} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^*_{0,1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = A^{*T}_{1,0}$$

$$A^*_{1,1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Contd..

- Inverse Transform gives

$$A^{*T}VA^{*}=\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}5 & -1\\ -2 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}3 & -1\\ 7 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

$$=U \Rightarrow \text{Original Image}$$

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Some Transformation Techniques

- Discrete Fourier Transform
- Discrete Cosine Transform
- Karhunen-Loeve Transform
- Haar Transform
- Walsh Transform
- Hadamard Transform

Fourier Transform (1D)

Continuous Fourier Transform (CFT)

$$W(f) = \mathcal{F}\{w(t)\} = \int_{-\infty}^{\infty} w(t)\, e^{-j2\pi ft}\, dt$$

$$W(f) = X(f) + j\,Y(f)$$

$$W(f) = |W(f)|\, e^{j\theta(f)}$$

Frequency, [Hz]

Phase Spectrum

Amplitude Spectrum

Inverse Fourier Transform (IFT)

$$w(t) = \mathcal{F}^{-1}\{W(f)\} = \int_{-\infty}^{\infty} W(f)\, e^{+j2\pi ft}\, df$$

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Discrete Fourier Transform (1D)

- Discrete Domains
  - Discrete Time: $n = 0, 1, 2, 3, \ldots\ldots\ldots\ldots, N\text{-}1$
  - Discrete Frequency: $k = 0, 1, 2, 3, \ldots\ldots\ldots\ldots, N\text{-}1$

Equal time intervals

Equal frequency intervals

- Discrete Fourier Transform

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk}; \qquad K = 0, 1, 2, \ldots.., N\text{-}1$$

- Inverse DFT

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[K]\, e^{j\left(\frac{2\pi}{N}\right)nk}; \qquad n = 0, 1, 2, \ldots.., N\text{-}1$$

2D – Discrete Fourier Transform

- Image: f(x, y), 0≤x≤N-1, 0≤y≤N-1
- Transform kernel: $g(x,y,u,v) = e^{-j2\pi\left(\frac{xu}{N}+\frac{yv}{N}\right)}$
- Discrete Fourier Transform of **f(x, y)**is

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)}$$

$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\left(\frac{ux}{N}+\frac{vy}{N}\right)}$$

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Contd..

- In general, Transform kernel can be represented as:

$$w_N = e^{-j\frac{2\pi}{N}} \qquad w_N^{ux+vy} = e^{-j\frac{2\pi}{N}(ux+vy)}$$

- Discrete Fourier Transform of ***f(x, y)*** can be represented as

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\, w_N^{ux+vy}$$

$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)\, w_N^{-(ux+vy)}$$

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Contd..

Let $R(u,v)$, $I(u,v)$ be real part and imaginary part of $F(u,v)$

Modulus:

$$|F(u,v)| = \left(R^2(u,v) + I^2(u,v)\right)^{1/2}$$

Phase:

$$\Phi(u,v) = \tan^{-1}\left(\frac{I(u,v)}{R(u,v)}\right)$$

Power Spectrum:

$$E(u,v) = R^2(u,v) + I^2(u,v)$$

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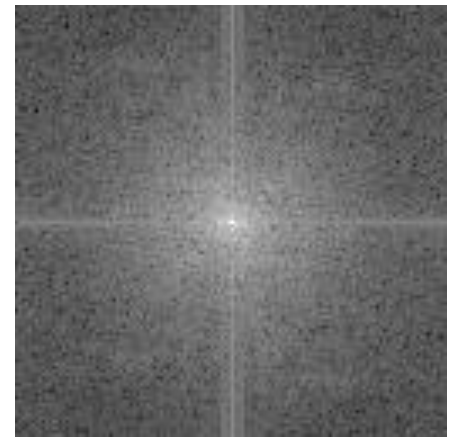
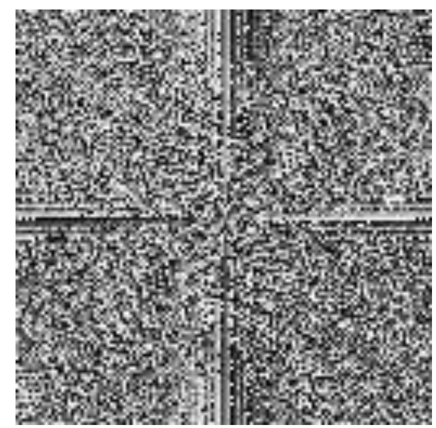
Amplitude and Phase

$$\left|F(x)\right|$$



amplitude

phase



original

$$\angle F(x)$$

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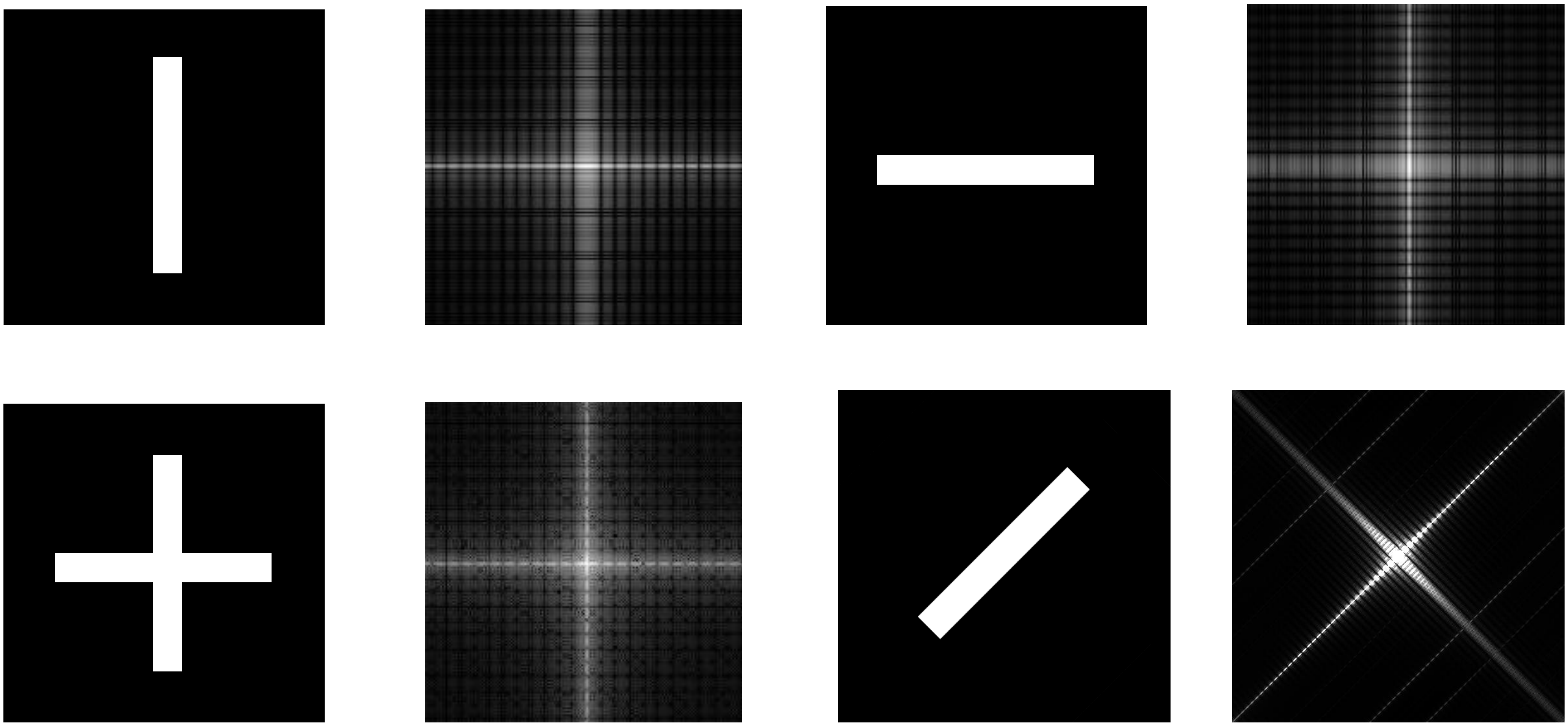
Images and their spectrums

Image domain | Frequency domain | Image domain | Frequency domain



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Properties of 2-D Fourier Transform

- Separability
- Linearity
- Scaling
- Shift Theorem
- Rotation
- Periodicity and conjugation
- Convolution
- Correlation
- Average

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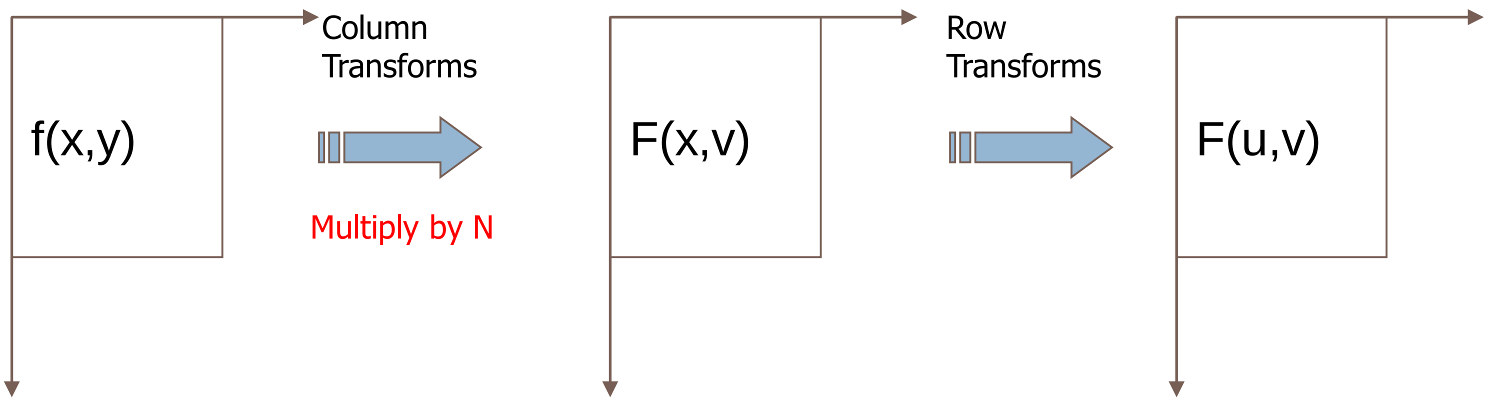
Separability

- The implementation steps for the two-dimensional DFT may be visualized as shown in the diagram below



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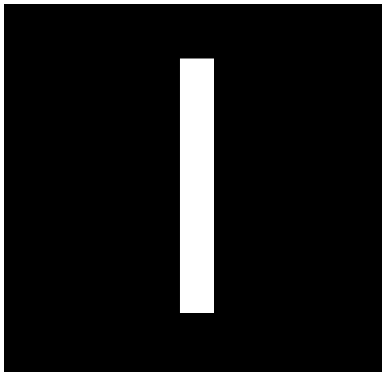
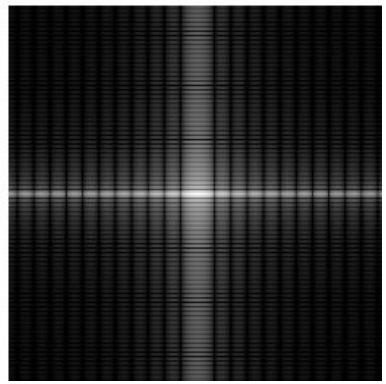
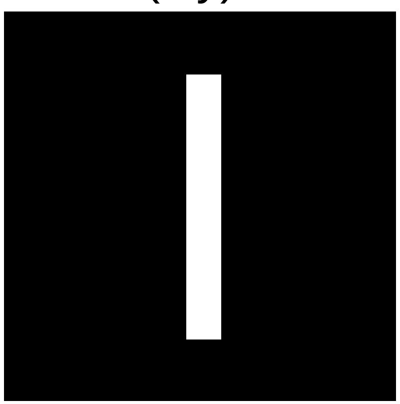
Separability (contd.)



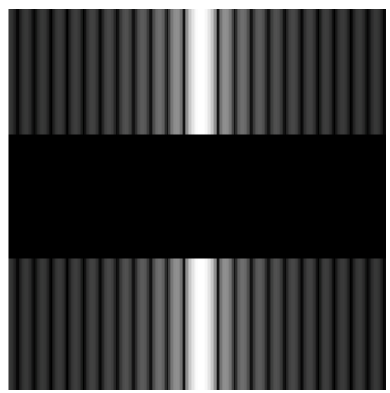
f(x,y)



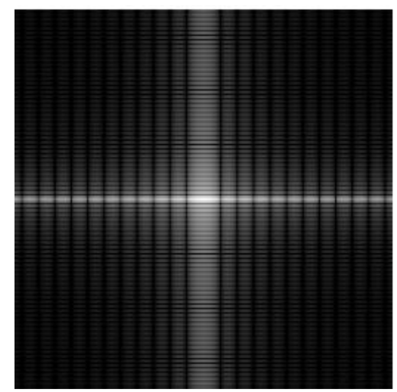
F(u,v)



f(x,y)



F(x,v)



F(u,v)

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Linearity

- DFT is a linear operator

$$F[\ a\, f_1(x,y)+b\, f_2(x,y)] = aF_1(u,v)+bF_2(u,v)$$

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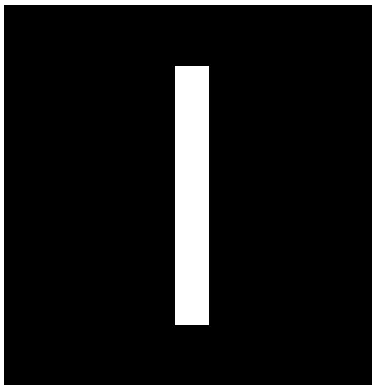
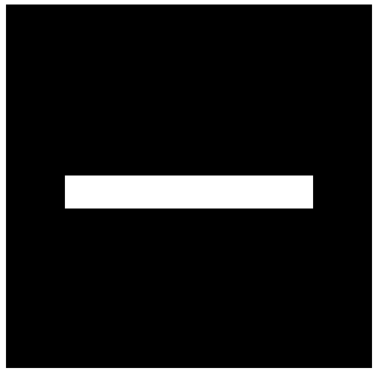
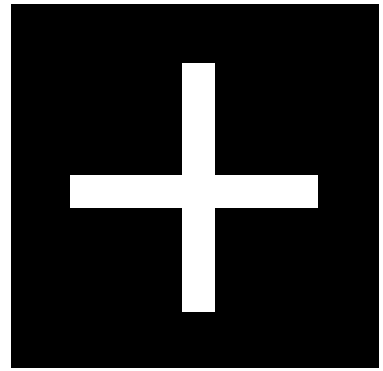
Linearity (contd..)



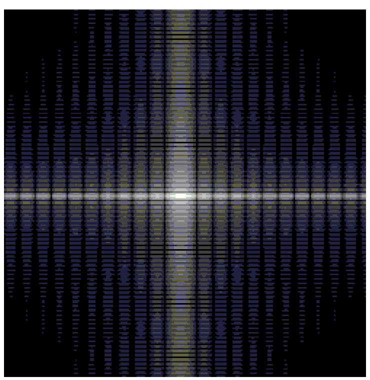
$f_1(x,y)$



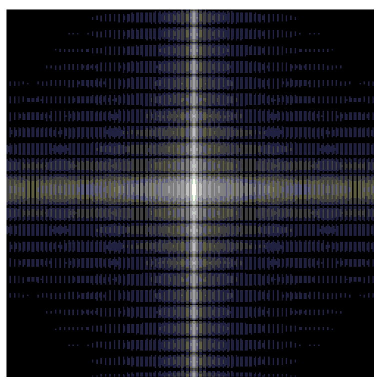
$f_2(x,y)$



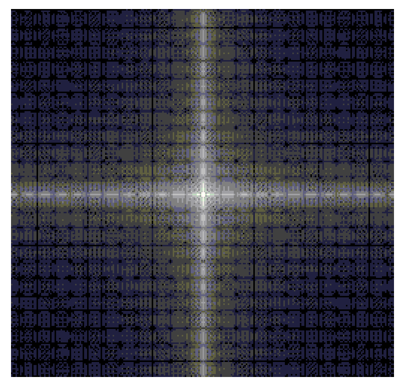
$2f_1(x,y)+3f_2(x,y)$



$F_1(u,v)$



$F_2(u,v)$



Phase Spectrum $(2F_1(u,v)+3F_2(u,v))$

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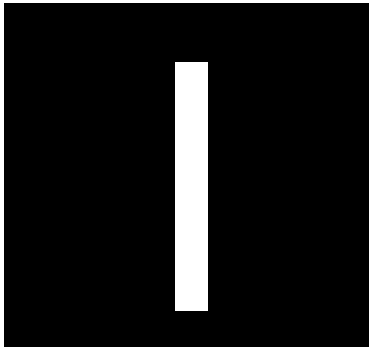
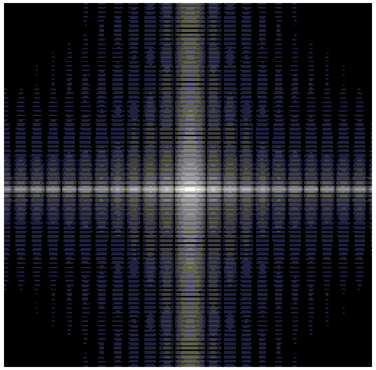
Scaling

$$f(ax, by) \xrightarrow{DFT} \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

Scaling (contd..)

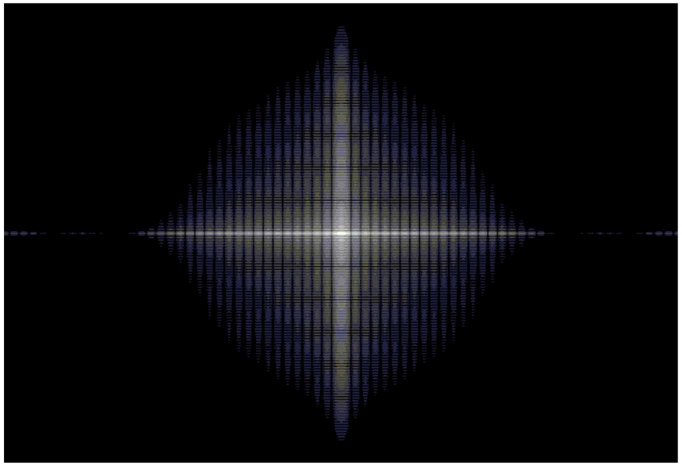


f(x,y)



F(u,v)

f(2x,3y)



F(u,v)

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Shift Theorem

$$f(x-x_0, y-y_0) \Longleftrightarrow F(u,v)\exp[-j2\pi(ux_0+vy_0)/N]$$

$$f(x,y)\exp[j2\pi(u_0x+v_0y)/N] \Longleftrightarrow F(u-u_0, v-v_0)$$

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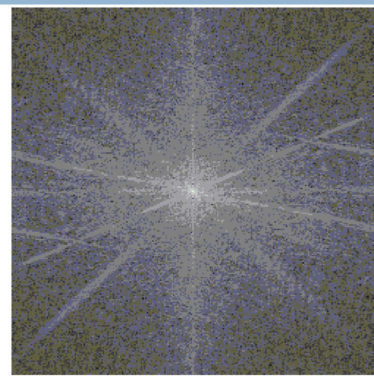
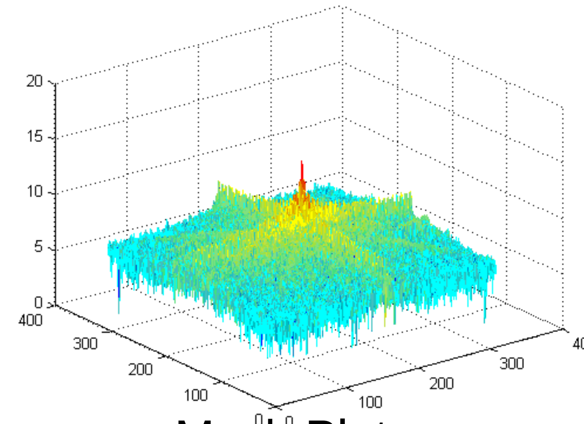
Shift Theorem (contd..)

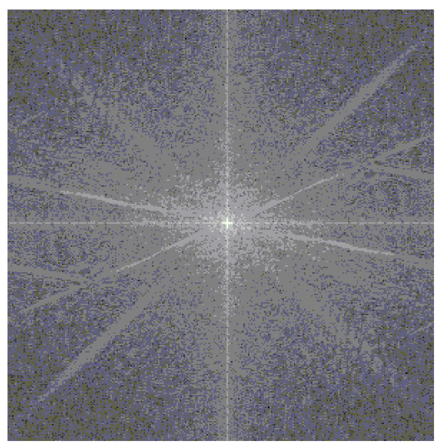

Original Image

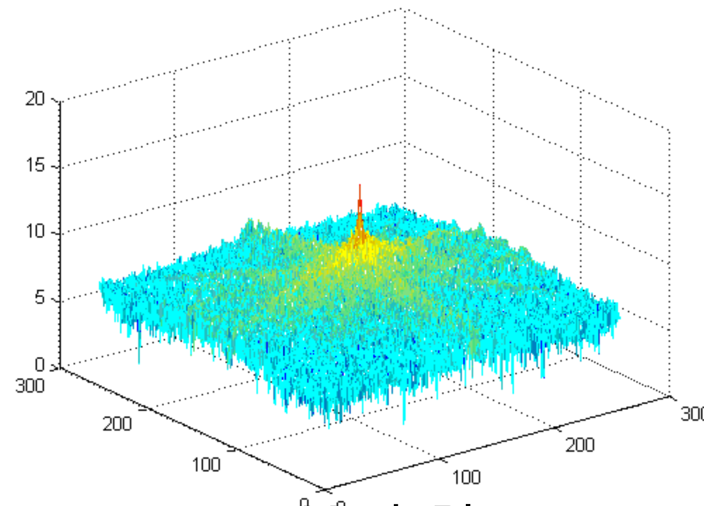

Spectrum Image



Mesh Plot


$T_x = 50, T_y = 50$


Spectrum Image



Mesh Plot

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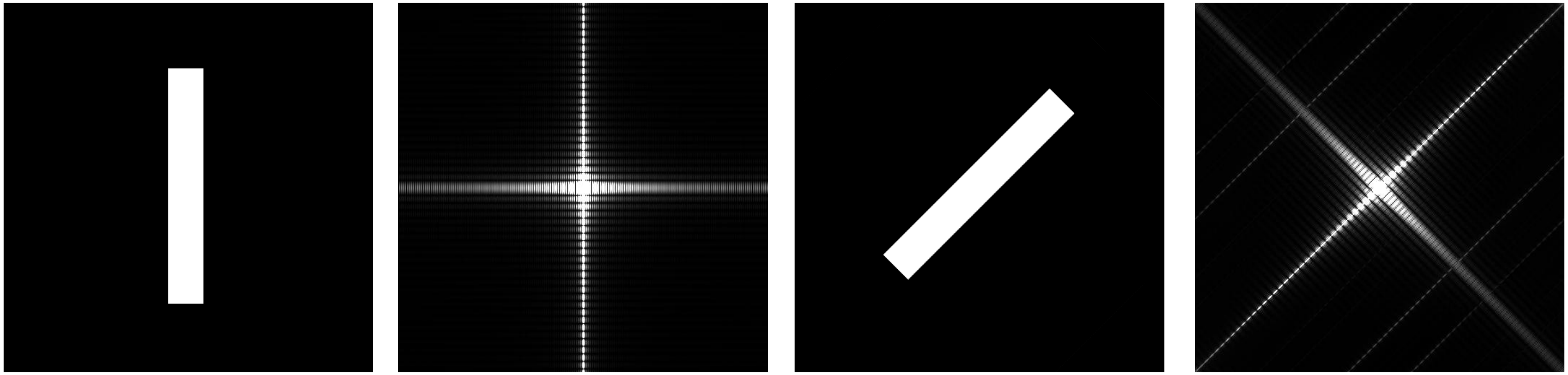
Rotation

In polar coordinates, $f(x,y)$ and $F(u,v)$ can be represented by $f(r, \theta)$ and $F(w,\varphi)$ alternatively. Then

$$f(r, \theta + \theta_0) \Longleftrightarrow F(w, \varphi + \theta_0)$$



(a) a sample image (b) its spectrum (c) rotated image (d) resulting spectrum

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Periodicity and conjugation

Periodicity：

$$F(u,v) = F(u+aN, v+bN)$$

$$f(x,y) = f(x+aN, y+bN)$$

Where a, b = 0, $\pm$1, $\pm$ 2,……

Conjugation：

$$F(u,v) = F^*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

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Convolution

2-D convolution is defined as:

$$f_e(x,y) * g_e(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_e(m,n)g_e(x-m,y-n)$$

Then: $f(x, y)*g(x, y) \Leftrightarrow F(u, v)G(u, v)$

$$f(x, y)g(x, y) \Leftrightarrow F(u, v)*G(u, v)$$

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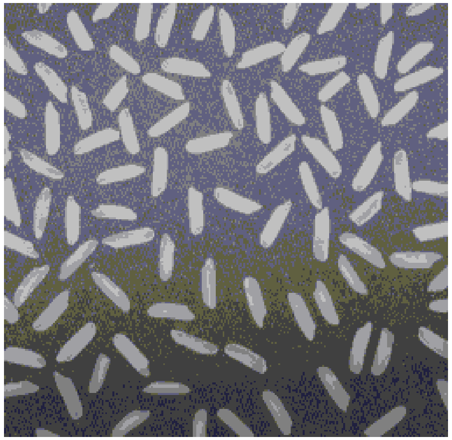
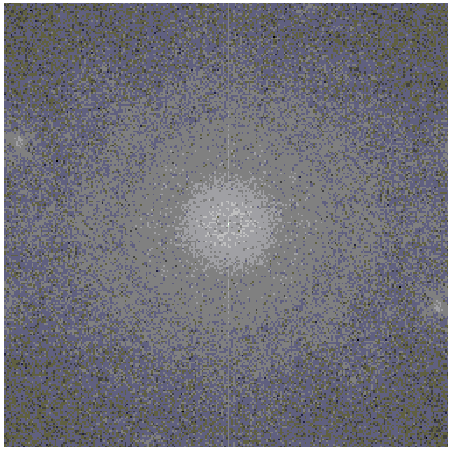
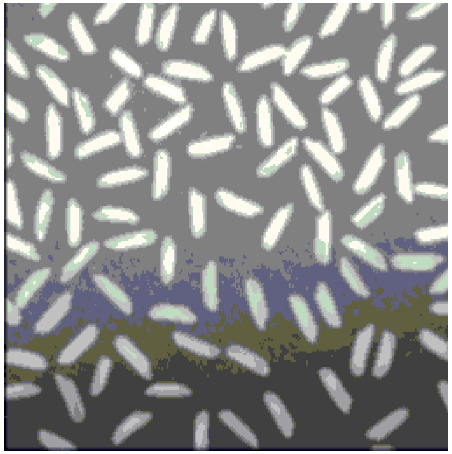
Convolution (contd..)



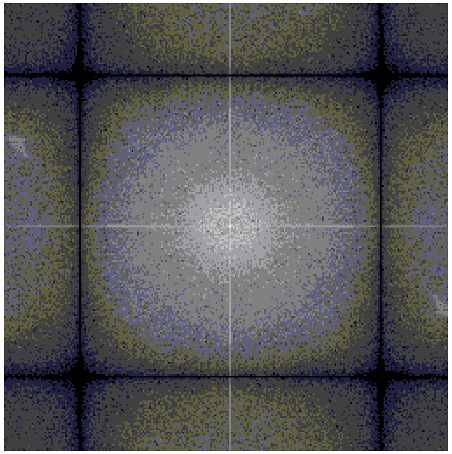
Original image



Spectrum



Convolved image(convolved with mean filter)



Spectrum

Correlation

2-D correlation is defined as：

$$f_e(x,y)\circ g_e(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f_e^*(m,n)g_e(x+m,y+n)$$

Correlation Theorems：

$$f(x,y)\circ g(x,y)\Leftrightarrow F^*(u,v)G(u,v)$$

$$f^*(x,y)g(x,y)\Leftrightarrow F(u,v)\circ G(u,v)$$

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Average

Average of 2D image *f* (x, y) is defined as:

$$\bar{f}(x,y) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

Let $u=v=0$, then:

$$F(0,0) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

$$\bar{f}(x,y) = \frac{1}{N} F(0,0)$$

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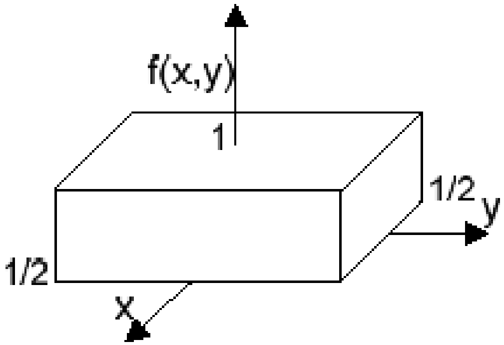
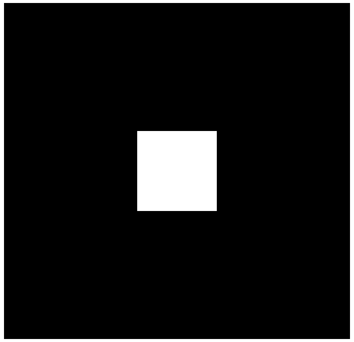
Contd..

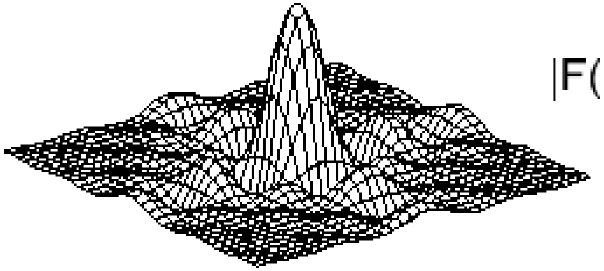




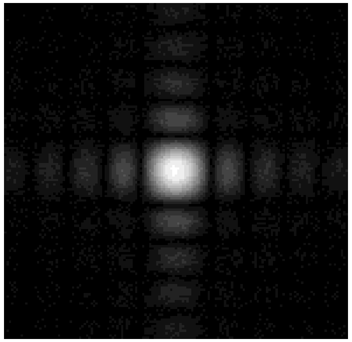
image

$$f(x,y) = \mathrm{rect}(x,y) = \begin{cases} 1 & |x| \quad 1/2,\ |y| \quad 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$F(u,v) = \ \mathrm{sinc}(u) \quad \mathrm{sinc}(v) = \mathrm{sinc}(u,v)$$

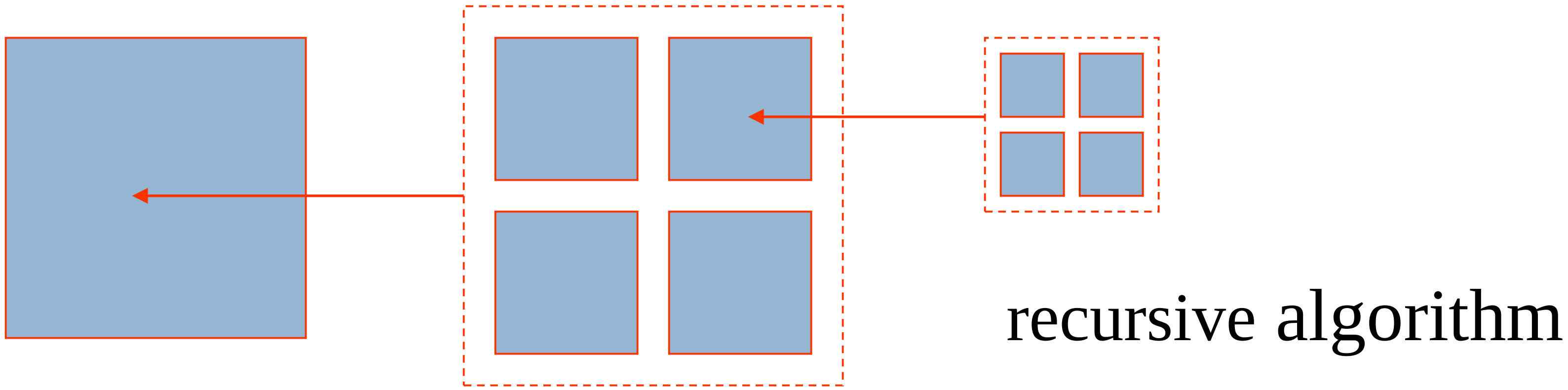


|F(u,v)|



and its
spectrum

Fast Fourier Transform



- decimation in time = odd even in freq. domain
- decimation in freq. domain = odd even in time

$$N^4 \rightarrow N^2 \log N$$

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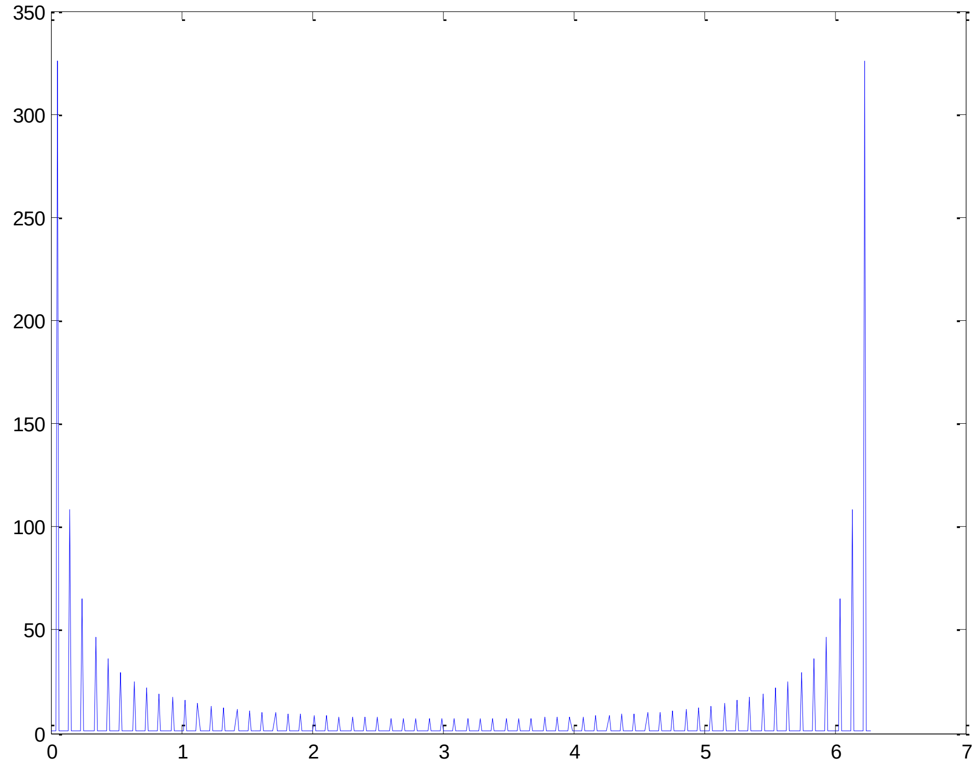
Visualizing the DFT

- Consider the power spectrum of the 1D square wave



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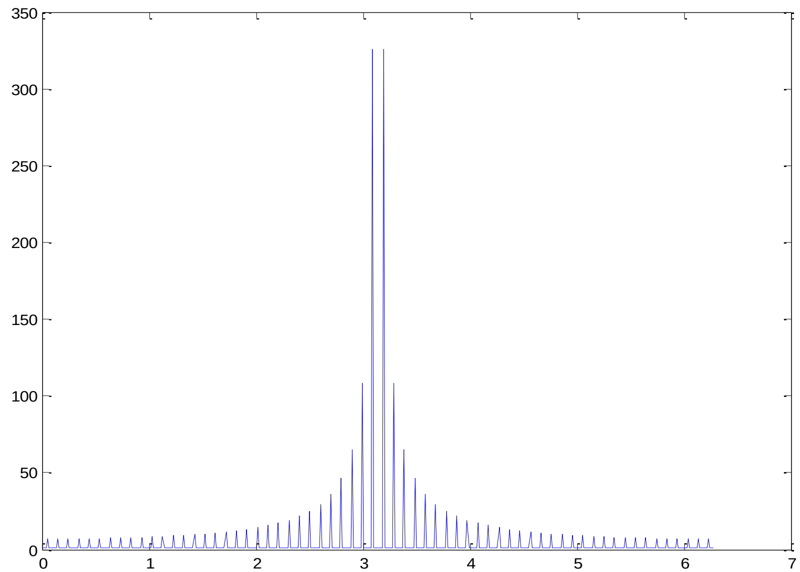
Visualizing the DFT(circular shifting)

- The FT is centered about the origin
- But the DFT is centered about N/2
- We need to correct with a circular shift operation
- Or, multiply by $(-1)^k$ prior to taking the transform



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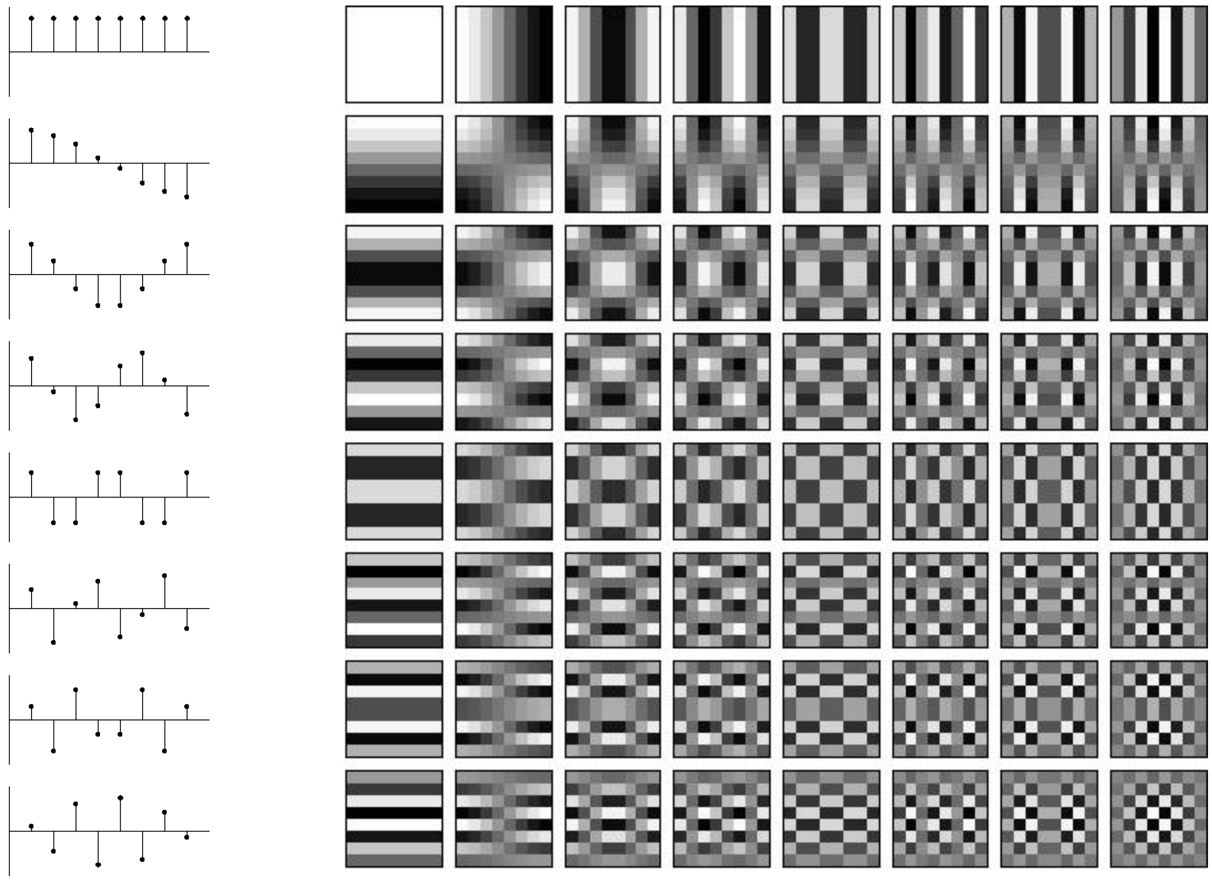
DFT basis images



Procedure for Filtering in the Frequency Domain

1. Multiply the input image by $(-1)^{x+y}$ to center the transform
2. Compute the DFT *F(u,v)* of the resulting image
3. Multiply F(u,v) by a filter G(u,v)
4. Computer the inverse DFT transform h*(x,y)
5. Obtain the real part h(x,y) of 4
6. Multiply the result by $(-1)^{x+y}$

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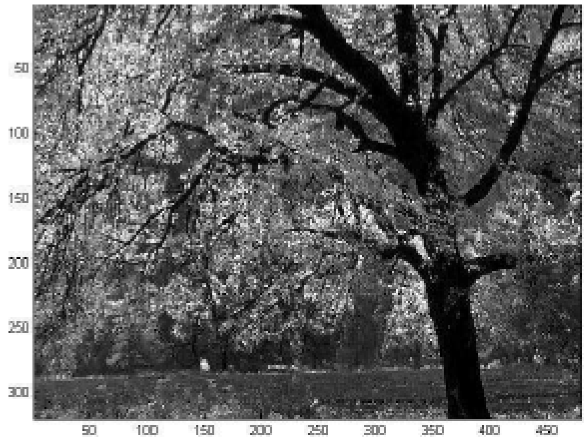
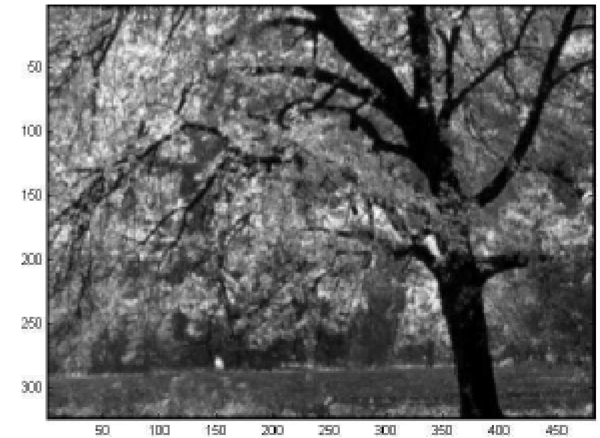
Filtering Example
Smooth an Image with a Gaussian Kernel

- Traditionally, we would just convolve the image with the a gaussian kernel



$$* \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} /256 =$$



- Instead, we will perform multiplication in the frequency domain to achieve the same effect

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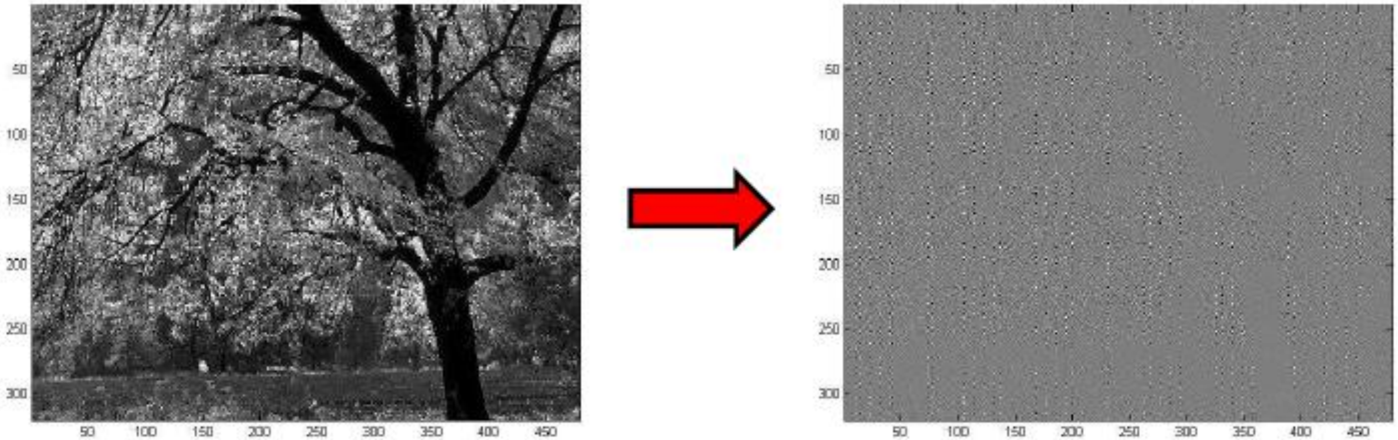
Filtering Example
Smooth an Image with a Gaussian Kernel

1. Multiply the input image by $(-1)^{x+y}$ to center the transform



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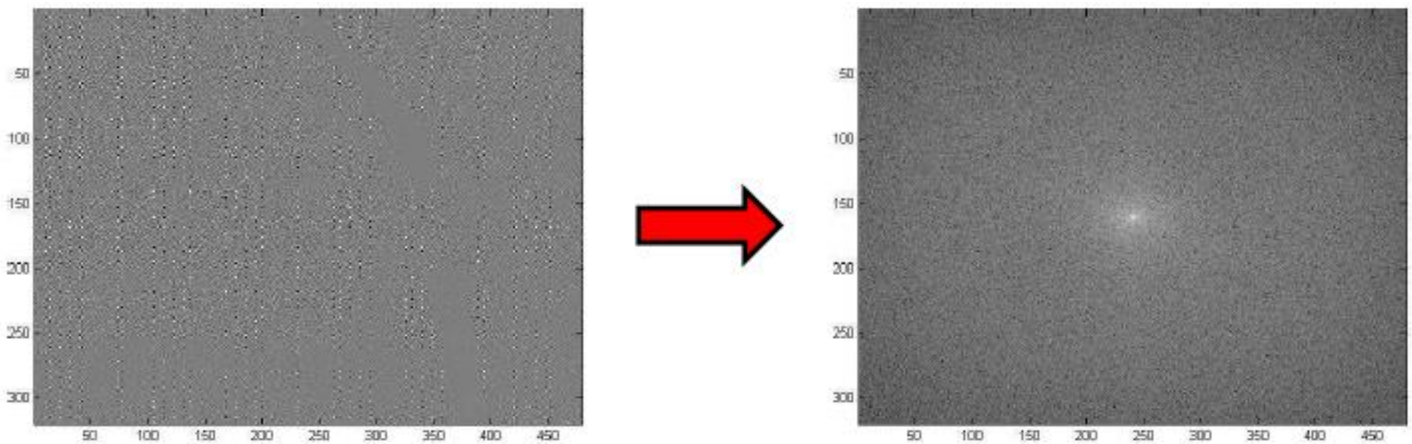
2. Compute the DFT *F(u,v)* of the resulting image



log transform

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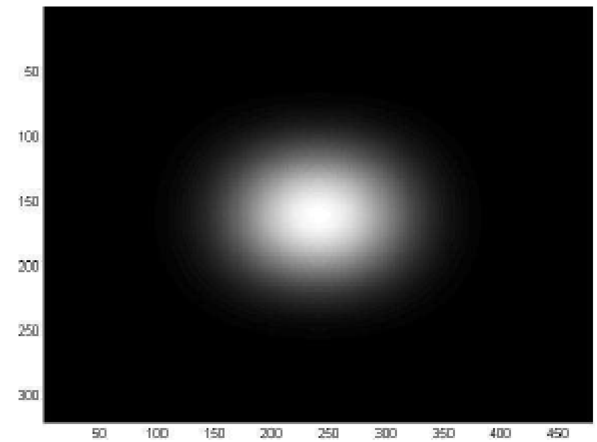
3. Multiply F(u,v) by a filter G(u,v)

$$\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} / 256$$

g(x,y)



G(u,v)

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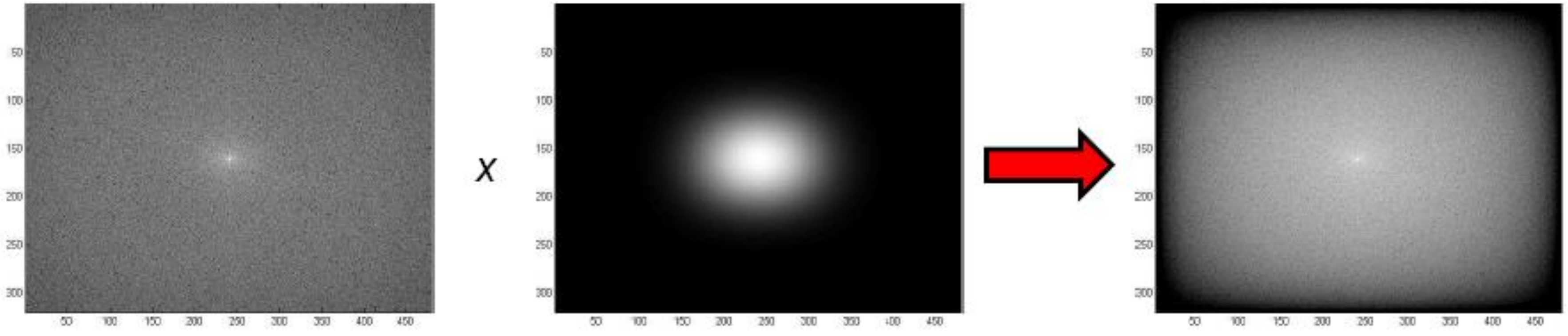
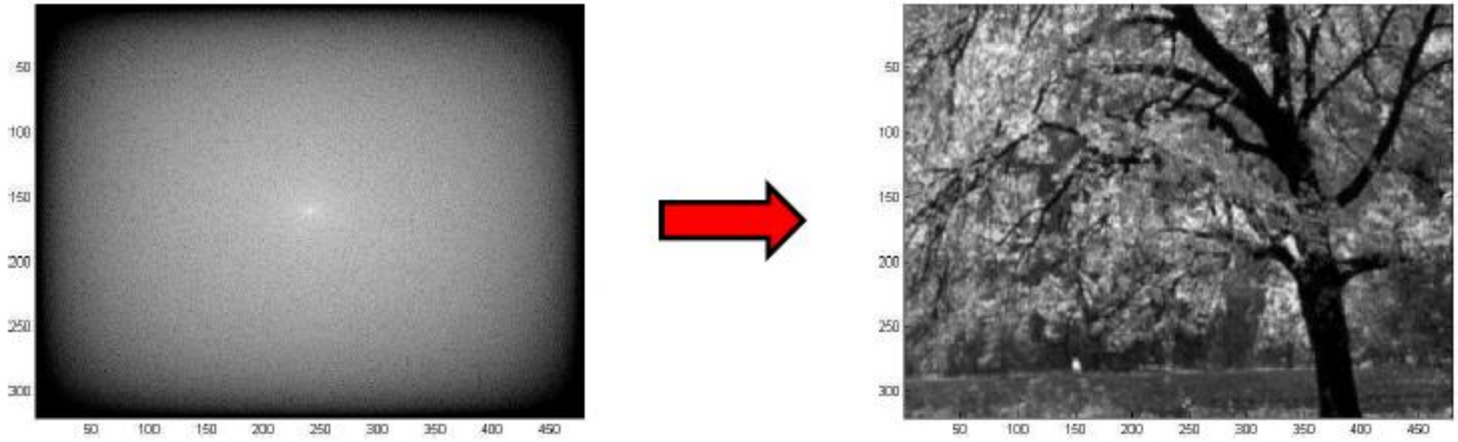
3. Multiply F(u,v) by a filter G(u,v)



4. Computer the inverse DFT transform $h^*(x,y)$
5. Obtain the real part $h(x,y)$ of 4
6. Multiply the result by $(-1)^{x+y}$

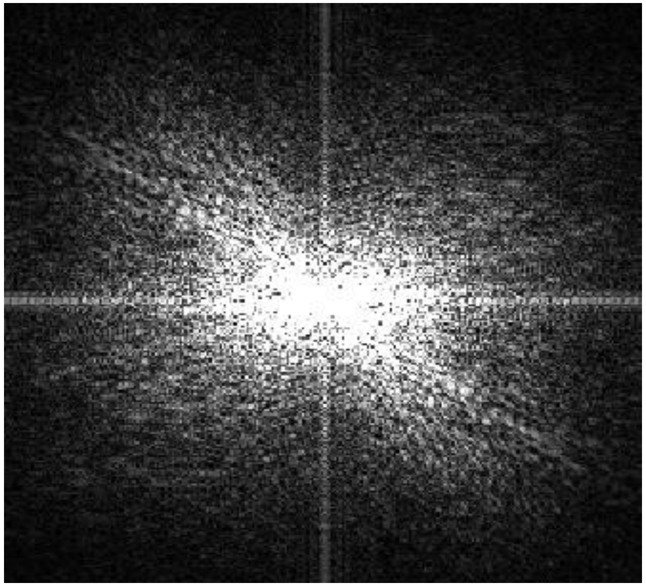


Example


```
img=imread('lena.bmp','bmp');
subplot(121);imshow(img);
title('original image ')
fimg=fftshift(fft2(img));
subplot(122); imshow(abs(fimg)/10000)
title(' transformed image ')
```

original image

transformed image



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Discrete Cosine Transform (DCT)

- DCT expresses a sequence of finite data points in terms of a sum of cosine functions oscillating at different frequencies.
- These cosine functions are treated as its basis function.
- Fast algorithm exists.
- Most popular in image compression application because of its energy compaction property.
- Adopted in JPEG.
- The periodicity implied by DCT implies that it causes less blocking effect than DFT.
- Can be implemented by 2*n* points FFT.

Contd..

Transform kernel of two-dimensional DCT is

$$C(x, y, u, v) = \alpha(u) \cdot \alpha(v) \cdot \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cdot \cos\left[\frac{\pi(2y+1)v}{2N}\right]$$

where

$$\alpha(0) = \sqrt{\frac{1}{N}}, \qquad \alpha(i) = \sqrt{\frac{2}{N}}, \; 1 \le i \le N$$

Clearly, the kernel for the DCT is both separable and symmetric, and hence the DCT may be implemented as a series of one dimensional DCTs.

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Contd..

Forward transform

$$F(u,v)=\alpha(u)\alpha(v)\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right]$$

Inverse Transform

$$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}\alpha(u)\alpha(v)F(u,v)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right]$$

where $u, v, x, y = 0, 1, 2, \ldots, N\text{-}1$

Forward and inverse transformations are same

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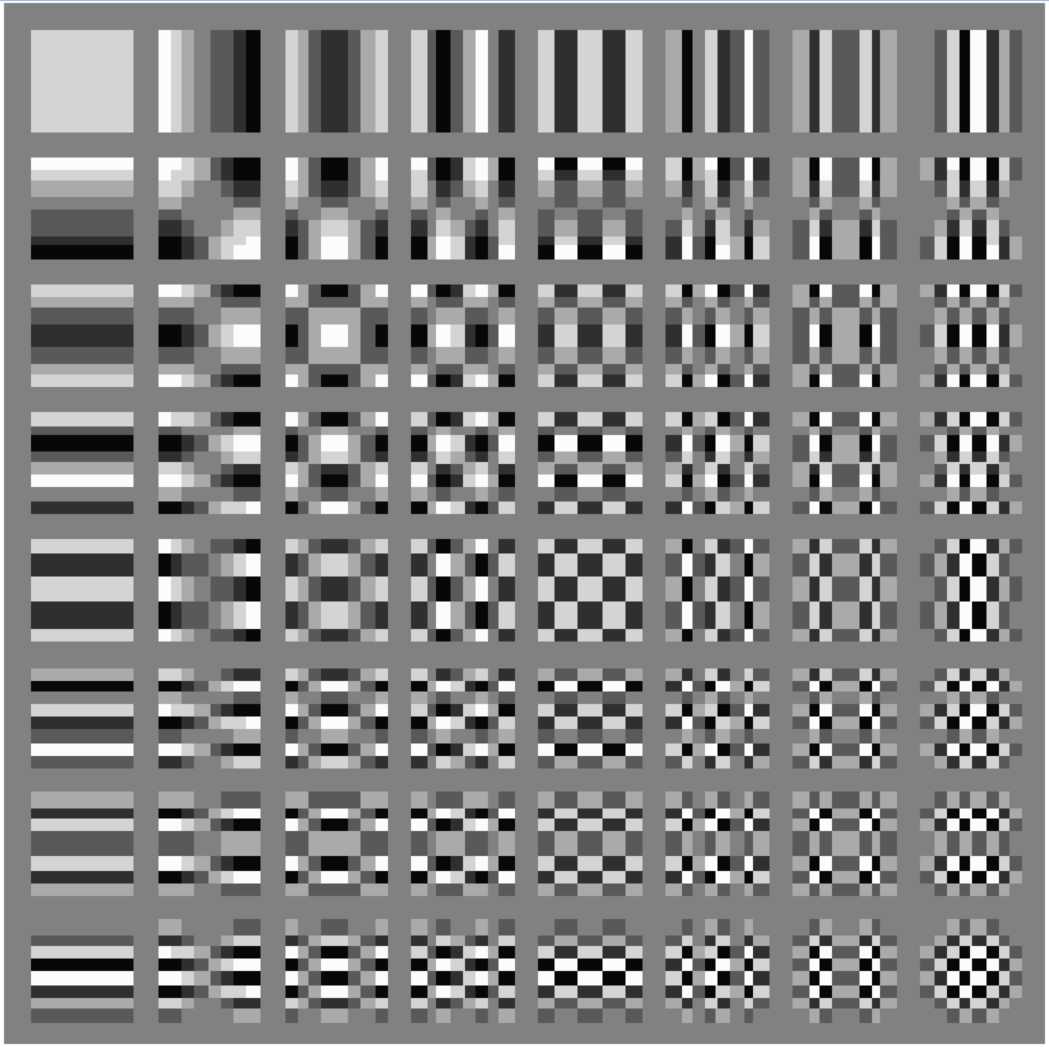
```
clc;
clear
N=8
x=(0:N-1)';
C=cos((2*x+1)*x'*pi/(2*N))*sqrt(2/N);
C(:,1)=C(:,1)/sqrt(2);
```

Kernel matrix to generate DCT Basis Images

0.353553	0.490393	0.46194	0.415735	0.353553	0.277785	0.191342	0.097545
0.353553	0.415735	0.191342	-0.09755	-0.35355	-0.49039	-0.46194	-0.27779
0.353553	0.277785	-0.19134	-0.49039	-0.35355	0.097545	0.46194	0.415735
0.353553	0.097545	-0.46194	-0.27779	0.353553	0.415735	-0.19134	-0.49039
0.353553	-0.09755	-0.46194	0.277785	0.353553	-0.41573	-0.19134	0.490393
0.353553	-0.27779	-0.19134	0.490393	-0.35355	-0.09755	0.46194	-0.41573
0.353553	-0.41573	0.191342	0.097545	-0.35355	0.490393	-0.46194	0.277785
0.353553	-0.49039	0.46194	-0.41573	0.353553	-0.27779	0.191342	-0.09755

Basis images of an 8x8 DCT



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Properties of DCT

- Separability

  2D DCT/ IDCT expression shows that they are separable.
  So they can be implemented as two 1-D DCT/IDCT

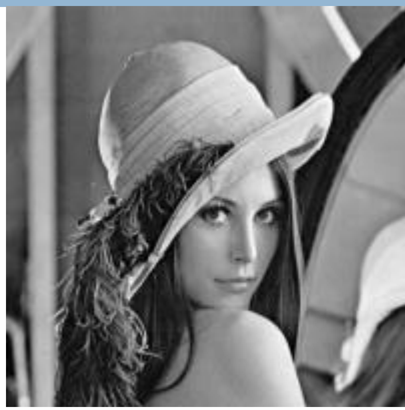
- Fast DCT

  FDCT is possible in the same manner as FFT

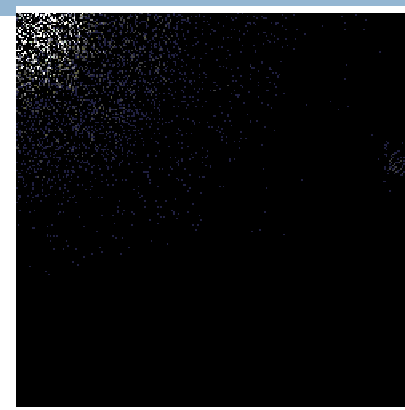
- Periodicity

  Magnitude of DCT coefficient is periodic with period 2N

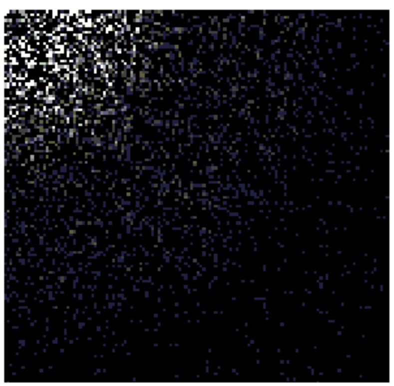
Example



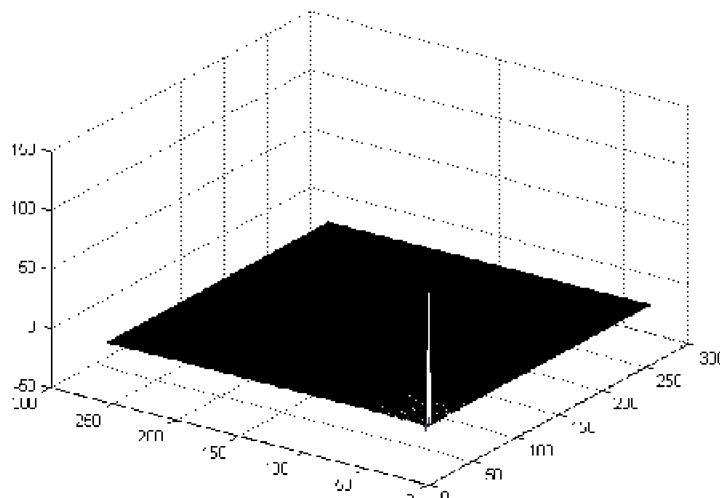
Original Image



DCT Coefficients



DCT Coefficients (Zoomed)



Mesh Plot

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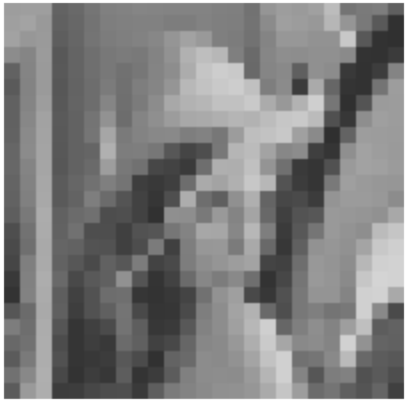
Image Reconstruction in DCT

Original image

Reconstructed image



Using only 1 DCT coefficient

Using 3 DCT coefficients

Coefficients are selected out of 8x8 window

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Contd..

Using 6 DCT coefficients

Using 15 DCT coefficients

Using 10 DCT coefficients

Using 64 DCT coefficients

Karhunen-Loeve Transform

- K-L transform is a linear combination of orthogonal functions of ensembles of input images.
- Basis functions are image dependent
- No fast algorithm exists
- Usually used for comparison
- Transform kernel is not fixed unlike other transforms
- Based on second order statistical properties of image
- Also called Hotelling transform or method of principal components

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Contd..

- Covariance matrix of a population of vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}$$

$$C_x = E\left\{ (X - \mu_x)(X - \mu_x)^T \right\}$$

$$\mu_x = \frac{1}{M} \sum_{k=1}^{M} X_k$$

$$C_x \Rightarrow \text{real \& symmetric}$$

So, Orthogonal vectors of C_x can be calculated

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Contd..

- Calculate eigenvectors and eigen values of C_x
- Sort eigen vectors depending on decreasing values of eigen values
- Select some of the eigenvectors
- These eigenvectors are considered as transformed matrix
- Calculate the transformed images

$$Y = A\left(X - \mu_x\right)$$

- Y is called as principal component coefficients.
- This transformation is called K-L transform

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Properties of Y

$$\mu_y = 0$$

$$C_y = AC_xA^T$$

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Example

Input matrix

$$x = \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \right\}$$

Mean Vector

$$\mu_x = \begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix}$$

Contd..

$$C_x = \begin{pmatrix} 0.75 & 0.375 \\ 0.375 & 0.75 \end{pmatrix}$$

Compute Eigen values

$$(0.75-\lambda)^2 = (0.375)^2$$

$$\Rightarrow 0.75-\lambda = \pm 0.375$$

$$\Rightarrow \lambda = 0.75 \pm 0.375$$

$$\lambda_1 = 1.125$$

$$\lambda_2 = 0.375$$

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Contd..

We get two Eigen vectors

$$e_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Transform Matrix

$$\rightarrow A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Contd.

- Reconstructed $$X = A^T Y + \mu_X$$

- K no of Eigen vectors

$$Y = A_k \left(X - \mu_X \right)$$

$$\hat{X} = A_k^T Y + \mu_X$$

$$A_k = k \times n$$

$$Y = k \times 1$$

$$X = n \times 1$$

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Properties of K-L Transform

- Decorrelation
  - $E[\,\underline{y}\,\underline{y}^H\,] = E[\,(U^H x)\,(U^H x)^H\,] = U^H\,E[\,x\,x^H\,]\,U = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$
  - Note: Other matrices (unitary or nonunitary) may also decorrelate the transformed sequence
- Minimizing MSE under basis restriction
  - If only allow to keep *m* coefficients for any $1 \le m \le N$, what's the best way to minimize reconstruction error?
  - → *Keep the coefficients w.r.t. the eigenvectors of the first m largest Eigen values*

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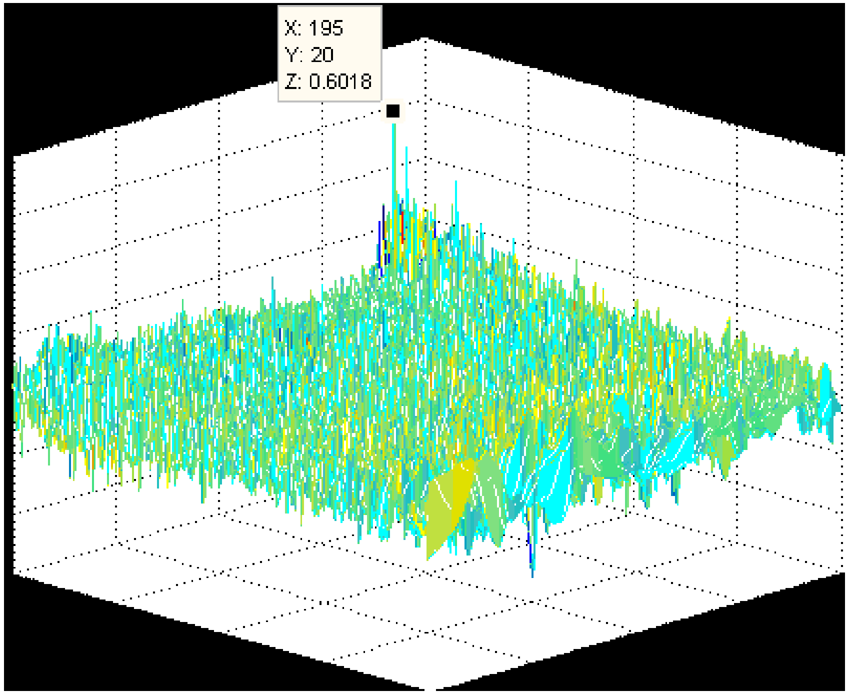
Principal Component Coefficients

Lena image



Energy compaction in lena image

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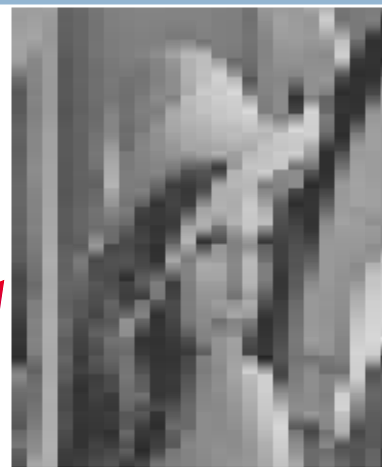
Image Reconstruction in K-L Transform

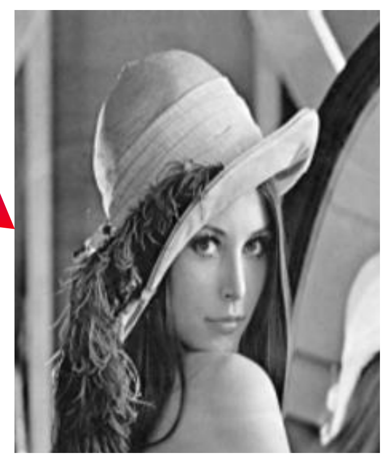


1 PC

10 PCs

20 PCs

Original image



50 PCs

All PCs

Haar Wavelet Transform

- One dimensional transformation on each row followed by one dimensional transformation of each column.
- Extracted coefficients would be
  - Approximation
  - Vertical
  - Horizontal
  - Diagonal
- Approximation coefficients are further decomposed into the next level
- 4 level decomposition is used

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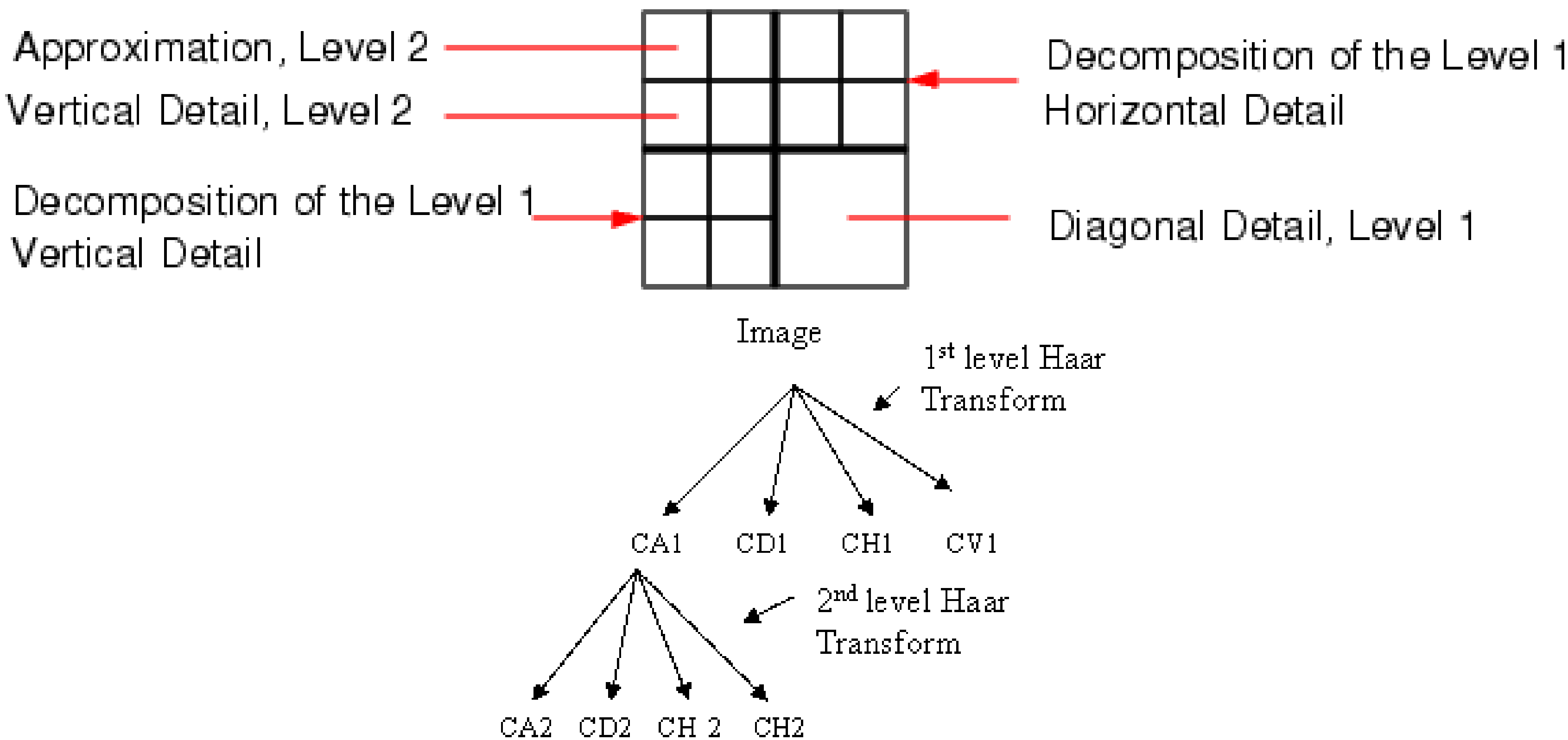
Contd..



Graphical Representation - Wavelet decomposition
(level = 2)

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Approach

- For a 2X2 matrix

$$x = \overrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \qquad x = \begin{pmatrix} a+b & a-b \\ c+d & c-d \end{pmatrix}\Big\downarrow$$

$$y = \frac{1}{2}\begin{pmatrix} a+b+c+d & a-b+c-d \\ a+b-c-d & a-b-c+d \end{pmatrix}$$

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Example

2	2	9	5
2	0	5	2
6	7	4	7
3	4	8	8

Column wise Summation

4	14	0	4
2	7	2	3
13	11	-1	-3
7	16	-1	0

Row wise Summation

6	21	2	7
20	27	-2	-3
2	7	-2	1
6	-5	0	-3

Finding Average

Approximation | Horizontal

3	10.5	1	3.5
10	13.5	-1	-1.5
1	3.5	-1	0.5
3	-2.5	0	-1.5

Vertical | Diagonal

Properties of Haar transform

- The Haar transform is real and orthogonal. Therefore,

  $\mathbf{H_r = H_r^*}$

- The Haar transform is a very fast transform.
- On an Nx1 vector it can be implemented in O(N) operations
- The Haar transform has poor energy compaction for Images

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- %Initialization
- H=[1]; NC=1/sqrt(2);%normalization constant
  LP=[1 1]; HP=[1 -1]; % iteration from H=[1]
- for i=1:Level
  H=NC*[kron(H,LP);kron(eye(size(H)),HP)];
- end
- HaarTransformationMatrix=H;

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$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{H}_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

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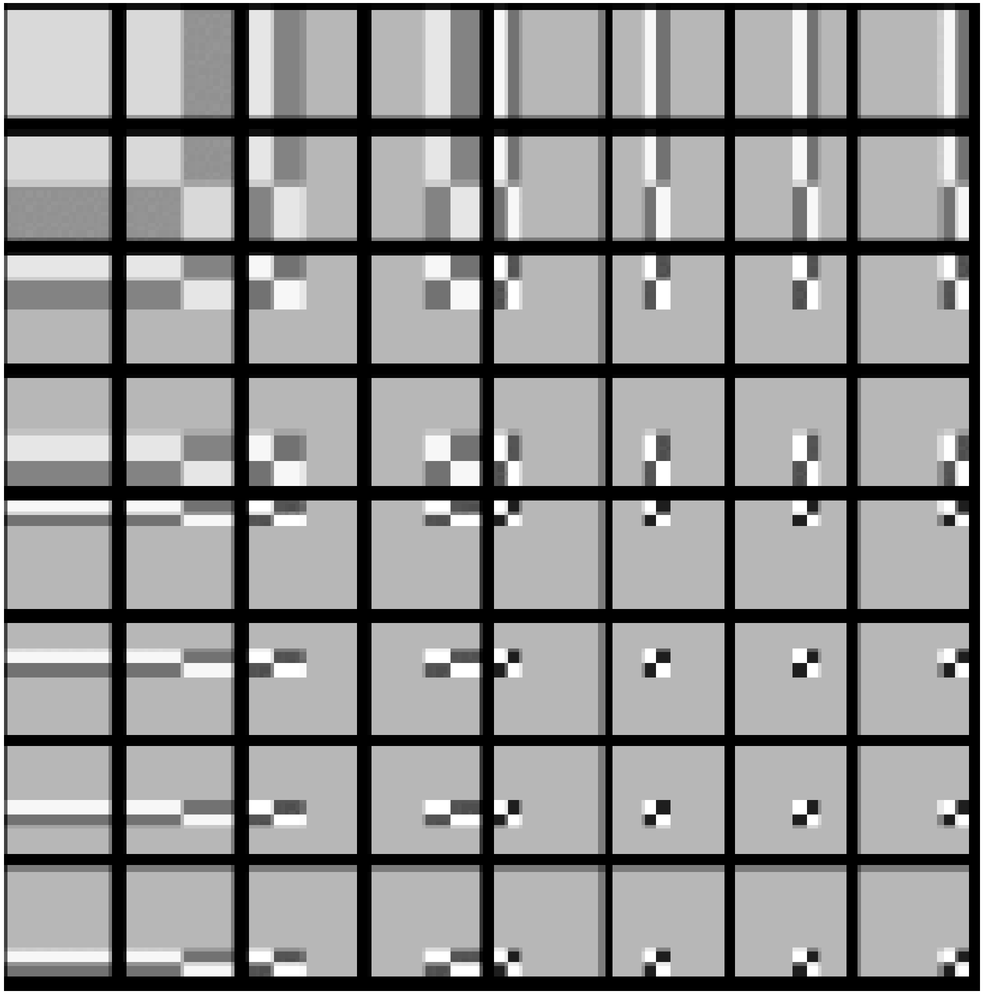
Basis images of Haar Transform



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1-step Haar Decomposition

Original image

Haar transformed image

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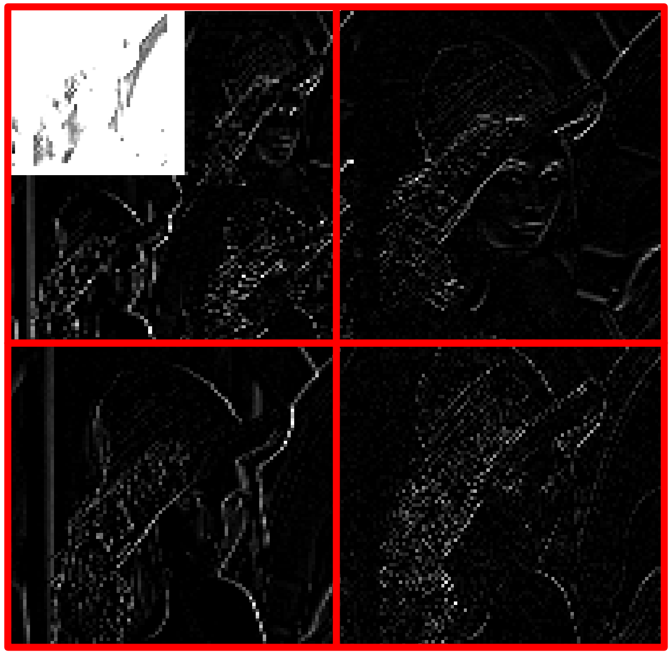
2-step Haar Decomposition



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Discrete Walsh Transform (1-D)

$$\underbrace{g(x,u) = \frac{1}{N}\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}}_{1-D\ Forward\ \text{Kernel}}$$

where,

$N \rightarrow$ No. of Samples

n $\rightarrow$ No. of bits needed to represent x and u

$\mathrm{b_k}(z) \rightarrow k^{th}$ bit in digital representation of z

$$\underbrace{W(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}}_{1-D\ \ \text{Forward}\ \ \text{Transform}}$$

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Contd..

$$\underbrace{h(x,u) = \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u)}}_{1-D \text{ Inverse Kernel}}$$

$$\underbrace{f(x) = \sum_{x=0}^{N-1} W(u) \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u)}}_{1-D \text{ Inverse Transform}}$$

Difference lies in Multiplicative Factor

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Discrete Walsh Transform (2-D)

$$\underbrace{g(x,y,u,v)=\frac{1}{N}\prod_{i=0}^{n-1}(-1)^{\{b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)\}}}_{2-D\ Forward\ \text{Kernel}}$$

$$\underbrace{W(u,v)=\frac{1}{N}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}f(x,y)\prod_{i=0}^{n-1}(-1)^{\{b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)\}}}_{2-D\ \text{Forward Transform}}$$

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Contd..

$$\underbrace{h(x,y,u,v)=\frac{1}{N}\prod_{i=0}^{n-1}(-1)^{\{b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)\}}}_{2-D\ \text{Inverse Kernel}}$$

$$\underbrace{f(x,y)=\frac{1}{N}\sum_{y=0}^{N-1}\sum_{x=0}^{N-1}W(u,v)\prod_{i=0}^{n-1}(-1)^{\{b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)\}}}_{2-D\ \text{Inverse Transform}}$$

Property

- **Separable**

$$W(u) = \frac{1}{2}[W_{even}(u) + W_{odd}(u)]$$

$$W(u+M) = \frac{1}{2}[W_{even}(u) - W_{odd}(u)]$$

$$where,$$

$$u = 0,1,\ldots,\frac{N}{2} - 1$$

$$M = \frac{N}{2}$$

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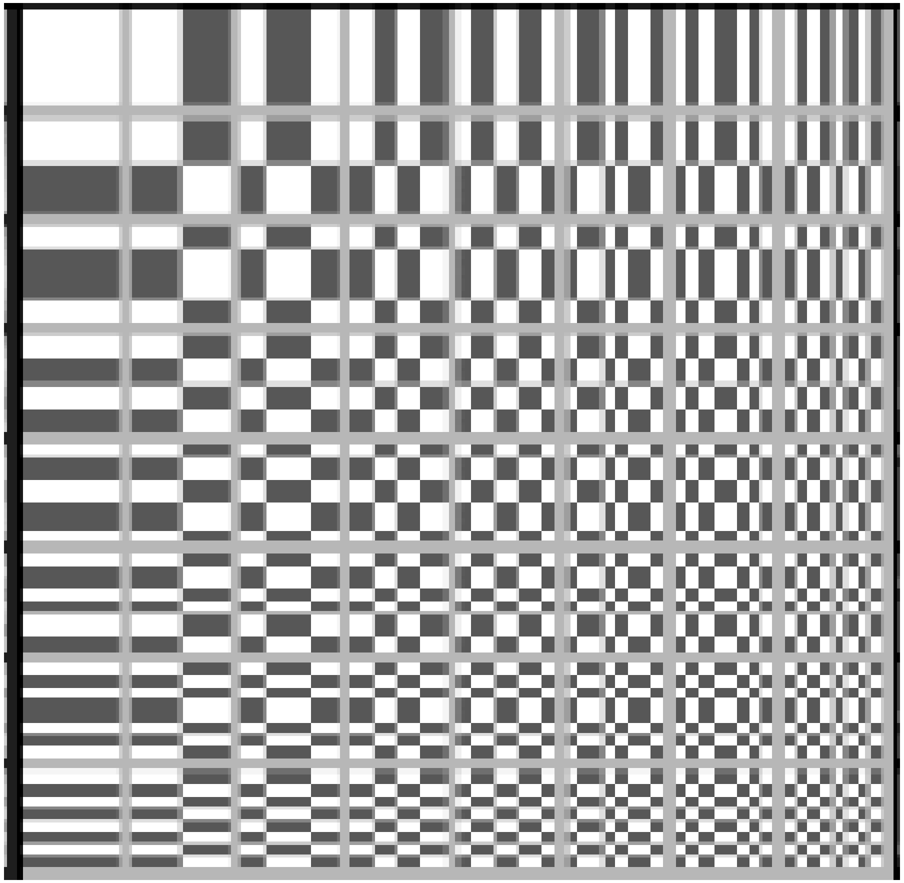
Basis images of Walsh Transform



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Hadamard Transfrom

$$\underbrace{g(x,y,u,v)=\frac{1}{N}(-1)^{\sum_{i=0}^{n-1}\{b_i(x)b_i(u)+b_i(y)b_i(v)\}}}_{2D\ \text{Forward Kernel}}$$

$$\underbrace{h(x,y,u,v)=\frac{1}{N}(-1)^{\sum_{i=0}^{n-1}\{b_i(x)b_i(u)+b_i(y)b_i(v)\}}}_{2D\ \text{Inverse Kernel}}$$

Hadamard Matrix

$$g(x,u)=\frac{1}{N}\underbrace{(-1)^{\sum_{i=0}^{n-1} b_i(x)b_i(u)}}_{Hadamard\ Matrix}$$

$u \longrightarrow$

$x \downarrow$	0(0)	1(7)	2(3)	3(4)	4(1)	5(6)	6(2)	7(5)
0	+	+	+	+	+	+	+	+
1	+	-	+	-	+	-	+	-
2	+	+	-	-	+	+	-	-
3	+	-	-	+	+	-	-	+
4	+	+	+	+	-	-	-	-
5	+	-	+	-	-	+	-	+
6	+	+	-	-	-	-	+	+
7	+	-	-	+	-	+	+	-

sign changes are not ordered (shown in backet)

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Hadamard transform

- Transform matrices can be recursively generated

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H_n = H_{n-1} \otimes H_1$$

$$H_3 = H_1 \otimes H_2, \;\; H_2 = H_1 \otimes H_1$$

Note: Hadamard Coefficients need reordering to concentrate energy

Contd..

- N=2

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & H_{-N} \end{bmatrix}$$

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Ordered Hadamard Transformation

$$g(x,u) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1} b_i(x)\, p_i(u)}$$

$$where,$$

$$p_0(u) = b_{n-1}(u)$$

$$p_1(u) = b_{n-1}(u) + b_{n-2}(u)$$

$$p_2(u) = b_{n-2}(u) + b_{n-3}(u)$$

.

.

.

$$p_{n-1}(u) = b_1(u) + b_0(u)$$

$b_i(u)$* is changed to *$p_i(u)$

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Contd..

N = 8

$u \rightarrow$; $x \downarrow$

	0	1	2	3	4	5	6	7
0	+	+	+	+	+	+	+	+
1	+	+	+	+	-	-	-	-
2	+	+	-	-	-	-	+	+
3	+	+	-	-	+	+	-	-
4	+	-	-	+	+	-	-	+
5	+	-	-	+	-	+	+	-
6	+	-	+	-	-	+	-	+
7	+	-	+	-	+	-	+	-

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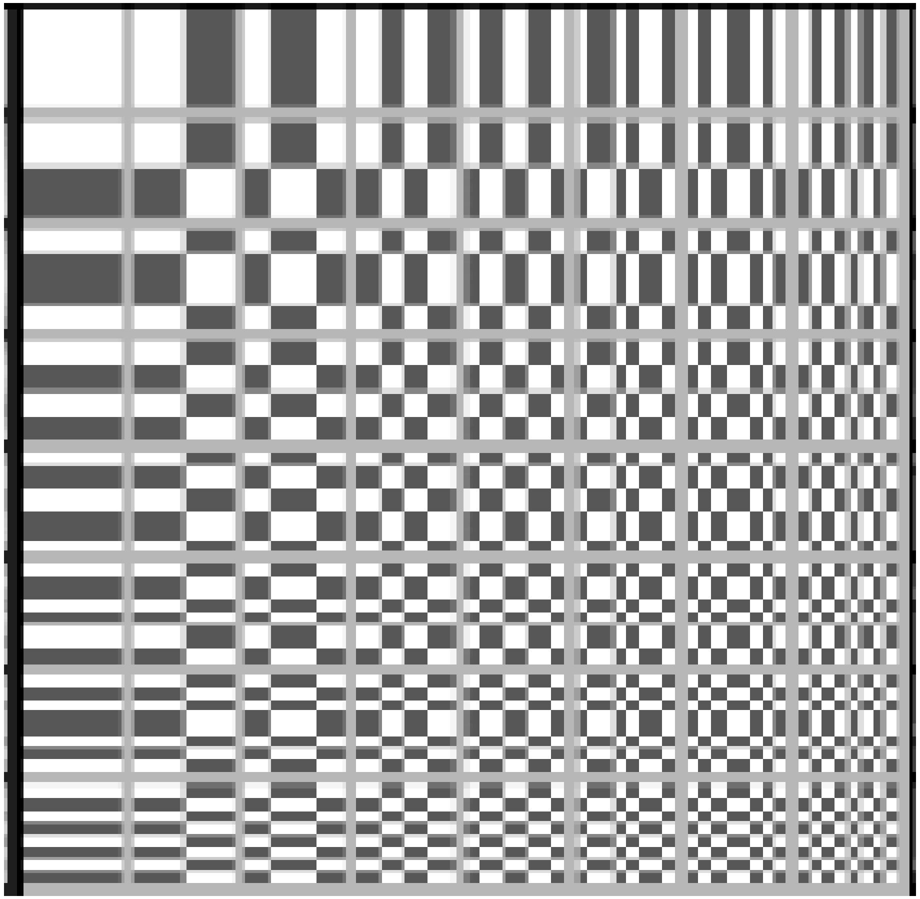
Basis images of Hadamard Transform



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Properties of Hadamard Transform

- Unlike Other Transform, the elements of the basis vectors of the Hadamard transform take only the binary values +1 , -1, therefore well suited for DSP.
- Hadamard transform H is real, symmetric, and orthogonal
- Hadamard transform is a fast transform. The one dimensional transform can be implemented in $O(N \log_2 N)$ additions and subtractions

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Similarity between Walsh and Hadamard Transform

- Basis images of Walsh and Hadamard transforms are same.
- So Hadamard transform is also known as Walsh transform or Hadamard-Walsh transform.

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Image Reconstruction

Original image

Reconstructed image

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Image Reconstruction

Original image

Reconstructed image

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Thank You!