The Analytic Class Number Formula and Some Uniform Distribution Results

by

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advised by

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as a part of

Visiting Students' Research Program 2024

Preface

This report is based on everything I read under Prof. Eknath Ghate as a part of the VSRP program conducted by TIFR. The results mentioned here can be found either in the text or in the exercises of the book *Number Fields* by Daniel A. Marcus. For a more detailed treatment of the Complex Analysis, one can also refer to *Complex Analysis* by Elias M. Stein and Rami Shakarchi.

I would like to thank Prof. Ghate for giving me this incredible opportunity to work under him and am grateful for the amount of time he spent with me listening to my presentations and also explaining different concepts.

I would further like to acknowledge the organisers of the program for giving me the opportunity to take part in VSRP-2024

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June 2024

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Chapter 1

Introduction

In this section, we introduce some basic notions and give an overview of the structure of this report.

Definition 1. Define an equivalence relation on the set of ideals of a number ring $\mathcal{O}_K$ by

$$I \sim J \iff \alpha I = \beta J \text{ for some non-zero } \alpha, \beta \in \mathcal{O}_K.$$

It is known that the ideal classes in this equivalence relation form a group under multiplication. This group is called the Class group. Our first goal will be to use geometric techniques to show this is a finite group.

The geometric techniques used in the proof of the finiteness of the class group turns out to be very strong. We further use those techniques to understand the structure of the group of units in a number field and the distribution of ideals in a number field.

In the later half of the report, we define the Dedekind zeta function of a number field and establish a connection between the order of the class group and the special values of the zeta function. We further simplify the formula in the case of abelian extensions of $\mathbb{Q}$ using Dirichlet characters and L-functions.

Finally, we define Polar density and show that it is the same as Dirichlet density and establish some uniform distribution results like the Frobenious Density Theorem and the Tchebotarov Density Theory. We also prove another abstract distribution theorem from which the Dirichlet's theorem on primes in an Arithmetic progression follows.

Chapter 2

Geometric Methods

Let $\sigma_1, \ldots, \sigma_r$ denote the real embeddings of K and let $\tau_1, \overline{\tau_1}, \ldots, \tau_s, \overline{\tau_s}$ denote the complex embeddings of K, where $r + 2s = n = [K : \mathbb{Q}]$. Embed K in $\mathbb{R}^n$ as

$$\alpha \mapsto \left(\sigma_1(\alpha), \ldots, \sigma_r(\alpha), \frac{\tau_1 + \overline{\tau_1}}{2}, \frac{\tau_1 - \overline{\tau_1}}{2}, \ldots, \frac{\tau_s + \overline{\tau_s}}{2}, \frac{\tau_s - \overline{\tau_s}}{2}\right).$$

Let $\alpha_1, \ldots, \alpha_n$ be a $\mathbb{Z}$ basis for $\mathcal{O}_K$. Their image under this embedding generates an n- dimensional lattice $\Lambda_{\mathcal{O}_K}$ in $\mathbb{R}^n$.

Theorem 2.1. The mapping $K \to \mathbb{R}^n$ sends any ideal I of $\mathcal{O}_K$ to an n-dimensional lattice Λ_I whose fundamental parallelotope has volume $\frac{1}{2^s}\sqrt{|\operatorname{disc}(\mathcal{O}_K)|}\|I\|$. In particular, the fundamental parallelotope for $\Lambda_{\mathcal{O}_K}$ has volume $\frac{1}{2^s}\sqrt{|\operatorname{disc}(\mathcal{O}_K)|}$.

Corollary 2.1.1. The image of K is dense in $\mathbb{R}^n$.

2.1 Finiteness of the Class Group

For $\mathbf{x} \in \mathbb{R}^n$ define a norm map as $N(\mathbf{x}) = x_1 \ldots x_r(x_{r+1}^2 + x_{r+2}^2) \ldots (x_{n-1}^2 + x_n^2)$. Note that if $\alpha \in \mathcal{O}_K$ gets mapped to $\mathbf{x}$ in $\mathbb{R}^n$ then $N(x) = N_{\mathbb{Q}}^K(\alpha)$. The finiteness of the class group follows from the following theorem:

Theorem 2.2. Every n-dimensional Lattice Λ in $\mathbb{R}^n$ contains a nonzero point $\mathbf{x}$ with $|N(x)| \leq \frac{n!}{n^n}\left(\frac{8}{\pi}\right)^s \operatorname{vol}(\mathbb{R}^n/\Lambda)$.

The proof of this theorem requires the following geometric lemma due to Minkowski:

Lemma 2.2.1 (Minkowski)**.** Let Λ be a n-dimensional Lattice in $\mathbb{R}^n$ and let E be a convex, measurable, centrally symmetric subset of $\mathbb{R}^n$ such that $\operatorname{vol}(E) > 2^n \operatorname{vol}(\mathbb{R}^n/\Lambda)$. Then E contains some nonzero point of Λ. If E is also compact, then the strict inequality can be weakened to $\geq$.

To prove the Theorem, we use Minkowski's Lemma on the subset of $\mathbb{R}^n$ given by

$$|x_1| + \ldots + |x_r| + 2\left(\sqrt{x_{r+1}^2 + x_{r+2}^2}\right) + \ldots + \left(\sqrt{x_{n-1}^2 + x_n^2}\right) \leq n.$$

Corollary 2.2.1. Every non-zero ideal of $\mathcal{O}_K$ has a non-zero element α with

$$|N_{\mathbb{Q}}^K(\alpha)| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|\operatorname{disc}(\mathcal{O}_K)|}\|I\|.$$

Corollary 2.2.2. Every Ideal Class of $\mathcal{O}_K$ contains an ideal with

$$\|J\| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|\operatorname{disc}(\mathcal{O}_K)|}.$$

Proof. Let C be an ideal class. Pick $I \in C^{-1}$. Choose $\alpha \in I$ as in Corollary 2.2.1 and let $J = (\alpha)I^{-1}$. Clearly, J is an ideal and $J \in C$. Now,

$$\|J\| = N_{\mathbb{Q}}^K(\alpha)\|I\|^{-1} \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|\operatorname{disc}(\mathcal{O}_K)|}.$$

□

Corollary 2.2.3. The ideal class group of a number field is finite.

Proof. This follows because only finitely many ideals satisfy $\|I\| \leq \lambda$ for $\lambda > 0$. This is because if P is a prime in $\mathcal{O}_K$ lying above $p \in \mathbb{Z}$ then $\|P\| \geq p$. □

Corollary 2.2.4. $|\operatorname{disc}(\mathcal{O}_K)| > 1$ whenever $K \neq \mathbb{Q}$.

2.2 Dirichlet's Unit Theorem

Define the following map $\log : \Lambda_{\mathcal{O}_K} \setminus \{0\} \to \mathbb{R}^{r+s}$ as follows

$$\mathbf{x} \mapsto \left(\log|x_1|, \ldots, \log|x_r|, \log(x_{r+1}^2 + x_{r+2}^2), \log(x_{n-1}^2 + x_n^2)\right).$$

We will also denote the composition of the embedding of K into $\mathbb{R}^n$ with log as log only. One easily checks the following properties:

1. $\log(\alpha\beta) = \log(\alpha) + \log(\beta)$.
2. Let U be the group of units in $\mathcal{O}_K$. Then $log(U) \subset H$, where H is the hyperplane defined by $\sum_{i=1}^{r+s} y_i = 0$.
3. Any bounded set in $\mathbb{R}^{r+s}$ has finite inverse image in U.

It follows from here that log is a group homomorphism and $\log(U) \subset H$ is a lattice, Λ_U of rank $\leq r + s - 1$. The following theorem gives a better characterization

Theorem 2.3 (Dirichlet's Unit Theorem)**.** $U \cong W \times V$, where W is a finite cyclic group consisting of the roots of unity in K and V is a free abelian group of rank $r + s - 1$.

Proof. From the third property of the log map listed above, it follows that the kernel of log is finite. As each root of unity is in the kernel it follows that the kernel consists of all the roots of unity in K and hence is cyclic. Let W denote the kernel.

Now, let Λ_U be of dimension $d \leq r + s - 1$. Fix $u_1, \ldots, u_d \in U$ map to a basis of Λ_U, then set $V =< u_1, \ldots, u_d >$. Using the definition of the log map it follows that $log(U) \cong V$ and hence V is free abelian of rank d. So the following exact sequence splits

$$0 \longrightarrow W \longrightarrow U \longrightarrow V \longrightarrow 0$$

giving $U = W \times V$. So all that remains to prove is $d = r+s-1$, which follows from the following two lemmas:

Lemma 2.3.1. For fixed k and $1 \leq k \leq r + s$, there exists $u \in U$ such that if $log(u) = (y_1, \ldots, y_{r+s})$ then $y_i < 0$ for all $i \neq k$.

Lemma 2.3.2. Let $A = (a_{ij})$ be an $m \times m$ matrix over $\mathbb{R}$ such that $a_{ii} > 0$ for all i, $a_{ij} < 0$ for all $i \neq j$ and each row-sum is 0. Then A has rank $m - 1$.

□

2.3 Distribution of Ideals

Let C be an ideal class of $\mathcal{O}_K$. For $t \geq 0$, let $i_C(t)$ denote the number of ideals I in the class C with $\|I\| \leq t$. Also denote by $i(t)$ the number of ideals I in $\mathcal{O}_K$ with $\|I\| \leq t$.

Theorem 2.4. $i_C(t) = \kappa t + \epsilon_C(t)$, where κ depends only on $\mathcal{O}_K$ and $\epsilon_C(t) = \mathcal{O}(t^{1-\frac{1}{n}})$; $n = [K : \mathbb{Q}]$.

Proof (Sketch). The proof begins with establishing a one-to-one correspondence between the following two sets:

$$\{I \in C : \|I\| \leq t\} \longleftrightarrow \left\{(\alpha) \subset J : J \in C^{-1}, \|(\alpha)\| \leq t\|J\|\right\}.$$

So it's enough to count the number of elements of J upto multiplication by units. Note that $U = W \times V$ from the unit theorem. So if $w = |W|$, we can count equivalently count the number of elements of J up to multiplication by elements of V and divide it by w. Also, identify $\mathbb{R}^n$ with $\mathbb{R}^r \times \mathbb{C}^s$ in the canonical way. So K embeds in $\mathbb{R}^r \times \mathbb{C}^s$ and we now have our map log from $\mathbb{R}^r \times \mathbb{C}^s$ to $\mathbb{R}^{r+s}$. So the problem is reduced to finding a set D of closet representative D of V in $(\mathbb{R}^*)^r \times (\mathbb{C}^*)^s$ and counting the number of elements in $\Lambda_J \cap D$ with $|N(x)| \leq t\|J\|$.

Lemma 2.4.1. If $f : G \to G'$ is a homomorphism of abelian groups such that for subgroups S, S' of G and G', $f(S) \cong S'$. If D' is a set of coset representatives of S' in G' then $f^{-1}(D')$ is a set of coset representatives of S in G.

Using the Lemma and the log map we get that

$$D = \{\mathbf{x} \in (\mathbb{R}^*)^r \times (\mathbb{C}^*)^s : \log(\mathbf{x}) \in F \bigoplus \mathbb{R}v\}$$

where F is the fundamental parallelotope for Λ_U in H and $v = (1, \ldots, 2)$ is a vector with first r entries as 1 and the rest as 2. Define $D_a = \{\mathbf{x} \in D : |N(\mathbf{x})| \leq a\}$. Then $D_a = \sqrt[n]{a} D_1$ and thus our problem reduces to finding $\Lambda_J \cap \sqrt[n]{t\|J\|}$. Now the theorem follows from the following lemma:

Lemma 2.4.2. If Λ is an n-dimensional lattice and B is a bounded subset of $\mathbb{R}^n$ whose boundary is $(n-1)$- Lipschitz parametrizable then

$$|\Lambda \cap aB| = \frac{\operatorname{vol}(B)}{\operatorname{vol}\mathbb{R}^n/\Lambda} a^n + \gamma(a)$$

where $\gamma(a) = \mathcal{O}(a^{n-1})$.

$\square$

Corollary 2.4.1. $i(t) = h\kappa t + \epsilon(t)$, where h is the class number and $\epsilon(t) = \mathcal{O}(t^{1-\frac{1}{n}})$; $n = [K : \mathbb{Q}]$.

Theorem 2.5.

$$\kappa = \frac{2^{r+s}\pi^s \operatorname{reg}(\mathcal{O}_K)}{w\sqrt{|\operatorname{disc}(\mathcal{O}_K)|}}$$

where $w = |W|$ and $\operatorname{reg}(\mathcal{O}_K) = \frac{\operatorname{vol}(H/\Lambda_U)}{\sqrt{r+s}}$.

Chapter 3

Analytic Methods

Let $s = x + iy$ denote a complex number.

Definition 2. Define the Dedekind zeta function of K by

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{j_n}{n^s}$$

where $x = Re(s) > 1$ and j_n is the number of ideals of $\mathcal{O}_K$ with $\|I\| = n$.

Lemma 1. Suppose $\sum_{n \leq t} a_n$ is $\mathcal{O}(t^r)$ for some real $r \geq 0$. Then the series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for all $s = x + iy$ with $x > r$ and is an analytic function of s in this half plane.

Lemma 2. For $i \geq 1$, let $a_i \mathbb{C}$, $|a_i| < 1$ and $\sum_{i=0}^{\infty} |a_i| < \infty$. Then

$$\prod_{i=1}^{\infty} (1 - a_i)^{-1} = 1 + \sum_{j=1}^{\infty} \sum_{r_1, \ldots r_j \geq 1} a_1^{r_1} \ldots a_j^{r_j}$$

Now Lemma 1 and Corollary 2.4.1 shows that $\zeta_K(s)$ is analytic on $x > 1$. Also from Lemma 2 it follows that

$$\zeta_K(s) = \sum_{I} \frac{1}{\|I\|^s} = \prod_{P} \left(1 - \frac{1}{\|P\|^s}\right)^{-1}.$$

Note that in the case $K = \mathbb{Q}$, $j_n = 1$ as the only ideal of norm n is (n). Hence in this case the Dedekind zeta function the the Riemann zeta function.

Theorem 3.1. The Riemann zeta function $\zeta(s)$ has an analytic continuation to a meromorphic function on $x > 0$, which is analytic everywhere except for a simple pole at $s = 1$.

3.1 The Analytic Class Number Formula

Theorem 3.2. $\zeta_K(s)$ has an analytic continuation to a meromorphic function on $x > 1 - \frac{1}{[K:\mathbb{Q}]}$ which is analytic everywhere expect for a simple pole at $s = 1$. Further

$$h\kappa = \lim_{s \to 1} \frac{\zeta_K(s)}{\zeta(s)}.$$

Proof. We have

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{j_n - h\kappa}{n^s} + h\kappa\zeta(s).$$

Now by Lemma 1 the series $\sum_{n=1}^{\infty} \frac{j_n - h\kappa}{n^s}$ converges to an analytic function on $x > 1 - \frac{1}{[K:\mathbb{Q}]}$. So by Theorem 3.1 the first part of the theorem follows. As $\sum_{n=1}^{\infty} \frac{j_n - h\kappa}{n^s}$ is analytic in a neighbourhood of $s = 1$, it has a finite limit as $s \to 1$. Also $\frac{1}{\zeta(s)} \to 0$ as $s \to 1$. Thus we get

$$h\kappa = \lim_{s \to 1} \frac{\zeta_K(s)}{\zeta(s)}.$$

$\square$

Using this, we will obtain a formula for h for an abelian extension of $\mathbb{Q}$. By Kronecker-Weber Theorem if $K/\mathbb{Q}$ is abelian then it is contained in some cyclotomic field $\mathbb{Q}(\omega), \omega = e^{\frac{2\pi i}{m}}$.

Now let χ be a character of $\mathbb{Z}_m^* \cong Gal(\mathbb{Q}(\omega)/\mathbb{Q})$. We then define the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \nmid m} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

If χ is non-trivial then the series converges for $x > 0$ by Lemma 1.

Let $G := Gal(K/\mathbb{Q})$, which by assumption is a finite abelian group. Let $\hat{G}$ be the group of characters of G. From the structure theorem it follows that $\hat{G} \cong G$.

Now $G \cong \mathbb{Z}_m^*/H$. So we have the following exact sequence

$$\mathbb{Z}_m^* \longrightarrow G \longrightarrow 0$$

$$0 \longrightarrow Hom_{\mathbb{Z}}(G, \mathbb{C}^*) \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}_m^*, \mathbb{C}^*)$$

So $\hat{G}$ can be considered as characters mod m.

Theorem 3.3. If f_p denote the inertia degree of p and r_p denote the number of primes in K lying above p then

$$\frac{\zeta_K(s)}{\zeta(s)} = \prod_{p|m}\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{f_p s}}\right)^{-r_p}\prod_{\substack{\chi\in\hat{G}\\ \chi\neq 1}} L(s,\chi).$$

This gives the following formula

Theorem 3.4.

$$h\kappa = \prod_{p|m}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{f_p}}\right)^{-r_p}\prod_{\substack{\chi\in\hat{G}\\ \chi\neq 1}} L(1,\chi).$$

Theorem 3.5. Let χ be a non-trivial character mod m. Then

$$L(1,\chi) = -\frac{1}{m}\sum_{k=1}^{m-1}\tau_k(\chi)\log(1-\omega^{-k})$$

where $\tau_k(\chi) = \sum_{a\in\mathbb{Z}_m^*}\chi(a)\omega^{ak}$ and $\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}$, wherever it converges.

Suppose that χ' is a character mod d for some $d \mid m$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{Z}_m^* \arrow[r, "mod\ d"] \arrow[dr, "\chi"'] & \mathbb{Z}_d^* \arrow[d, "\chi'"] \\ & \mathbb{C}^*. \end{tikzcd}$$

Then we say that χ' induces χ. If χ is not induced by any $\chi' \neq \chi$ then χ is called a primitive character.

It follows that if χ' induces χ then

$$L(1,\chi) = \prod_{\substack{p|m\\ p\nmid d}}\left(1-\frac{\chi'(p)}{p}\right)L(1,\chi').$$

So it's sufficient to determine $L(1,\chi)$ for primitive characters.

Also note that in Theorem 3.4, $h\kappa > 0$. So it's sufficient to calculate $|L(1,\chi)|$ for primitive characters χ. This is given by the following theorem

Theorem 3.6. If χ is a primitive character mod $m \geq 3$, then

$$|L(1,\chi)| = \begin{cases} \frac{2}{\sqrt{m}} \left| \sum_{\substack{k \in \mathbb{Z}_m^* \\ k < \frac{m}{2}}} \chi(k) \log \sin \frac{k\pi}{m} \right| & if\ \chi(-1) = 1 \\ \frac{\pi}{|2-\chi(2)|\sqrt{m}} \left| \sum_{\substack{k \in \mathbb{Z}_m^* \\ k < \frac{m}{2}}} \chi(k) \right| & if\ \chi(-1) = -1 \end{cases}$$

3.2 Polar Density

Let A be a set of primes in $\mathcal{O}_K$ and $[A]$ denote the set ideals whose prime factorization contains only the primes of A. Define $\zeta_{K,A}(s)$ as

$$\zeta_{K,A}(s) = \sum_{I \in [A]} \frac{1}{\|I\|^s} = \prod_{P \in A} \left(1 - \frac{1}{\|P\|^s}\right)^{-1}.$$

Definition 3 (Polar Density)**.** If $\zeta_{K,A}^n$ can be extended to a meromorphic function in a neighbourhood of $s = 1$, having a pole of order m at $s = 1$, then the ratio $\frac{m}{n}$ is called the polar density of A.

The following theorem explains why polar density is a natural density to consider

Theorem 3.7. If A is a set of primes having polar density $\frac{m}{n}$ then

$$\frac{\sum_{P \in A} \frac{1}{\|P\|^s}}{\sum_{\text{all } P} \frac{1}{\|P\|^s}} \to \frac{m}{n}$$

as $s \to 1^+$, $s \in \mathbb{R}$.

Theorem 3.8. If two sets of primes of $\mathcal{O}_K$ differ only by primes for which $\|P\|$ is not a prime in $\mathbb{Z}$, then the polar density of one set exists iff that of the other does, and they are equal.

This leads us to the following theorem

Theorem 3.9. Let L/K be normal where L and K are number fields. Then the set of primes of K that splits completely in L has polar density $\frac{1}{[L:K]}$.

Theorem 3.10 (Frobenious Density Theorem)**.** Let L/K be a normal extension where L and K are number fields with Galois Group G. Let $\sigma \in G$ be of order n and c denote the number of distinct conjugates $\tau < \sigma > \tau^{-1}$ of $< \sigma >$, $\tau \in G$. Let A denote the set of primes P of K satisfying

1. P is unramified in K.

2. $\|P\|$ is a prime.

3. $\phi(Q|P) = \sigma^k$ where ϕ denotes the Frobenius element for some prime Q of L lying over P and $(k, n) = 1$.

Then A has polar density $\frac{c\varphi(n)}{[L:K]}$.

Now we will prove one result in a more abstract setting and deduce many important results as its corollary.

Let X be a countably infinite set and G be a finite abelian group. Let $\phi : X \to G$ be a function. Also, assign a real number $\|P\| > 1$ for every $P \in X$. Define

$$\Pi = \left\{ \prod_{P \in X} P^{i_P} : i_P \geq 0, i_P \in \mathbb{Z}, i_P = 0 \text{ for all but finitely many P} \right\}.$$

So, if we assume $\sum_{P\in X} \frac{1}{\|P\|^s} < \infty$ for $Re(s) > 1$ then by Lemma 2 the following is well-defined

$$L(s, \chi) = \sum_{I \in \Pi} \frac{\chi(I)}{\|I\|^s} = \prod_{P \in X} \left(1 - \frac{\chi(P)}{\|P\|^s}\right)^{-1}$$

where $\chi(I) := \chi(\phi(I))$.

Theorem 3.11 (Abstract Distribution Theorem)**.** Suppose for all $\chi \in \hat{G}$, $L(s, \chi)$ has a meromorphic continuation in a neighbourhood of $s = 1$. Further suppose $L(s, 1)$ has a pole at $s = 1$ while $L(s, \chi)$ has a finite non-zero value at $s = 1$ for $\chi \neq 1$. Then for any fixed $a \in G$ and $s \in \mathbb{R}$

$$\lim_{s \to 1^+} \left(\sum_{\phi(P)=a} \frac{1}{\|P\|^s} - \frac{1}{|G|} \sum_{\text{all } P} \frac{1}{\|P\|^s} \right)$$

exists and is finite.

Corollary 3.11.1. Under the assumptions of the theorem, for $s \in \mathbb{R}$

$$\lim_{s \to 1^+} \frac{\sum_{\phi(P)=a} \frac{1}{\|P\|^s}}{\sum_{\text{all } P} \frac{1}{\|P\|^s}} = \frac{1}{|G|}.$$

In particular, there are infinitely many $P \in X$ such that $\phi(P) = a$.

Corollary 3.11.2. For each $a \in \mathbb{Z}_m^*$ and $s \in \mathbb{R}$,

$$\lim_{s \to 1^+} \frac{\sum_{p \equiv a \pmod m} \frac{1}{p^s}}{\sum_{\text{all } p} \frac{1}{p^s}} = \frac{1}{\varphi(m)}.$$

Proof. Here X is the set of primes of $\mathbb{Z}$ with $p \nmid m$. Also G is $\mathbb{Z}_m^*$ and ϕ is the reduction modulo m map. So $L(1, \chi)$ coincides with the L-function defined in the previous section. Note that $L(s, 1)$ differs from $\zeta(s)$ by a finite factor and thus has an analytic continuation to $x > 0$ with a simple pole at $s = 1$. Also $L(s, \chi)$ for $\chi \neq 1$ are analytic for $x > 0$ and $L(1, \chi) \neq 0$ because of Theorem 3.4. Hence the result follows. □

Now Theorem 3.11 and some results from Class field theory give us the following:

Theorem 3.12 (Tchebotarov Density Theorem)**.** Let L/K be normal where L and K are number fields and G be their Galois group. Fix $\sigma \in G$. Then the set of primes P of K which are unramified in L and such that $\phi(Q|P) = \sigma$ for some prime Q of L lying over P has Dirichlet density $\frac{c}{[L:K]}$, where c is the number of conjugates of σ.