The Analytic Class Number Formula

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Notations

1. L and K will denote number fields, i.e, finite extensions of $\mathbb{Q}$ with the convention that $K \subset L$.
2. $\mathcal{O}_K$ will denote the ring of integers in K and U_K will denote the group of units in $\mathcal{O}_K$. We will omit the subscript when the number field we are working with is clear from the context.
3. N_K^L and T_K^L will denote the field norms for the extension $K \subset L$.
4. For an ideal I in $\mathcal{O}_K$ we will denote by $\|I\|$ the finite number $\mid \mathcal{O}_K/I \mid$.
5. If $[K : \mathbb{Q}] = n$, σ_i the embeddings of K in $\mathbb{C}$ and $\{\alpha_i\}$ an integral basis of $\mathcal{O}_K$, $1 \leq i \leq n$. Then we will denote by disc$(\mathcal{O}_K)$ the determinant $det((\sigma_i(\alpha_j))$.

Embedding in $\mathbb{R}^n$

Let $\sigma_1, \ldots, \sigma_r$ be the real embeddings of K and $\tau_1, \overline{\tau}_1, \ldots, \tau_s, \overline{\tau}_s$ be the complex embeddings, where $r + 2s = n = [K : \mathbb{Q}]$. Then we can map K to $\mathbb{R}^n$ by sending α to
$(\sigma_1(\alpha), \ldots, \sigma_r(\alpha), Re(\tau_1(\alpha)), Im(\tau_1(\alpha)), \ldots, Re(\tau_s(\alpha)), Im(\tau_s(\alpha)))$.

Theorem 1

The mapping $K \to \mathbb{R}^n$ sends any ideal I of $\mathcal{O}_K$ to an n-dimensional lattice Λ_I whose fundamental parallelotope has volume $\frac{1}{2^s}\sqrt{|\operatorname{disc}(\mathcal{O}_K)|}\|I\|$. In particular, the fundamental parallelotope for $\Lambda_{\mathcal{O}_K}$ has volume $\frac{1}{2^s}\sqrt{|\operatorname{disc}(\mathcal{O}_K)|}$.

For $\mathbf{x} \in \mathbb{R}^n$, define a norm on $\mathbb{R}^n$ by setting $N(\mathbf{x}) = x_1 \ldots x_r(x_{r+1}^2 + x_{r+2}^2) \ldots (x_{n-1}^2 + x_n^2)$. Note that under the embedding, if $\alpha \in K$ gets mapped to $\mathbf{x} \in \mathbb{R}^n$ then $N(\mathbf{x}) = N_{\mathbb{Q}}^K(\alpha) = \|(\alpha)\|$.

Embedding in $\mathbb{R}^n$

Theorem 2

Every n-dimensional Lattice Λ in $\mathbb{R}^n$ contains a nonzero point $\mathbf{x}$ with $|N(x)| \leq \frac{n!}{n^n}\left(\frac{8}{\pi}\right)^s \mathrm{vol}(\mathbb{R}^n/\Lambda)$.

Lemma 2.1 (Minkowski)

Let Λ be a n-dimensional Lattice in $\mathbb{R}^n$ and let E be a convex, measurable, centrally symmetric subset of $\mathbb{R}^n$ such that $\mathrm{vol}(E) > 2^n \mathrm{vol}(\mathbb{R}^n/\Lambda)$. Then E contains some nonzero point of Λ. If E is also compact, then the strict inequality can be weakened to $\geq$.

To prove Theorem 2, use Minkowski's Lemma on the subset of $\mathbb{R}^n$ given by

$$|x_1| + \ldots + |x_r| + 2\left(\sqrt{x_{r+1}^2 + x_{r+2}^2}\right) + \ldots + \left(\sqrt{x_{n-1}^2 + x_n^2}\right) \leq n.$$

Embedding in $\mathbb{R}^n$

Corollary 2.1

Every Ideal Class of $\mathcal{O}_K$ contains an ideal with $\|J\| \leq \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|\operatorname{disc}(\mathcal{O}_K)|}$

Corollary 2.2

There are finitely many ideal classes in $\mathcal{O}_K$.

Corollary 2.3

$|\operatorname{disc}(\mathcal{O}_K)| > 1$ whenever $K \neq \mathbb{Q}$.

Theorem 3 (Dirichlet)

$U_K = W \times V$ where W is a finite cyclic group consisting of the roots of unity in K, and V is a free abelian group of rank $r + s - 1$.

Distribution of Ideals

For $t \geq 0$, let $i(t)$ denote the number of ideals I of $\mathcal{O}_K$ with $\|I\| \leq t$. Also for each ideal class C, let $i_C(t)$ denote the number of ideals in the ideal class C with $\|I\| \leq t$.

Theorem 4

$i_C(t) = \kappa t + \epsilon_C(t)$, where

$$\kappa = \frac{2^{r+s}\pi^s \operatorname{reg}(\mathcal{O}_K)}{w\sqrt{|\operatorname{disc}(\mathcal{O}_K)|}}$$

and $\epsilon_C(t) = \mathcal{O}(t^{1-\frac{1}{n}})$; $n = [K : \mathbb{Q}]$.

Corollary 4.1

$i(t) = h\kappa t + \epsilon(t)$, where h is the class number of $\mathcal{O}_K$ and $\epsilon(t) = \mathcal{O}(t^{1-\frac{1}{n}})$; $n = [K : \mathbb{Q}]$.

Let $s = x + iy$ denote a complex number.

Definition 1

Define the Dedekind zeta function of K by

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{j_n}{n^s}$$

where $x = Re(s) > 1$ and j_n is the number of ideals of $\mathcal{O}_K$ with $\|I\| = n$.

Theorem 4 and some results from Complex Analysis show that $\zeta_K(s)$ has a meromorphic extension on the half plane $x > 1 - \frac{1}{[K:\mathbb{Q}]}$ with a simple pole at $s = 1$. Also one can show that for $x > 1$

$$\zeta_K(s) = \sum_{I} \frac{1}{\|I\|^s} = \prod_{P} \left(1 - \frac{1}{\|P\|^s}\right)^{-1}$$

Note that we can write $\zeta_K(s) = \sum_{n=1}^{\infty} \frac{j_n - h\kappa}{n^s} + h\kappa\zeta(s)$ from which we get

$$h\kappa = \lim_{s\to 1} \frac{\zeta_K(s)}{\zeta(s)}.$$

Using this, we will obtain a formula for h for an abelian extension of $\mathbb{Q}$. By Kronecker-Weber Theorem if $K/\mathbb{Q}$ is abelian then it is contained in some cyclotomic field $\mathbb{Q}(\omega), \omega = e^{\frac{2\pi i}{m}}$.

Now let χ be a character of $\mathbb{Z}_m^* \cong Gal(\mathbb{Q}(\omega)/\mathbb{Q})$. We then define the series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p\nmid m} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

If χ is non-trivial then the series converges for $x > 0$.

Let $G := Gal(K/\mathbb{Q})$, which by assumption is a finite abelian group. Let $\hat{G}$ be the group of characters of G. From the structure theorem it follows that $\hat{G} \cong G$. Also as $K \subset \mathbb{Q}(\omega)$, characters of G can be treated as characters mod m.

If f_p denote the inertia degree of p and r_p denote the number of primes in K lying above p then

$$\frac{\zeta_K(s)}{\zeta(s)} = \prod_{p|m} \left(1 - \frac{1}{p^s}\right) \left(1 - \frac{1}{p^{f_p s}}\right)^{-r_p} \prod_{\substack{\chi \in \hat{G} \\ \chi \neq 1}} L(s, \chi).$$

This gives the following formula

Theorem 5

$$h_K = \prod_{p|m} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^{f_p}}\right)^{-r_p} \prod_{\substack{\chi \in \hat{G} \\ \chi \neq 1}} L(1, \chi).$$

Theorem 6

Let χ be a non-trivial character mod m. Then

$$L(1,\chi) = -\frac{1}{m}\sum_{k=1}^{m-1} \tau_k(\chi) log(1-\omega^{-k})$$

where $\tau_k(\chi) = \sum_{a\in\mathbb{Z}_m^*}\chi(a)\omega^{ak}$ and $log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}$.

Suppose that χ' is a character mod d for some $d \mid m$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{Z}_m^* \arrow[r, "mod\ d"] \arrow[dr, "\chi"'] & \mathbb{Z}_d^* \arrow[d, "\chi'"] \\ & \mathbb{C}^*. \end{tikzcd}$$

Then we say that χ' induces χ. If χ is not induced by any $\chi' \neq \chi$ then χ is called a primitive character.

It follows that if χ' induces χ then

$$L(1,\chi) = \prod_{\substack{p|m \\ p\nmid d}} \left(1 - \frac{\chi'(p)}{p}\right) L(1,\chi').$$

So it's sufficient to determine $L(1,\chi)$ for primitive characters.

Also note that in Theorem 5, $h\kappa > 0$. So it's sufficient to calculate $|L(1,\chi)|$ for primitive characters χ. This is given by the following theorem

Theorem 7

If χ is a primitive character mod $m \geq 3$, then

$$|L(1,\chi)| = \begin{cases} \frac{2}{\sqrt{m}} \left| \sum_{\substack{k\in\mathbb{Z}_m^* \\ k<\frac{m}{2}}} \chi(k) \log\sin\frac{k\pi}{m} \right| & \text{if } \chi(-1) = 1 \\ \frac{\pi}{|2-\chi(2)|\sqrt{m}} \left| \sum_{\substack{k\in\mathbb{Z}_m^* \\ k<\frac{m}{2}}} \chi(k) \right| & \text{if } \chi(-1) = -1 \end{cases}$$

A Worked Example

We use the results established to calculate the class number of $\mathbb{Z}\left[1+\frac{\sqrt{-23}}{2}\right]$ which is the ring of integers of $\mathbb{Q}(\sqrt{-23})$. This is an abelian extension of $\mathbb{Q}$ contained in the cyclotomic extension $\mathbb{Q}(\omega)$ where $\omega = e^{\frac{2\pi i}{23}}$.

Also, as 23 is ramified we have $f_p = r_p = 1$ and so by Theorem 5 we have $h\kappa = L(1,\chi)$, where χ is the unique non trivial character of $Gal(\mathbb{Q}(\sqrt{-23})/\mathbb{Q})$. And in this case $\kappa = \frac{\pi}{\sqrt{23}}$. One can further show that χ is primitive and $\chi(-1) = -1$.

Also for $K = \mathbb{Q}(\sqrt{d})$, d squarefree the non trivial character is determined as $\chi(n) = \left(\frac{d}{n}\right)$ for n coprime with $disc(\mathcal{O}_K)$ where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol.

Now Theorem 7 gives

$$h = \frac{1}{1-\chi(2)}\left|\sum_{k=1}^{11}\chi(k)\right| = \frac{1}{2-1}|1+1+1+1-1+1-1+1+1-1-1| = 3.$$