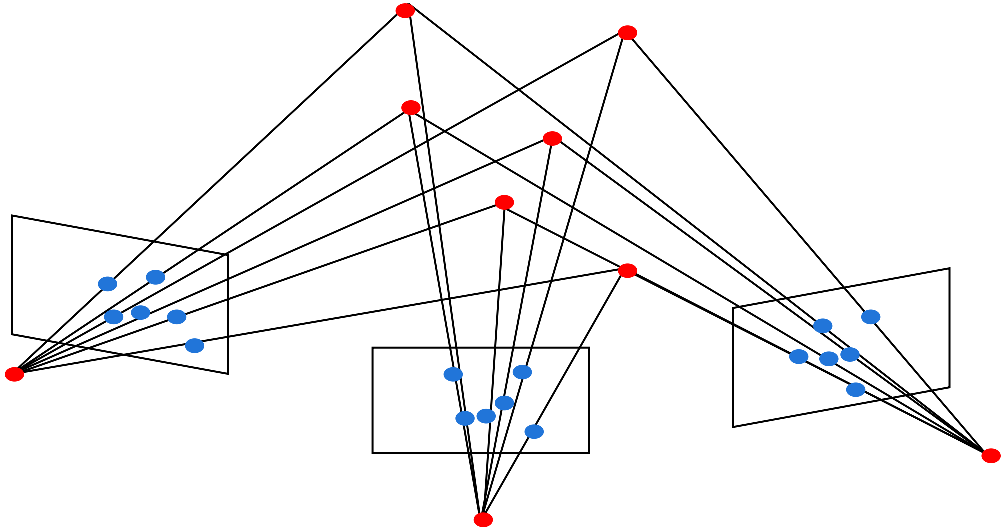
Bundle Adjustment

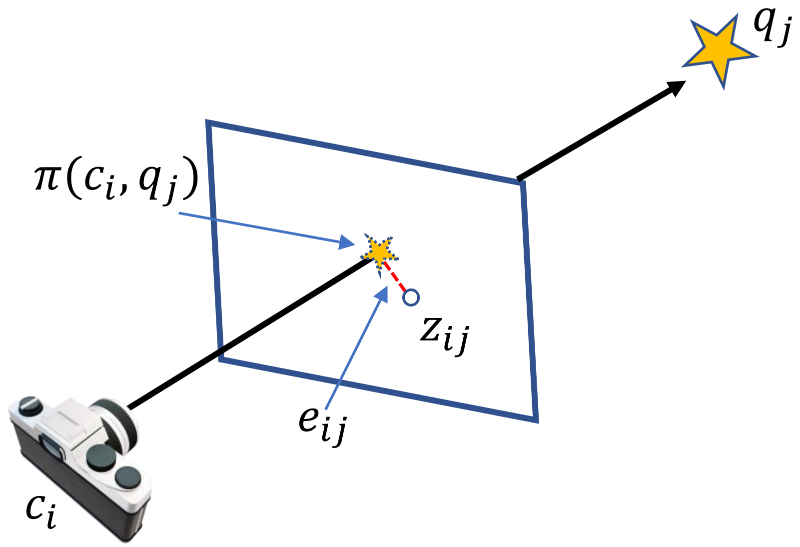
David Arnon

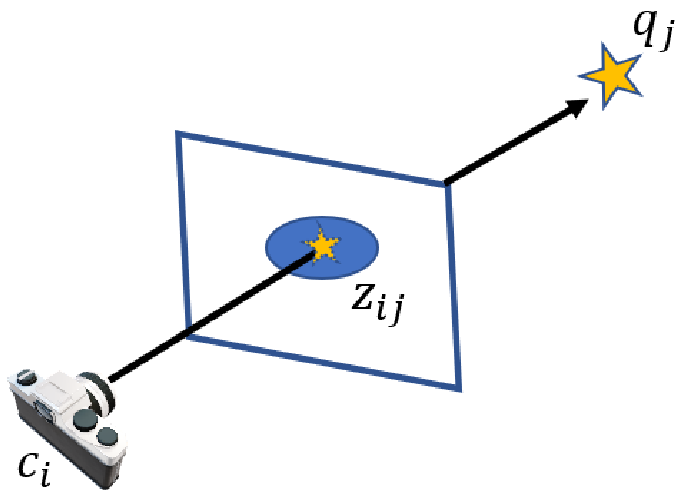
Bundle Adjustment

- Refines a visual reconstruction to produce jointly optimal 3D structure (world) and viewing parameters (cameras)
- *'bundle'* refers to the bundle of light rays leaving each 3D feature and converging on each camera center.
- Developed in the field of photogrammetry in the 1950´s



Measurement Model





$$p(z_{ij} | c_i, q_j) \sim N(\pi(c_i, q_j), \Sigma)$$

$$z_{ij} = \pi(c_i, q_j) + w, \qquad w \sim N(0, \Sigma)$$

Bayes

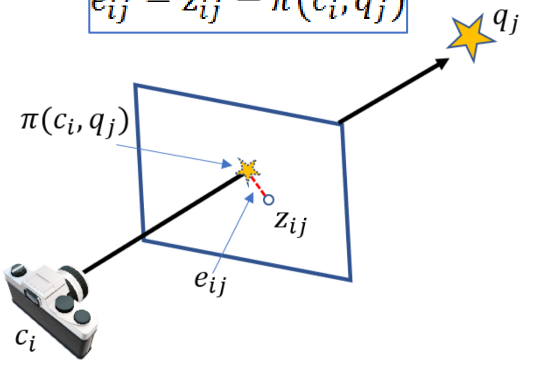
- $p\left(z_{ij} \middle| c_i, q_j\right) \sim N\left(\pi\left(c_i, q_j\right), \Sigma\right)$

$$\boxed{N_{\mu,\Sigma}(z) \propto exp\left(-\frac{1}{2}\|z-\mu\|_{\Sigma}^{2}\right)}$$

- $p\left(c_i, q_j \middle| z_{ij}\right) = \frac{1}{p(z_{ij})} p\left(z_{ij} \middle| c_i, q_j\right) p\left(c_i, q_j\right)$
- $p\left(c_i, q_j \middle| z_{ij}\right) \propto p\left(z_{ij} \middle| c_i, q_j\right) p\left(c_i, q_j\right)$
- $p\left(c_i, q_j \middle| z_{ij}\right) \propto p\left(z_{ij} \middle| c_i, q_j\right)$
- $p\left(c_i, q_j \middle| z_{ij}\right) \propto \exp\left(-\frac{1}{2}\left\|z_{ij} - \pi\left(c_i, q_j\right)\right\|_{\Sigma}^{2}\right)$
- $p\left(c_i, q_j \middle| z_{ij}\right) \propto \exp\left(-\frac{1}{2}\left\|e_{ij}\right\|_{\Sigma}^{2}\right)$

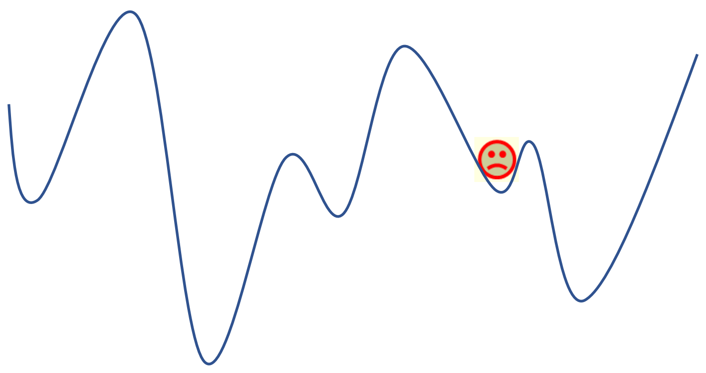
Reprojection error:

$$\boxed{e_{ij} \doteq z_{ij} - \pi\left(c_i, q_j\right)}$$



Bundle Adjustment

- Maximum likelihood for normally distributed measurements
- Sensitive to outliers
  - The Gaussian has extremely small tail compared to most real measurement error distribution
- Non-linear least squares problem
- Solved using an iterative process
- General problem is non-convex,
  can settle in a local minima
- Requires a reasonable starting point



Jacobian

- $f(x + \Delta x) \cong f(x) + J(x)\Delta x$

$$\underbrace{\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_p} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_p} \end{bmatrix}}_{J_f(x)} \underbrace{\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_p \end{bmatrix}}_{\Delta x}$$

Bundle Adjustment
Linear Approximation

- $e(x) = \frac{1}{2}(f(x) - z)^T \Sigma^{-1} (f(x) - z)$
- $\left(\frac{\partial e(x)}{\partial x}\right)^T = J(x)^T \Sigma^{-1} (f(x) - z) = J(x)^T \Sigma^{-1} \Delta z$
- $e(x + \Delta x) \cong e(x) + \frac{\partial e(x)}{\partial x} \Delta x$
- $e(x + \Delta x) \cong e(x) - \frac{1}{\lambda} \left\| \frac{\partial e(x)}{\partial x} \right\|_2^2 < e(x)$
- $\Delta x = -\frac{1}{\lambda} J(x)^T \Sigma^{-1} \Delta z$

$$\boxed{\Delta z \doteq f(x) - z}$$

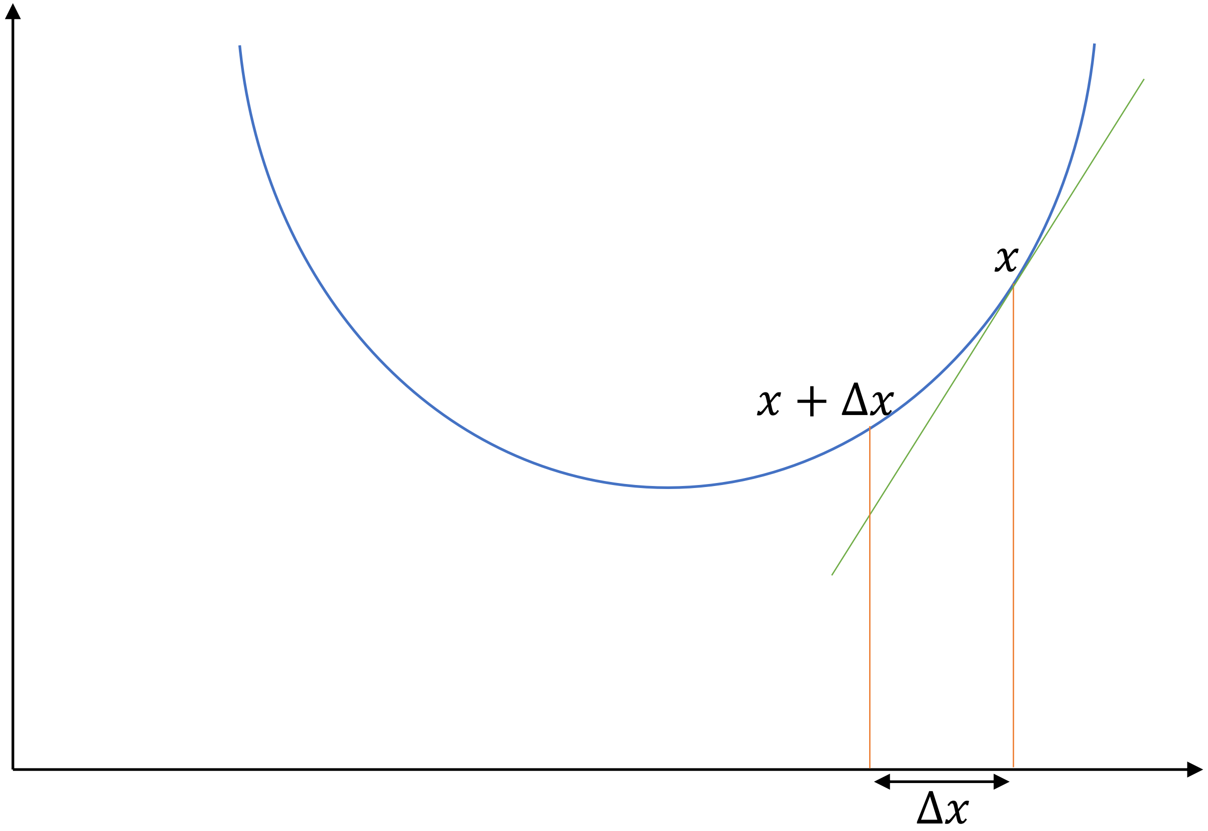
$$\boxed{\Delta x = -\frac{1}{\lambda} \left(\frac{\partial e(x)}{\partial x}\right)^T}$$

$$\boxed{g \doteq J(x)^T \Sigma^{-1} \Delta z}$$

$$\boxed{\Delta x = -\frac{1}{\lambda} g}$$

Bundle Adjustment

Gradient Decent



Bundle Adjustment
Gauss – Newton Algorithm

- Set starting point
- Linearize the measurement function
- Solve linear least squares problem
- Iterate

- Quadratic Approximation!

Bundle Adjustment
Quadratic Approximation

- $argmin_{\Delta x}\|f(x_i+\Delta x)-z\|_\Sigma^2 \quad \cong$
- $argmin_{\Delta x}\|f(x_i)+J(x_i)\Delta x-z\|_\Sigma^2$
- $argmin_{\Delta x}\|J(x_i)\Delta x+f(x_i)-z\|_\Sigma^2$
- $argmin_{\Delta x}\|J(x_i)\Delta x+\Delta z_i\|_\Sigma^2$
- $argmin_{\Delta x}\left\|\Sigma^{-½}J(x_i)\Delta x+\Sigma^{-½}\Delta z_i\right\|_2^2$
- $J(x_i)^T\Sigma^{-1}J(x_i)\Delta x=-J(x_i)^T\Sigma^{-1}\Delta z_i$
    - $J(x_i)^T\underbrace{\Sigma^{-½T}\Sigma^{-½}}_{\Sigma^{-1}}J(x_i)\Delta x=-J(x_i)^T\underbrace{\Sigma^{-½T}\Sigma^{-½}}_{\Sigma^{-1}}\Delta z_i$

$$\boxed{H\Delta x=-g}$$

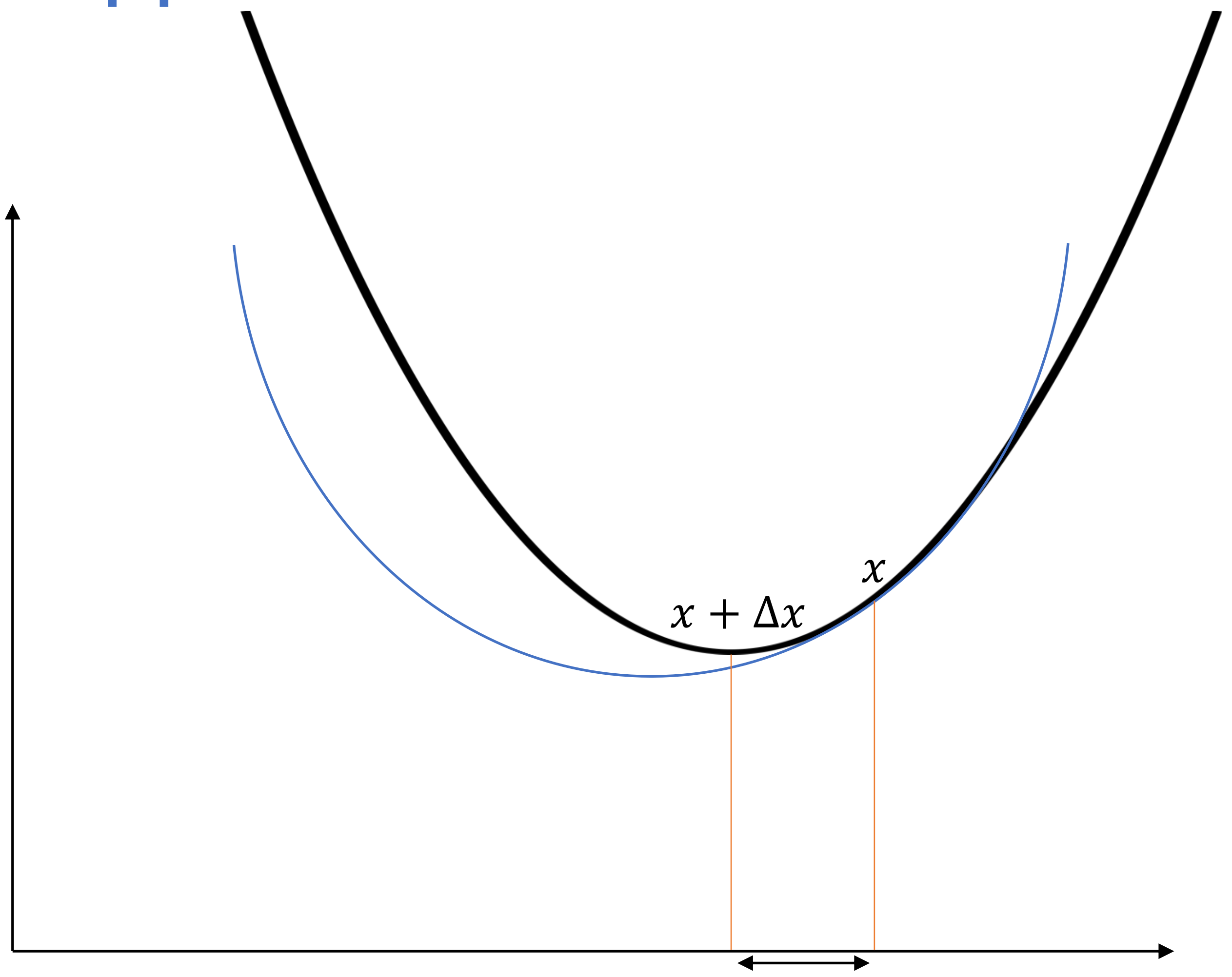
$$\boxed{\begin{gathered}H\doteq J(x_i)^T\Sigma^{-1}J(x_i)\\ g\doteq J(x_i)^T\Sigma^{-1}\Delta z_i\end{gathered}}$$

$$\boxed{\Delta z_i\doteq f(x_i)-z}$$

$$\boxed{\begin{gathered}\Sigma=(\Sigma^{½})(\Sigma^{½})^T\\ \Sigma^{-1}=\Sigma^{-½\mathrm{T}}\Sigma^{-½}\end{gathered}}$$

$$\boxed{argmin_x\|Ax-b\|_2^2 \quad\Longrightarrow\quad A^TAx=A^Tb}$$

Bundle Adjustment
Quadratic Approximation



Bundle Adjustment
Gauss – Newton

- Converges in one iteration for quadratic functions

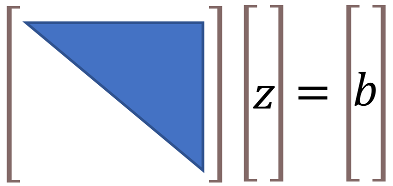
- For general functions, the asymptotic convergence is quadratic 

- Inverting H is expensive 

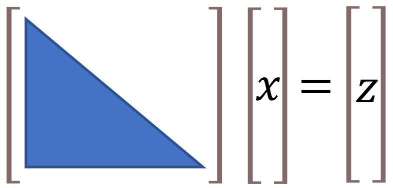
Bundle Adjustment
Cholesky Decomposition

$$\boxed{H = CC^T}$$

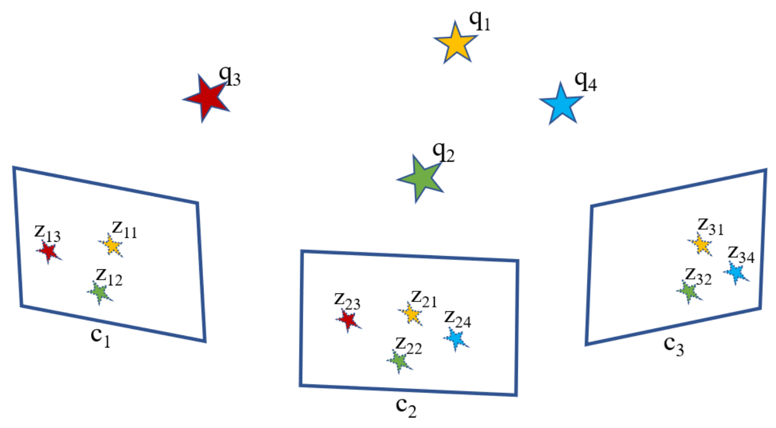
- $Hx = b$
- $C\,\underbrace{C^T x}_{z} = b$
- $Cz = b$



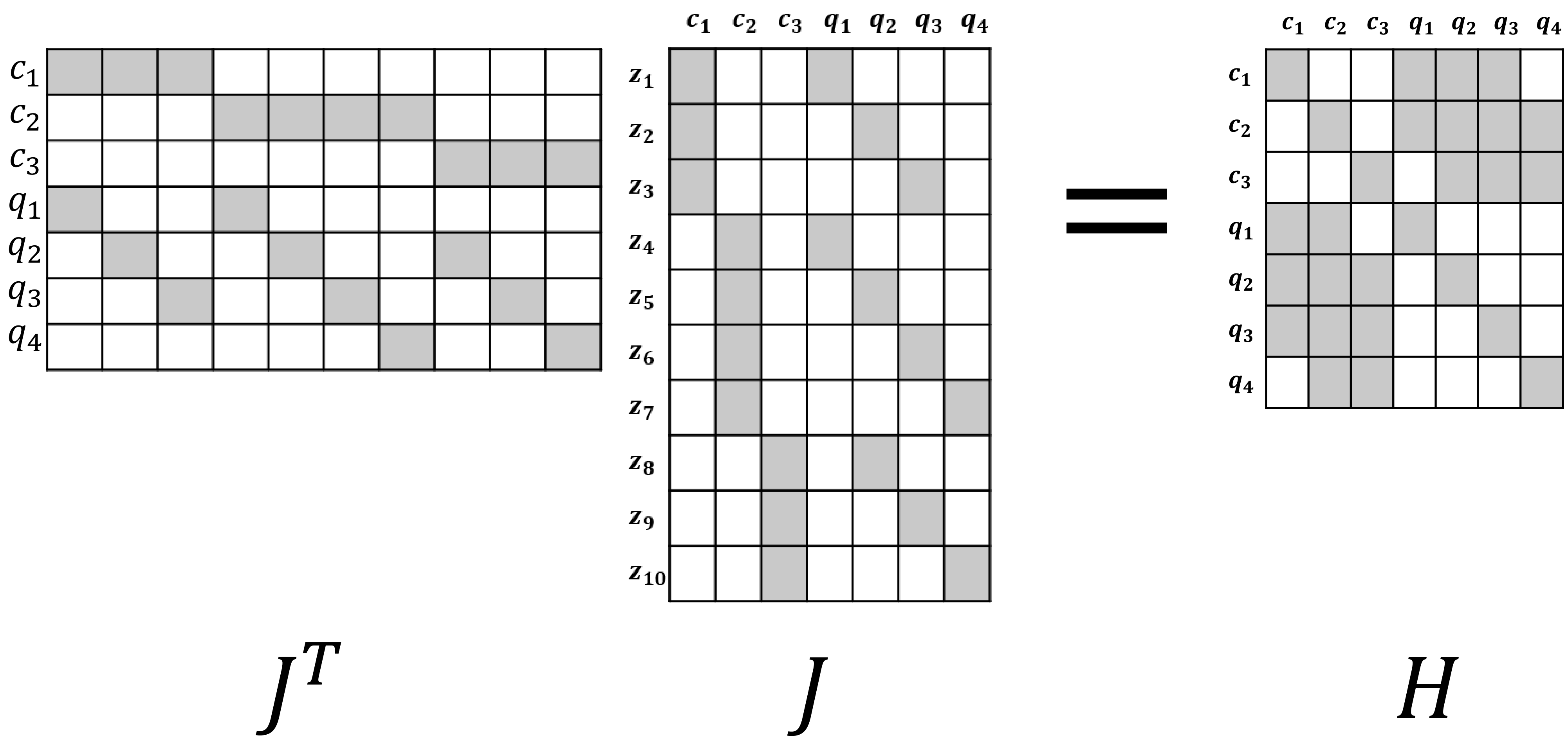
- $C^T x = z$



Bundle Adjustment
Sparsity



$$f(x) \doteq \begin{bmatrix} \pi(c_1, q_1) \\ \pi(c_1, q_2) \\ \pi(c_1, q_3) \\ \pi(c_2, q_1) \\ \pi(c_2, q_2) \\ \pi(c_2, q_3) \\ \pi(c_2, q_4) \\ \pi(c_3, q_2) \\ \pi(c_3, q_3) \\ \pi(c_3, q_4) \end{bmatrix}$$



Bundle Adjustment
Uncertainty

- H is the information matrix
  - Inverse of the covariance matrix of the estimated Δx
  - Approximation of the hessian -
    second-order partial derivatives matrix

$$H^{-1} = \begin{bmatrix} \Sigma_1 & & & * \\ & \Sigma_2 & & \\ & & \ddots & \\ * & & & \Sigma_n \end{bmatrix}$$

- Can be used to estimate the uncertainty of the result
  - Marginal covariances
- Conditioning $p(x_j | x_i)$
  It is possible to estimate the relative uncertainty between x_j and x_i
  - erase row and column i
  - invert and use diagonal block j