VAN course
Lesson 11

Dr. Refael Vivanti
refael.vivanti@mail.huji.ac.il

Today’s topics:

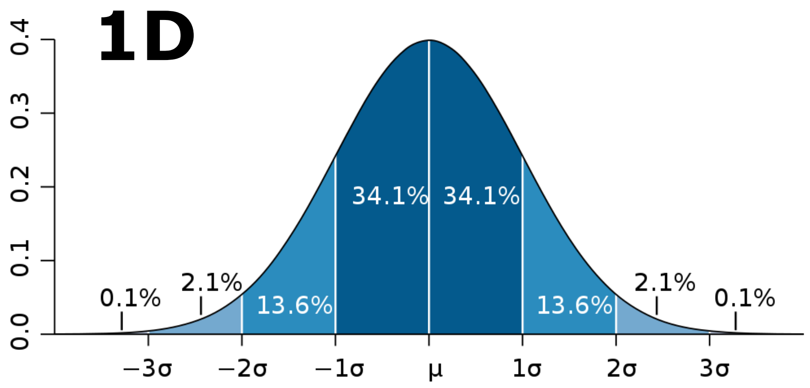
- Back to some statistics:
  - Information matrix and vector
  - Marginalization vs conditioning
- Compromises in our Pose Graph
- Our Pose Graph – how to
- Loop closure

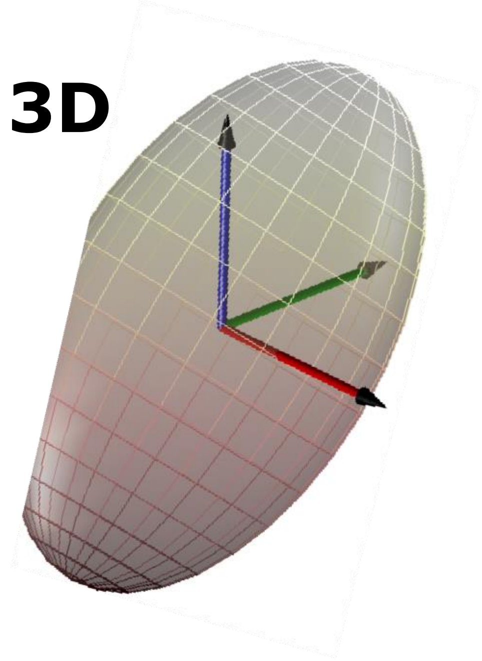
Back to
some statistics

Gaussians

- Gaussian described by **moments** μ, Σ

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big)$$





Canonical Parameterization

- Alternative representation for Gaussians
- Described by **information matrix** Ω and **information vector** ξ

Canonical Parameterization

- Alternative representation for Gaussians
- Described by **information matrix** Ω

$$\Omega = \Sigma^{-1}$$

- and **information vector** ξ

$$\xi = \Sigma^{-1}\mu$$

Complete Parameterizations

moments

$$\boxed{\begin{aligned} \Sigma &= \Omega^{-1} \\ \mu &= \Omega^{-1}\xi \end{aligned}}$$

canonical

$$\boxed{\begin{aligned} \Omega &= \Sigma^{-1} \\ \xi &= \Sigma^{-1}\mu \end{aligned}}$$

Towards the Information Form

$$
\begin{aligned}
& p(x) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
& p(x) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(\underline{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)} \Big) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu \Big)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
& p(x) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) \Big) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(\underline{-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu} - \underline{\frac{1}{2}\mu^T\Sigma^{-1}\mu} \Big) \\
= \quad & \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mu^T\Sigma^{-1}\mu \Big) \\
& \qquad \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu \Big)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
& p(x) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\Big) \\
&\qquad\qquad \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big) \\
&= \eta \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\Big)
\end{aligned}
$$

Towards the Information Form

$$
\begin{aligned}
p(x) & \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu\right) \\
&= \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mu^T\Sigma^{-1}\mu\right) \\
&\qquad \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + x^T\Sigma^{-1}\mu\right) \\
&= \eta \exp\left(-\frac{1}{2}x^T\underline{\Sigma^{-1}}x + x^T\underline{\Sigma^{-1}\mu}\right) \\
&= \eta \exp\left(-\frac{1}{2}x^T\Omega x + x^T\xi\right)
\end{aligned}
$$

Dual Representation

$$p(x) = \frac{\exp(-\frac{1}{2}\mu^T\xi)}{\det(2\pi\Omega^{-1})^{\frac{1}{2}}} \exp\left(-\frac{1}{2}x^T\Omega x + x^T\xi\right)$$

canonical parameterization

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

moments parameterization

Marginalization vs. conditioning

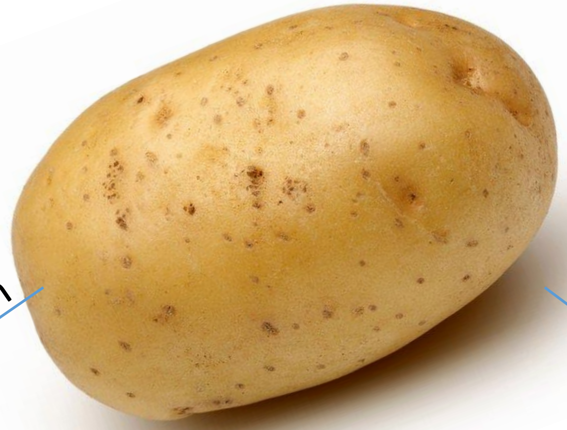
- Both are dimension reduction: P(*x, y*) -> P(*x*)
  - **Marginalization** - summing over all y:

$$p(x) = \sum_y p(x, y)$$
$$= \sum_y p(x|y)\, p(y)$$

  - **Conditioning** – probability given a specific *y*:

$$p(x \mid y = y_0)$$

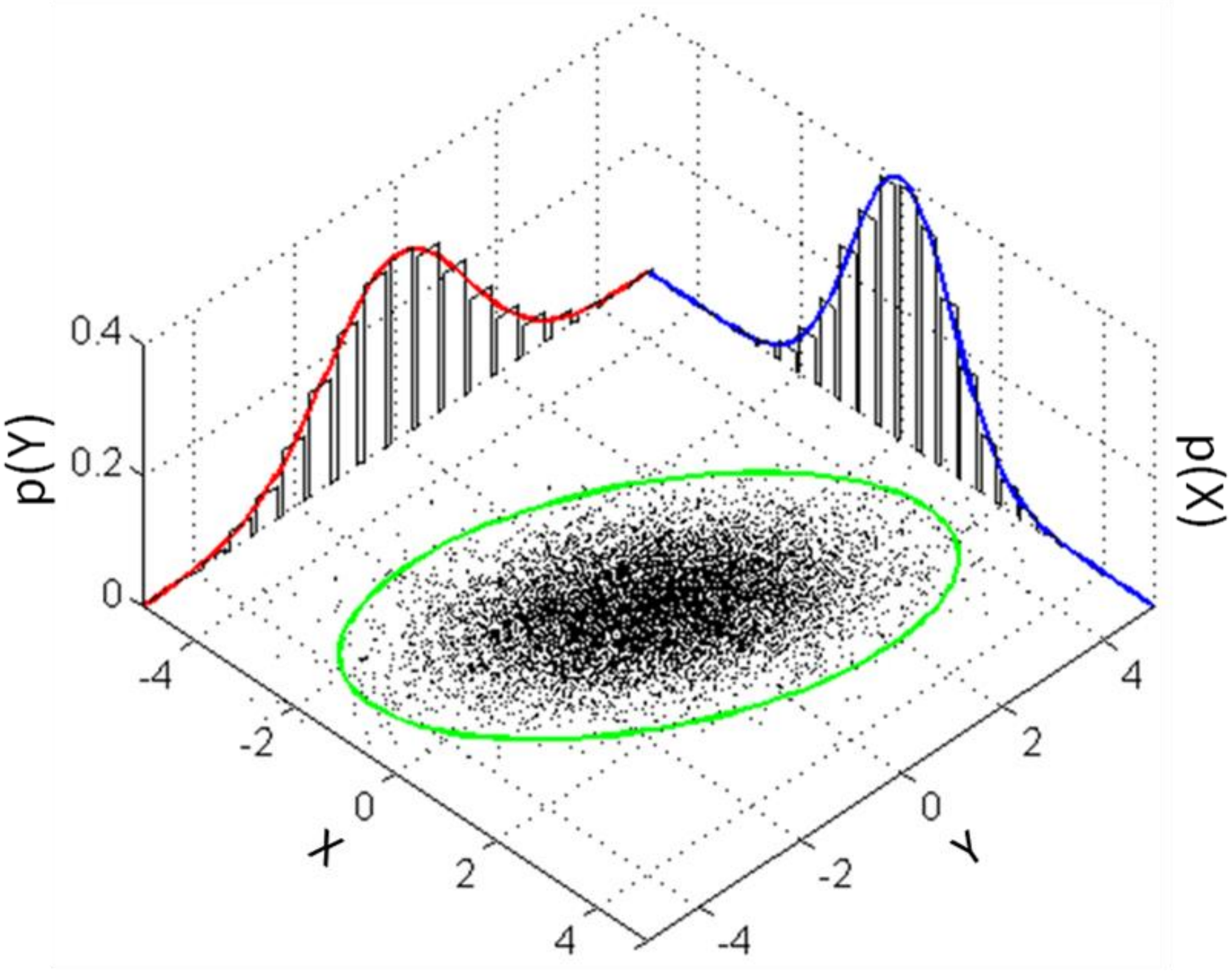
Marginalization vs. conditioning



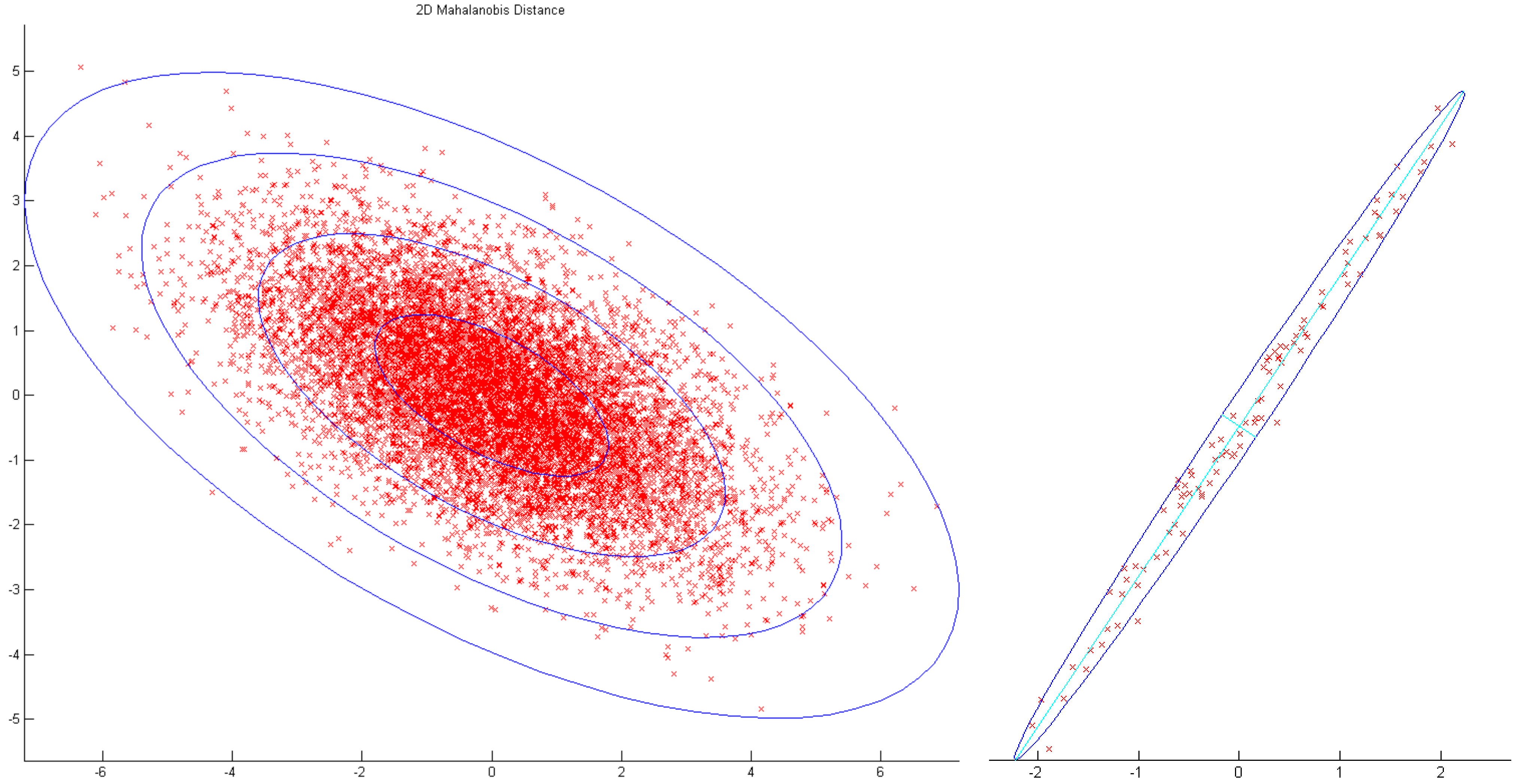
marginalization

conditioning

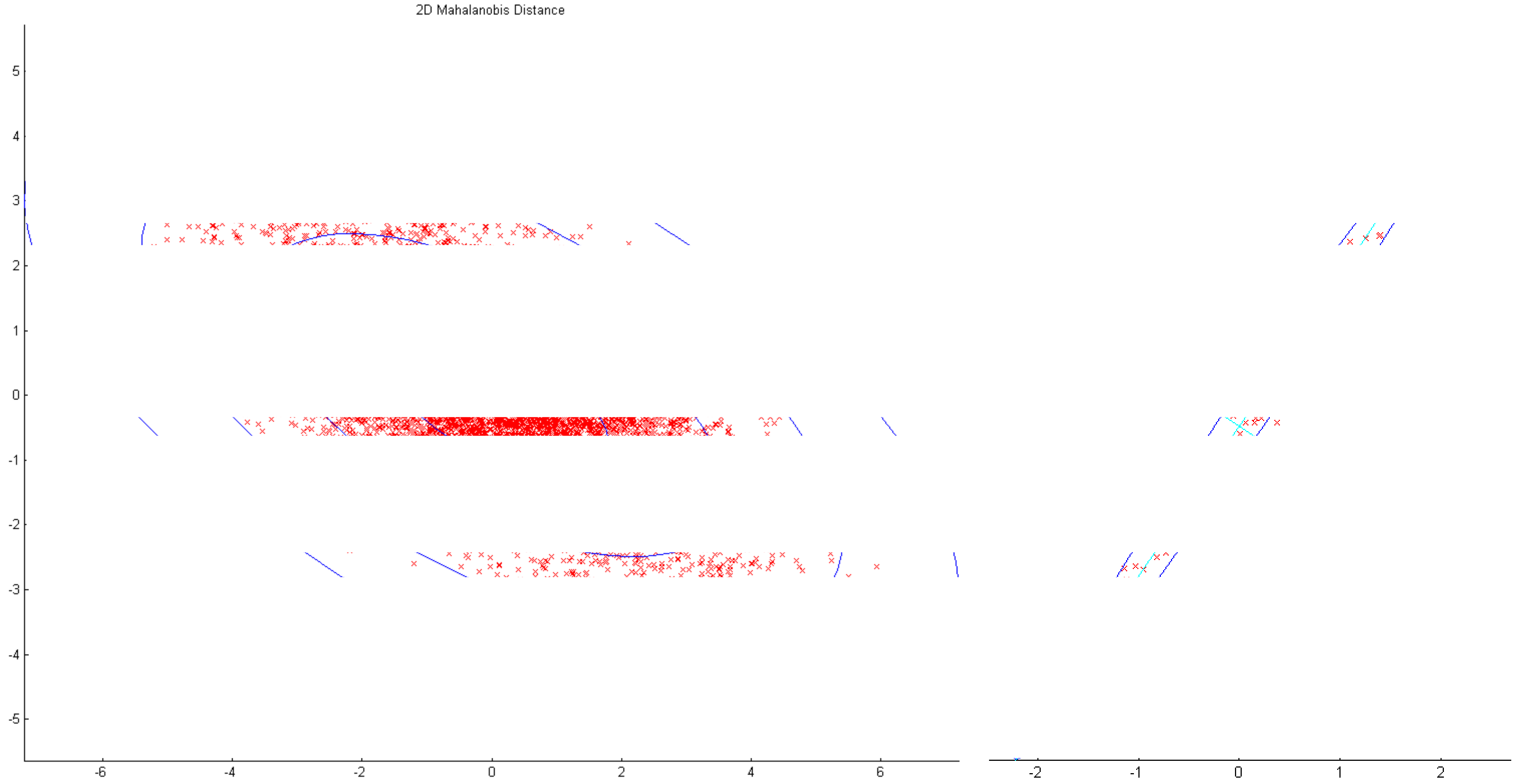
Marginalization – geometric intuition



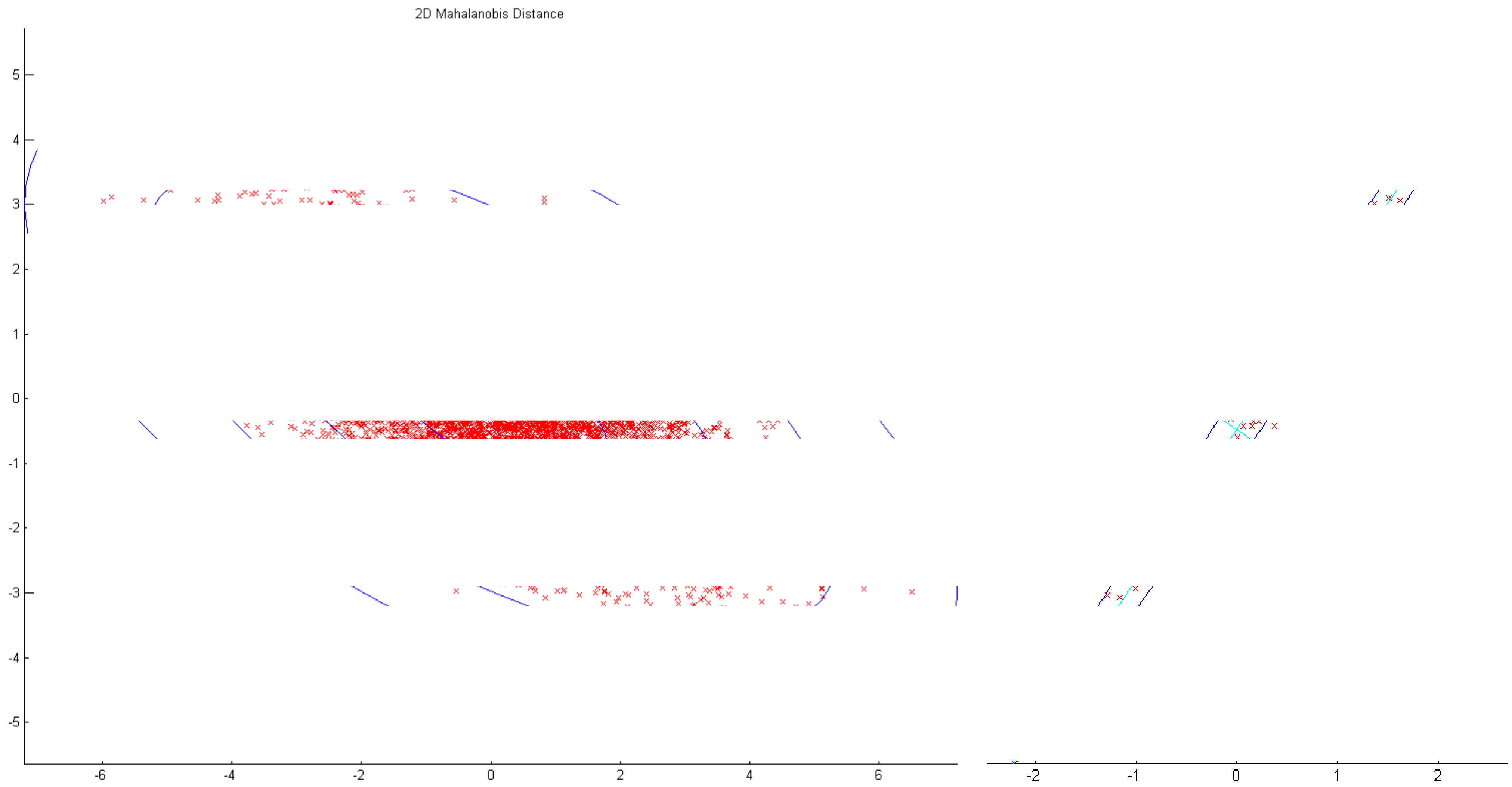
Conditioning – geometric intuition



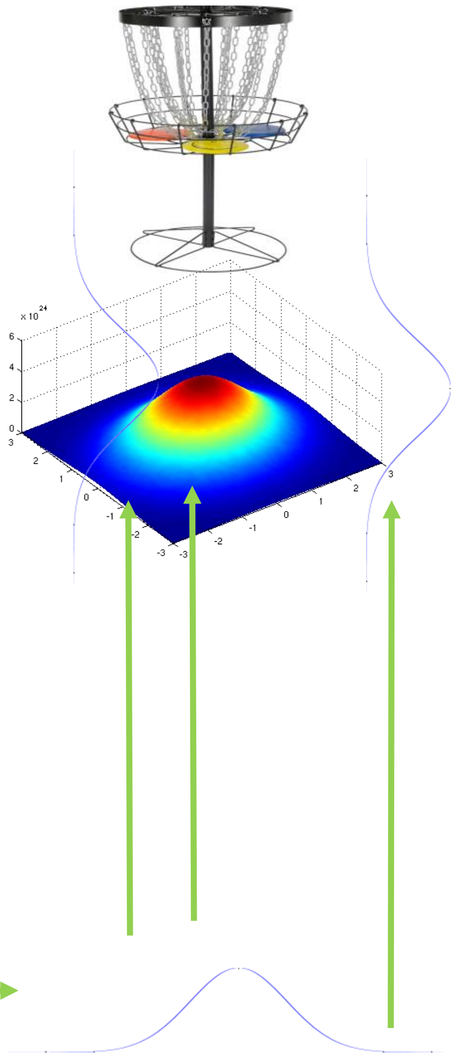
Conditioning – geometric intuition



Conditioning – geometric intuition



Why covariance is constant?



Marginalization and Conditioning – how to

$\Lambda = \Omega$

$$p(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_\alpha \\ \boldsymbol{\mu}_\beta \end{bmatrix}, \begin{bmatrix} \Sigma_{\alpha\alpha} & \Sigma_{\alpha\beta} \\ \Sigma_{\beta\alpha} & \Sigma_{\beta\beta} \end{bmatrix}\right) = \mathcal{N}^{-1}\left(\begin{bmatrix} \boldsymbol{\eta}_\alpha \\ \boldsymbol{\eta}_\beta \end{bmatrix}, \begin{bmatrix} \Lambda_{\alpha\alpha} & \Lambda_{\alpha\beta} \\ \Lambda_{\beta\alpha} & \Lambda_{\beta\beta} \end{bmatrix}\right)$$

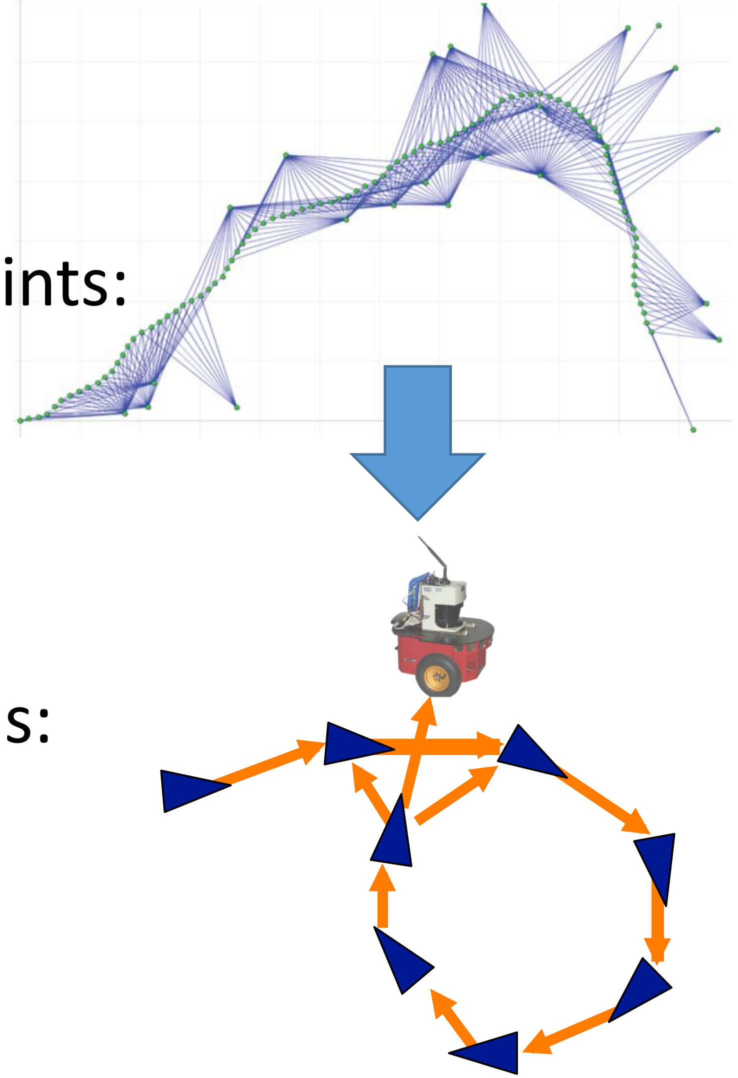
	MARGINALIZATION $p(\boldsymbol{\alpha}) = \int p(\boldsymbol{\alpha}, \boldsymbol{\beta}) d\boldsymbol{\beta}$	CONDITIONING $p(\boldsymbol{\alpha} \mid \boldsymbol{\beta}) = p(\boldsymbol{\alpha}, \boldsymbol{\beta}) / p(\boldsymbol{\beta})$
COV. FORM	$\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$ $\Sigma = \Sigma_{\alpha\alpha}$	$\boldsymbol{\mu}' = \boldsymbol{\mu}_\alpha + \Sigma$ [illegible] $- \boldsymbol{\mu}_\beta)$ $\Sigma' =$ [illegible] $\Sigma_{\alpha\beta}\Sigma_{\beta\beta}^{-1}\Sigma_{\beta\alpha}$ expensive
INFO. FORM	$\boldsymbol{\eta} = \boldsymbol{\eta}_\alpha - \Lambda$ [illegible] β Λ [illegible] $\Lambda_{\alpha\beta}\Lambda_{\beta\beta}^{-1}\Lambda_{\beta\alpha}$ expensive	$\boldsymbol{\eta}' = \boldsymbol{\eta}_\alpha - \Lambda_{\alpha\beta}\boldsymbol{\beta}$ $\Lambda' = \Lambda_{\alpha\alpha}$

Courtesy: R. Eustice

Pose Graph - compromises

- We replaced our big factor graph with a pose graph
- How did we compromise?
- From each small factor graph of K cameras and P points:
  - We removed all points
  - We removed most cameras
  - $P(C_1,\ldots,C_K,p_1,..p_P) \underbrace{\text{->}}_{\text{marginalization}} P(C_1,C_K) \underbrace{\text{->}}_{\text{conditioning}} P\,(C_K|C_1)$
- In the full factor graph, point p might have F cameras:
  - We used this track by parts, in each small factor graph:
  - $P(C_1,\ldots,C_F,p_1) \text{ -> } P(KF_2,p|KF_1) \cdot\cdot P(KF_N, p|KF_{N-1})$

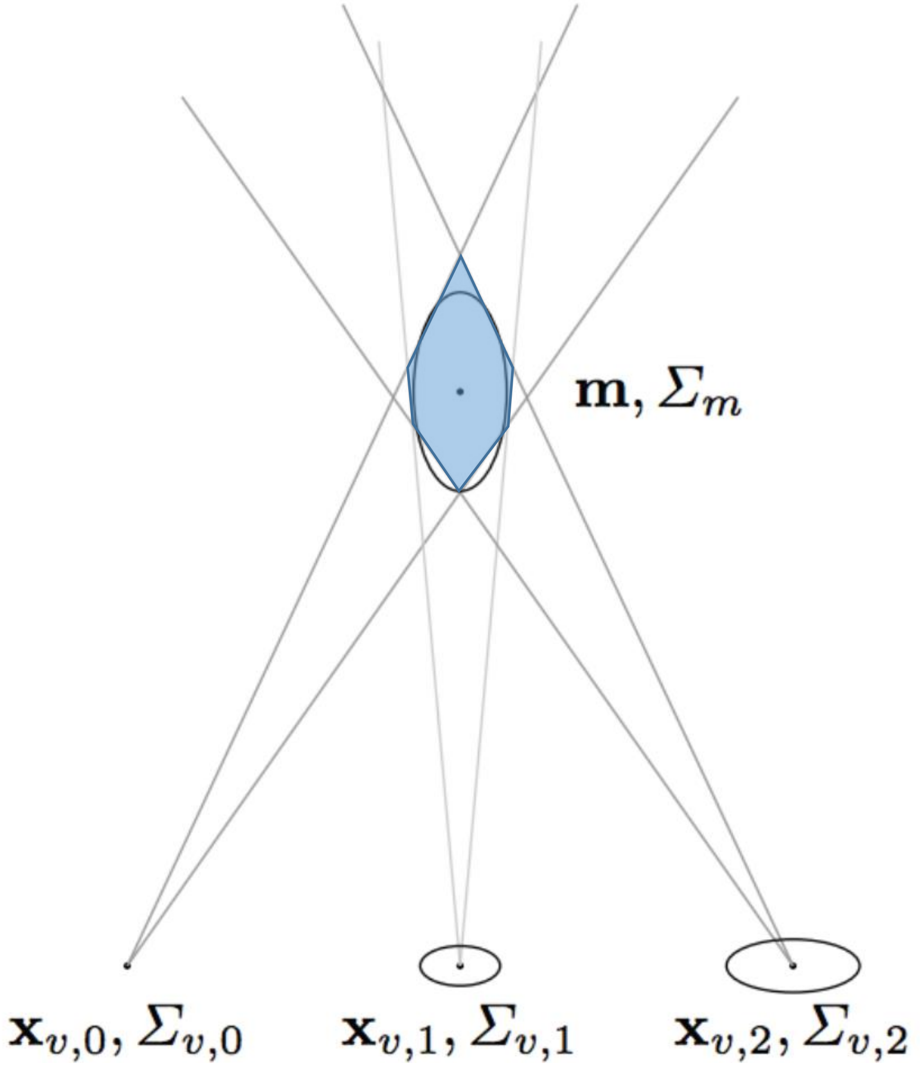
Courtesy of Dellaert06ijrr: "Square Root SAM"



Pose Graph - compromises

- Covariance approximation:
  - The small factor graph information is represented as a normal distribution
  - If all operations were linear it's ok
  - But we do non-linear projection
  - The covariance is an **approximation**

- We ignored **a lot** of information
  - Surprisingly, the results are often pretty accurate!



Pose Graph – how to

- To build and initialize the pose graph we need:
  - Each KF will be a vertex (symbol)
    - Initialization: global-coords pose of each KF camera.
  - Each successive KF pair, which participated in one small factor graph, will define a factor between two vertices:

$$\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) = \mathrm{t2v}(\underline{\mathbf{Z}_{ij}^{-1}}(\underline{\mathbf{X}_i^{-1}\mathbf{X}_j}))$$

  - It also needs a covariance..

Pose Graph – how to

- How to calculate the covariance:

$$\Sigma_{all} \xrightarrow{marg.} \Sigma_{1N} \xrightarrow{inv.} \Omega_{1,N} \xrightarrow{cond.} \Omega_{N|1} \xrightarrow{inv.} \Sigma_{N|1}$$

$$\begin{array}{c} C_1 \\ \vdots \\ C_N \\ P_1 \\ \vdots \\ P_M \end{array} \begin{bmatrix} C_{11} & \cdots & C_{1N} & C_1P_1 & \cdots & C_1P_M \\ \vdots & \ddots & & & & \\ C_{N1} & & C_{NN} & C_NP_1 & & C_NP_M \\ C_1P_1 & & C_NP_1 & P_{11} & & P_{1M} \\ & & & & \ddots & \\ C_1P_M & & C_NP_M & P_{M1} & & P_{MM} \end{bmatrix} \rightarrow \begin{bmatrix} C_{11} & C_{1N} \\ C_{N1} & C_{NN} \end{bmatrix} \rightarrow \begin{bmatrix} I_{11} & I_{1N} \\ I_{N1} & I_{NN} \end{bmatrix} \rightarrow \begin{bmatrix} I_{NN} \end{bmatrix} \rightarrow \begin{bmatrix} C_{NN} \end{bmatrix}$$