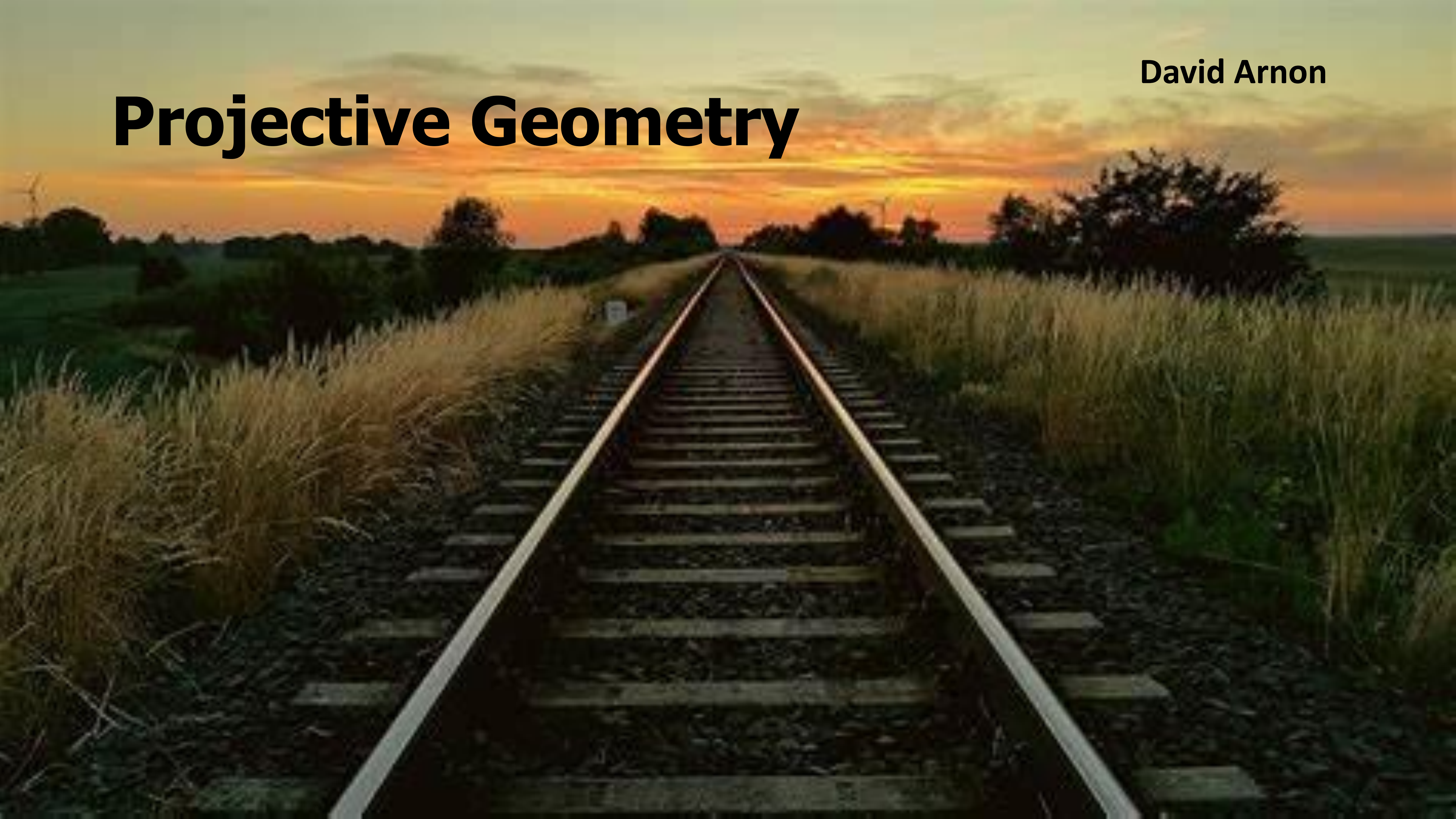
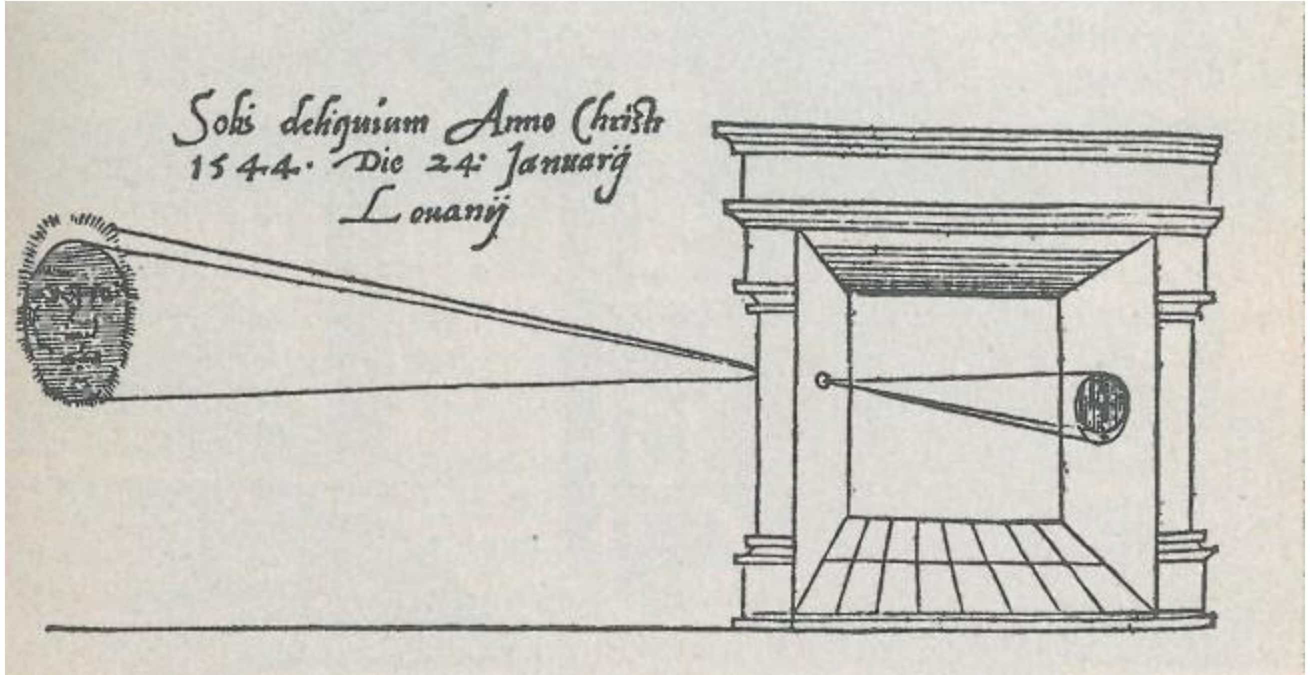
Projective Geometry

David Arnon

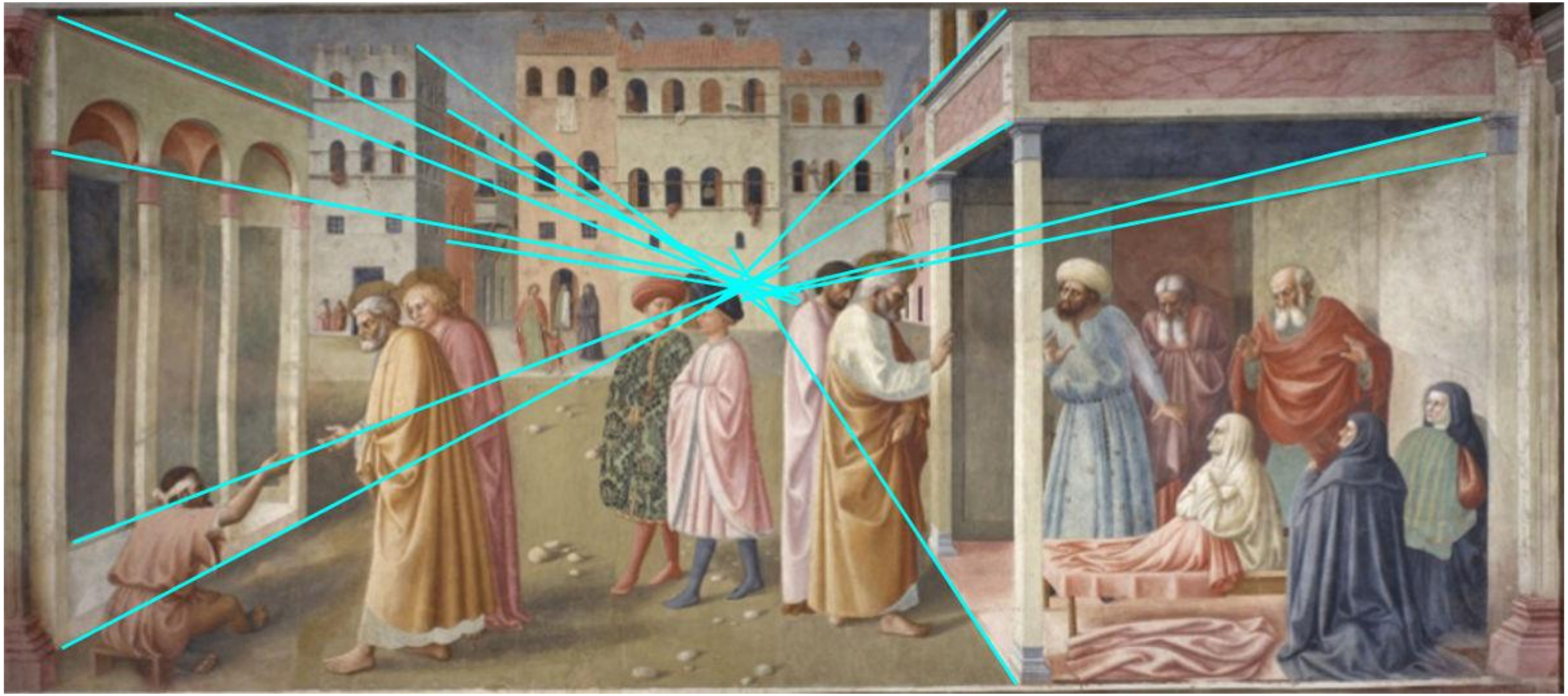


Projection



Gemma Frisius - camera obscura De Radio Astronomica et Geometrica 1545

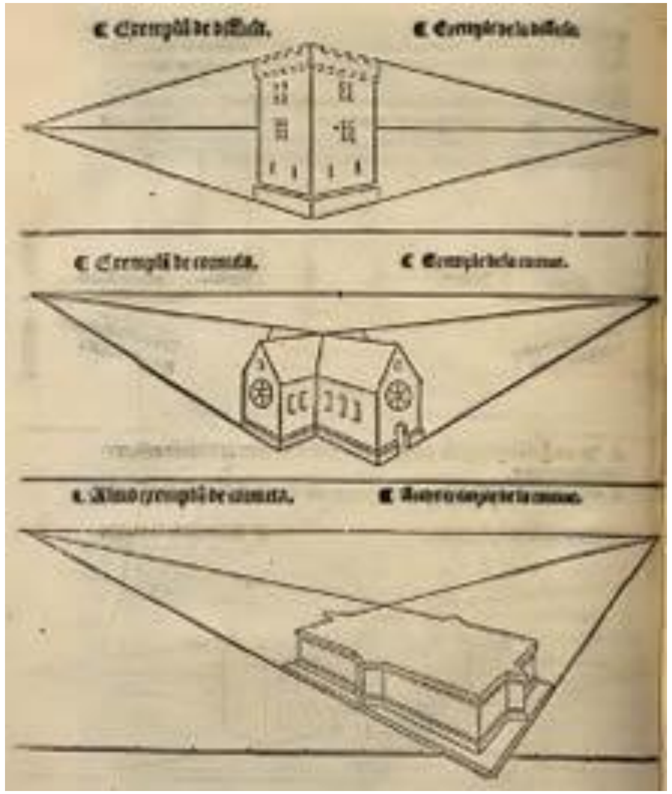
Perspective projection
Perspective drawing



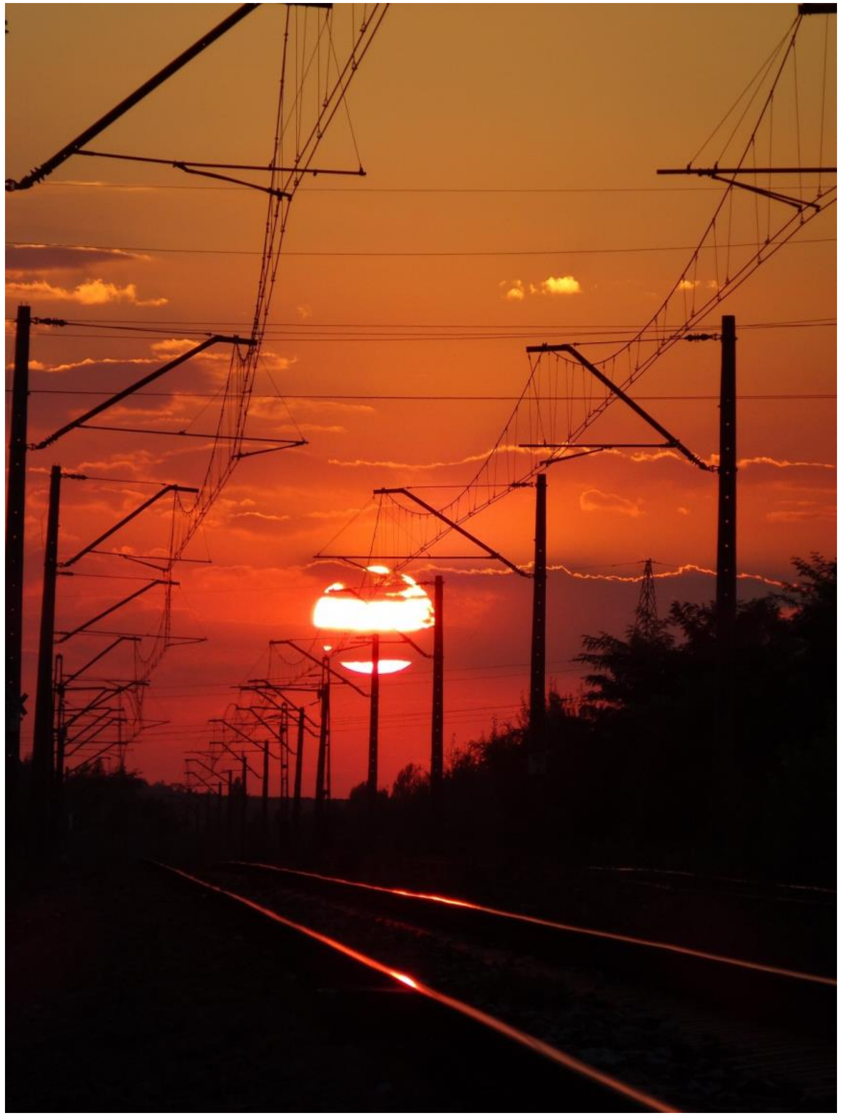
The Healing of the Cripple and Raising of Tabitha Masolino 1426

Projective Geometry

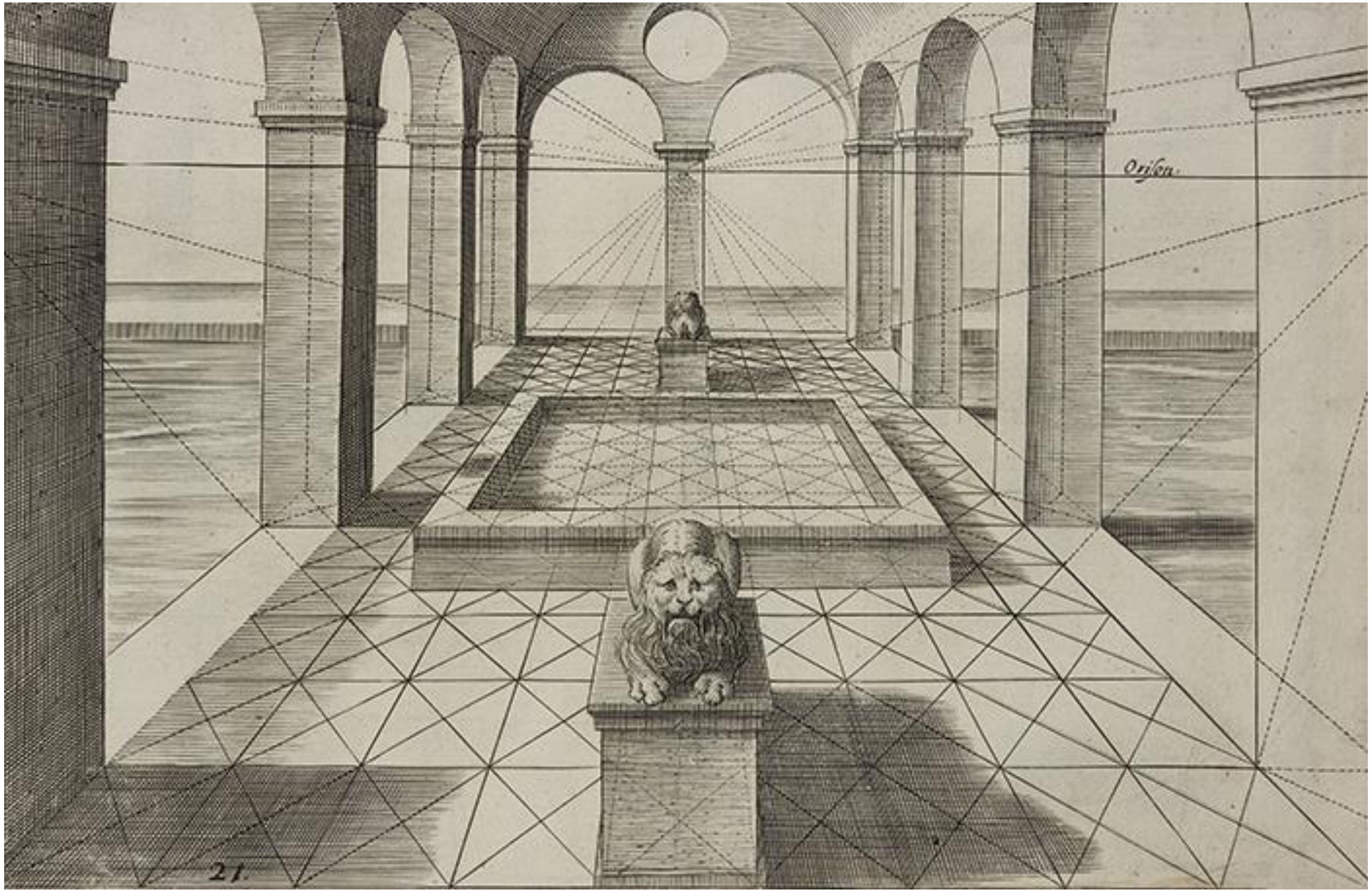
- Two vanishing points



De Artificiali perspectiva Jean Pelerin 1505



Projective Geometry



Hans Vredeman de Vries 1604 RIBA Collections

Projective Geometry

- In Euclidian geometry things get difficult
- Projection of a plane into the image plane
- Projective transformations
- Projective plane $\mathbb{P}_2(\mathbb{R})$
- Points at infinity
  - Line at infinity
  - Point-Line duality

Homogeneous Coordinates

- Möbius 1827
- Used as a coordinate system for projective geometry
- Often much simpler to use
- Can represent points at infinity
- Surprisingly many things can be represented as linear operations (matrix)

Homogeneous Coordinates

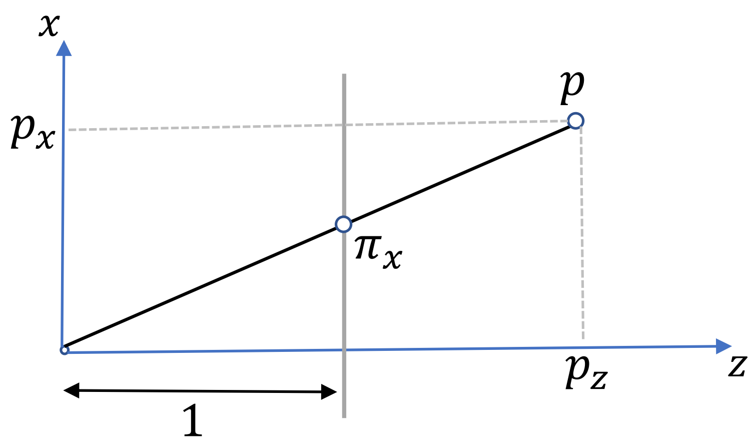
- Homogeneous representation: $x = \lambda x$ for $\lambda \neq 0$
  - $x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix}$

<u>Homogeneous</u> <u>Euclidian</u>

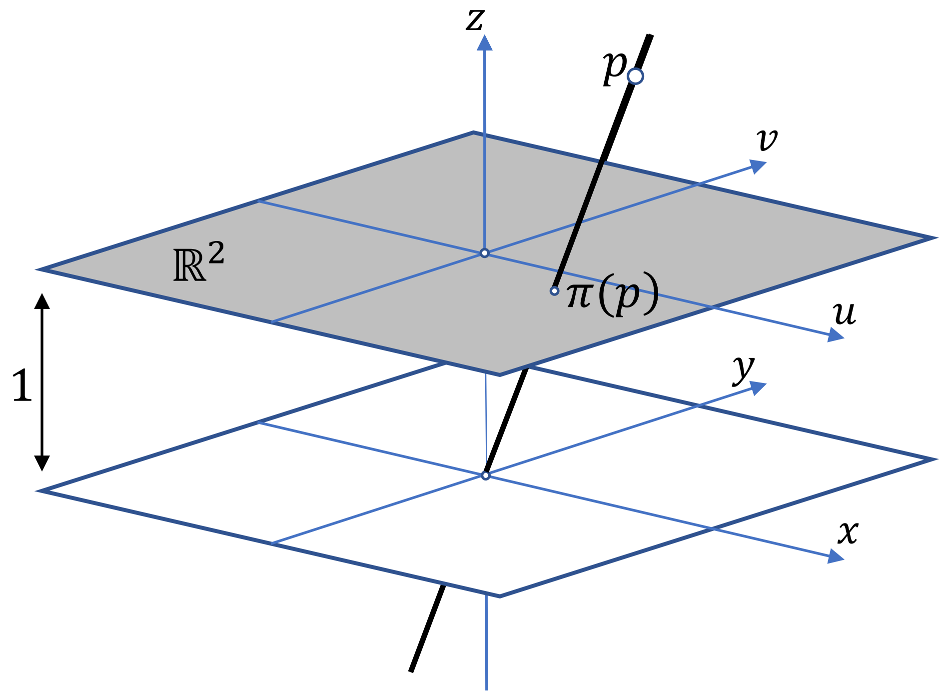
- $p = \begin{bmatrix} \lambda p_x \\ \lambda p_y \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} \Rightarrow p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$

Homogeneous Coordinates

- Projection = division by z



$$\frac{\pi_x}{1} = \frac{p_x}{p_z}$$



Homogeneous Coordinates

- Points at infinity (ideal points)

$$p_\infty = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Homogeneous Coordinates

- Homogeneous representation of 2D points and lines

$$ax + by + c = 0$$

$$[a, b, c]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- Point p lies on line l iff $\quad l^T p = 0$
- Invariant to scale, only 2dof
- Line at infinity: $\quad l_\infty = [0,0,1]^T$
- $\mathbb{P}_2 = \mathbb{R}^2 \cup l_\infty$

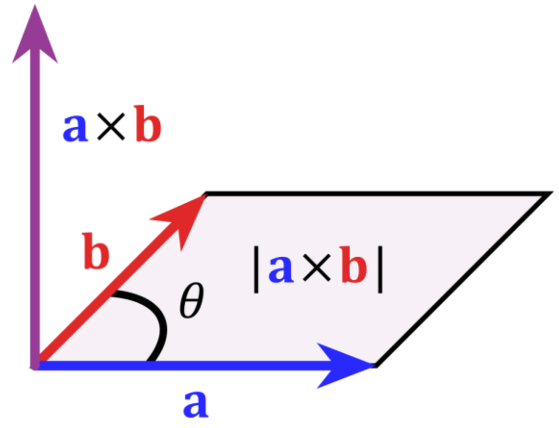
Homogeneous Coordinates

- Dot product $a \cdot b = a^T b = \cos(\theta) \, \|a\| \|b\|$

- Cross product $a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

$$[\mathbf{a}]_\times \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$



Homogeneous Coordinates

- Intersection of lines:

$$p = l \times \tilde{l}$$

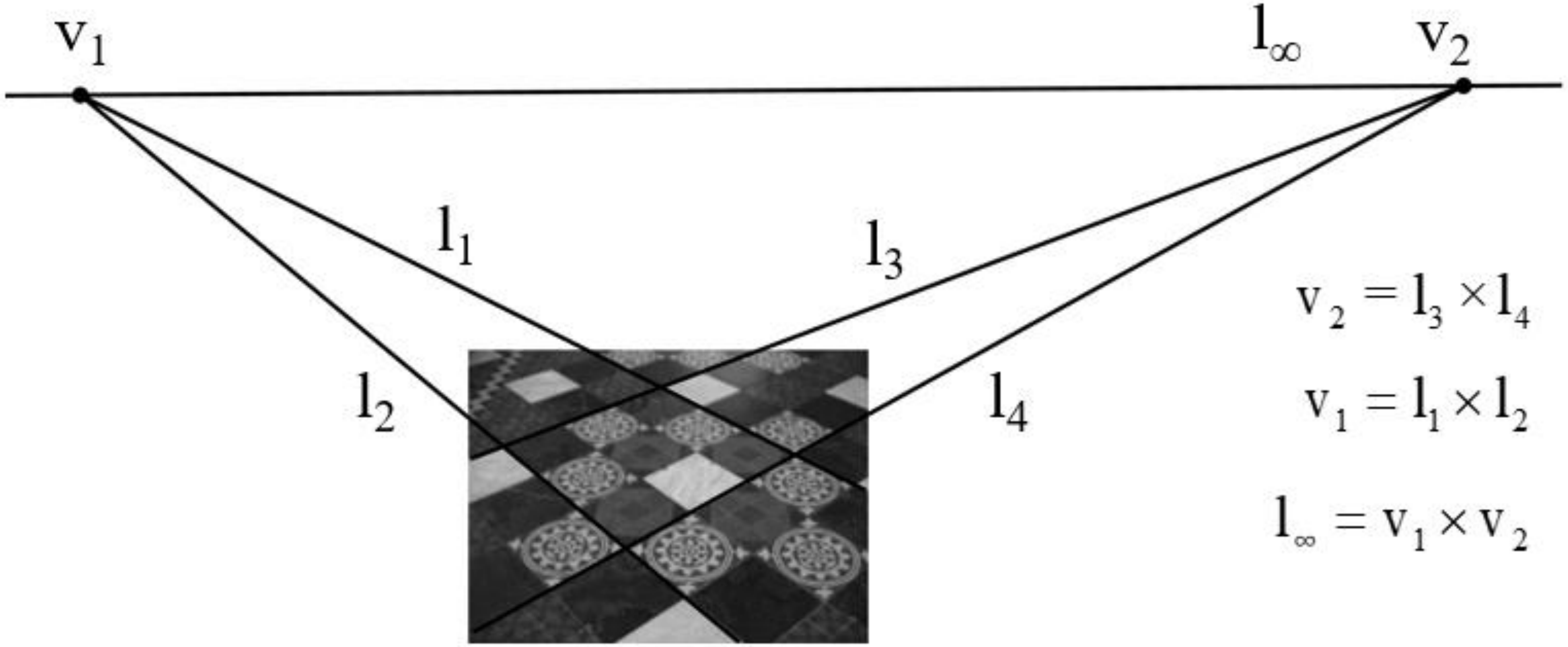
- Connecting two points:

$$l = p \times \tilde{p}$$

- Examples

Homogeneous Coordinates

- Find horizon:



Homogeneous Coordinates

- Definition:

  A *projectivity* is an invertible mapping $h: \mathbb{P}_2 \rightarrow \mathbb{P}_2$ such that three points x_1, x_2, x_3 lie on the same line iff $h(x_1), h(x_2), h(x_3)$ do.

- Theorem:

  Any *projectivity* can be represented in homogeneous coordinates as a non-singular 3x3 matrix.
  (and vice-versa)

Homogeneous Coordinates

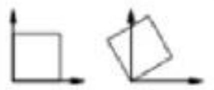
- *Homography* *(projectivity, planar transformation,…)*

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{or} \quad y = Hx$$

  - Only 8 dof

- Transformation for lines: $\tilde{l} = H^{-T} l$

Homogeneous Coordinates

2D Transformation	Figure	d. o. f.	H	H
Translation		2	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Mirroring at y-axis		1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} Z & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Rotation		1	$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Motion		3	$\begin{bmatrix} \cos\varphi & -\sin\varphi & t_x \\ \sin\varphi & \cos\varphi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Similarity		4	$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda R & \boldsymbol{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Affinity		6	$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$
Projectivity		8	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{p}^\top & 1/\lambda \end{bmatrix}$

Courtesy of K. Schindler

Summary

- Homogeneous coordinates simplify math for projections
- Equivalence up to scale $x = \lambda x$ with $\lambda \neq 0$
- Extra dimension
- Duality between points and lines
- Easy chaining and inversion
- Worth the price of adding 1 dimension
  - Simple
  - Linear
  - Avoid division
  - Less bugs
- Models projection of plane to camera
  - Between two cameras that see a common plane
  - Between two cameras with only orientation change

Literature

- **Multiple View Geometry** **in computer vision**
  Hartley and Zisserman