# VAN course
# Lesson 4

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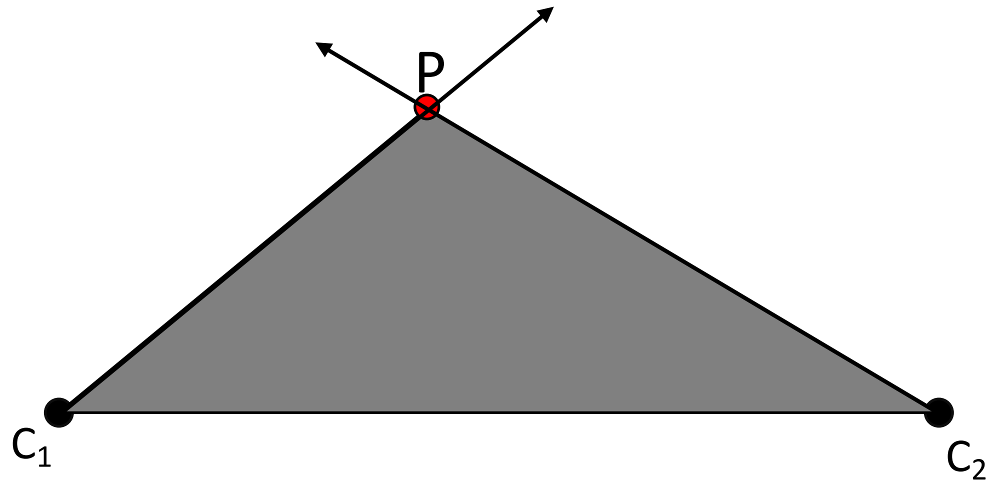
# Today’s topics

- Epipolar geometry
- Epipolar lines, Epipole
- Fundamental Matrix Calculation
- Rectification
- RANSAC

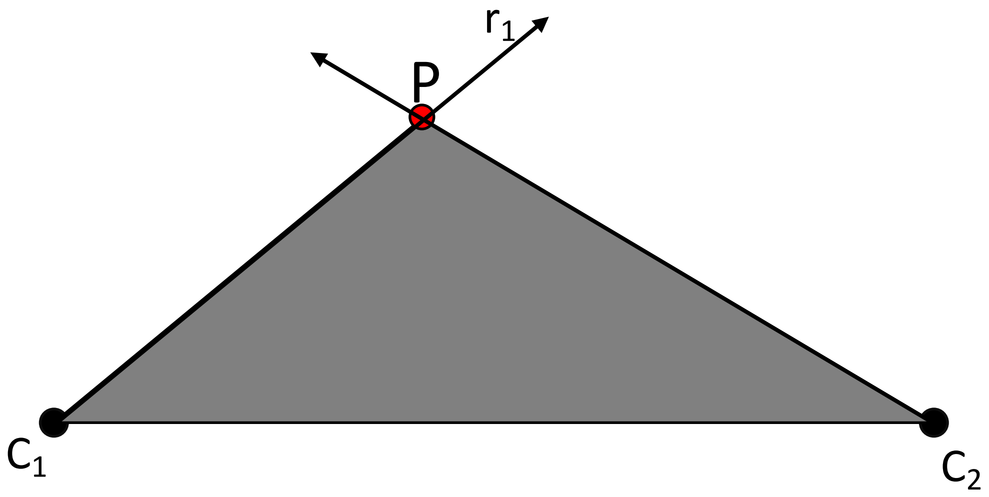
# Epipolar geometry

- The relations between views as appeared in the image



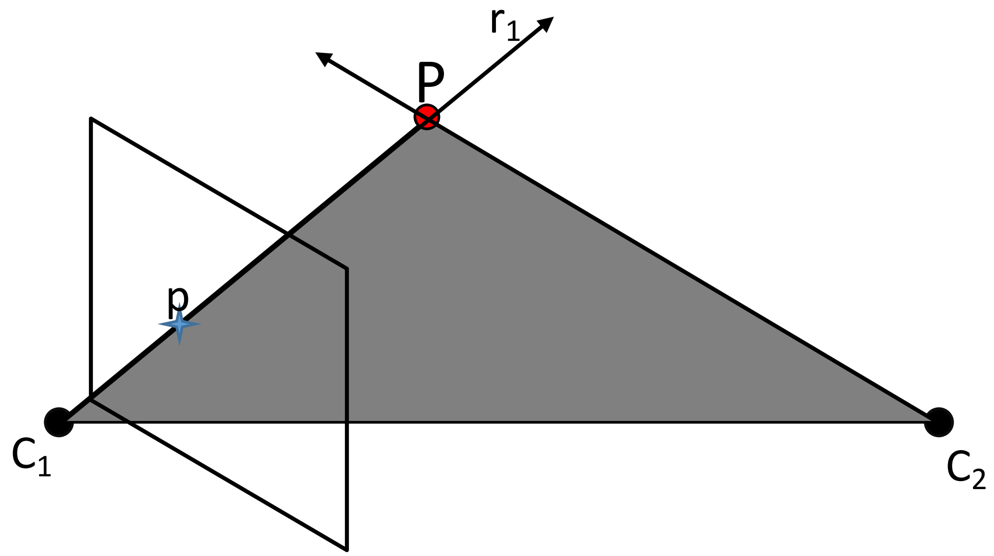
# Epipolar geometry

- The relations between views as appeared in the image
- **In 3D:** Ray $r_1$ intersects P



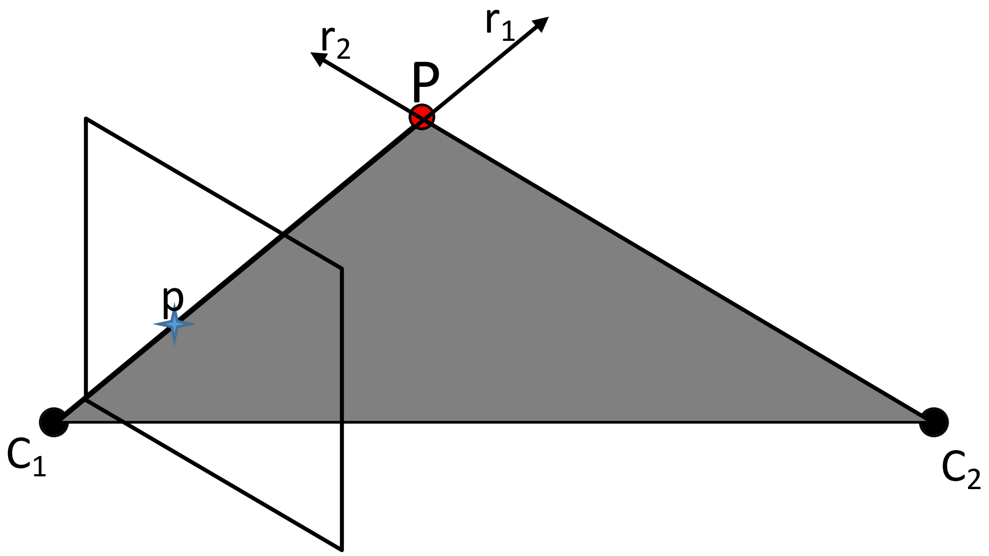
# Epipolar geometry

- The relations between views as appeared in the image
- **In 3D:** Ray $r_1$ intersects P **In image:** P is projected to p



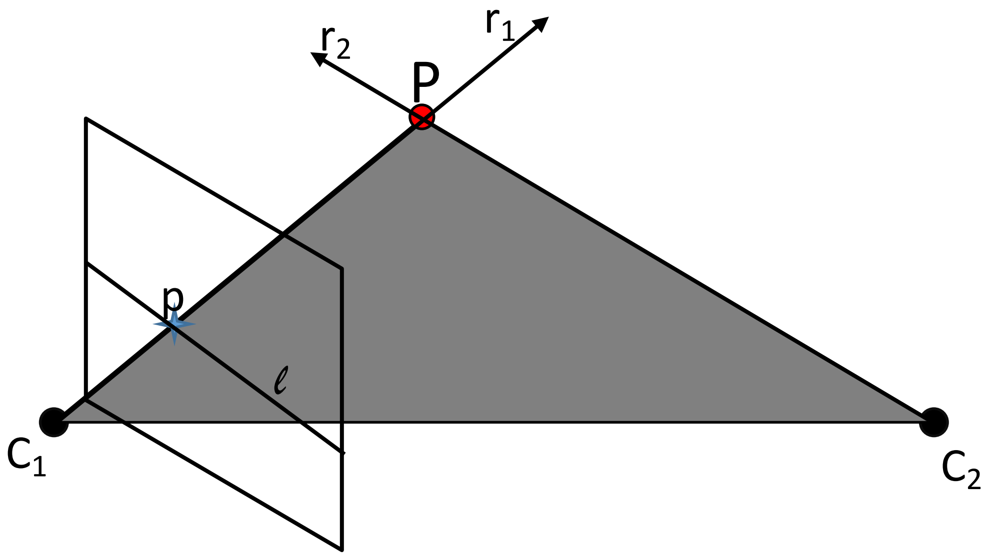
# Epipolar geometry

- The relations between views as appeared in the image
- **In 3D:** Ray $r_1$ intersects P **In image:** P is projected to p
- **In 3D:** Ray $r_2$ intersect P



# Epipolar geometry

- The relations between views as appeared in the image
- **In 3D:** Ray $r_1$ intersects P **In image:** P is projected to p
- **In 3D:** Ray $r_2$ intersect P **In Image:** $r_2$ is projected to line $\ell$



# Epipolar geometry

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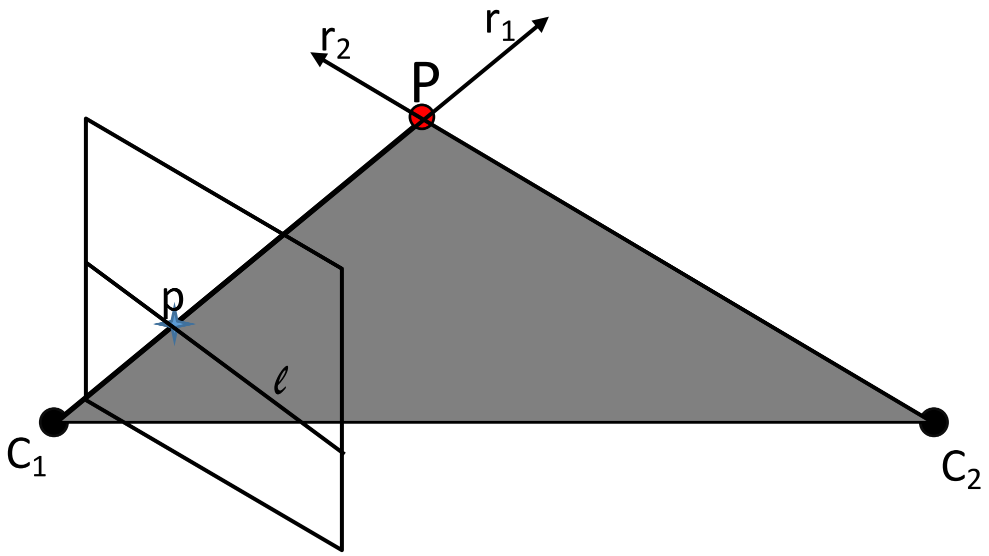
- **In 3D:** Ray $r_1$ intersects P  **In image:** P is projected to p
- **In 3D:** Ray $r_2$ intersect P  **In Image:** $r_2$ is projected to line $\ell$
- $\ell$ is the **epipolar line**



# Epipolar geometry

- The relations between views as appeared in the image
- **In 3D:** Ray $r_1$ intersects P **In image:** P is projected to p
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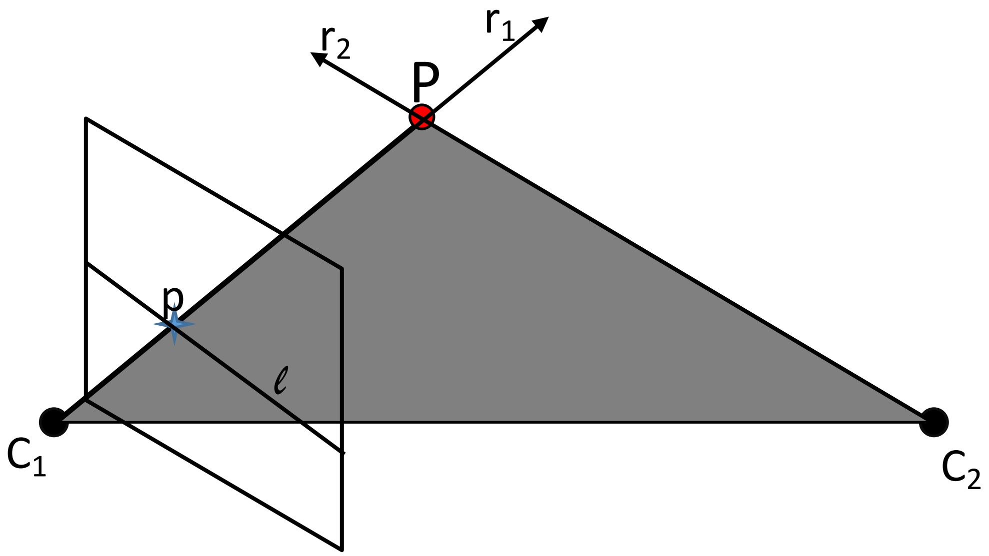
- $\ell$ is the **epipolar line**
- $\ell$ intersects p in image 1

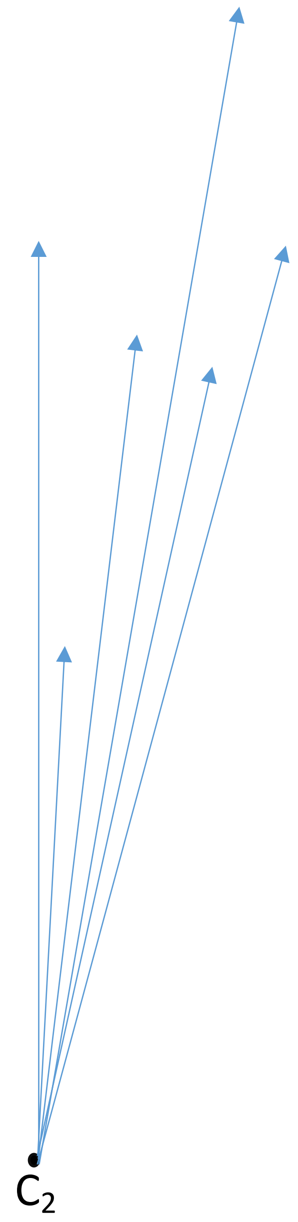


# Epipolar geometry

- What epipolar line is good for?
- If we search for a match for $p_1$
  - It will be on the epipolar line $\ell$
- If we suspect the match is wrong
  - We can decide it is an outlier if it's not on $\ell$

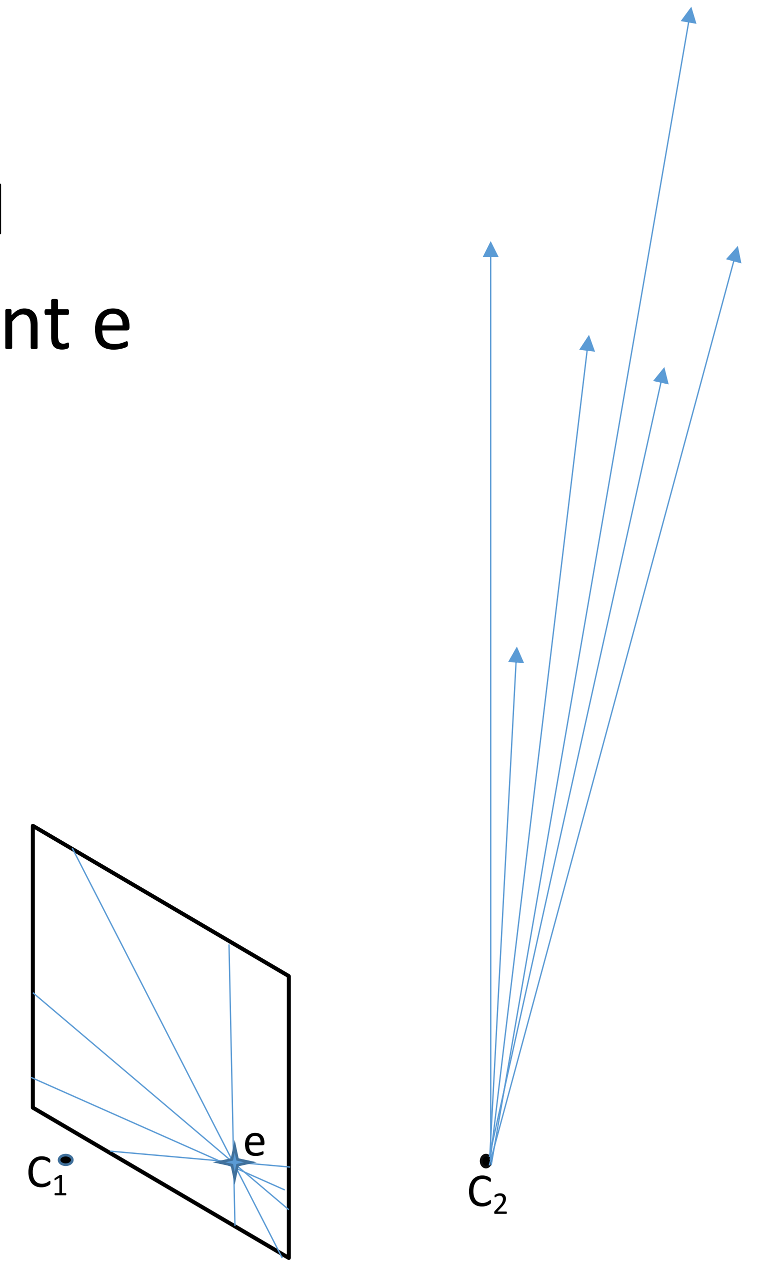
# Epipolar geometry

- **In 3D:** All 3D rays coming from $C_2$ create a pencil



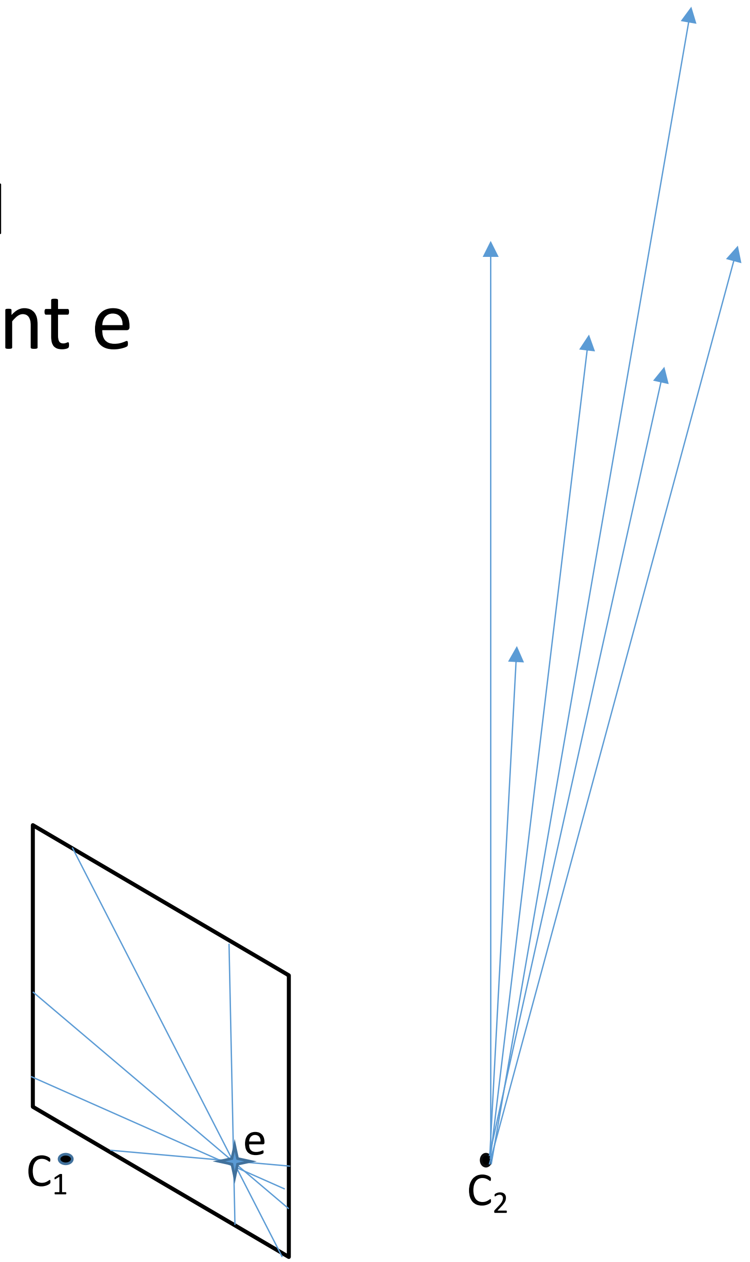
# Epipolar geometry

- **In 3D:** All 3D rays coming from $C_2$ create a pencil
- **In image**: all epipolar lines intersect at point e



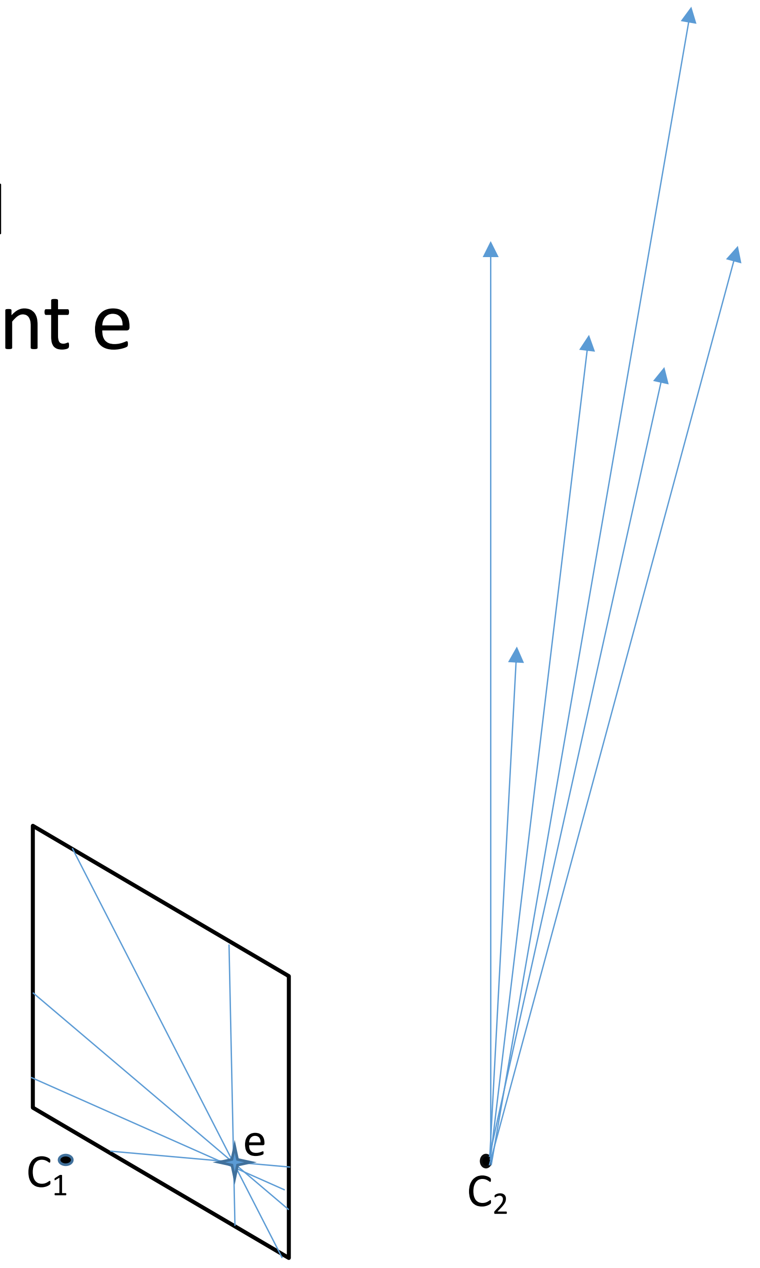
# Epipolar geometry

- **In 3D:** All 3D rays coming from $C_2$ create a pencil
- **In image**: all epipolar lines intersect at point e
- e is the **epipole**



# Epipolar geometry

- **In 3D:** All 3D rays coming from $C_2$ create a pencil
- **In image**: all epipolar lines intersect at point e
- e is the **epipole**
- e is the projection of $c_2$ 3D location



# Epipolar geometry

- How can we calculate the epipolar line?
  - Project pixel to ray, rotate ray, project ray to line.
- Using the Fundamental Matrix F:
  - Algebraic representation of the epipolar geometry
  - Mapping from points in Im1 to lines in Im2:

    $l^T_{1x3} = F_{3x3}x_{3x1} \quad \Leftrightarrow \quad x_2^T F x_1 = 0$
- Properties:
  - The null space is the Epipole:

$$\forall x: x^T F e = 0$$
$$Fe = 0$$

  - F has 7 Degrees Of Freedom
    - It has rank 2
    - It is determined up to scale

# Epipolar geometry

$$\mathrm{Y} = \mathrm{RX} + \mathrm{t}$$

$$[\mathrm{t}]_\times \mathrm{Y} = [\mathrm{t}]_\times (\mathrm{RX} + \mathrm{t})$$

$$[\mathrm{t}]_\times \mathrm{Y} = [\mathrm{t}]_\times \mathrm{RX}$$

$$\mathrm{Y}^\intercal [\mathrm{t}]_\times \mathrm{Y} = \mathrm{Y}^\intercal [\mathrm{t}]_\times \mathrm{RX}$$

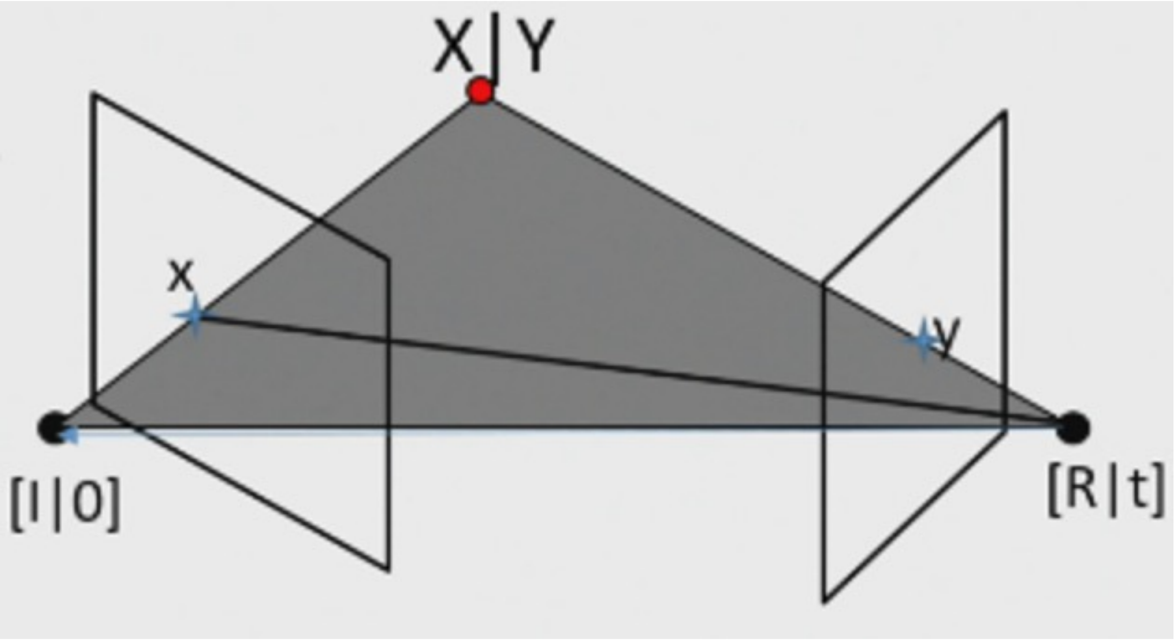
$$\mathrm{Y}^\intercal [\mathrm{t}]_\times \mathrm{RX} = 0$$

$$q = K_1 \mathrm{X}, p = K_2 \mathrm{Y}$$

$$\mathrm{X} = \mathrm{K}_1^{-1} q, \mathrm{Y} = \mathrm{K}_2^{-1} p$$

$$p^\intercal \mathrm{K}_2^{-\intercal} [\mathrm{t}]_\times \mathrm{R} \mathrm{K}_1^{-1} q = 0$$

$$p^\intercal F q = 0$$



# Epipolar geometry

- If we don’t know $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{R}$, or $\mathbf{t}$, can we still estimate $\mathbf{F}$?
- Yes, given enough correspondences.
- **Many algorithms:**
    - Linear (the normalized 8-point algorithm)
    - Minimal (7-point)
    - Robust (RANSAC)
    - Non-linear refinement (MLE, Algebraic minimization)
- We use 8-point algorithm
    - Although it’s inaccurate
    - Because it’s fast

# Estimating F: 8-point algorithm

- The fundamental matrix F is defined by

$$\mathbf{x}'^{\mathrm{T}}\mathbf{F}\mathbf{x} = 0$$

for any pair of matches x and x' in two images.

- Let x=$(u,v,1)^{\mathrm{T}}$ and x'=$(u',v',1)^{\mathrm{T}}$,

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

each match gives a linear equation

$$uu' f_{11} + vu' f_{12} + u' f_{13} + uv' f_{21} + vv' f_{22} + v' f_{23} + uf_{31} + vf_{32} + f_{33} = 0$$

# 8-point algorithm

$$\begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & v_nu_n' & u_n' & u_nv_n' & v_nv_n' & v_n' & u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

# 8-point algorithm

$$\begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & v_nu_n' & u_n' & u_nv_n' & v_nv_n' & v_n' & u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

$$\mathbf{A}\,\mathbf{f} = \mathbf{0}$$

# 8-point algorithm

- We solve it as before, using SDV:

  Af = 0 s.t. $\|\mathrm{f}\| = 1$

  A = U$\Sigma$V$^T$

  f = $V^T{}_N$

- $f_{9x1}$ => $F_{3x3}$
- Same method if N>8
  - We minimize $\|\mathrm{Af}\|$ s.t. $\|\mathrm{f}\| = 1$

# 8-point algorithm

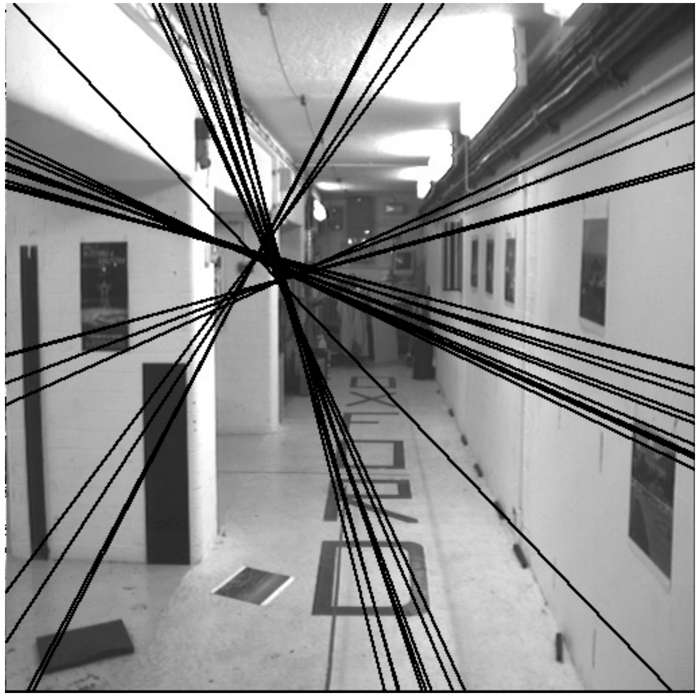
- $\mathbf{F}_{3x3}$ should have rank 2. It doesn't.
- To enforce that **F** is of rank 2, F is replaced by $F'_{3x3}$ that minimizes $\|\mathbf{F}-\mathbf{F}'\|$ .

- This too is achieved by SVD. Let $\mathbf{F}=\mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$, where

$$\Sigma=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix} \qquad \Sigma'=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

then $\mathbf{F}'=\mathbf{U}\Sigma'\mathbf{V}^{\mathrm{T}}$ is the solution.

# 8-point algorithm



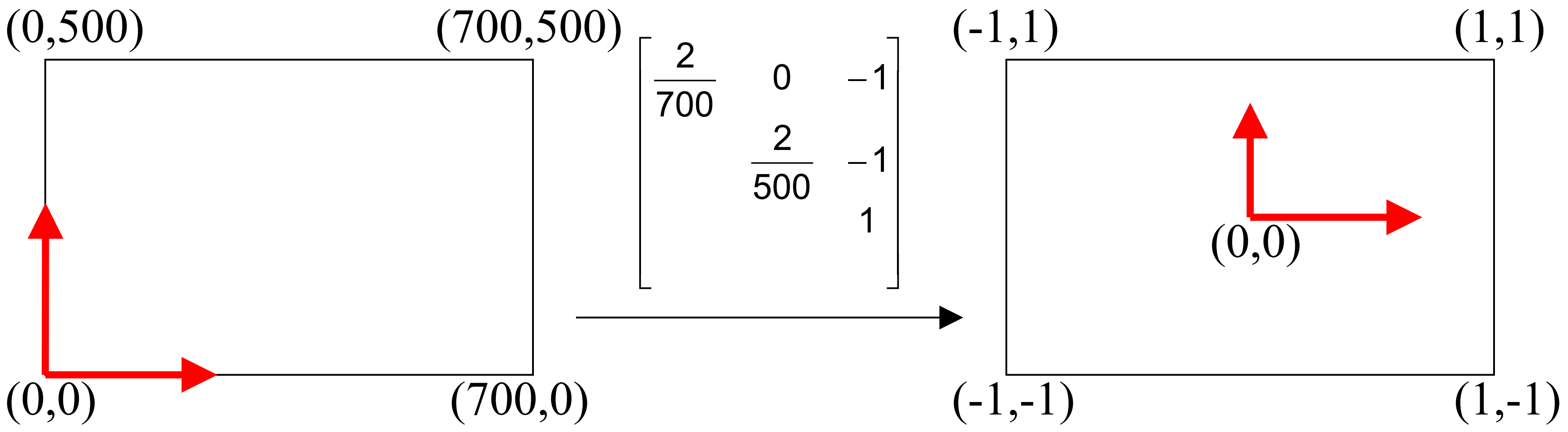
# 8-point algorithm

$$\begin{array}{c}\left[\begin{array}{ccccccccc} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & v_nu_n' & u_n' & u_nv_n' & v_nv_n' & v_n' & u_n & v_n & 1 \end{array}\right] \\ \begin{array}{ccccccccc} \sim 10000 & \sim 10000 & \sim 100 & \sim 10000 & \sim 10000 & \sim 100 & \sim 100 & \sim 100 & 1 \end{array}\end{array} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

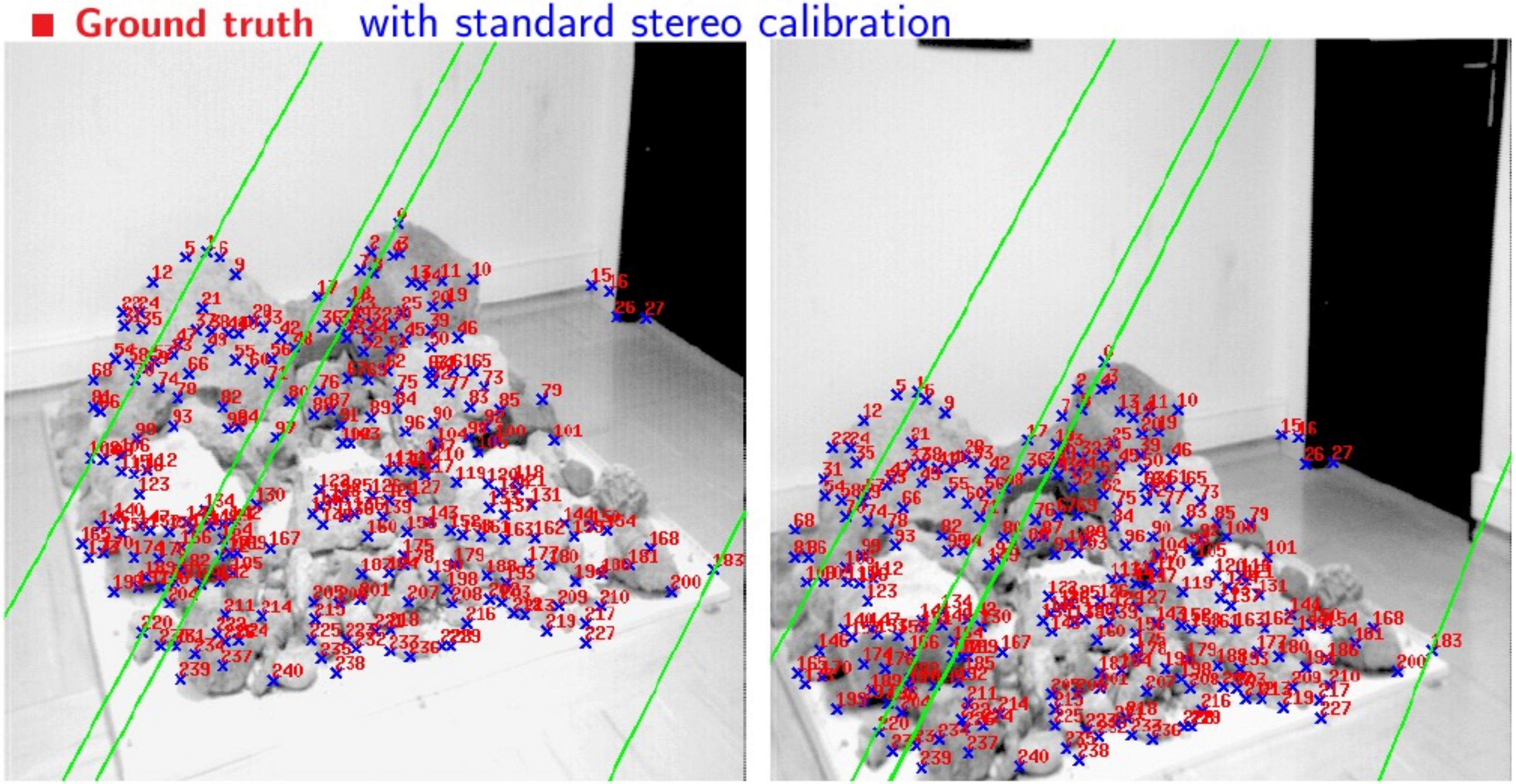
Orders of magnitude difference
between column of data matrix
$\rightarrow$ least-squares yields poor results

# 8-point algorithm

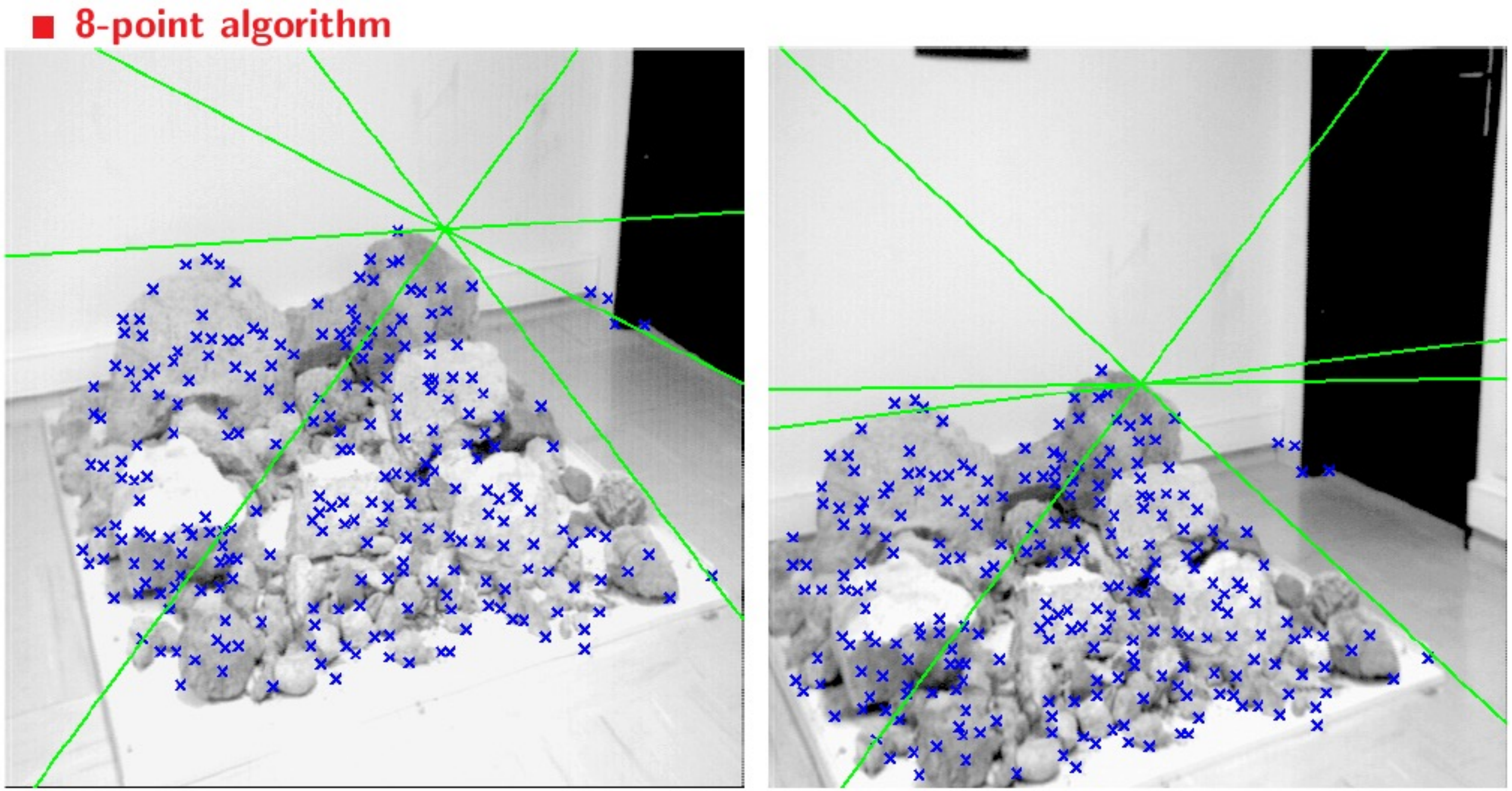
- normalized least squares yields good results
- Transform image to ~[-1,1]x[-1,1]



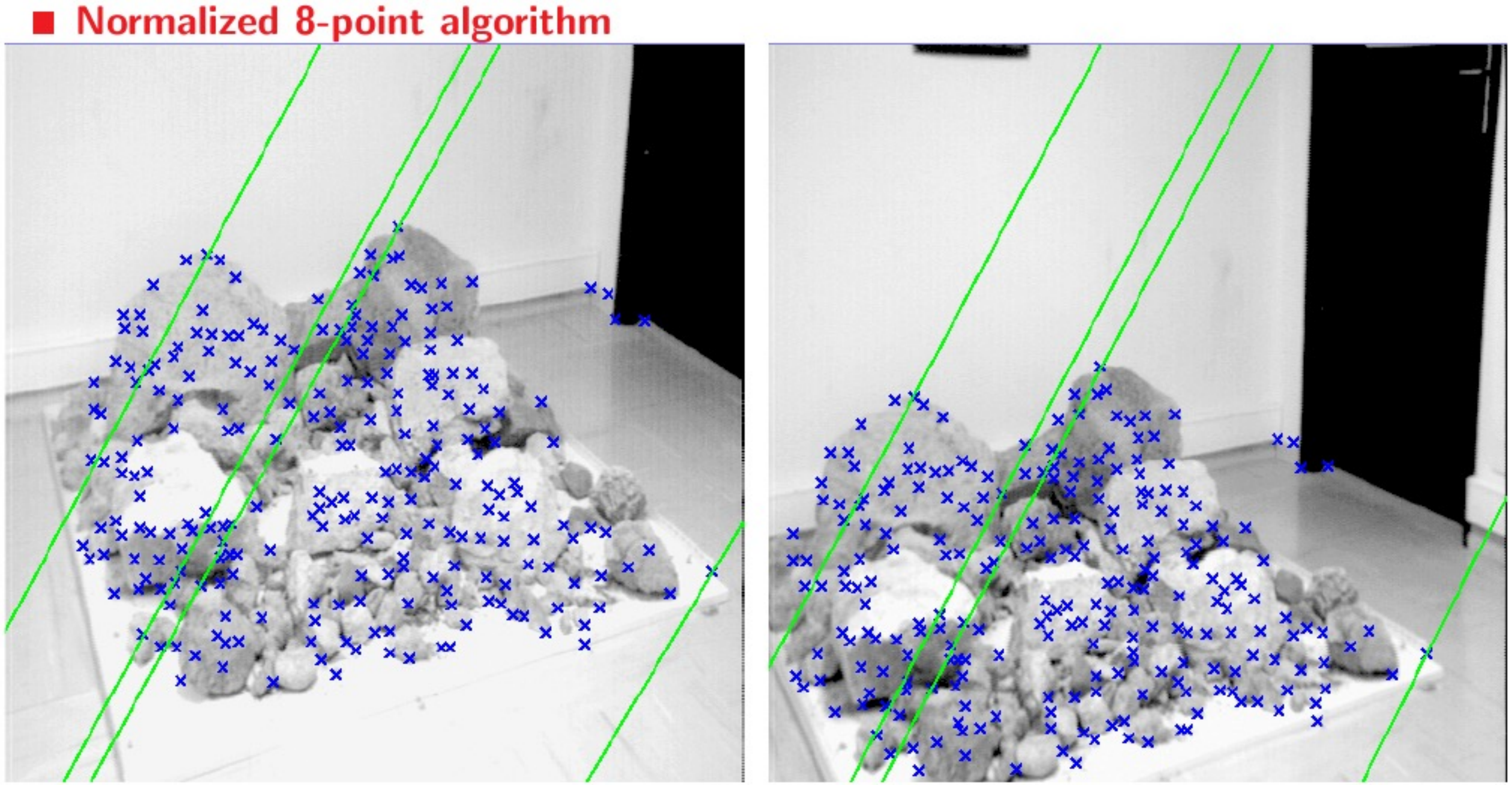
# Results (ground truth)



# Results (8-point algorithm)

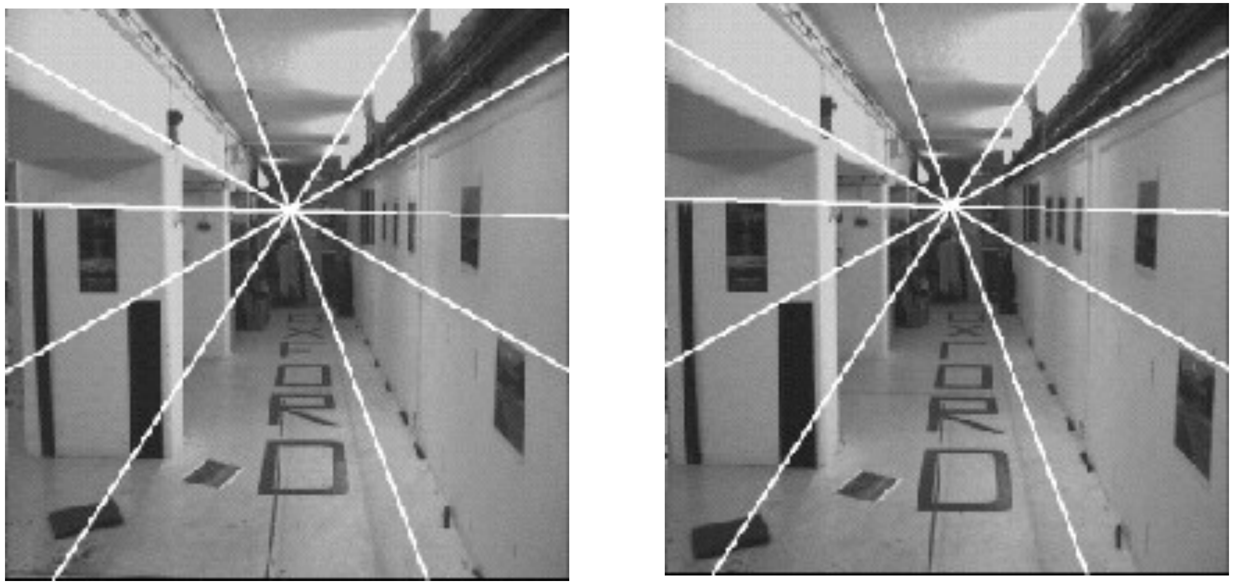


# Results (normalized 8-point algorithm)



# Epipolar geometry

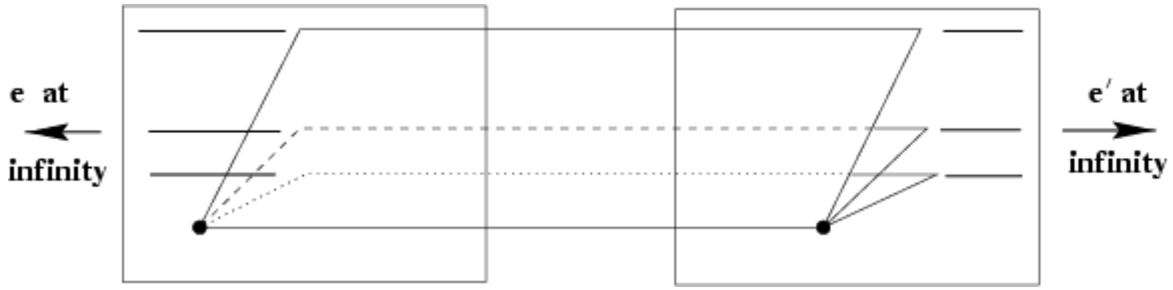
**Example: forward motion**

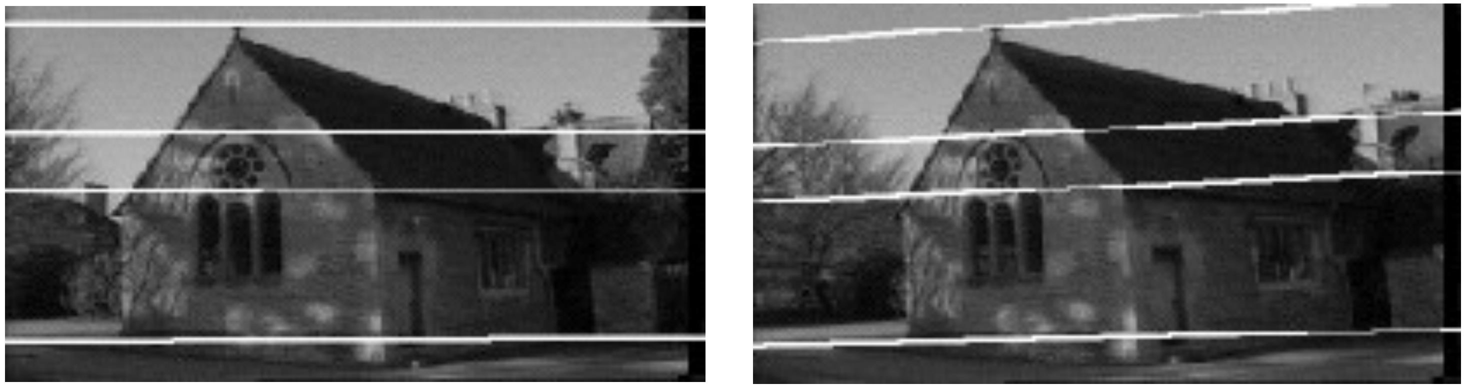


Courtesy of Marc Pollefeys

# Epipolar geometry

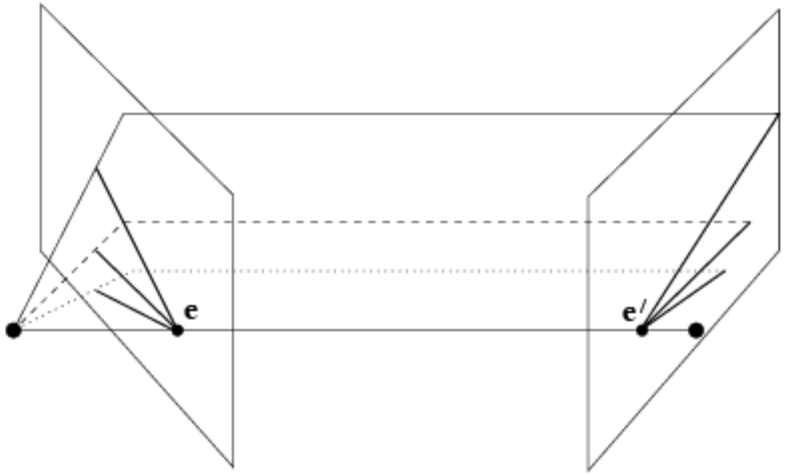
**Example: motion parallel with image plane**

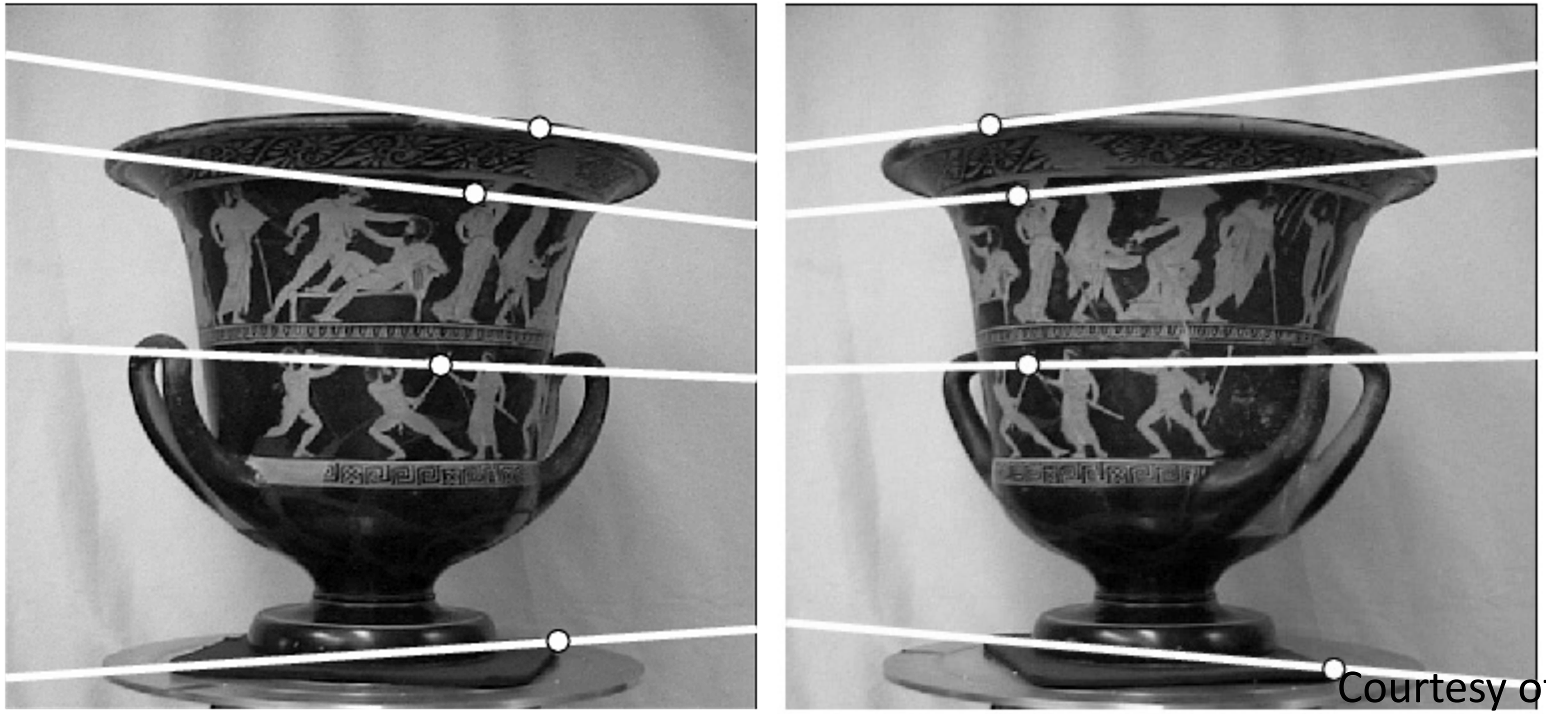


Courtesy of Marc Pollefeys

# Example: converging cameras



Courtesy of Marc Pollefeys

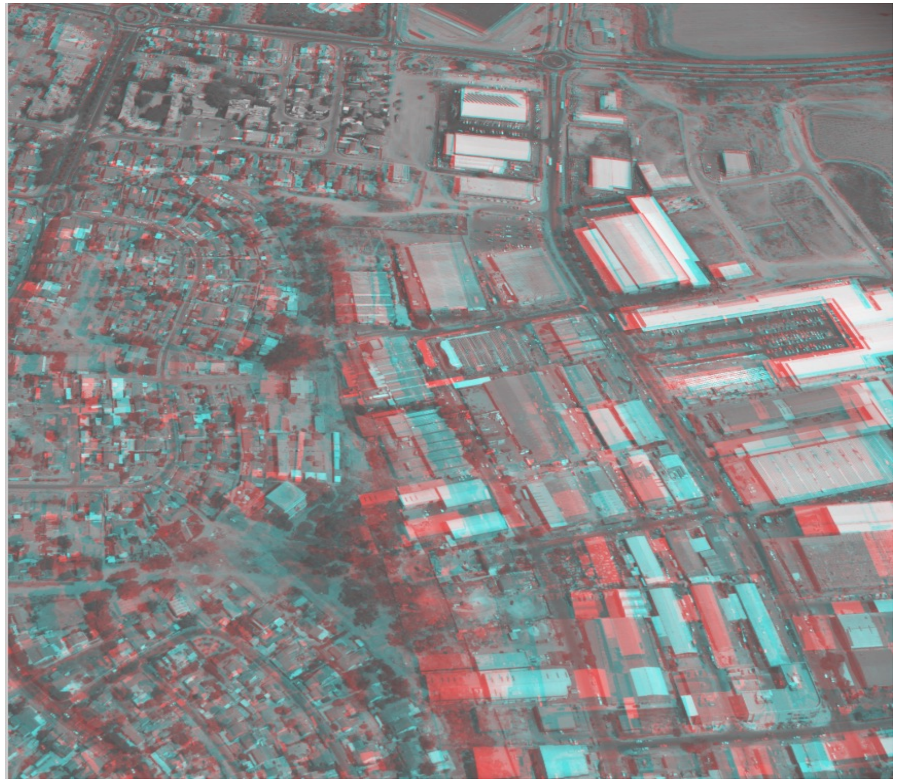
# Epipolar geometry

- After stereo rectification:

$$F\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = [0,1,-y]$$

So F is from the shape:

$$F = \begin{bmatrix} & & \\ & & 1 \\ & -1 & \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\times} = [t]_{\times}$$



# RANSAC

- RANdom SAmple Consensus
- Problem:
  - All inliers obey some model
  - But there are some unknown outliers.
- Example: inliers are on a curve
- Chicken and egg situation:
  - If we had the curve, we could spot the outliers
  - If we knew the inliers, we could estimate the curve
- Key to solution:
  - The model can be estimated using a small set


Courtesy of: ET Wales

# RANSAC

- Algorithm:
  1. Repeat:
     1. Sample a minimal set
     2. Estimate a model
     3. Check how many points obey the model
  2. Choose model with maximal #points
  3. Repeat:
     1. Estimate model from all inliers
     2. Calc inliers of new model
- Output: inliers, outliers, and model

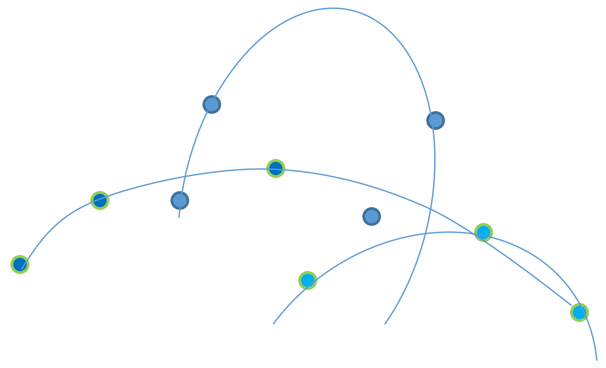
# RANSAC

- Example:
- #inliers = 3. max #inliers = 3
- #inliers = 5. **max #inliers = 5**
- #inliers = 3. max #inliers = 5

.

.

- Output: #inliers = 5



# RANSAC

- When to stop the first loop of RANSAC?
- Goal: one sample that will have only inliers, with high prob p.
- Prob of being an outlier: $\epsilon$
- P(being an inlier) = $1-\epsilon$
- P(all inliers-sample) = $(1-\epsilon)^s$
- P(bad sample)= $1-(1-\epsilon)^s$
- P(All samples are bad) = $(1-(1-\epsilon)^s)^I$
- We wish it to be small: $(1-(1-\varepsilon)^s)^I < 1-p$

$$\log(1-(1-\varepsilon)^s)^I < \log(1-p)$$

$$I\log(1-(1-\varepsilon)^s) < \log(1-p)$$

$$I > \log(1-p)/\log(1-(1-\varepsilon)^s)$$

# RANSAC

- This can get high:

| s \ ϵ | 25% | 50% | 60% | 70% | 80% | 85% |
|---|---|---|---|---|---|---|
| **2** | 6 | 16 | 26 | 49 | 113 | 202 |
| **3** | 8 | 34 | 70 | 168 | 573 | 1362 |
| **7** | 33 | 588 | 2808 | 21055 | 2.5E05 | 2.6E06 |

- What if we don't know $\varepsilon$?
- We can estimate it online:
  - We calc #inliers at each sample
  - This gives an ever-decreasing upper-bound on $\varepsilon$
  - Hence the needed iteration number $\mathrm{I}$ is also decreasing

# RANSAC

- Which models are used with RANSAC?
- 2D points matching:
  - Fundamental matrix
  - Homography transformation
  - Essential Matrix
  - Trifocal Tensor
- 3D points:
  - Point cloud registration
  - Perspective-n-Point (PNP)
  - Plane fitting
  - Curve fitting

# RANSAC

- Limitations of RANSAC with FM:
  - Efficiency: unknown
    - because outlier ratio ε in unknown
  - Accuracy
    - Even good sample may give a bad model
    - Sensitive to inlier threshold
  - Degeneracy
    - The plain+paralax problem
  - Many tricks and extensions:
    - PROSAC
    - USAC