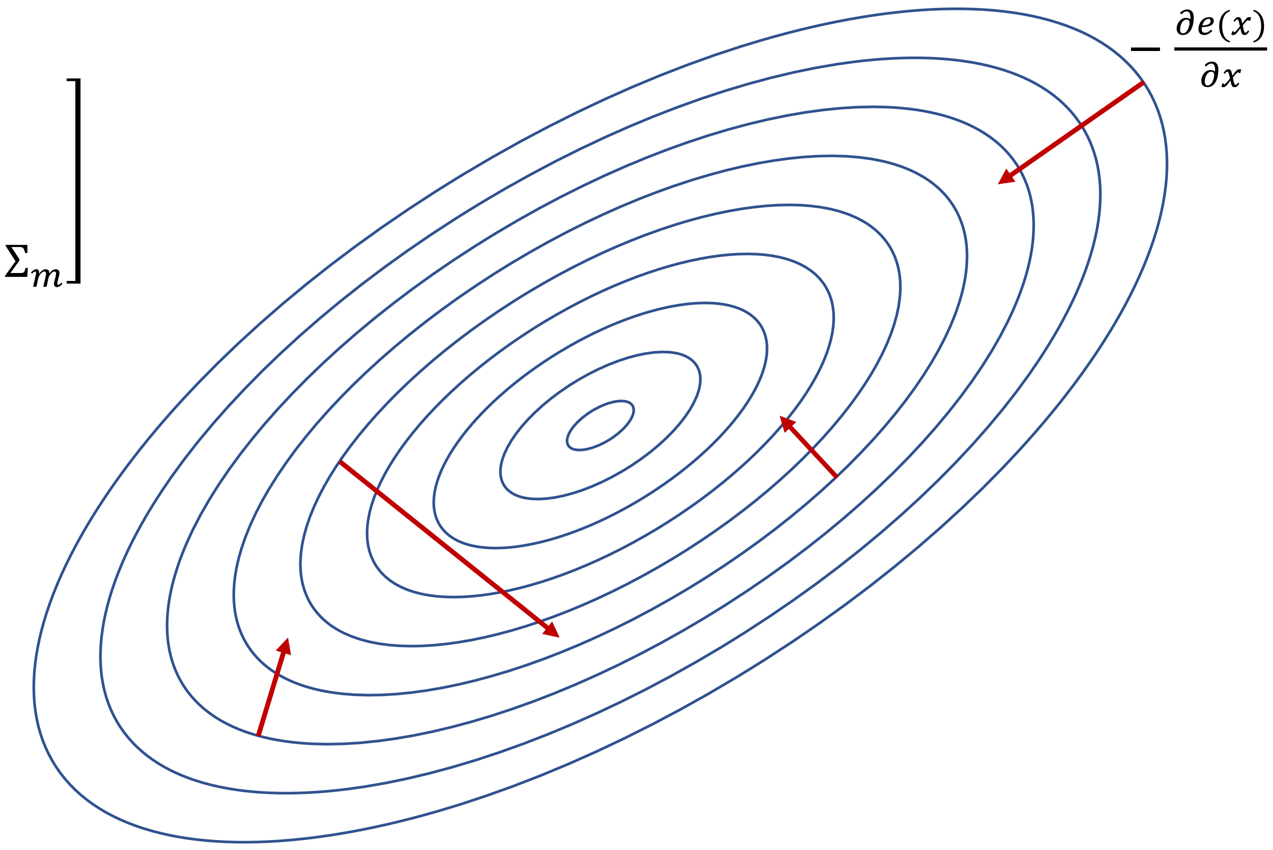
Bundle Adjustment

David Arnon

Bundle Adjustment
Gradient Decent

- Error function $e(x) = \|f(x) - z\|_\Sigma^2$:

$$\Sigma = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & \ddots & \\ & & & \Sigma_m \end{bmatrix}$$



Bundle Adjustment

Linear Approximation

- $e(x) = \frac{1}{2}(f(x) - z)^T \Sigma^{-1} (f(x) - z)$

- $\left(\frac{\partial e(x)}{\partial x}\right)^T = J(x)^T \Sigma^{-1} (f(x) - z) = J(x)^T \Sigma^{-1} \Delta z$

$$\boxed{\Delta z \doteq f(x) - z}$$

- $e(x + \Delta x) \cong e(x) + \frac{\partial e(x)}{\partial x} \Delta x$

$$\boxed{\Delta x = -\frac{1}{\lambda}\left(\frac{\partial e(x)}{\partial x}\right)^T}$$

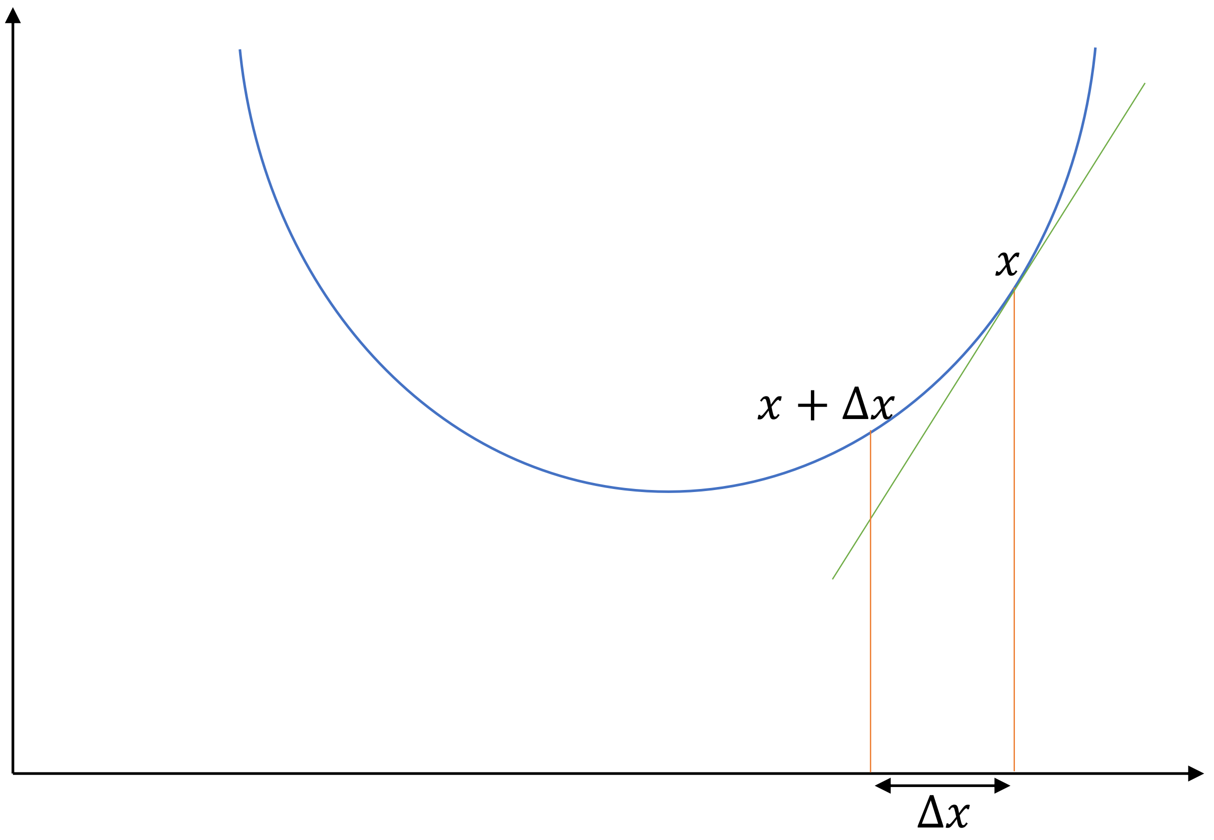
- $e(x + \Delta x) \cong e(x) - \frac{1}{\lambda}\left\|\frac{\partial e(x)}{\partial x}\right\|_2^2 < e(x)$

$$\boxed{g \doteq J(x)^T \Sigma^{-1} \Delta z}$$

- $\Delta x = -\frac{1}{\lambda} J(x)^T \Sigma^{-1} \Delta z$

$$\boxed{\Delta x = -\frac{1}{\lambda} g}$$

Bundle Adjustment
Gradient Decent



Bundle Adjustment
Gauss – Newton Algorithm

- Set starting point
- Linearize the measurement function
- Solve linear least squares problem
- Iterate

- Quadratic Approximation!

Bundle Adjustment
Quadratic Approximation

- $argmin_{\Delta x} \|f(x_i + \Delta x) - z\|_\Sigma^2 \quad \cong$
- $argmin_{\Delta x} \|f(x_i) + J(x_i)\Delta x - z\|_\Sigma^2 \quad =$
- $argmin_{\Delta x} \|J(x_i)\Delta x + f(x_i) - z\|_\Sigma^2 \quad =$
- $argmin_{\Delta x} \|J(x_i)\Delta x + \Delta z_i\|_\Sigma^2 \quad =$
- $argmin_{\Delta x} \left\|\Sigma^{-½} J(x_i)\Delta x + \Sigma^{-½}\Delta z_i\right\|_2^2$
- $J(x_i)^T \Sigma^{-1} J(x_i)\Delta x = -J(x_i)^T \Sigma^{-1} \Delta z_i$
  - $J(x_i)^T \underbrace{\Sigma^{-½T}\Sigma^{-½}}_{\Sigma^{-1}} J(x_i)\Delta x = -J(x_i)^T \underbrace{\Sigma^{-½T}\Sigma^{-½}}_{\Sigma^{-1}} \Delta z_i$

$$\boxed{H\Delta x = -g}$$

$$\boxed{\begin{array}{c} H \doteq J(x_i)^T \Sigma^{-1} J(x_i) \\ g \doteq J(x_i)^T \Sigma^{-1} \Delta z_i \end{array}}$$

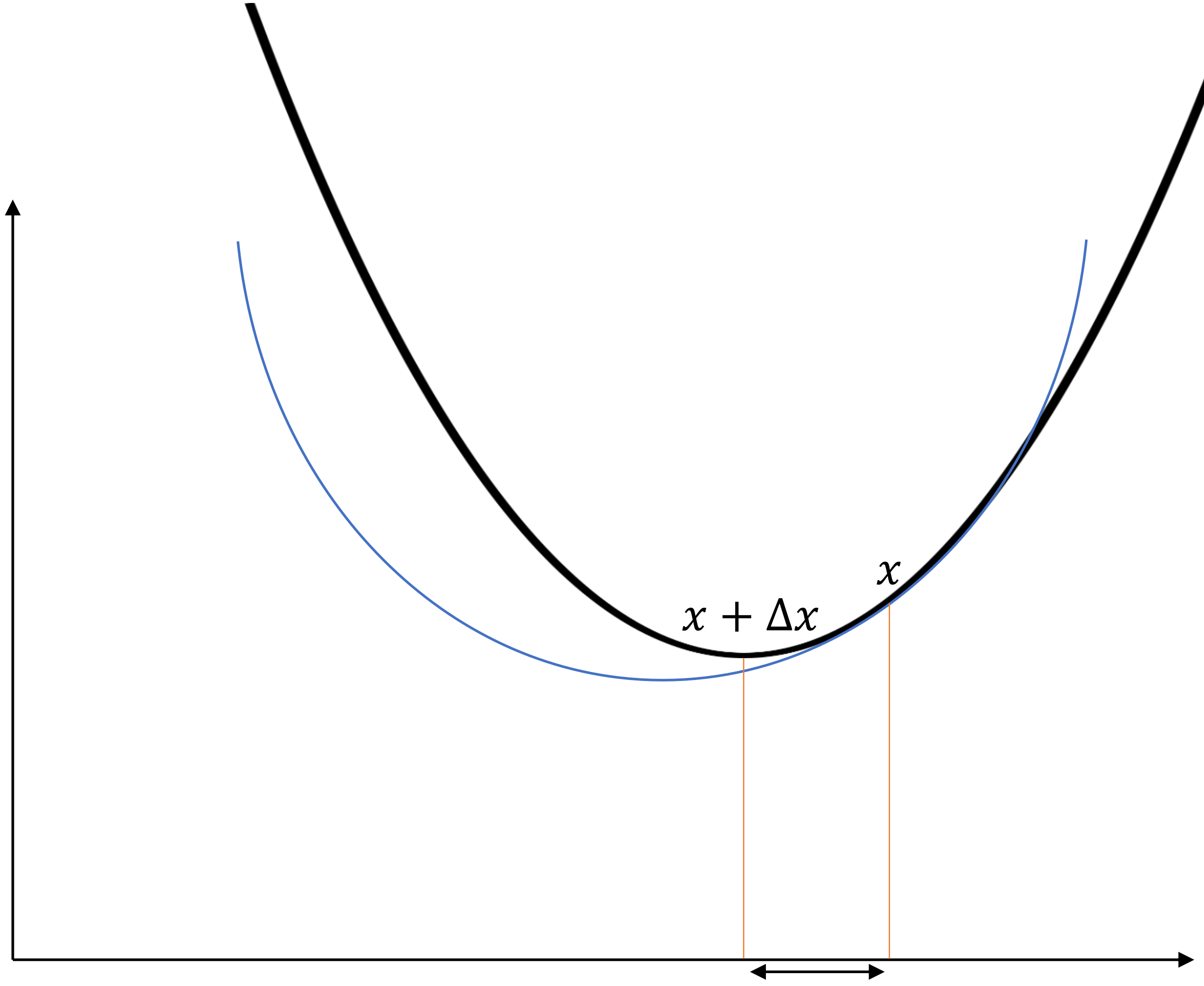
$$\boxed{\Delta z_i \doteq f(x_i) - z}$$

$$\boxed{\begin{array}{c} \Sigma = (\Sigma^{½})(\Sigma^{½})^T \\ \Sigma^{-1} = \Sigma^{-½T}\Sigma^{-½} \end{array}}$$

$$\boxed{argmin_x \|Ax - b\|_2^2 \quad \Longrightarrow \quad A^T A x = A^T b}$$

Bundle Adjustment

Quadratic Approximation



Bundle Adjustment
Gauss – Newton

- Converges in one iteration for quadratic functions

- For general functions, the asymptotic convergence is quadratic 

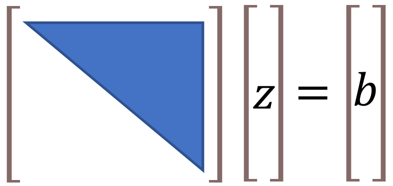
- Inverting H is expensive 

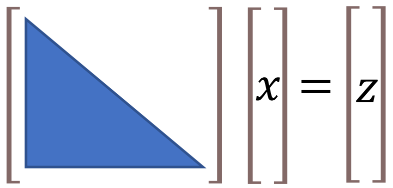
Bundle Adjustment

Cholesky Decomposition

- $Hx = b$
- $C \underbrace{C^T x}_{z} = b$
- $Cz = b$
- $C^T x = z$

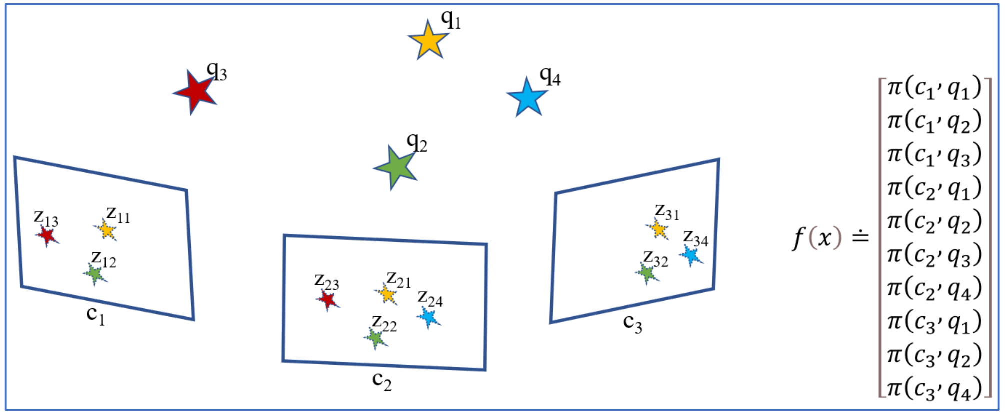
$$H = CC^T$$

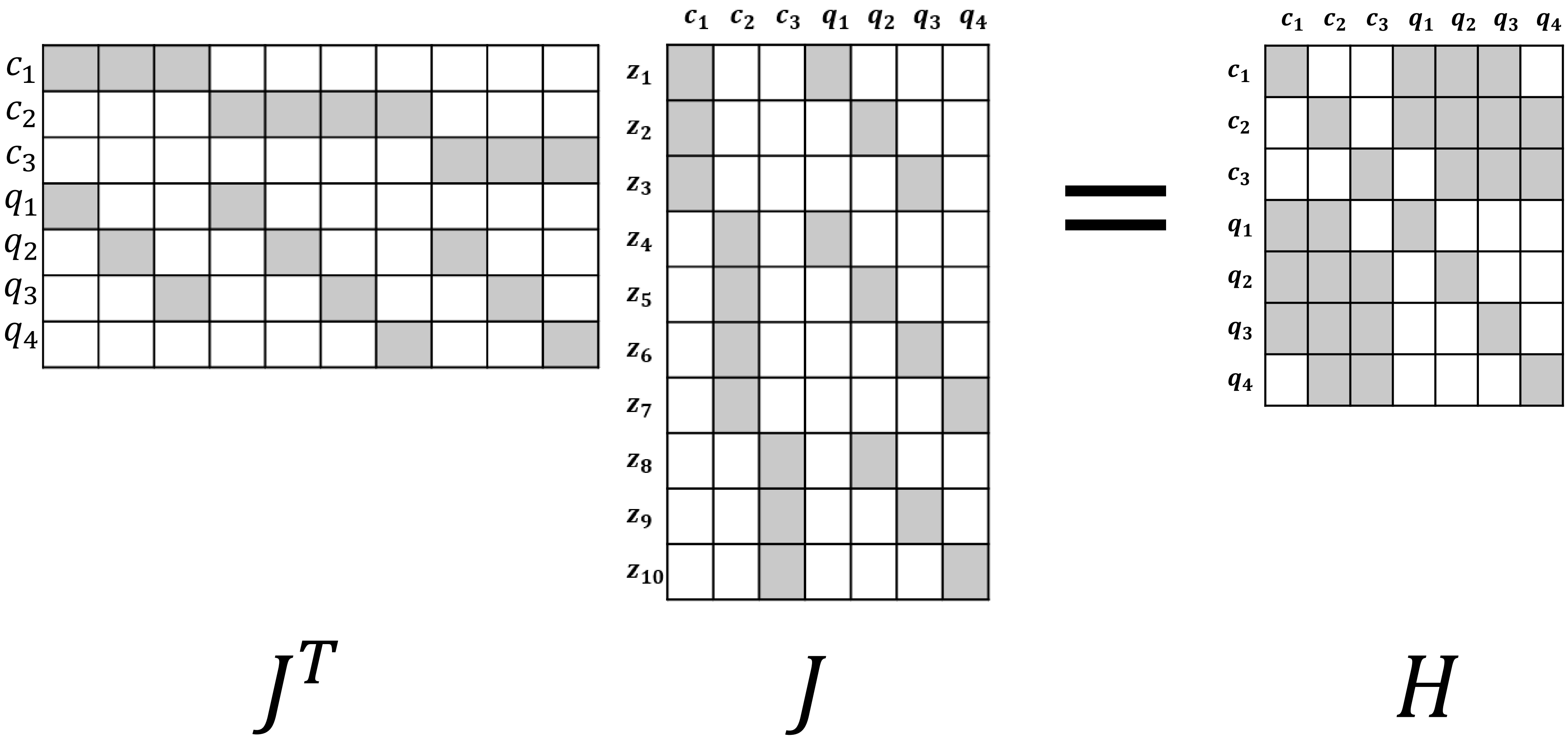




Bundle Adjustment

Sparsity





Bundle Adjustment
Uncertainty

- H is the information matrix
  - Inverse of the covariance matrix of the estimated Δx
  - Approximation of the hessian -
    second-order partial derivatives matrix

$$H^{-1} = \begin{bmatrix} \Sigma_1 & & & * \\ & \Sigma_2 & & \\ & & \ddots & \\ * & & & \Sigma_p \end{bmatrix}$$

- Can be used to estimate the uncertainty of the result
  - Marginal covariances

- Conditioning $p(x_j | x_i)$
  It is possible to estimate the relative uncertainty between x_j and x_i
  - erase row and column i
  - invert and use diagonal block j

Gauss-Newton Example

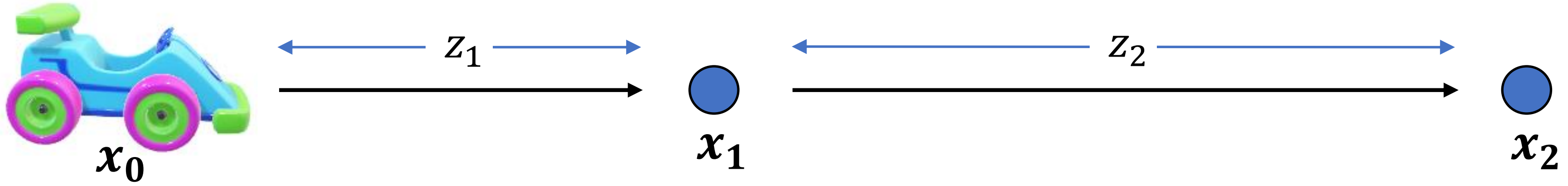
David Arnon

Gauss-Newton
Example

- $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \Sigma = \begin{bmatrix} ½ & \\ & ¼ \end{bmatrix} \quad \Rightarrow \quad \Sigma^{-1} = \begin{bmatrix} 2 & \\ & 4 \end{bmatrix}$

- $f\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_0 \\ x_2 - x_1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

- $\mathrm{J} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$



Gauss-Newton
Example

- $$-g = -\begin{bmatrix}-1 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & \\ & 4\end{bmatrix}\left(\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}-2\\-6\\8\end{bmatrix}$$

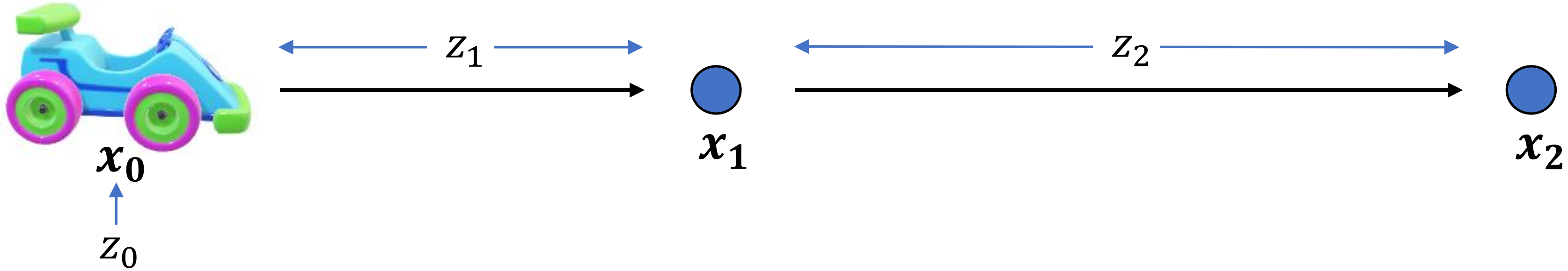
- $$H = \begin{bmatrix}-1 & 0\\ 1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & \\ & 4\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} = \begin{array}{ccc} \multicolumn{3}{c}{\begin{bmatrix}2 & -2 & 0\\ -2 & 6 & -4\\ 0 & -4 & 4\end{bmatrix}} \\ \hline 0 & 0 & 0\end{array}$$

Singular!

Gauss-Newton

Take 2

- $z = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \Sigma = \begin{bmatrix} 1 & & \\ & ½ & \\ & & ¼ \end{bmatrix} \quad \Rightarrow \quad \Sigma^{-1} = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 4 \end{bmatrix}$
- $f\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 - x_0 \\ x_2 - x_1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$
- $\mathrm{J} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$



Gauss-Newton

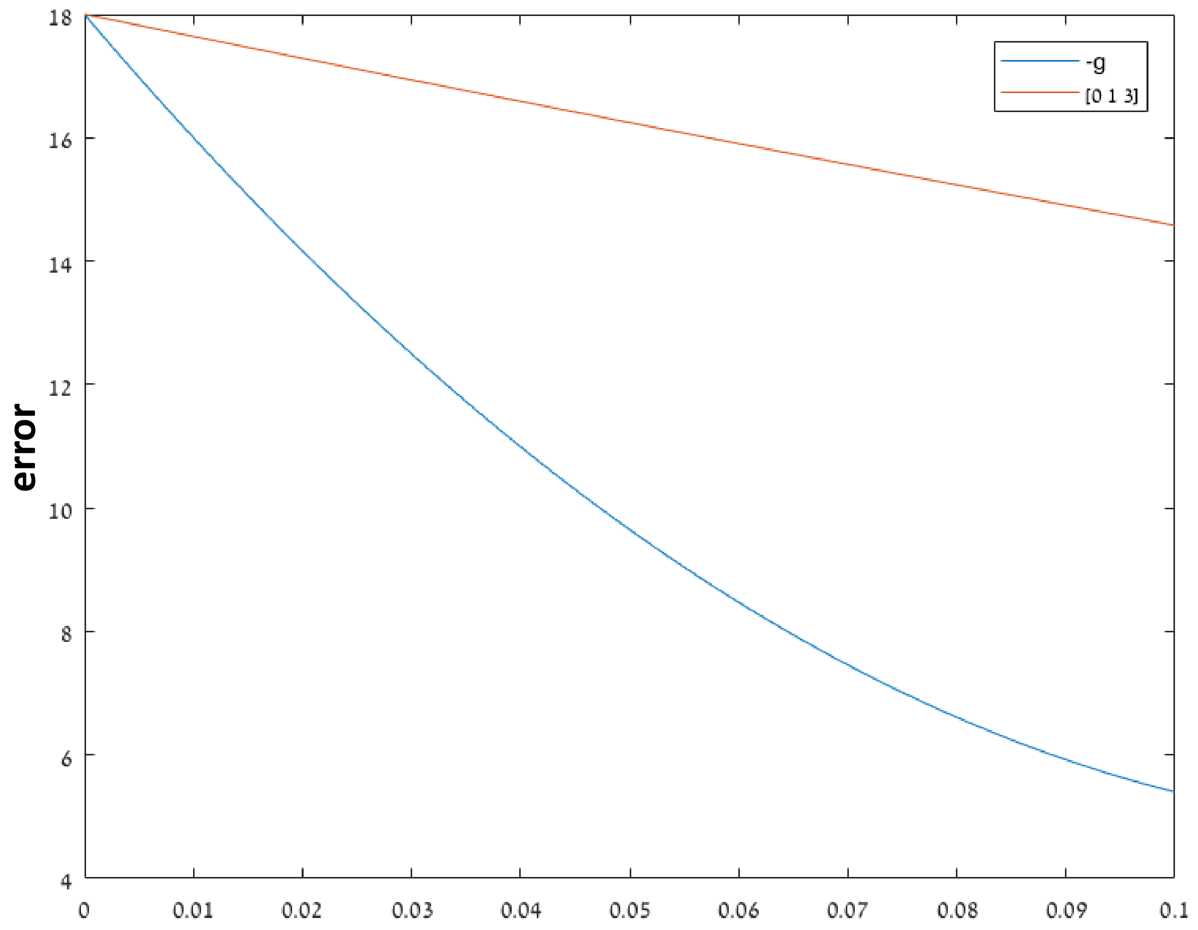
Take 2

- $$-g = -\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & & \\ & 2 & \\ & & 4\end{bmatrix}\left(\begin{bmatrix}0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\2\end{bmatrix}\right) = \begin{bmatrix}-2\\-6\\8\end{bmatrix}$$

- $$H = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & & \\ & 2 & \\ & & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}3 & -2 & 0\\ -2 & 6 & -4\\ 0 & -4 & 4\end{bmatrix}$$

- $$\Delta x = -H^{-1}g = \begin{bmatrix}1 & 1 & 1\\ 1 & 1\tfrac{1}{2} & 1\tfrac{1}{2}\\ 1 & 1\tfrac{1}{2} & 1\tfrac{3}{4}\end{bmatrix}\begin{bmatrix}-2\\-6\\8\end{bmatrix} = \begin{bmatrix}0\\1\\3\end{bmatrix}$$

Gauss-Newton
Example



Gauss-Newton
Example