Worksheet 3: Numerical integration

Create a folder having your ID number and let this be your working directory. After you solve the problem, create a zipped (compressed) archive of your work and upload this to WeLearn.

Problems

1. **Trapezoidal rule**: We know from the last class that the trapezoidal rule gives exact result when $f(x)$ is a polynomial of order 1. The rule can be summarized by the formula below:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(a)+f(b)+2\sum_{i=1}^{n-1} f(a+ih)\right], \quad \text{where, } h=\frac{b-a}{n}.$$

where, n is the number of equal-width strips into which the interval $[a,b]$ is divided. Write a program which uses the above rule to evaluate

$$\int_0^1 f(x)dx \quad \text{and, } f(x)=x^9$$

for $n = 10, 100, \ldots, 100,000$.

- Modify the function to $f(x) = 2x$ and verify that the rule gives exact result for polynomials of order 1. After verification, switch $f(x)$ to x^9.
- The program should print `log10` of h, the computed integral, and the `log10` of the absolute difference between the exact and the computed integrals.
- The estimated error of trapezoidal method is about $\sim k_1h^3$. Since there are N such strips, the accumulated error is $\sim N \times k_1h^3 \sim h^2$.
- To verify the above (please do NOT copy-paste):
  - Start `gnuplot`. Define a fitting function `f(x) = a*x + b`.
  - Fit the data (log of error vs log of h) using `fit f(x) 'output.dat' u 1:3 via a, b`.
  - Plot the error and the fit using `plot 'output.dat' u 1:3 with point pointtype 7 pointsize 2, f(x) title 'fit' with line linewidth 2`
  - Save the plot in `prob1.png` file.
  - The value of `a` is the order of the error ($\sim h^a$)! Report in a text file the value of `a`.

2. **Simpson's 1/3 rule**: An improvement over the trapezoidal rule is obtained by using the Simpson's 1/3 rule.

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a)+f(b)+4\sum_{j=1,j\to\text{odd}}^{n-1} f(a+jh)+2\sum_{j=1,j\to\text{even}}^{n-1} f(a+ih)\right], \quad \text{where } h=\frac{b-a}{n}$$

- Copy and modify the previous program and check if the 1/3 rule gives exact result for $f(x) = 2x$ and $f(x) = 3x^2$ when integrated from 0 to 1.
- Use 1/3 rule to integrate $f(x) = 4x^3$ from 0 to 1 (use n values as in question 1).
- Apply the `gnuplot` tricks mentioned in question 1 to estimate the order of error (as a function of h) for this rule. Report the results (`a` value and goodness of the fit) in a text file.

3. **Simpson's 3/8 rule**: As mentioned in the class a better method than Simpson's 1/3 rule can be found by using a cubic spline made to pass through three successive points of any given function. The rule is given below:

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + 2f(x_6) + \ldots + f(x_n)\right]$$

where, $x_i = a + ih$ and $h = \frac{b-a}{n}$ (n should be a multiple of 3).

- As before modify the last program to check if the 3/8 rule gives exact result for $f(x) = 2.0x$, $f(x) = 3x^2$ and $f(x) = 4x^3$ when integrated from 0 to 1.
- Use 3/8 rule to integrate $f(x) = 4x^3$ from 0 to 1 (use n values as in question 1).
- Apply the `gnuplot` tricks mentioned in question 1 to estimate the order of error (as a function of h) for this rule. Report the results (`a` value and goodness of the fit) in a text file.

4. Plot the 1st and the 3rd columns (log of h and log of absolute difference of exact and computed integrals) for the integral

$$\int_0^1 5x^4 dx$$

using all the three rules (reuse your programs). Save the plot as `prob4.png`. Create a text file to record your observations about the relative accuracies of the above methods as evident from the plot.