Laplace Approximation for Bernoulli-Beta Model

(i) Developing the Laplace Approximation

The posterior distribution is proportional to:

$$p(\mu|x,a,b) \propto p(x|\mu)p(\mu|a,b) \tag{1}$$

Taking the log:

$$\ln p(\mu|x,a,b) = \ln p(x|\mu) + \ln p(\mu|a,b) + const \tag{2}$$

For Bernoulli likelihood and Beta prior:

$$\begin{aligned}\ln p(\mu|x,a,b) &= x\ln\mu + (1-x)\ln(1-\mu) + (a-1)\ln\mu + (b-1)\ln(1-\mu) + const \qquad (3)\\ &= (x+a-1)\ln\mu + (b-x)\ln(1-\mu) + const \qquad (4)\end{aligned}$$

Let's call this log posterior $f(\mu)$. The Laplace approximation uses a second-order Taylor expansion around the mode μ_0:

$$f(\mu) \approx f(\mu_0) + f'(\mu_0)(\mu-\mu_0) + \frac{1}{2}f''(\mu_0)(\mu-\mu_0)^2 \tag{5}$$

First, we find μ_0 by solving $f'(\mu_0) = 0$:

$$f'(\mu) = \frac{x+a-1}{\mu} - \frac{b-x}{1-\mu} = 0 \tag{6}$$

Solving:

$$\mu_0 = \frac{x+a-1}{a+b-1} \tag{7}$$

The second derivative is:

$$f''(\mu) = -\frac{x+a-1}{\mu^2} - \frac{b-x}{(1-\mu)^2} \tag{8}$$

At the mode μ_0, $f'(\mu_0) = 0$, so our Taylor expansion becomes:

$$f(\mu) \approx f(\mu_0) + \frac{1}{2}f''(\mu_0)(\mu-\mu_0)^2 \tag{9}$$

Therefore:

$$p(\mu|x,a,b) \approx \exp\{f(\mu_0) + \frac{1}{2}f''(\mu_0)(\mu-\mu_0)^2\} \tag{10}$$

Define $\lambda = -f''(\mu_0)$ (the negative second derivative at the mode). Then:

$$p(\mu|x,a,b) \propto \exp\{-\frac{\lambda}{2}(\mu-\mu_0)^2\} \tag{11}$$

This is the kernel of a Gaussian distribution with mean μ_0 and precision λ:

$$q(\mu|x,a,b) = \mathcal{N}(\mu|\mu_0, \lambda^{-1}) \tag{12}$$

(ii) Concrete Case: a=4, b=2, x=1

For $a = 4$, $b = 2$, $x = 1$:

$$\mu_0 = \frac{1+4-1}{4+2-1} = \frac{4}{5} = 0.8 \tag{13}$$

$$\lambda = -f''(\mu_0) = \frac{4}{\mu_0^2} + \frac{1}{(1-\mu_0)^2} = \frac{4}{0.64} + \frac{1}{0.04} = 31.25 \tag{14}$$

Therefore:

$$q(\mu|x=1, a=4, b=2) = \mathcal{N}(\mu|0.8, \frac{1}{31.25}) \tag{15}$$

(iii) Comparison of True Posterior and Approximation

The true Beta posterior is:

$$p(\mu|x=1, a=4, b=2) = \mathrm{Beta}(\mu|5, 2) \tag{16}$$

The Gaussian approximation from our Taylor expansion is:

$$q(\mu) = \mathcal{N}(\mu|0.8, 0.032) \tag{17}$$