Suppose X, Y are univariate R.V. sampled uniformly from $[0, 1]$.

Now for a uniformly distributed R.V. from $[a, b]$

$$p_x = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \, p_x \, dx = \int_a^b \frac{x}{(b-a)} dx = \frac{1}{2}\left[\frac{x^2}{b-a}\right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} = \frac{a+b}{2}$$

$$Var(X) = E[(X-\mu)^2] = E[X^2] - \mu^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_x \, dx = \int_a^b \frac{x^2}{(b-a)} dx$$

$$= \frac{1}{3} \frac{b^3 - a^3}{b-a} = \frac{a^2 + b^2 + ab}{3}$$

$$\Rightarrow Var(X) = \frac{a^2 + b^2 + ab}{3} - \frac{a^2 + b^2 + 2ab}{4}$$

$$= \frac{a^2 + b^2 - 2ab}{} = \frac{(b-a)^2}{}$$

$$12 \qquad\qquad 12$$

Now in Our Case $a = 0$ $b = 1$

$$\Rightarrow \quad E[X] = E[Y] = \frac{1}{2} \quad , \quad Var(X) = Var(Y) = \frac{1}{12}$$

Now there is a R.V. $Z = (X-Y)^2$

We need to find $E[Z], Var(Z)$

$$E[Z] = E[(X-Y)^2] = E[X^2+Y^2-2XY]$$

$$= E[X^2] + E[Y^2] - 2\,E[X]\,E[Y]$$

↳ As X and Y are independently sampled

$$= \frac{1}{3} + \frac{1}{3} - 2\times\left(\frac{1}{2}\right)^2 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$E[Z^2] = E[(X-Y)^4]$$

$$= E[X^4] - 4\,E[X^3]\,E[Y] + 6\,E[X^2][Y^2] - 4\,E[X]\,E[Y^3] + E[Y^4]$$

So we need to find $E[X^3], E[X^4]$

$$E[X^3] = \frac{1}{4}\left[\frac{x^4}{b-a}\right]_a^b = \frac{(a+b)(b^2+a^2)}{4} = \frac{1}{4}$$

$$E[X^4] = \frac{1}{5}\left[\frac{x^5}{b-a}\right]_b^a = \frac{(b-a)(a^4+a^3b+a^2b^2+ab^3+b^4)}{5}$$

$$= \frac{1}{5}$$

$$\Rightarrow \quad E[z^2] = \frac{1}{5} - 4\times\frac{1}{4}\times\frac{1}{2} + 6\times\frac{1}{3}\times\frac{1}{3} - 4\times\frac{1}{4}\times\frac{1}{2} + \frac{1}{5}$$

$$= \frac{2}{5} - 1 + \frac{2}{3} = \frac{2}{3} - \frac{3}{5} = \frac{1}{15}$$

$$Var(z) = E[z^2] - \mu_z^2 = \frac{1}{15} - \frac{1}{36} = \frac{7}{180}$$

Now Suppose we have dimensions from $1, 2 \,.\, d$

We define $S = Z_1 + Z_2 + \cdots + Z_d$

$$E[S] = E[Z_1] + E[Z_2] + \cdots + E[Z_d]$$

$$= \frac{1}{6}\big[\underbrace{1 + 1 + \cdots\cdots 1}_{d}\big]$$

$$= \frac{d}{6}$$

$$Var(S) = E[S^2] - \left(\frac{d}{6}\right)^2$$

$$E[S^2] = E[(Z_1 + Z_2 + Z_3 + \cdots Z_d)^2]$$

$$= E[\underbrace{Z_1^2 + \cdots Z_d^2}_{d \text{ terms}} + \underbrace{2Z_1Z_2 + \cdots\cdots}_{\frac{d(d-1)}{2} \text{ terms}}]$$

$$= d\,E(z^2) + d(d-1)\left(E[z]\right)^2$$

$$= d \times \frac{1}{15} + \frac{d^2 - d}{36} = \frac{d^2}{36} + \frac{7d}{180}$$

$$Var(S) = \cancel{\frac{d^2}{36}} + \frac{7d}{180} - \cancel{\frac{d^2}{36}} = \frac{7d}{180}$$