Code Listings

Implementing the metric matrix and its inverse

```
Σ[x_, y_] := x^2 + a^2 Cos[y]^2;
Δ [x_] := x^2 - 2 m x + a^2;
g [x_, y_] := {{1 - (2 m x) / Σ[x, y], (2 a m x Sin [y]^2) / Σ[x, y] , 0 , 0},
  {(2 a m x Sin [y]^2) / Σ[x, y] , -(Δ[x] + (2 m x (x^2 + a^2)) / Σ[x, y]) Sin [y]^2, 0 , 0},
  {0 , 0 , -Σ[x, y] / Δ[x], 0} , {0, 0, 0, -Σ[x, y]}}
ginverse[x_, y_] := Simplify[Inverse[g[x, y]]]
```

Effective Potential

```
V[x_, y_, En_, Lz_] := (g[x, y][[2, 2]] En^2 + 2 g[x, y][[1, 2]] En Lz + g[x, y][[1, 1]] Lz^2) /
    (g[x, y][[1, 1]] × g[x, y][[2, 2]] - g[x, y][[1, 2]]^2) - 1;
V[x, π / 2, 0.97, 4]
Plot[V[x, π / 2, 0.97, 4], {x, 0, 30}]
Solve[V[x, π / 2, 0.97, 4] == 0, x]
```

Christoffel Symbols

```
Γ[t_, ϕ_, r_, θ_] := Simplify[
   Table[(1 / 2) Sum[ginverse[r, θ][[i, l]] (D[ g[r, θ][[j, l]], {t, ϕ, r, θ}[[k]]] +
         D[ g[r, θ][[k, l]], {t, ϕ, r, θ}[[j]]] - D[ g[r, θ][[j, k]], {t, ϕ, r, θ}[[l]]]) ,
      {l, 4}], {i, 1, 4}, {j, 1, 4}, {k, 1, 4}]];
```

Numerical Solutions of Geodesic Equations

```
In[•]:= α = 0; β = 10000;
Sol[r0_, θ0_, R0_, En_, Lz_] :=
  NDSolve[{ D[t[τ], τ] == T[τ], D[ϕ[τ], τ] == Φ[τ], D[r[τ], τ] == R[τ], D[θ[τ], τ] == Θ[τ],
     D[T[τ], τ] == -Sum[Sum[Γ[t[τ], ϕ[τ], r[τ], θ[τ]][[1, j, k]]
          {T[τ], Φ[τ], R[τ], Θ[τ]}[[j]] {T[τ], Φ[τ], R[τ], Θ[τ]}[[k]], {j, 4}], {k, 4}],
     D[Φ[τ], τ] == -Sum[Sum[Γ[t[τ], ϕ[τ], r[τ], θ[τ]][[2, j, k]]
          {T[τ], Φ[τ], R[τ], Θ[τ]}[[j]] {T[τ], Φ[τ], R[τ], Θ[τ]}[[k]], {j, 4}], {k, 4}],
     D[R[τ], τ] == -Sum[Sum[Γ[t[τ], ϕ[τ], r[τ], θ[τ]][[3, j, k]]
          {T[τ], Φ[τ], R[τ], Θ[τ]}[[j]] {T[τ], Φ[τ], R[τ], Θ[τ]}[[k]], {j, 4}], {k, 4}],
     D[Θ[τ], τ] == -Sum[Sum[Γ[t[τ], ϕ[τ], r[τ], θ[τ]][[4, j, k]]
          {T[τ], Φ[τ], R[τ], Θ[τ]}[[j]] {T[τ], Φ[τ], R[τ], Θ[τ]}[[k]], {j, 4}], {k, 4}],
     t[α] == 0, ϕ[α] == 0, r[α] == r0, θ[α] == θ0, T[α] ==
      (g[r[α], θ[α]][[2, 2]] En + g[r[α], θ[α]][[1, 2]] Lz) /
       (g[r[α], θ[α]][[1, 1]] × g[r[α], θ[α]][[2, 2]] - (g[r[α], θ[α]][[1, 2]])^2),
     Φ[α] == -(g[r[α], θ[α]][[1, 2]] En + g[r[α], θ[α]][[1, 1]] Lz) /
       (g[r[α], θ[α]][[1, 1]] × g[r[α], θ[α]][[2, 2]] - (g[r[α], θ[α]][[1, 2]])^2),
     R[α] == R0,
     Θ[α] == Sqrt[
       (-V[r[α], θ[α], En, Lz] - g[r[α], θ[α]][[3, 3]] (R[α])^2) / g[r[α], θ[α]][[4, 4]]]},
    {t, ϕ, r, θ, T, Φ, R, Θ}, {τ, α, β}] ;

S1 = Sol[9, π / 2, 0, 0.95, 3]; S2 = Sol[12, π / 2, 0, 0.95, 3];
S3 = Sol[14, π / 2, 0, 0.95, 3];
```

Plots of Solutions

```
t1 = Evaluate[t[τ] /. S1];
ϕ1 = Evaluate[ϕ[τ] /. S1]; r1 = Evaluate[r[τ] /. S1]; θ1 = Evaluate[θ[τ] /. S1];
T1 = Evaluate[T[τ] /. S1];
Φ1 = Evaluate[Φ[τ] /. S1]; R1 = Evaluate[R[τ] /. S1]; Θ1 = Evaluate[Θ[τ] /. S1];
t2 = Evaluate[t[τ] /. S2];
ϕ2 = Evaluate[ϕ[τ] /. S2]; r2 = Evaluate[r[τ] /. S2]; θ2 = Evaluate[θ[τ] /. S2];
T2 = Evaluate[T[τ] /. S2];
Φ2 = Evaluate[Φ[τ] /. S2]; R2 = Evaluate[R[τ] /. S2]; Θ2 = Evaluate[Θ[τ] /. S2];
t3 = Evaluate[t[τ] /. S3];
ϕ3 = Evaluate[ϕ[τ] /. S3]; r3 = Evaluate[r[τ] /. S3]; θ3 = Evaluate[θ[τ] /. S3];
T3 = Evaluate[T[τ] /. S3];
Φ3 = Evaluate[Φ[τ] /. S3]; R3 = Evaluate[R[τ] /. S3]; Θ3 = Evaluate[Θ[τ] /. S3];

Plot[{t1, t2, t3}, {τ, α, β}]
Plot[{ϕ1, ϕ2, ϕ3}, {τ, α, β}]
Plot[{r1, r2, r3}, {τ, α, β}]
Plot[{θ1, θ2, θ3}, {τ, α, β}]
```

Numerical Errors

```
Plot[{Evaluate[Φ[τ] /. S2] × g[Evaluate[r[τ] /. S2], Evaluate[θ[τ] /. S2]][[1, 2]] +
   Evaluate[T[τ] /. S2] × g[Evaluate[r[τ] /. S2], Evaluate[θ[τ] /. S2]][[1,
   1]] - En}, {τ, α, β}]
Plot[
 {-(Evaluate[Φ[τ] /. S2] × g[Evaluate[r[τ] /. S2], Evaluate[θ[τ] /. S2]][[2, 2]]) +
  Evaluate[T[τ] /. S2] (-g[Evaluate[r[τ] /. S2], Evaluate[θ[τ] /. S2]][[1, 2]]) - Lz},
 {τ, α, β}]
Plot[{Evaluate[R[τ] /. S2]^2 * g[Evaluate[r[τ] /. S2], Evaluate[θ[τ] /. S2]][[3, 3]] +
   Evaluate[Θ[τ] /. S2]^2 ×
  g[Evaluate[r[τ] /. S2], Evaluate[θ[τ] /. S2]][[4, 4]] +
   V[Evaluate[r[τ] /. S2], Evaluate[θ[τ] /. S2], En, Lz]}, {τ, α, β}]
```

Poincare Maps

```
Lr = List[]; LR = List[];
Do[root = FindRoot[Evaluate[θ[τ] /. S3] == π / 2, {τ, 100 i}];
 rval = Evaluate[r[τ] /. S3] /. root ;
 Rval = Evaluate[R[τ] /. S3] /. root;
 Θval = Evaluate[Θ[τ] /. S3] /. root;
 If[Θval[[1]] > 0, Lr = Append[Lr, rval[[1]]];
  LR = Append[LR, Rval[[1]]]], {i, 1, 95}]
NumberForm[Lr[[2 ;; 4]], 3];
NumberForm[Lr[[-3 ;;]], 3];
NumberForm[LR[[2 ;; 4]], 3];
NumberForm[LR[[-3 ;;]], 3];
data = Transpose@{Lr, LR};
ListPlot[data]
```

Conservation of Q

In[21]:= $Q = (a\,En\,Sin[\theta] - Lz\,Csc[\theta])^2 + \left(r^2 + a^2\,Cos[\theta]^2\right)^2 (\Theta)^2 + \delta\, a^2\,Cos[\theta]^2$

```
In[23]:= Qtheta = D[Q, θ]

In[24]:= Qr = D[Q, r]

In[25]:= QTheta = D[Q, Θ]

In[26]:= DDtheta =
  -Sum[Sum[Γ[t, ϕ, r, θ][[4, j, k]] {T, Φ, R, Θ}[[j]] {T, Φ, R, Θ}[[k]], {j, 4}], {k, 4}]
```

In[27]:= DerQ = Simplify[Qtheta Θ + Qr R + QTheta DDtheta /.
 $\{R^2 \to (1 - g[r, \theta][[1, 1]]\, T^2 - 2\, g[r, \theta][[1, 2]]\, T\, \Phi - g[r, \theta][[2, 2]]\, \Phi^2 - g[r, \theta][[4, 4]]\, \Theta^2) /$
 $g[r, \theta][[3, 3]],$
 $Lz \to -g[r, \theta][[1, 2]]\, T - g[r, \theta][[2, 2]]\, \Phi,\ En \to g[r, \theta][[1, 1]]\, T + g[r, \theta][[1, 2]]\, \Phi\}]$