1.2 Ordinary Differential Equations

This project builds on theory covered in Part IA Differential Equations.

1 Background Theory

The aim in the first part of this project (§2) is to study the performance of three different numerical methods for step-by-step integration of a first-order ordinary differential equation (ODE)

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

with a given initial condition

$$y = Y_0 \text{ at } x = x_0 \tag{2}$$

for specified x_0 and Y_0. A case has been chosen where the exact solution $y(x)$ can be found in simple analytic form. In the second part of this project (§3), one of the methods is extended to solve a second-order problem.

The numerical methods to be investigated are as follows.

(a) The **Euler** method is the simplest method. It employs the scheme

$$Y_{n+1} = Y_n + hf(x_n, Y_n) \tag{3}$$

where Y_n denotes the numerical solution at $x_n \equiv x_0 + nh$, that is, at the nth step with step length h. The Euler method has first-order accuracy, which means that the local truncation error e_{n+1} is $O(h^2)$ as $h \to 0$. The local truncation error is found by setting $Y_n = y(x_n)$ (the exact solution at x_n), computing Y_{n+1} using equation (3), then calculating $e_{n+1} = Y_{n+1} - y(x_{n+1})$. On the other hand, the global error in the numerical solution using $n + 1$ steps starting from the initial condition (2) is denoted by E_{n+1}. The Euler method is called a *single-step* method, since Y_{n+1} is obtained from the previous step Y_n.

(b) The **Leapfrog** (LF) method employs the scheme

$$Y_{n+1} = Y_{n-1} + 2hf(x_n, Y_n) \tag{4}$$

and has second-order accuracy, i.e. e_{n+1} is $O(h^3)$ as $h \to 0$. It is a multi-step method, using both Y_{n-1} and Y_n to obtain Y_{n+1}, and the first step must be taken by a single-step method, e.g. the Euler method.

(c) The fourth-order **Runge–Kutta** (RK4) method employs the scheme

$$Y_{n+1} = Y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{5}$$

where

$$k_1 = hf(x_n,\ Y_n) \tag{6}$$
$$k_2 = hf(x_n + \tfrac{1}{2}h,\ Y_n + \tfrac{1}{2}k_1) \tag{7}$$
$$k_3 = hf(x_n + \tfrac{1}{2}h,\ Y_n + \tfrac{1}{2}k_2) \tag{8}$$
$$k_4 = hf(x_n + h,\ Y_n + k_3) \tag{9}$$

and has fourth-order accuracy, i.e. e_{n+1} is $O(h^5)$ as $h \to 0$. When the RK4 method is used on coupled ODEs, each of f, Y_n, k_1, k_2, k_3 and k_4 become vectors of the same dimension as the number of coupled ODEs.

The theoretical background for the accuracy and stability of these methods is set out in, for example, *An Introduction to Numerical Methods and Analysis* by J.F.Epperson, *An Introduction to Numerical Methods* by A.Kharab and R.B.Guenther and *Numerical Recipes* by Press *et al.*

2 Comparison of the numerical methods for solving ODEs

The specific case to be studied in detail in this section is equation (1) with

$$f(x, y) = -4y + 3\mathrm{e}^{-x} \tag{10}$$

and initial condition

$$y(0) = 0. \tag{11}$$

This has the exact solution

$$y(x) = \mathrm{e}^{-x} - \mathrm{e}^{-4x}. \tag{12}$$

Programming Task: Write program(s) to implement each of the methods (a), (b) and (c) above.

2.1 Stability

Question 1 Starting with $x_0 = 0$, $Y_0 = 0$, use the LF method (with the first step taken by the Euler method) to integrate the ODE (1), (10) with initial condition (11) numerically from $x = 0$ to $x = 10$ with $h = 0.4$ [i.e for n up to 25]. Tabulate the values of x_n, the numerical solution Y_n, the analytic solution $y(x_n)$ from (12) and the global error $E_n = Y_n - y(x_n)$. You should find that the numerical result is unstable over this range of x: the error oscillates wildly with magnitude ultimately growing exponentially, proportional to $\mathrm{e}^{\gamma x}$ where the 'growth rate' γ is a constant which you should estimate.

Repeat with $h = 0.2$, 0.1 and 0.05 [i.e. for n up to 50, 100 and 200 respectively], not necessarily tabulating the output at every step. Comment on the effect of reducing h on the size of the instability, and on its growth rate.

Question 2 (i) Find the analytic solution to the LF *difference* equation

$$Y_{n+1} = Y_{n-1} + 2h\left[-4Y_n + 3\left(\mathrm{e}^{-h}\right)^n\right] \tag{13}$$

with

$$Y_0 = 0\ , \quad Y_1 = 3h \quad \text{(from the Euler method).} \tag{14}$$

(ii) Hence explain why instability occurs, and how its growth rate depends on h.

(iii) Show that in the limit $h \to 0$, $n \to \infty$ with $x_n \equiv nh$ fixed, the solution of the LF-difference-equation problem (13)–(14) found in part (i) converges to the solution (12) of the differential-equation problem (1), (10), (11). Does this mean that the instability can be suppressed by using a sufficiently small value for h?

2.2 Accuracy

Question 3 Integrate the ODE (1), (10) with initial condition (11) numerically up to $x = 4$, using both the Euler and the RK4 method with $h = 0.4$. Tabulate both numerical solutions Y_n against x_n, and plot them with the exact solution $y(x_n)$ superimposed.

Question 4 For each of the Euler, LF and RK4 methods, tabulate the global error E_n at $x_n = 0.4$ against $h = 0.4/n$ for $n = 2^k$, $k = 0, 1, 2, \ldots, 15$, and plot a log–log graph of $|E_n|$ against h over this range. Comment on the relationship of these results to the theoretical accuracy of the methods.

3 Numerical solutions of second-order ODEs

This section investigates the response of a simple harmonic oscillator with (possibly nonlinear) damping to a driving force.

The equation to be studied is

$$\frac{d^2y}{dt^2} + \frac{d}{dt}\left(\gamma y + \tfrac{1}{3}\delta^3 y^3\right) + \Omega^2 y = a\sin(\omega t) \tag{15}$$

where γ, δ, Ω, ω and a are non-negative real constants and t and y are real variables. In the case of purely linear damping, $\delta = 0$, it can of course be solved analytically.

Question 5 Find the analytic general solution to equation (15) for the linear, lightly damped case with $\delta = 0$, $0 < \gamma < 2\Omega$. Show that

$$y \to A_s \sin(\omega t - \phi_s) \quad \text{as } t \to \infty \tag{16}$$

and write down expressions for the 'steady-state' amplitude A_s and the 'steady-state' phase shift ϕ_s in terms of γ, Ω, ω and a.

Equation (15) can be rewritten as a pair of coupled first-order ODEs for

$$y^{(1)}(t) \equiv y(t) \qquad \text{and} \qquad y^{(2)}(t) \equiv \frac{dy(t)}{dt}\ , \tag{17}$$

namely

$$\frac{dy^{(1)}}{dt} = f^{(1)}(t, y^{(1)}, y^{(2)}) \equiv y^{(2)}\ , \tag{18}$$

$$\frac{dy^{(2)}}{dt} = f^{(2)}(t, y^{(1)}, y^{(2)}) \equiv -\gamma y^{(2)} - \delta^3\left[y^{(1)}\right]^2 y^{(2)} - \Omega^2 y^{(1)} + a\sin(\omega t)\ , \tag{19}$$

which can then be solved using either the Euler or the RK4 method. In this part of the project you are to use RK4, and take as initial conditions

$$y = \frac{dy}{dt} = 0 \quad \text{at } t = 0\ . \tag{20}$$

Programming Task: Write a program to solve equation (15) with initial conditions (20) using the RK4 method.

The next two questions are concerned with the particular case of equation (15) with $\delta = 0$, $\Omega = 1$ and $a = 1$, i.e.,

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + y = \sin(\omega t). \tag{21}$$

Question 6 Write down the analytic solution of (21) for general γ (< 2) and ω subject to the initial conditions (20). Taking $\gamma = 1$ and $\omega = \sqrt{3}$, use your program to compute Y_n for t up to 10 with $h = 0.4$ [i.e. for n up to 25], and tabulate the numerical solution Y_n, the analytic solution $y(t_n)$ and the global error $E_n \equiv Y_n - y(t_n)$ against t_n. Repeat with both $h = 0.2$ and $h = 0.1$ [integrating up to $t = 10$, i.e. for n up to 50 and 100 respectively], not necessarily presenting all the output. Comment on the errors.

Question 7 Use your RK4 program (with suitable h) to generate and plot numerical solutions of (20)–(21) up to $t = 40$ for $\omega = 1$ and $\gamma = 0.25$, 0.5, 1.0 and 1.9, checking that they agree with the analytic solutions. Do likewise for $\omega = 2$ and the same values of γ. Explain the differences between the various cases in terms of the mathematics and the physics of the system under investigation.

The last question considers a case with nonlinear damping,

$$\frac{d^2y}{dt^2} + \frac{d}{dt}\left(\tfrac{1}{3}\delta^3\, y^3\right) + y = \sin t \ , \tag{22}$$

for which an analytic solution is not available. The initial conditions are as before,

$$y = \frac{dy}{dt} = 0 \quad \text{at } t = 0 \ . \tag{23}$$

Question 8 For $\delta = 0.25$, 0.5, 1.0 and 20, use your RK4 program to generate and plot numerical solutions to (22)–(23) for t up to 60, using suitable value(s) of h (justify your choice). Comment on the solutions, comparing them with each other and with those of Question 7 for $\omega = 1$.

Hint: it *may* be helpful to observe that when δ is 'small', equation (22) has a 2π-periodic solution of the form

$$y = \sum_{n=-1}^{\infty} \delta^n y_n(t) \tag{24}$$

where each $y_n(t)$ is periodic in t with period 2π and

$$y_{-1}(t) = A\cos t \ , \quad y_0(t) = B\sin t + C\sin 3t \tag{25}$$

for suitable values of the constants A, B and C [recall that $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$, and note that to determine y_0 completely it is necessary to consider terms of order δ]. What if δ is 'large'?

Project 1.2: Ordinary Differential Equations

Marking Scheme and additional comments for the Project Report

The purpose of these additional comments is to provide guidance on the structure and length of your CATAM report. Use the same concepts to write the rest of the reports. To help you assess where marks have been gained/lost, this marking scheme will be completed and returned to you during Lent Term. You are advised to keep a copy of your write-up in order to correlate your answers to the marks awarded.

Question no.	**marks available**[1]	**marks awarded**[2]
Programming task *Program:* for instructions regarding printouts and what needs to be in the write-up, refer to the introduction to the project manual.		
Question 1 *Tables:* for presentation and layout, refer to the introduction. [*approx. 2 lines*][3]	C2+M0	
Question 2 *Analytic solution:* do not include trivial steps in your worked answer. [*approx. 12 lines*][3]	C0+M3.5	
Question 3 *Graphs:* you may use one graph or two.	C1+M0	
Question 4 *Graphs:* you may use one graph, or two, or three. *Comments:* what can be said about how the global error of each method depends on h? How is this reflected in the plots? [*approx. 4 lines*][3]	C1+M0 C0+M1	
Question 5 *Analytic solution:* do not include trivial steps in your worked answer; be sure to specify A_s and ϕ_s unambiguously. [*approx. 4 lines*][3]	C0+M1	
Question 6 *Analytic and numerical solutions compared:* the purpose of this step is to check that the program works and gives accurate answers ('validation'). Do the errors behave as expected when h is decreased? [*approx. 2 lines*][3]	C2+M1	
Question 7 *Comments:* first identify the salient features of the plots. Examine the nature of the functions that you are plotting: what are their components and how do these contribute to the overall solutions? Then use mathematical arguments (cf. the Part IA course *Differential Equations*) to explain the behaviour of the plots; link to the theory of the physical system under investigation. [*approx. 30 lines*][3]	C1+M2	
Question 8 *Numerical solutions:* explain why you are satisfied that your chosen value(s) of h will deliver sufficiently accurate results. *Comments:* identify the key similarities and differences between the various solutions, and with the help of the hint, or otherwise, try to explain them mathematically and/or physically. [*approx. 20 lines*][3]	C1+M0 C0+M1.5	
Excellence marks awarded for, *among other things*, mathematical clarity and good, clear output (graphs and tables) — see the introduction to the project manual.	E2	
Total Raw Marks	20	
Total Tripos Marks	**40**	

[1] C#+M# : computational and mathematical marks
[2] For use by the assessor
[3] This figure is only meant to be indicative of the length of your answer, rather than the exact number of lines you are expected to write

