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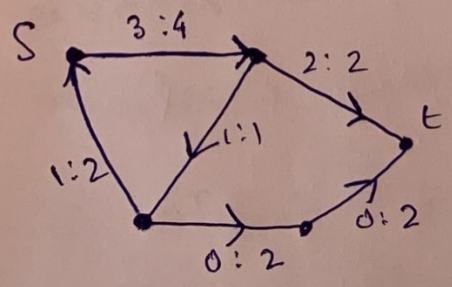
Problem 1

To describe an algorithm S.t in given graph $G_2 = (V, E, c, s, t)$, $f(e) = 1$, to come up with new flow S.t $f'(e) \neq 0$ and $|f'| \geq |f|$

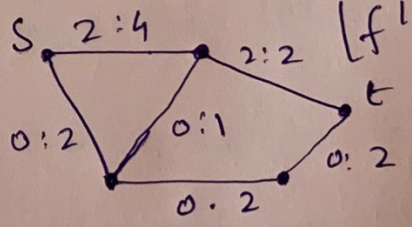
~~S.t → ~~

S.t → such that.

Here in the problem it says that there exists an edge with $f(e) = 1$ and head s, this ~~the~~ set up creates a cycle in the graph. We have to make $f'(e) = 0$, thus get rid of the cycle in the graph. To get rid of cycle, we use DFS and substract a value of 1 for the flows involved in the cycle.



$|f| = 2$

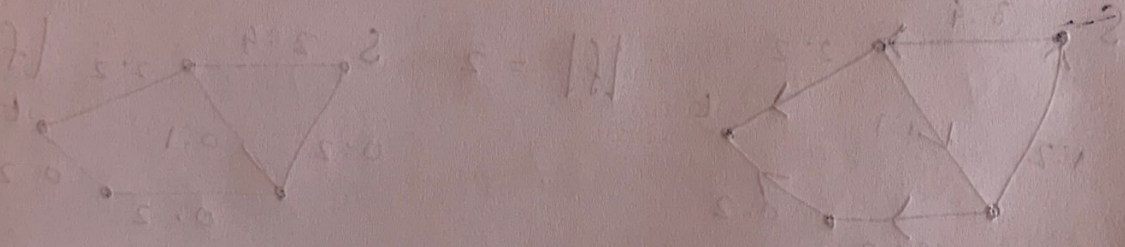


$|f'| = 2$

Therefore, resulting in $f'(e) = 0$ and $|f'| = |f|$ in this case.

Proof of correctness ⇒

We know the graph G_r has a flow > 0 i.e. $f(s,u) > 0$, with a path P. Now, assume there is another path p' which leads back to S. $S \to u \to s$., thus forming a cycle. We know that, the cycle has a minimum flow value of 1, thus by substracting 1, one or more edges in the cycle will have flow value = 0 and thus still satisfies the properties of the flow network, and $|f'| = |f|$. Hence proved.



Problem 2

Here in the above problem we have multiple sources and multiple sinks.

1) Call the set of source nodes as "supply Nodes" and set of sink Nodes as "Demand Nodes".

2) Next add a single numeric value to both supply and demand nodes. Consider positive for all demand node cost and negative for all supply node cost. If a node is neither a supply nor a demand node, assign the cost as 0.

3) S denotes the set of supply nodes ($dv < 0$), and let T denotes the set of demand nodes ($dv > 0$).

4) Given a flow f and a node V, $f_{in}(v)$ is the sum of flows along incoming edges to v and $f_{out}(v)$ is the sum of flows along outgoing edges from V.

5) We define a circulation in G to be a function f that assigns a non-negative real number to each edge that satisfies the following two conditions.

Capacity constraints: For each $(v_1, v_2) \in E$,

$$0 \leq f(v_1, v_2) \leq c(v_1, v_2)$$

$$\sum_{e \in \delta^+(s)} f(e) - \sum_{e \in \delta^-(s)} f(e) = p_x$$

and such that for all $t \in T$: $\sum_{e \in \delta^-(t)} f(e) - \sum_{e \in \delta^+(t)} f(e) = q_t$

Supply constraints: For vertex $v \in V$, $f_{in}(v) - f_{out}(v) = d_v$

$$\sum_{e \in \delta^-(u)} f(e) = \sum_{e \in \delta^+(u)} f(e);$$

6) Now reduce this decision problem on any graph G_1 which maintains the above constraint into a network flow problem.

The input to our reduction is a network $G_1 = (V, E)$. For each vertex v, let d_v denote the demand value.

A. For each edge (u, v), let $c(u, v)$ denote capacity.

a) If (Supply Sum != demand total)

i) Return "No"

b) If (Supply Sum == demand total)

i) create a new network $G_1' = (V', E')$ that has all the same vertices and edges as G_1

ii) Add to V' a super-source S^* and a super-sink T^*

iii) For each supply node $v \in S$

1) Add a new edge (S^*, v) of capacity $-dv$

iv) For each demand node $u \in T$,

1) Add a new edge (u, T^*) of capacity dv

B. Invoke Ford Fulkerson on G'. Define D as total demand,

a) If (max(flow) == D)

i) Return "Yes", G has a feasible circulation.

b) If (max (flow) != D)

i) Return "No", G has no feasible circulation

Time complexity

In $O(m+f)$ time reduction can be done. Thus, the overall running time is dominated by time to compute the network flow $O(M*f)$ which is the running time estimation for Ford-Fulkerson on graphs with integer capacities, where M is the number of edges and f is the value of maximum flow.

Proof of correctness $\Rightarrow$

Let us consider there is a feasible circulation f in G_1. The net flow coming out of all supply nodes is total demand D. If we saturate all the edges coming out of S^* and all the edges coming into T^*, we can create flow f' of value D in G'. We claim that this is a valid flow for G' and it satisfies all the capacity constraints.

For each vertex $v \in V$, there can be 3 cases—

a) $(v \in S)$ The flow into v from S^* matches the supply coming out of v from the circulation.

b) $(v \in T)$ The flow out of v to t^* matches the demand coming into v from the circulation.

c) $(v \in V \setminus (S \cup T))$ We have $dv = 0$, which means that it satisfied flow conservation by the supply/demand constraints.

Conversely, let us suppose, we have a flow f' of value D in G'. It must be that each edge leaving S^* and each

edge entering t^* is saturated. Therefore, by the flow conservation of f_0, all the supply nodes and all the demand nodes have achieve their desired supply/demand constraints. All the other nodes satisfy their supply/demand constraints because by the flow conservation of f_0 the incoming flow equals the outgoing flow. There fore, the resulting flow is a circulation for G.

Problem 3

We are going to construct a directed graph G_r^* from G_r by replacing each edge u, v in G_r by two directed edge (u, v) in G_r^*. Let us consider $F^*(u, v)$ be the maximum flow value from u to v through G_r^* with all edges capacities equal to 1.

Let's pick an ~~abritrary~~ arbitrary node u and compute $F^*(u, v)$ for all $v \neq u$.

We are affirming that edge connectivity equals

$$c^* = \min_{v \neq u} F^*(u, v).$$

Hence the edge connectivity of G_r can be computed by running the maximum flow algorithm $|V|-1$ times on flow networks each having $|V|$ vertices and $2|E|$ edges.

Let us suppose K is the edge connectivity of the graph and S is the set of K edges such that removal of S will disconnect the graph into 2 non-empty subgraphs G_{r1} and G_{r2}. We assume the node $u \in G_{r1}$. Let w be a node in G_{r2}. Since $u \neq w$, the value $F^*(u, w)$ will be computed by the algorithm. By the max-flow min-cut theorem $F^*(u, w)$ equals the min cut size between the pair (u, w), which is at most K since

S disconnects u and w. Therefore we have

$$c^* \leq F^*(u, w) \leq k$$

But c^* can not be smaller than k since that would imply a cut set of size smaller than k, contradicting the fact that k is the edge connectivity. Therefore $c^* = k$ and the algorithm returns the edge connectivity of the graph correctly.