Wave Optics

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1 Huygen's Principle

Wavefront: Defined as a surface of particles oscillating simultaneously with the same phase difference.

Speed of the Wave: Speed of the wave is defined as the speed with which a wavefront moves away from its source.

Types of wavefronts:

1. If the source emits waves uniformly in all directions: Wavefronts are concentric spheres centered at the source.
2. If the source is at a large distance from point of observation: Wavefronts are parallel planes.

Huygen's Principle: Huygen's principle states that each point on a wavefront acts as a secondary source, producing secondary wavelets. A common tangent to all these wavelets after time $t = \tau + t_i$ represents the wavefront after τ seconds.

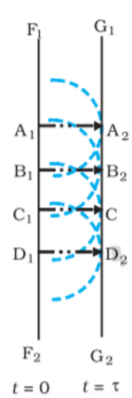


Figure 1: Sample Construction

1.1 Proving Snell's Law using Huygen's Principle

In Figure 2, let $A'A$ and BC be the incident rays and AB be the incident wavefront. Clearly, if light takes τ time to reach C from B, the distance $BC = v_1\tau$ where v_1 is the speed of light in medium 1. Meanwhile, light that was at A while light from the other ray was at B must have travelled a distance $v_2\tau$ in this time. Constructing a wavelet of radius $v_2\tau$ from A and constructing its tangent from C gives us the refracted wavefront. We call this point of tangency E.

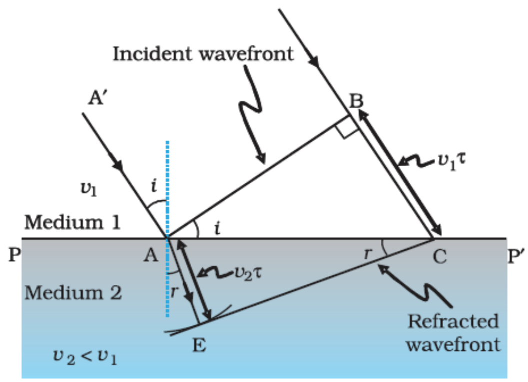


Figure 2: Construction

In ΔABC,

$$\sin i = \frac{BC}{AC} = \frac{v_1\tau}{AC}$$

In ΔAEC,

$$\sin r = \frac{AE}{AC} = \frac{v_2\tau}{AC}$$

By dividing the two expressions, we obtain

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2} \tag{1}$$

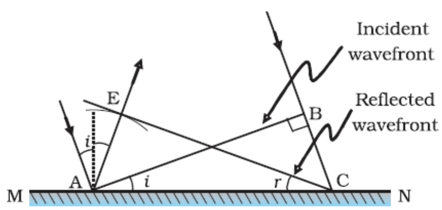
If $n_1 = \frac{c}{v_1}$ is the refractive index of medium 1, and $n_2 = \frac{c}{v_2}$ is the refractive index of medium 2, then we can rewrite (1) as:

$$\frac{\sin i}{\sin r} = \frac{n_2}{n_1} \tag{2}$$

This is what is known as the *Snell's Law of Refraction*

1.2 Proving Laws of Reflection using Huygen's Principle

We consider two parallel incident rays of light striking the surface at A and C. We construct the normal to the first incident ray intersecting the second incident ray (BC). BC represents our incident wavefront. Then, we realize that in the time it takes from light BC to strike the surface, the light from A would have been reflected the same distance as BC. So, we construct an arc of radius BC from A. Meanwhile, the Huygen source at C would just have been activated. So, we per Huygen's Principle, we construct a tangent from C to the arc to obtain the reflected wavefront. Now, we simply construct the normal from the reflected wavefront to A to obtain the reflected ray.



Reflection of a plane wave AB by the reflecting surface MN.
AB and CE represent incident and reflected wavefronts.

Figure 3: Construction

If light took τ time to reach C from B, then $BC = v\tau$, similarly $AE = v\tau$. By RHS congruency, ΔAEC is congruent to ΔCBA and hence $\angle i = \angle r$

2 Coherent and Incoherent Addition of Waves

Superposition Principle: At a particular point in the medium, the resultant displacement produced by a number of waves is the vector sum of displacements produced by each of the waves. For waves $w_1...w_n$, if the displacement produced at P are $\vec{y_1}...\vec{y_n}$, then the net displacement at P, $\vec{y_{net}}$ is:

$$\vec{y}_{net} = \sum_{i=1}^{n} \vec{y_i}$$

Coherent Sources: Two sources are said to be coherent if the phase difference between the waves they produce does not change in time.

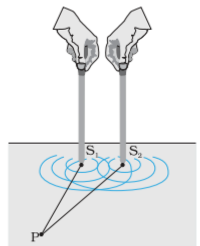


Figure 4: S_1P and S_2P

We have two identical waves (equal amplitude and wavelength) emanating from two points S_1 and S_2 in phase. We consider the intersection of the two waves at point Q (equidistant from S_1 and S_2). Since the initial phase difference of the two waves was 0, and both the waves now travel the same distance to P, the waves that arrive will also be in phase. Therefore, the displacements due the waves can be written as:

$$y_1 = a\cos\omega t$$

$$y_2 = a\cos\omega t$$

Now, the resultant displacement at P is given by

$$y_{net} = y_1 + y_2 = 2a\cos\omega t$$

Since the intensity of a wave $I \propto a^2$, the net intensity of the wave at P, I is given by:

$$I = 4I_0$$

Here, I_0 represents the intensity of each of the individual waves. Now, if we consider a point Q such that wave S_1Q arrives at Q 2 cycles before S_2Q, the path difference is:

$$S_2Q - S_1Q = 2\lambda$$

Since a path difference of 2λ represents a phase difference of 4π, if S_1Q is given by:

$$y_1 = a\cos\omega t$$

Then the expression for S_2Q is written as:

$$y_2 = a\cos(\omega t - 4\pi) = a\cos\omega t$$

Once again, the net intensity will be $4I_0$, giving rise to *Constructive Interference.*

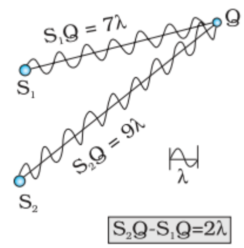


Figure 5: S_1Q and S_2Q

We now consider a point R such that S_1R reaches R 2.5 cycles after S_2R. In this case, if the wave from S_1 is written as:

$$y_1 = a \cos \omega t$$

Then, since a path difference of 2.5λ corresponds to a phase difference of 5π, we can express the second wave as:

$$y_2 = a \cos (\omega t + 5\pi) = -a \cos \omega t$$

In this case, we realize that the net displacement **and** intensity at R is 0. We call this as *Destructive Interference.*

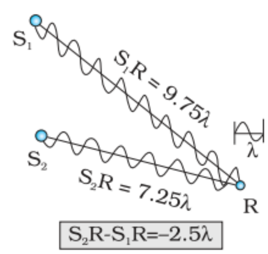


Figure 6: S_1R and S_2R

In general, for two coherent sources S_1 and S_2, then for a point P, if the path difference between the waves is:

$$S_2P \sim S_1P = n\lambda$$

for $n \in N$, we will have constructive interference and the resultant intensity will be $4I_0$. On the other hand if

$$S_2P \sim S_1P = \left(n + \frac{1}{2}\right)\lambda$$

for $n \in N$, we will have destructive interference and the net intensity will be 0.

If the phase difference between two waves emerging from two coherent sources is ϕ, then if the first wave is:

$$y_1 = a\cos\omega t$$

then, the second wave can be written as:

$$y_2 = a\cos(\omega t + \phi)$$

The net resultant then becomes:

$$y_{net} = y_1 + y_2 = a\cos\omega t + a\cos(\omega t + \phi) = 2a\cos\frac{\phi}{2}\cos(\omega t + \frac{\phi}{2})$$

The resultant amplitude is $2a\cos\frac{\phi}{2}$ and the resultant intensity is $4I_0\cos^2\frac{\phi}{2}$ If $\phi = 0, \pm 2\pi, \pm 4\pi, ...$ we will have the condition of constructive interference, and if $\phi = \pm\pi, \pm 3\pi, \pm 5\pi, ...$ we will have the condition for destructive interference.

Now, when the two sources are not coherent (i.e. the phase of difference between the waves they emit changes over time). For the cases in which the phase difference changes very rapidly over time, we consider a "time-averaged" intensity and its expression is:

$$\langle\ I\ \rangle = 4I_0 \cdot \langle\ \cos^2\frac{\phi}{2}\ \rangle$$

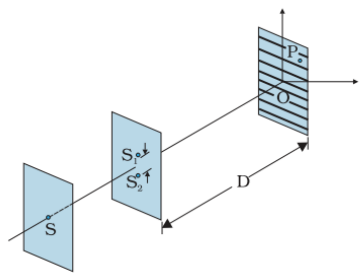
If $\cos^2\frac{\phi}{2}$ varies between 0 and 1 very rapidly and randomly over time, intuitively, we can observe that its time-average will be $\frac{1}{2}$. In this case, the intensity becomes:

$$I = 2I_0$$

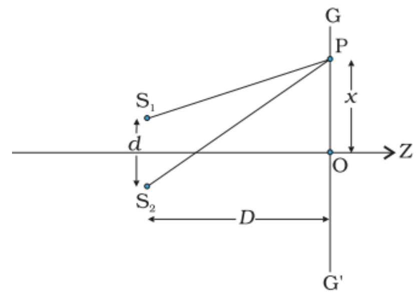
In general, when the sources are incoherent and the phase difference varies randomly and rapidly, it is observed that the resultant intensity is simply the sum of individual waves' intensities.

3 Interference of Light and Young's Double Slit Experiment

To eliminate phase differences in two ordinary sources of light, Young pointed a single light source S at two pinholes made on an opaque screen. The light from these pinholes was made to pass on another screen GG'. This set-up locked the two sources of light in phase to be coherent and indulge in proper interference. The arrangement for Young's experiment and a schematic diagram is included in Figure 7.



(a) Arrangement



(b) Schematic Diagram

Figure 7: Young's Double Slit Experiment

We know that intensity to correspond to a maximum in interference, we must have:

$$S_1P - S_2P = n\lambda;\ n \in \{0, 1, 2...\}$$

From Figure 7(b) it is clear that:

$$S_2P^2 = D^2 + \left(x + \frac{d}{2}\right)^2$$

$$S_1P^2 = D^2 + \left(x - \frac{d}{2}\right)^2$$

So for $S_2P^2 - S_1P^2$ we have:

$$S_2P^2 - S_1P^2 = \left[D^2 + \left(x + \frac{d}{2}\right)^2\right] - \left[D^2 + \left(x - \frac{d}{2}\right)^2\right] = 2xd$$

$$S_2P - S_1P = \frac{2xd}{S_1P + S_2P}$$

If $x, d <<< D$, we can replace $S_2P + S_1P$ with $2D$ to obtain:

$$S_2P - S_1P \approx \frac{xd}{D}$$

We will have constructive interference when:

$$x = x_n = n\frac{\lambda D}{d}$$

We will have destructive interference when:

$$x = x_n = \left(n + \frac{1}{2}\right)\frac{\lambda D}{d}$$

These dark and white bands observed on the GG' screen as seen in Figure 8 are called *fringes* and they are equidistant. The distance between two adjacent dark fringes (called *fringe width*) is given by:

$$\beta = x_{n+1} - x_n = \frac{\lambda D}{d}$$

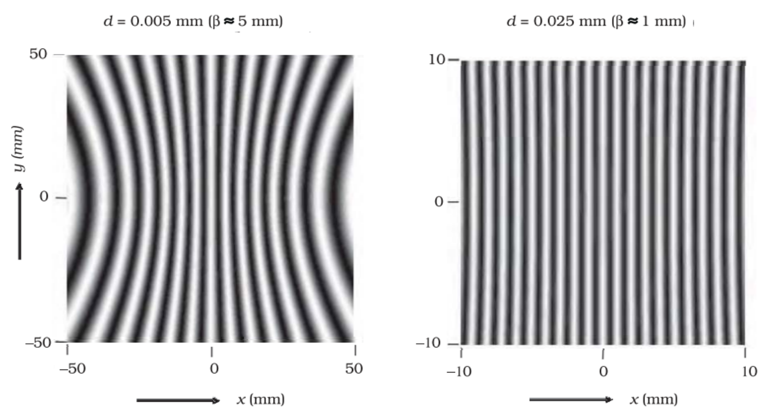


Figure 8: Fringe pattern

4 Diffraction

4.1 The Single Slit

In single slit diffraction, an intuitive idea about the wave intensity distribution on the scree can be obtained arithmetically. In figure 8, the path difference between NP and LP is as follows:

$$NP - LP = a \sin\theta \approx a\theta$$

In figure 9, consider θ of the form $\frac{n\lambda}{a}$ (say $\theta = \frac{\lambda}{a}$). Now, for each point M_1 in LM, there is a point M_2 in MN such that $M_1 - M_2 = \frac{a}{2}$. Then, the path difference between waves from M_1 and M_2 is:

$$(M_1 - M_2)\theta = \frac{a}{2}\theta = \frac{\lambda}{2}$$

We realise that the above path difference satisfies the condition for *destructive interference* and for all rays from points between LM, a point between MN cancels the intensity out. Therefore, whenever $\theta = \frac{n\lambda}{2}$, we observe a minimum ($I = 0$) intensity on the screen.

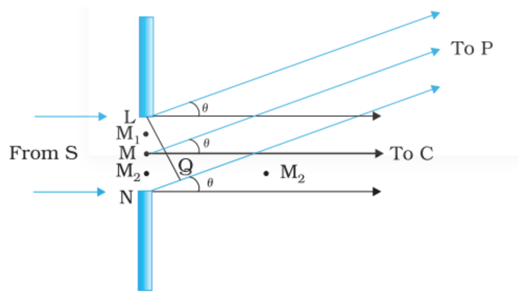


Figure 9: Beams from the Slit

We now discuss some of the key features of diffraction from a single slit:

1. The interference pattern has a number of equally spaced bright and dark bands. The pattern has a central maxima **twice as wide as the other maxima** $(\frac{2\lambda D}{a})$

2. We calculate the interference pattern by superposing two waves from two infinitesimally narrow slits. The overall pattern originates from the super position of a collection of waves throughout the slit.

3. For a single slit of width a, the first minima occurs at $\theta = \frac{\lambda}{a}$, however for a double slit, we obtain a maxima for $\theta = \frac{\lambda}{a}$.

4.2 Resolving Power of Telescopes