ACM 104 Problem Set 5

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Problem 2: Hermite Polynomials

Solution. In a similar fashion to the Legendre polynomials, the Hermite polynomials may be calculated as following with respect to the given inner product,

$$
\begin{aligned}
h_0(x) &= 1 \\
h_1(x) &= x - \frac{\langle x, 1 \rangle}{||1||^2} = x - 0 = x \\
h_2(x) &= x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} - \frac{\langle x^2, x \rangle}{\|x\|^2} \cdot x \\
&= x^2 - \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = x^2 - 1 \\
h_3(x) &= x^3 - \frac{\langle x^3, 1 \rangle}{\|1\|^2} - \frac{\langle x^3, x \rangle}{\|x\|^2} \cdot x - \frac{\langle x^3, x^2 - 1 \rangle}{\|x^2 - 1\|^2} \cdot \left(x^2 - 1\right) \\
&= x^3 - \frac{3\sqrt{2\pi}}{\sqrt{2\pi}} \cdot x = x^3 - 3x \\
h_4(x) &= x^4 - \frac{\langle x^4, 1 \rangle}{\|1\|^2} - \frac{\langle x^4, x \rangle}{\|x\|^2} \cdot x - \frac{\langle x^4, x^2 - 1 \rangle}{\|x^2 - 1\|^2} \cdot \left(x^2 - 1\right) \\
&- \frac{\langle x^4, x^3 - 3x \rangle}{\|x^3 - 3x\|^2} \cdot \left(x^3 - 3x\right) \\
&= x^4 - \frac{3\sqrt{2\pi}}{\sqrt{2\pi}} - \frac{12\sqrt{2\pi}}{2\sqrt{2\pi}} \cdot \left(x^2 - 1\right) \\
&= x^4 - 6x^2 + 3
\end{aligned}
$$

Problem 3: Orthogonal Compliments

Solution. For $W_1^{\perp}$, conveniently note that vectors that are perpendicular to both x and y will naturally be perpendicular to all possible linear combinations of x and y, which means, the span of $v_1 \times v_2$ will be complimenting W under this product. More specifically, we will have

$$W_1^{\perp} = \text{span}\{v_1 \times v_2\} = \text{span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}2\\0\\1\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}2\\5\\-4\end{pmatrix}\right\}$$

For $W_2^{\perp}$, note that the basis u will be such that $\langle v_1, u\rangle = \langle v_2, u\rangle$. For general u,

$$\langle v_1, u\rangle = \langle [1, 2, 3]^T, u\rangle = u_x + 4u_y + 9u_z$$

Similarly for the second vector, we have

$$\langle v_2, u\rangle = \langle [2, 0, 1]^T, u\rangle = 2u_x + u_z$$

So we need u_x, u_y, u_z that simultaneously satisfy

$$\begin{aligned} u_x + 4u_y + 9u_z &= 0 \\ 2u_x + u_z &= 0 \end{aligned}$$

In terms of u_z, we simply the above expressions to get $u_x = \frac{-3}{2}u_z$ and $u_y = \frac{-15}{8}u_z$. This means

$$W_2^{\perp} = \text{span}\left\{\begin{pmatrix}\frac{-3u_z}{2}\\ \frac{-15u_z}{8}\\ u_z\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}\frac{-3}{2}\\ \frac{-15}{8}\\ 1\end{pmatrix}\right\}$$

Problem 4: Complete Matrices

Solution. The matrix A is complete if and only if all eigenvalues have algebraic and geometric multiplicities of 1. First consider for an eigenvalue $\det(A - \lambda I) = 0$. We essentially have

$$\det(A - \lambda I) = \det\begin{pmatrix}-\lambda & 0 & -1\\ 0 & 1-\lambda & 0\\ 1 & 0 & -\lambda\end{pmatrix} = 0$$

This implies $-\lambda^3 + \lambda^2 - \lambda + 1 = 0$. By trial and error, we find $\lambda \neq 0$. However $\lambda = 1$ is a root. Then factoring $\lambda - 1$ out, we obtain the other coefficient as

$$(\lambda - 1)(-\lambda^2 - 1) = 0$$

This implies $\lambda = \pm i$. At this point, note that all three roots (eigenvalues) have algebraic multiplicities of 1.

Now, we check if all three eigenvalues have geometric multiplicity of 1.

1. When $\lambda = 1$,
$$Av = \lambda v \implies Av = v \implies v = \langle 0, k, 0 \rangle$$

2. When $\lambda = i$,
$$Av = \lambda v \implies Av = iv \implies = v = \langle ki, 0, k \rangle$$

3. When $\lambda = i$,
$$Av = \lambda v \implies Av = -iv \implies v = \langle -ki, 0, k \rangle$$

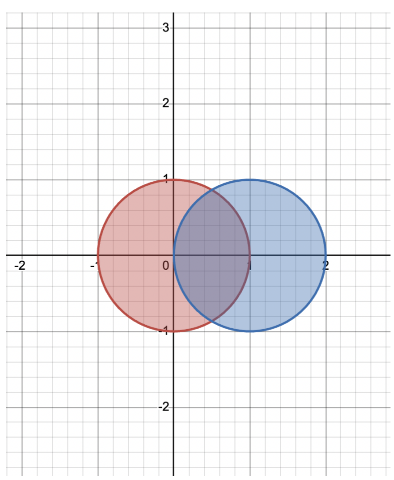
In each of the cases, the basis for the subspace spanned by v is defined completely using only 1 vector (setting $k = 1$ for example). Since the dimension is 1, in each of the cases, the span of the eigenvector has geometric multiplicity 1, so A is therefore complete.

Problem 5

Solution. (a) From the definition of the Gershgorin disk, we have

$$\mathcal{D}_1 = \left\{ z \in \mathbb{C} \;\middle|\; |z - b_{11}| \le \sum_{j \ne 1} |b_{1j}| \right\} = \{ z \in \mathbb{C} \mid |z| \le 1 \}$$
$$\mathcal{D}_2 = \left\{ z \in \mathbb{C} \;\middle|\; |z - b_{22}| \le \sum_{j \ne 2} |b_{2j}| \right\} = \{ z \in \mathbb{C} \mid |z - 1| \le 1 \}$$
$$\mathcal{D}_3 = \left\{ z \in \mathbb{C} \;\middle|\; |z - b_{33}| \le \sum_{j \ne 3} |b_{3j}| \right\} = \{ z \in \mathbb{C} \mid |z - 1| \le 1 \}$$

This makes it clear that there will be two discs, centered at $(0, 0)$ and $(1, 0)$. The domain will be as follows:



(b) First of all, from Gershgorin's Theorem, we know that spec$(A) \subset D_A$. Note for all λ such that $\det(A - \lambda I) = 0$, then we have $\det((A - \lambda I)^T) = 0 \implies \det(A^T - \lambda I) = 0$. But all λ that satisfy this form D_{A^T}. This implies spec$(A) \subset D_{A^T}$. Since spec$(A) \subset D_A$ and spec$(A) \subset D_{A^T}$, we trivially have spec$(A) \subset D_A^*$.

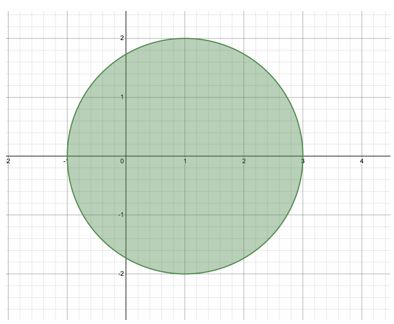
(c) For the refined domain, let's first find the domain for B^T.

$$B^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then we simply have

$$\mathcal{D}_1 = \{z \in \mathbb{C} \mid |z| \leq 0\}$$
$$\mathcal{D}_2 = \{z \in \mathbb{C} \mid |z - 1| \leq 2\}$$
$$\mathcal{D}_3 = \{z \in \mathbb{C} \mid |z - 1| \leq 1\}$$

We can ignore $\mathcal{D}_1$ (because it's a point that's anyways contained in a disk for B). Note that $\mathcal{D}_3$ is also identical to a disk for B. For $\mathcal{D}_2$, we have a circle centered at (1,0) with radius 2. This circle encapsulates all remaining disks of B, so the domain D_B^* finally is:



(d) The eigenvalues satisfy $\det(B - \lambda I) = 0$. We have

$$\det(B - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{pmatrix} = -\lambda((1-\lambda)^2 + 1)$$

This makes it clear (by observation) that the eigenvalues are $\lambda = 0, 1 - i, 1 + i$. Clearly all of them lie in the green disk above, so they do belong to D_B^*.

(e) Consider the matrix

$$B = \begin{pmatrix} 0 & 3 \\ 7 & 1 \end{pmatrix}$$

This matrix has $\mathcal{D}_1 = \{z \in \mathbb{C} \mid |z| \leq 3\}$ and $\mathcal{D}_2 = \{z \in \mathbb{C} \mid |z - 1| \leq 7\}$. Clearly the point (0,0) is contained in the first disc, so the domain $\mathcal{D}_1 \cup \mathcal{D}_2$ contains zero and it's also easy to see that the determinant is not zero.

ACM/IDS 104 - Problem Set 5 - MATLAB Problems

Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

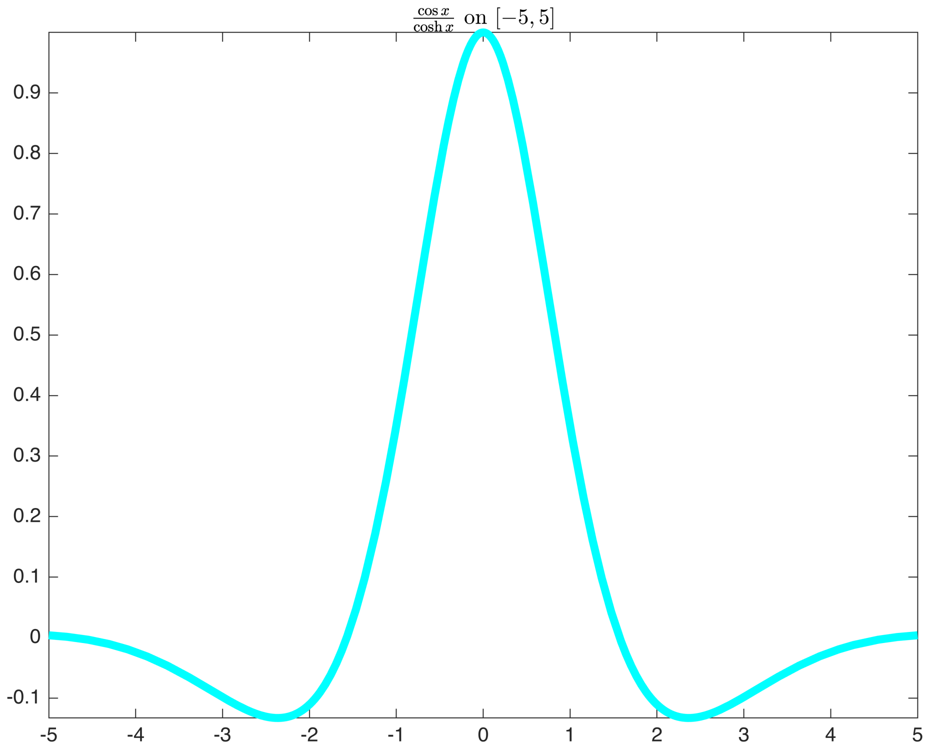
Problem 1 (10 points) Application of Projections to Approximation

In Problem 4 of PS4, we saw that even higher degree interpolating polynomials may not be accurate approximations to complex functions. We have the function:

$$f(x) = \frac{\cos x}{\cosh x}, \quad \text{on } [-a, a], \quad a = 5$$

Let us recall how this function looks like and how its interpolating polynomials of degree $(n-1)$ for $n = 3, 5, 10, 15$ behave:

```
%{
Setup
%}
f = @(x) cos(x)./cosh(x); % our function
a = 5; % setting the value of a
n = [3 5 10 15]; % setting the number of points
sub = 1; % subplot index

%{
How f(x) looks like on [-5, 5]
%}
figure;
fplot(f, [-a, a], "-c", "lineWidth", 4);
title("$\frac{\cos{x}}{\cosh{x}}$ on $[-5, 5]$","Interpreter","latex");
```

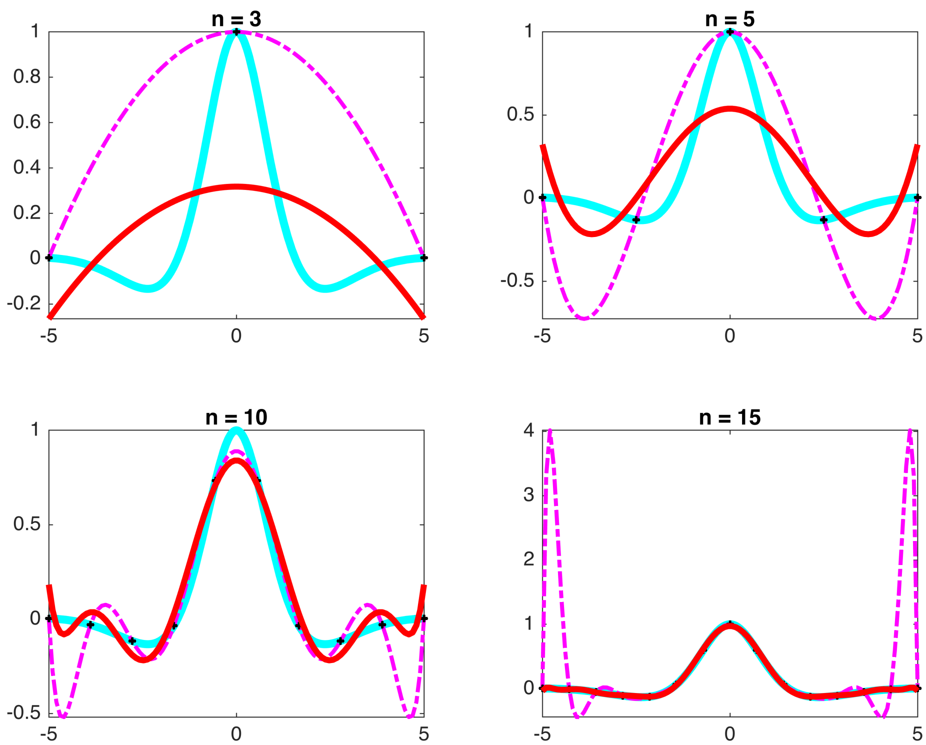



```
%{
Read the discussion below and complete the code
%}
figure;
for ival = a
    for degree = n-1
        %{
        INTERPOLATING POLYNOMIALS -- no changes needed
        -> Select degree+1 points in the interval
        -> Evaluate f(x) on these points
        -> Find the polynomial coefficients
        %}
        pts = ones(degree+1, 2); % initializing the points
        pts(:, 1) = linspace(-ival, ival, degree+1); % setting the x-values
        for i = 1 : degree+1
            pts(i, 2) = f(pts(i, 1)); % evaluating cos(x) /
cosh(x)
        end
        coeffs = polyfit(pts(:, 1), pts(:, 2), degree); % coefficients
        %{
        ORTHOGONAL PROJECTIONS -- TODO
        -> Get transformed Legendre polynomials
        -> Find alpha_k using L^2 inner product
        -> Evaluate alpha_k*Q_k
        %}
```

```
        x = linspace(-a, a);
        y = zeros(100 ,1);
        alpha_k = zeros(degree, 1);

        for d=0:degree
            numrgrand = @(x) (f(x) .* legendreP(d, x/a));
            denrgrand = @(x) (legendreP(d, x /a) .* legendreP(d,x/a));
            alpha_k(d+1) = integral(numrgrand, -a,a)/integral(denrgrand,-a,
a);
        end
        alpha_k

        for i = 1:100
         valati = 0;

         for d = 0:degree
            valati = valati + legendreP(d, x(i)/a) * alpha_k(d + 1);
         end

         y(i) = valati;
        end

        %{
        PLOTTING
        Plot f(x), the sampled points, interpolating and approximating
        polynomials
        Please use different colors and linestyles
        %}
        subplot(2, 2, sub);
        fplot(f, [-ival, ival], "-c", "lineWidth", 4);
        hold on
        interpoints = linspace(-ival, ival);
        p = polyval(coeffs, interpoints); % evaluating coeffs in interval
        plot(interpoints, p, "-.m", "lineWidth", 2);
        plot(pts(:, 1), pts(:, 2), "ok", "MarkerSize", 2, "lineWidth", 3);
        plot(x, y, "-r", "MarkerSize", 3, "lineWidth", 3);
        title(strcat("n = ", int2str(degree+1)));
        sub = sub + 1; % increase subplot index
    end
end
```

```
alpha_k = 3×1
    0.1235
   -0.0000
   -0.3888
alpha_k = 5×1
    0.1235
   -0.0000
   -0.3888
   -0.0000
    0.5881
alpha_k = 10×1
    0.1235
```

```
   -0.0000
   -0.3888
   -0.0000
    0.5881
   -0.0000
   -0.5815
    0.0000
    0.4415
    0.0000
alpha_k = 15×1
    0.1235
   -0.0000
   -0.3888
   -0.0000
    0.5881
   -0.0000
   -0.5815
    0.0000
    0.4415
    0.0000
      ⋮
```



Now, instead of interpolating polynomials, let us approximate $f(x)$ by its orthogonal projection onto the inner space $\mathscr{P}^{(n-1)}_{[-a,a]}$ of polynomials on $[-a, a]$, equipped with the L^2 inner product:

$$f(x) \approx p(x) = \mathrm{pr}_{\mathscr{P}^{(n-1)}_{[-a,a]}} f(x)$$

Recall (Lecture 10) that $p(x)$ is the closest (in the L^2 sense) polynomial to $f(x)$ in $\mathscr{P}^{(n-1)}_{[-a,a]}$, i.e.

$$p(x) = \arg \min_{q \in \mathscr{P}^{(n-1)}_{[-a,a]}} \|f(x) - q(x)\|$$

We know that the transformed Legendre polynomials $\widetilde{Q}_0(x), \cdots, \widetilde{Q}_{n-1}(x)$ form an orthogonal basis of $\mathscr{P}^{(n-1)}_{[-a,a]}$, and, therefore:

$$p(x) = \sum_{k=0}^{n-1} \alpha_k \widetilde{Q}_k(x)$$

where α_k are the coordinates of $p(x)$ in that basis.

Modify the above code to find the approximating polynomials as well. Plot each approximating polynomial on its corresponding subplot. Useful functions for this problem:

```
legendreP(), integral()
```