ACM 104 Problem Set 6

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Problem 3: Matrix Diagonalization

Solution. We are given

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

First, let's find the eigenvalues of F. We will have

$$\det(F - \lambda I) = 0 \implies \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

This means $\lambda^2 - \lambda - 1 = 0$. On solving this quadratic, we obtain values

$$\begin{aligned} \lambda &= \frac{1+\sqrt{5}}{2} \\ &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

For eigenvectors $v = \langle x, y \rangle$ we will have

$$Fv = \lambda v$$

This produces the following system of linear equations

$$\begin{aligned} x + y &= \lambda x \\ x &= \lambda y \end{aligned}$$

Depending on λ, we get the two eigenvectors (putting $y = 1$) as

$$v_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

So in the diagonalised form, F simply becomes

$$F_d = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}, B = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \implies B^{-1} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$$

Now we can see that

$$F = BF_dB^{-1} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$$

Problem 4: Symmetric Matrices

Solution. (a) Yes, $\text{Sym}(n)$ is a vector space, this is because the operations have the same properties on this set as they do on $\mathbb{M}_{n\times n}$, additionally, note that for any symmetric matrix, the additive inverse is guaranteed to be symmetric as well, also, the additive identity $\mathbf{0}$ is obviously symmetric.

(b) Note that the diagonals can be arbitrary in symmetric matrices. We must have n matrices to account for arbitrary values on the diagonal. Additionally, we need to account for each "flipped" position across the diagonal *once*. We count all i, j such that $i \neq j$. This is $n^2 - n$. Removing double counting, we get $(n^2 - n)/2$. So finally, the dimension is $(n^2 + n)/2 = n(n+1)/2$. These are the number of linearly independent matrices we need to generate all matrices in $\text{Sym}(n)$. The basis will simply be the set of all where all entries on the diagonal get a matrix with 1 on that entry and 0 elsewhere and each "flipped" element gets an entry 1 with 0 everywhere else except for the corresponding flipped position.

ACM/IDS 104 - Problem Set 6 - MATLAB Problems

Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

NOTE: Start with Problem 1, at the *bottom of this livescript*. (It's at the bottom because it's a MATLAB live function)

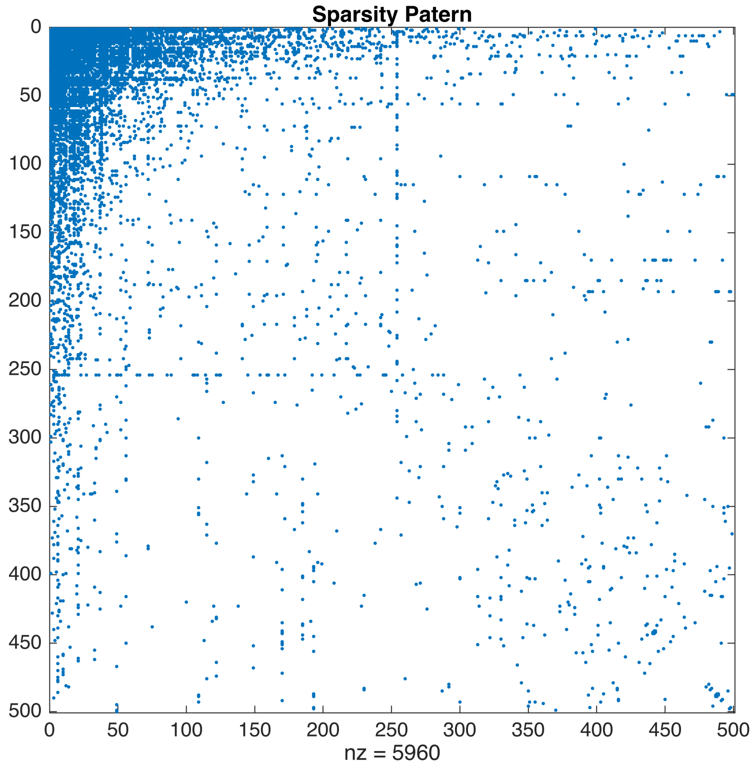
Problem 2 (10 points) Ranking US Airports using PageRank

The PageRank algorithm can be used for ranking not only web pages, but any "entities" organized into a network. The **`USAirTransportation.mat`** file contains the adjacency matrix A for the network of (anonymized) US airports: nodes represent airports and links represent air travel connections among them. That is, airports i and j are connected $(A_{ij} = 1)$ if there are direct flights between them. The network in **`USAirTransportation.mat`** is obtained by considering the 500 US airports with the largest amount of traffic from publicly available data. Implement the PageRank algorithm (see Lecture 12) for ranking the airports. Use your function from Problem 1 for finding the importance score vector. What are the ID numbers of 10 most important airports for $\alpha = 0.10, 0.15, 0.20$?

Functions that may be useful for this problem:

`repmat(), sort(), disp(), strcat(), num2str()`

```
load USAirTransportation.mat % dataset
%{
Let us see how matrix A looks like
%}
figure;
spy(A);
title('Sparsity Patern');
```




```
alpha = [0.10 0.15 0.20]; % values of alpha
[~,n]=size(A); % size of A
%{
Implement the algorithm below
You can use bar() to visualize the scores
Report the top 10 airports (in descending order) for each alpha
%}

% Recall Page Rank only works on stochastic matrices. This will be A_prime
A_prime = zeros(n, n);
for i = 1:n
    column_i = A(:, i);
    A_prime(:, i)= column_i/sum(column_i);
end

S = zeros(n, n);
S(:) = 1/n;
alpha_n = alpha';

for j = 1:size(alpha_n)
    Apb = (1 - alpha_n(j))*A_prime*alpha_n(j)*S;
    [lambda, v] = power_method(Apb);
    v_top = sort(v, 'descend');
    maxVals = v_top(1:10, 1);
    indices = zeros(0,1);
    for k = 1:10
```

```
        idx_int = find(v == maxVals(k, 1));
        indices = [indices; idx_int];
    end
    fprintf('Top 10 matrices in descending order for alpha = %.2f,\n',
alpha_n(j));
    for p = 1:size(indices)
        disp(indices(p));
    end
    fprintf('\n');
end
```

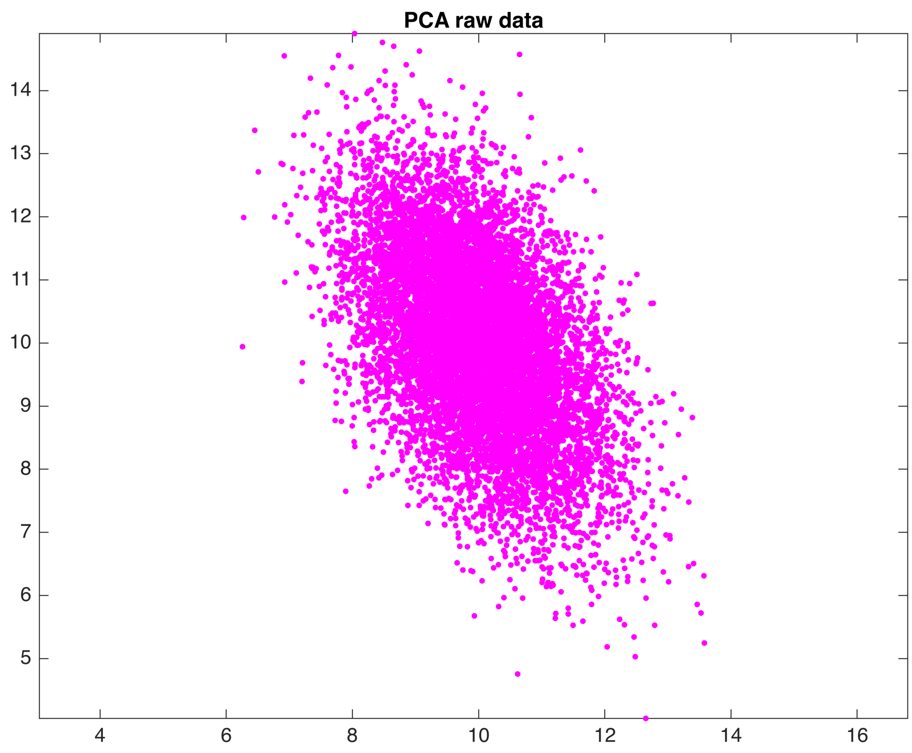
```
Top 10 matrices in descending order for alpha = 0.10,
     6

     3

     1

    21

     7

    10

    11

    12

     2

    14
Top 10 matrices in descending order for alpha = 0.15,
     6

     3

     1

    21

     7

    10

    11

    12

     2

    14
Top 10 matrices in descending order for alpha = 0.20,
     6

     3

     1

    21

     7
```

```
10

11

12

 2

14
```

Problem 5 (10 points) Principal Component Analysis

In this problem, you will find the principal components of the dataset **pca_data.mat**, which contains the $m \times n$ matrix X. Columns of X correspond to $n = 10^4$ measurements and rows of X correspond to $m = 2$ quantities (features). Let us see what the data looks like:

```
load pca_data.mat
figure;
plot(X(1,:),X(2,:),'.m','MarkerSize',10);
title("PCA raw data");
axis equal
```



Part (a)

Find the $m \times m$ covariance matrix of X. Use **disp()** to display it.

```
[m, n] = size(X);
X0 = zeros(m, n);
```

```
ct = mean(X, 2);
X0(1, :) = X(1, :) - ct(1);
X0(2, :) = X(2, :) - ct(2);
covariance = (1/n) * (X0) * (X0');
disp('Covariance matrix of X: ');
```

```
Covariance matrix of X:
```

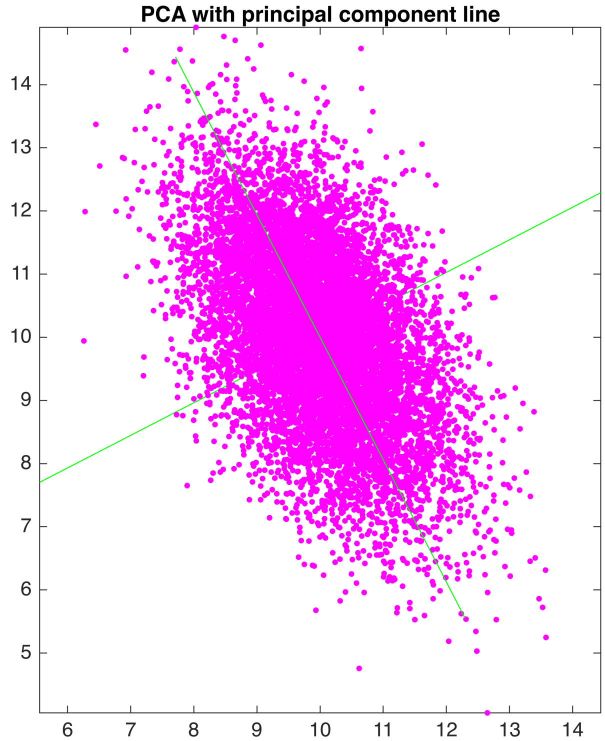
```
disp(covariance);
```

```
    0.9829   -0.6808
   -0.6808    1.9508
```

Part (b)

Find the m principal components of X and show them on top of the original data. That is, if v is a principal component, then plot the line $l(t) = \mu + vt$ over the scatter plot shown above, where $\mu \in \mathbb{R}^2$ is the centroid of the data.

```
%{
Find the eigenvectors
Normalize
Obtain the centroids
Create the lines
Plot
%}
[v, useless] = eig(covariance); %not using power_method in case it's off.
syms t
figure;
x = ct(1) + t * v(1, 1);
y = ct(2) + t * v(2, 1);
fplot(x, y, '-g');
hold on
x = ct(1) + t * v(1, 2);
y = ct(2) + t * v(2, 2);
plot(X(1,:),X(2,:),'.m','MarkerSize',10);
title("PCA with principal component line");
axis equal
fplot(x, y, '-g');
hold off
```



Part (c)

If we change the basis of $\mathbb{R}^m$ from the standard basis to the basis of the principal components $[v_1, \cdots, v_m] = V$ of X, the data matrix will transform to $Y = V^{-1}X$. Find the covariance of the transformed data Y. Use `disp()` to display it.

```
%fetching Y
Y = inv(v) * X;
Y0 = zeros(m ,n);
ct2 = mean(Y, 2);
Y0(1,:) = Y(1, :) - ct2(1);
Y0(2, :) = Y(2, :) - ct2(2);
Covy = (1/n) * (Y0) * (Y0');
disp('Covariance matrix of Y:');
```

```
Covariance matrix of Y:
```

```
disp(Covy)
```

```
    0.6316   -0.0000
   -0.0000    2.3021
```

Part (d)

Principal component analysis is a very popular data analysis technique and it is often used in various applications. As such, it is implemented in most numerical software packages. In MATLAB, this is a built-in function `pca()`. In the simplest form, `V = pca(X)` returns the principal components of the data matrix X, where rows of X correspond to the *measurements* and columns correspond to the *features*. Find the principal components of X using `pca()` and compare them with the ones found in part (b). Are they the same?

```
builtinPCA = pca(X');
disp('Built-in MATLAB PCA Results: ');
```

```
Built-in MATLAB PCA Results:
```

```
disp(builtinPCA);
```

```
   -0.4586    0.8886
    0.8886    0.4586
```

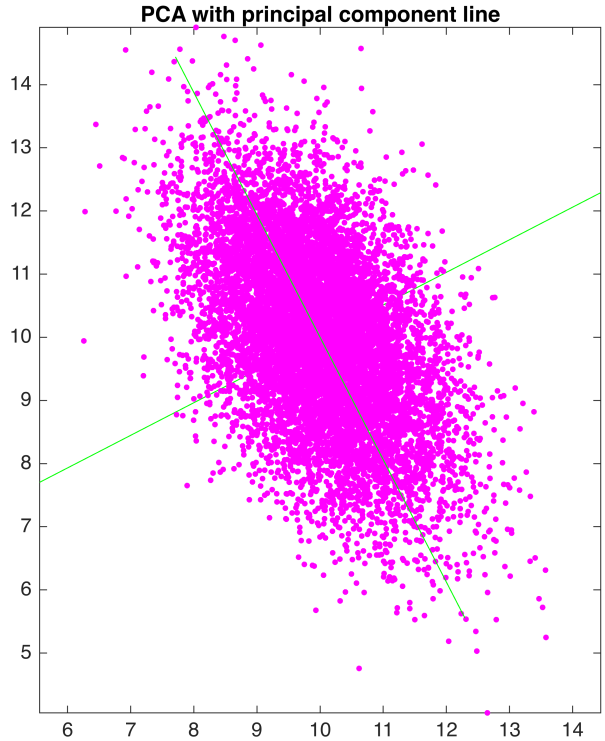
```
disp('PCA results from part (B): ');
```

```
PCA results from part (B):
```

```
disp(v)
```

```
   -0.8886   -0.4586
   -0.4586    0.8886
```

```
syms t
figure;
x = ct(1) + t * v(1, 1);
y = ct(2) + t * v(2, 1);
fplot(x, y, '-g');
hold on
x = ct(1) + t * v(1, 2);
y = ct(2) + t * v(2, 2);
plot(X(1,:),X(2,:),'.m','MarkerSize',10);
title("PCA with principal component line");
axis equal
fplot(x, y, '-g');
x = ct(1) + t * builtinPCA(1, 1);
y = ct(2) + t * builtinPCA(2, 1);
fplot(x, y, '-g');
x = ct(1) + t * v(1, 2);
y = ct(2) + t * v(2, 2);
fplot(x, y, '-g');
hold off
```




```
%as we can see, there are only 2 lines still, which means our results
%perfectly align with the built in PCA! :)
```

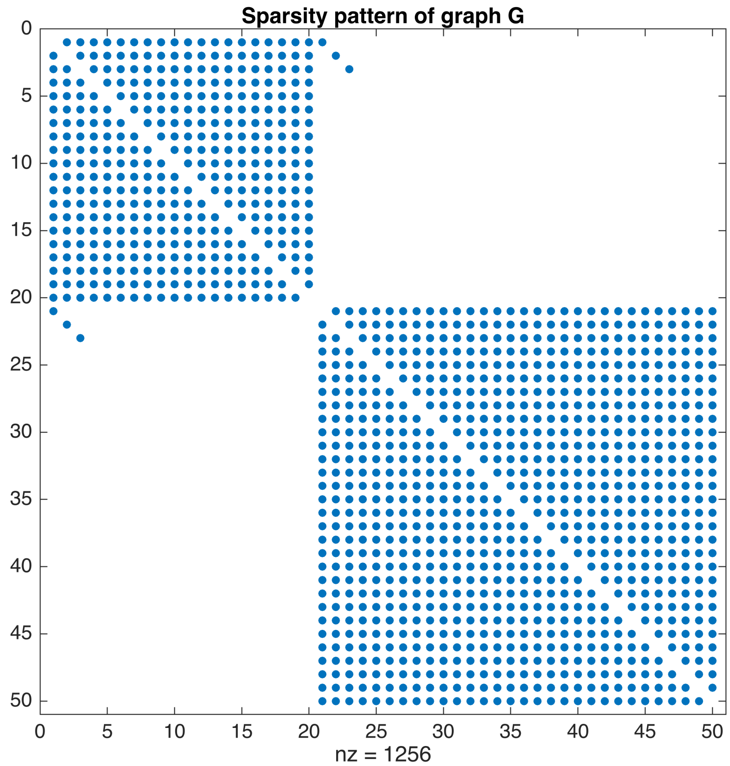
Problem 6 (10 points) Spectral Method for Graph Partitioning

A complete graph K_n with n vertices is a graph in which each pair of graph vertices is connected by an edge. Let G be a barbell-like graph obtained by connecting K_{n_1} and K_{n_2} with $m = 3$ bridges, where $n_1 = 20$ and $n_2 = 30$.

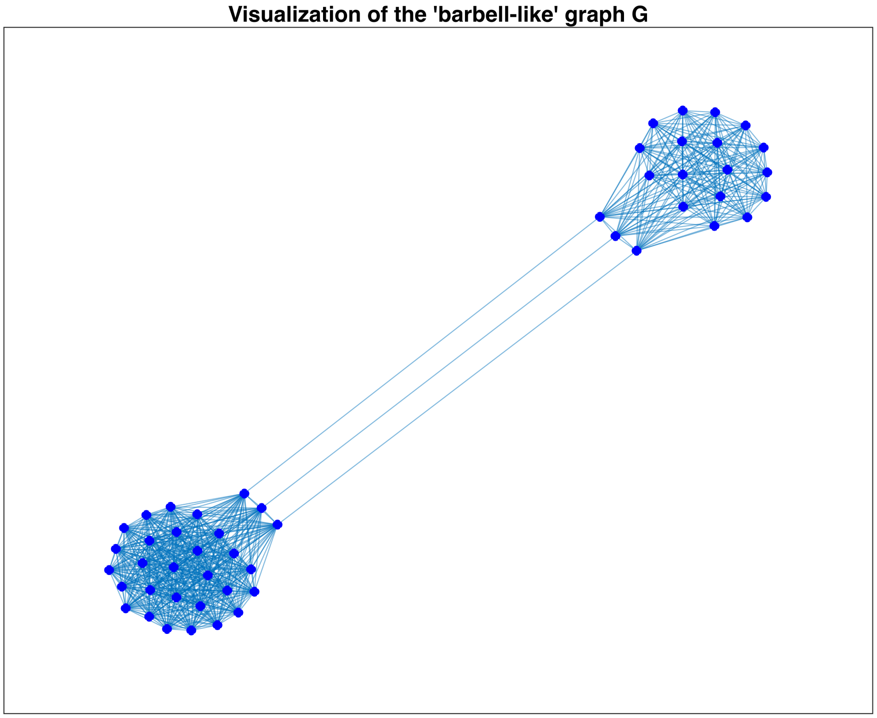
The graph G and the sparsity pattern of its adjacency matrix are shown:

```
%{
Adjacency matrix
%}
n1=20;
n2=30;
B1=ones(n1,n1)-diag(ones(1,n1));
B2=ones(n2,n2)-diag(ones(1,n2));
B=[B1, zeros(n1,n2);zeros(n2,n1), B2]; % adjacency matrix B of graph G
for i=1:3
    B(i,n1+i)=1; B(n1+i,i)=1; % bridges
end
%{
Visualization
%}
figure;
spy(B);
```

```
title("Sparsity pattern of graph G");
```




```
figure;
G = graph(B);
plot(G,'NodeLabel',{},'NodeColor','b');
title("Visualization of the 'barbell-like' graph G");
```



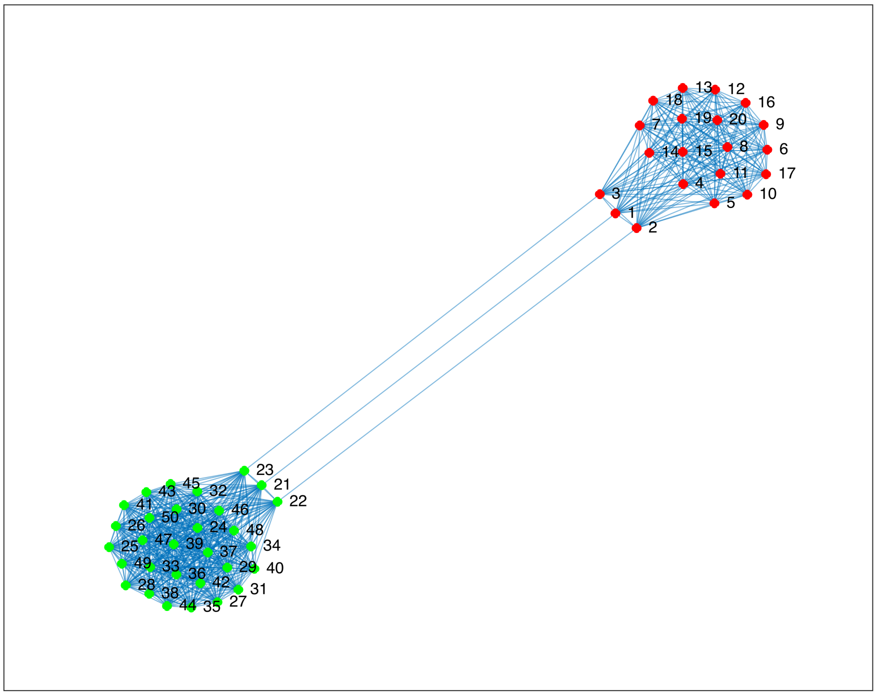
Use the spectral partitioning method to find a division of the graph G into two subgraphs of size $n_1 = 20$ and $n_2 = 30$ such that the number of edges between the subgraphs (the cut size) is minimized. Visualize the obtained partition and find the corresponding cut size. Useful functions for this problem:

eigs(), find(), highlight(), plot()

```
%{
-> Find the graph Laplacian
-> Find the Fiedler vector
-> Determine the 2 candidate partitions
-> Visualize
-> Report the cut size using disp()
%}

% LAPLACIAN & FIEDLER VECTOR
n = 50;
Dv = degree(G);
D = zeros(n);
for i = 1:n
    D(i, i) = Dv(i);
end
L = D - B;
[v, ld] = eigs(L, 2, 'SA');
fiedler = v(:, 2);
% PARTITIONS
```

```
b1 = (n1 - n2)/sqrt(n);
b2 = 2*(sqrt(n1*n2/n));
p1 = b1 * ones(n, 1) + b2 * fiedler;
p2 = b1 * ones(n, 1) - b2 * fiedler;

p1s = sort(p1, 'descend');
p2s = sort(p2, 'ascend');
maxvals = p1s(1:n1, 1);
minvals = p2s(1:n1, 1);
maxindices = zeros(0, 1);
minindices = zeros(0, 1);

for j = 1:n1
    if length(maxindices) < n1
        maxsofar = find(p1 == maxvals(j, 1));
        if ~ismember(maxsofar, maxindices)
            maxindices = [maxindices; maxsofar];
        end
    else
        maxindices = maxindices(1:n1, 1);
    end
    if length(minindices) < n1
        minxsofar = find(p2 == minvals(j, 1));
        if ~ismember(minxsofar, minindices)
            minindices = [minindices; minxsofar];
        end
    else
        minindices = minindices(1:n1, 1);
    end
end

for i = 1:n
    if ismember(i, maxindices)
        p1(i) = 1;
    else
        p1(i) = -1;
    end
    if ismember(i, minindices)
        p2(i) = 1;
    else
        p2(i) = -1;
    end
end

cutsizep1 = 1/4 * p1' * L * p1;
cutsizep2 = 1/4 * p2' * L * p2;
syms cutsizevar;
if cutsizep1 < cutsizep2
    cutsizevar = p1;
else
```

```
    cutsizevar = p2;
end

% VISUALIZATION
gf = plot(G, 'NodeColor', 'g');
locations = find(cutsizevar == 1);
highlight(gf, locations, 'NodeColor', 'r');
```



```
% REPORTING CUT SIZE
disp(cutsizep1);
```

```
    3
```

```
disp(cutsizep2);
```

```
    3
```

```
%theyre equal
```

Problem 1 (10 points) The Power Method (aka Von Mises Iteration)

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ such that:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$$

The eigenvalue λ_1 is called the dominant eigenvalue and the corresponding eigenvector v_1 is called the dominant eigenvector. Write a function that takes A as an argument and uses the Power method (Lecture 12) to compute λ_1 and v_1. We can check that λ_1 is correct by comparing it with the `eig()` function for a random matrix. An example is provided below:

```
E = [9,8,7;6,5,4;3,2,1];
[b, d] = eig(E);
[J, VE] = power_method(E);
disp(d(1));
```

```
16.1168
```

```
disp(J);
```

```
16.4236
```

```
function [lambda, v] = power_method(A)
[n, ~] = size(A); % size of nxn matrix A
v_prev = rand(n, 1); % random starting vector
tol = Inf; % relative error
%{
Implement the algorithm in the while loop
%}
while tol > 1e-15
    intermediate = A * v_prev;
    intermediate = intermediate / norm(intermediate, inf);
    if (norm(intermediate-v_prev, Inf) < tol)
        v_prev = intermediate;
        break;
    end
    v_prev = intermediate;
end
template = A*v_prev;
lambda = template(1)/v_prev(1);
v = v_prev/norm(v_prev);
end
```