ACM 104 Problem Set 3

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Problem 1: Inner Products vs Norms

(a) We need to show that we can construct the arbitrary inner produce $\langle u, v \rangle$ using only the norms of some vectors.

$$||u+v|| = \sqrt{\langle u+v, u+v \rangle}$$

Then we have

$$||u+v||^2 = \langle u+v, u+v \rangle$$

On expanding the right hand

$$||u+v||^2 = \langle u+v, v \rangle + \langle v, u+v \rangle = \langle u, v \rangle + \langle v, v \rangle + \langle v, u \rangle + \langle v, v \rangle = \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle$$

Now we know $\langle u, u \rangle = ||u||^2, \langle v, v \rangle = ||v||^2$, so we get for $\langle u, v \rangle$,

$$\langle u, v \rangle = \frac{||u+v||^2 - ||u||^2 - ||v||^2}{2}$$

(b) We will show that two inner products that induce the same norm must necessarily be non-distinct. Assume for any u, v, that the norms are

$$||u||_1 = ||u||_2, ||v||_1 = ||v||_2 \implies \langle u, u \rangle_1 = \langle u, u \rangle_2, \langle v, v \rangle_1 = \langle v, v \rangle_2$$

for some inner products $\langle u, u \rangle_1, \langle u, u \rangle_2$. Now consider the sum such that

$$\langle u+v, u+v \rangle_1 = \langle u, u \rangle_1 + 2\langle u, v \rangle_1 + \langle v, v \rangle_1$$

$$\langle u+v, u+v \rangle_2 = \langle u, u \rangle_2 + 2\langle u, v \rangle_2 + \langle v, v \rangle_2$$

By assumption $||u+v||_1 = ||u+v||_2$, we must also have

$$\langle u, u \rangle_1 - \langle u, u \rangle_2 + 2\langle u, v \rangle_1 - 2\langle u, v \rangle_2 + \langle v, v \rangle_1 - \langle v, v \rangle_2 = 0$$

Then we simply get $\langle u, v \rangle_1 = \langle u, v \rangle_2$. Thus these are the same inner products for all arbitrary u, v.

Problem 2: Continuously Differentiable Functions

(a) Notice for the first inner product $\langle f, g\rangle_1$ that if the inner product $\langle f, f\rangle_1 = k$ for any $k \neq \mathbf{0} \in \mathbb{R}^n$ then it follows that

$$\langle f, f\rangle_1 = \int_0^1 f'(x)f'(x)\mathrm{d}x = \int_0^1 0 \cdot 0\mathrm{d}x = 0$$

But we know that $\langle f, f\rangle_1 \neq 0$ unless $f = \mathbf{0}$, thus this isn't positive definite. Since the first one isn't an inner product, and one of them has to be an inner product as per the question, we conclude that it's the second one.

(b) Let f and g be two functions, then the inner product

$$\langle f, g\rangle = \int_0^1 (fg + f'g')\mathrm{d}x$$

obeys the following Cauchy-Schwartz Inequality:

$$|\langle f, g\rangle|^2 \leq \langle f, f\rangle \cdot \langle g, g\rangle$$

Now by definition, we have

$$\langle f, f\rangle = \int_0^1 (f^2 + f'^2)\mathrm{d}x$$
$$\langle g, g\rangle = \int_0^1 (g^2 + g'^2)\mathrm{d}x$$

Thus we have

$$|\langle f, g\rangle|^2 \leq \left(\int_0^1 (f^2 + f'^2)\mathrm{d}x\right) \cdot \left(\int_0^1 (g^2 + g'^2)\mathrm{d}x\right)$$

It also obeys the following Triangle inequality:

$$||\langle f+g, f+g\rangle|| \leq ||\langle f, f\rangle|| + ||\langle g, g\rangle||$$

Note that

$$||\langle f+g, f+g\rangle|| = \sqrt{\langle f+g, f+g\rangle} = \sqrt{\int_0^1 ((f+g)^2 + (f'+g')^2)\mathrm{d}x}$$

So finally we have

$$\sqrt{\int_0^1 ((f+g)^2 + (f'+g')^2)\mathrm{d}x} \leq \sqrt{\left(\int_0^1 (f^2 + f'^2)\mathrm{d}x\right)} + \sqrt{\left(\int_0^1 (g^2 + g'^2)\mathrm{d}x\right)}$$

(c) For this part, recall that for two elements $f, g \in C^1[0,1]$, the angle between them with respect to an inner product $\langle -, -\rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{\langle f, g\rangle}{||\langle f, f\rangle|| \cdot ||\langle g, , g\rangle||}\right) = \cos^{-1}\left(\frac{\int_0^1 e^x \mathrm{d}x}{\left(\sqrt{\int_0^1 1\mathrm{d}x}\right) \cdot \left(\sqrt{\int_0^1 2e^{2x}\mathrm{d}x}\right)}\right) = \cos^{-1}\left(\frac{e-1}{\sqrt{e^2-1}}\right) = \cos^{-1}(0.68) \approx 47^\circ$$

Problem 4: Gram Matrices

(a) The Gram matrix G of $v_1, v_2, \dots v_n$ in an inner product space with the inner product $\langle -, - \rangle$ is given by

$$g_{ij} = \langle v_i, v_j \rangle$$

Thus for $1, e^x, e^{2x}$, we obtain

$$G = \begin{pmatrix} \langle 1, 1 \rangle & \langle 1, e^x \rangle & \langle 1, e^{2x} \rangle \\ \langle e^x, 1 \rangle & \langle e^x, e^x \rangle & \langle e^x, e^{2x} \rangle \\ \langle e^{2x}, 1 \rangle & \langle e^{2x}, e^x \rangle & \langle e^{2x}, e^{2x} \rangle \end{pmatrix}$$

When the inner product is L^2, we get

$$G = \begin{pmatrix} \int_0^1 1 \mathrm{d}x & \int_0^1 e^x \mathrm{d}x & \int_0^1 e^{2x} \mathrm{d}x \\ \int_0^1 e^x \mathrm{d}x & \int_0^1 e^{2x} \mathrm{d}x & \int_0^1 e^{3x} \mathrm{d}x \\ \int_0^1 e^{2x} \mathrm{d}x & \int_0^1 e^{3x} \mathrm{d}x & \int_0^1 e^{4x} \mathrm{d}x \end{pmatrix} = \begin{pmatrix} 1 & e-1 & \frac{e^2-1}{2} \\ e-1 & \frac{e^2-1}{2} & \frac{e^3-1}{3} \\ \frac{e^2-1}{2} & \frac{e^3-1}{3} & \frac{e^4-1}{4} \end{pmatrix}$$

(b) The Gram matrix is positive-definite when the elements involved in it are linearly independent, since $1, e^x, e^{2x}$ are obviously linearly independent, G in this case, is positive-definite.

(c) Using the inner product in problem (2), we have

$$G = \begin{pmatrix} \int_0^1 1 \mathrm{d}x & \int_0^1 e^x \mathrm{d}x & \int_0^1 e^{2x} \mathrm{d}x \\ \int_0^1 e^x \mathrm{d}x & \int_0^1 2e^{2x} \mathrm{d}x & \int_0^1 3e^{3x} \mathrm{d}x \\ \int_0^1 e^{2x} \mathrm{d}x & \int_0^1 3e^{3x} \mathrm{d}x & \int_0^1 5e^{4x} \mathrm{d}x \end{pmatrix} = \begin{pmatrix} 1 & e-1 & \frac{e^2-1}{2} \\ e-1 & e^2-1 & e^3-1 \\ \frac{e^2-1}{2} & e^3-1 & 5\frac{e^4-1}{4} \end{pmatrix}$$

Since $(1, e^x, e^{2x})$ are still linearly independent, G is positive definite.

(d) The linear independence of $1, e^x, e^{2x}$ is invariant of the inner product the space is under, so as long as these are the elements involved in G_1 and G_2, they will both have to be positive-definite.

ACM/IDS 104 - Problem Set 3 - MATLAB Problems

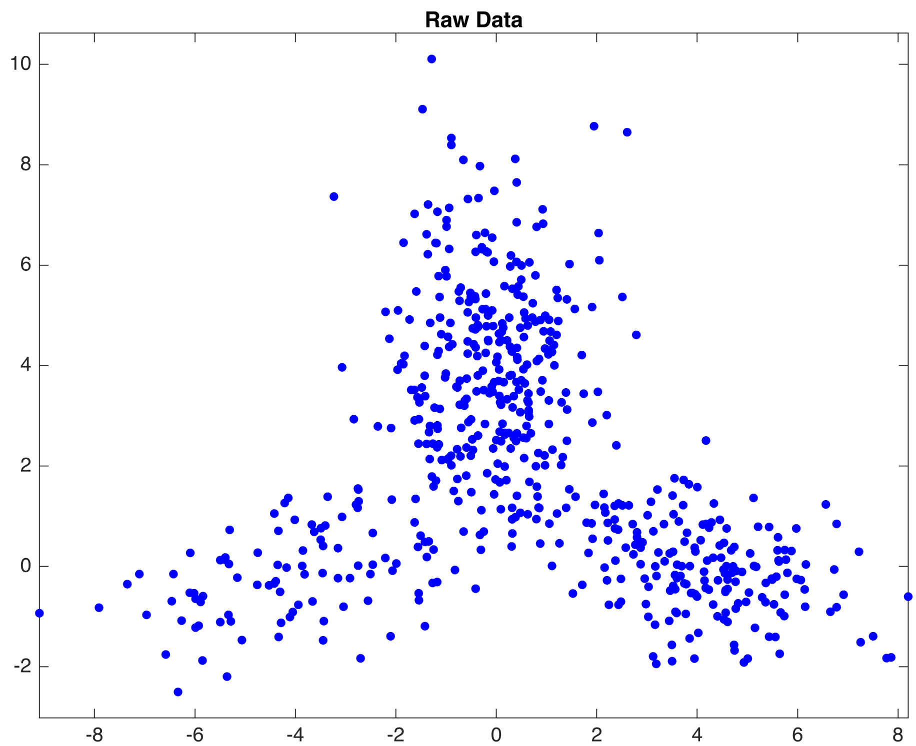
Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

Problem 3 (10 points) The K-Means Algorithm for Clustering

In this problem, we will be implementing the K-Means Algorithm for clustering vectors in $\mathbb{R}^2$. To start, let us load and visualize our dataset.

```
load clustering_data.mat
figure;
plot(x(:,1),x(:,2),'.b','MarkerSize',15);
title("Raw Data")
axis equal;
```




```
rng(2016); % for reproducibility
```

Part (a) The Algorithm

In this part, we will implement the algorithm for $K = 3$, which seems to be a reasonable number of clusters for this dataset. Show the clusters after the 1st, 5th, 10th and last iteration. This will illustrate the cluster evolution as the algorithm runs. `subplot()` is very useful here.

To further aid our understanding of the algorithm, we will be keeping track of a few things:

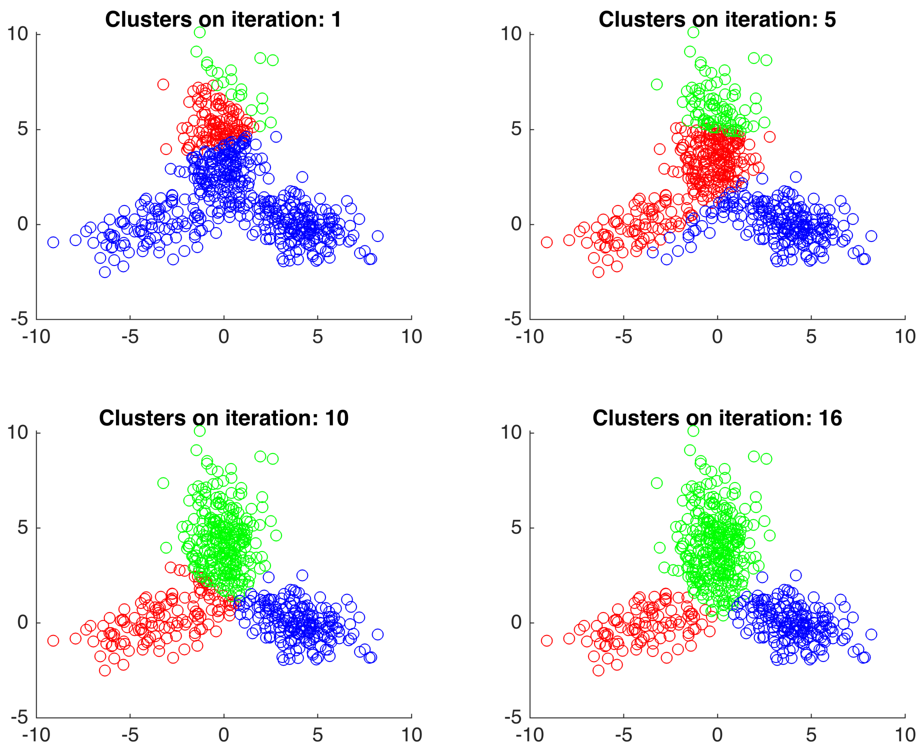
1. Keep track of the evolution of the representatives for each cluster. You can do so by updating the variable `z` after each iteration.
2. Keep track of the objective function value at each iteration.

Once you are done implementing the algorithm, feel free to change the value of `K` and see how the clusterings are altered.

```
%{
Some useful setup done for you
%}
K = 3; % number of clusters
n = length(x); % number of vectors
c = zeros(n, 1); % clustering : c(i) is the cluster number for vector
x(i, :)
iter = 0; % iteration number
z = datasample(x, K); % initial representatives

%{
Step 1: Partitioning into clusters given our initial representatives
%}
for i = 1:n
    c_min = -1;
    dist_min = Inf;
    for k = 1:K
        d = norm(x(i, :) - z(k, :));
        %{calculating distance from initial centroids
        if d < dist_min
            c_min = k;
            dist_min = d;
        end
    end
    %{classifying the point based on the closest centroid
    c(i) = c_min;
end
%{
Increment the iteration number
Update the objective function
%}
plots_done = 1;
objective_new = [];
z_prev = 0;

journey1 = zeros(0,2);
journey2 = zeros(0,2);
journey3 = zeros(0,2); %{initializing jounreys for cluster centroids
```

```
while z_prev ~= z %{while there is still a change in the hypothesis
    z_prev = z(:, :);
    % Incrementing the iteration number, updating cluster centroids
    iter = iter + 1;
    for k = 1:K
        x_k = x(c == k, :); %{all the x that were classified under cluster k
        if ~isempty(x_k)
            z(k, :) = mean(x_k); %{new k is the mean of all these x
        end
    end
    journey1 = [journey1; z(1, :)];
    journey2 = [journey2; z(2, :)];
    journey3 = [journey3; z(3, :)]; %{keep track of centroid journeys after
each update
    for i = 1:n
        c_min = 10;
        dist_min = Inf;
        for k = 1:K
            d = norm(x(i, :) - z(k, :));
            if d < dist_min
                c_min = k;
                dist_min = d;
            end
        end
        c(i) = c_min; %{reclassifying points based on new centroid positions
    end

    p_tot = 0;
    for k = 1:K
        x_k = x(c == k, :);
        for i = 1:length(x_k)
            p_tot = p_tot + (norm(x_k(i, :) - z(k, :))^2);
        end
    end
    p_tot = p_tot/n;

    objective_new = [objective_new, p_tot]; %{appending the new objective

    colors = ['r', 'g', 'b'];

    if ismember(iter,[1,5,10,16]) %{if we have reached an
        subplot(2, 2, plots_done);
        hold on;
        for k = 1:K
            x_k = x(c == k, :);
            color = colors(k);
            scatter(x_k(:, 1), x_k(:, 2), 30, color);
        end
        title(['Clusters on iteration: ' num2str(iter)]);
        hold off;
```

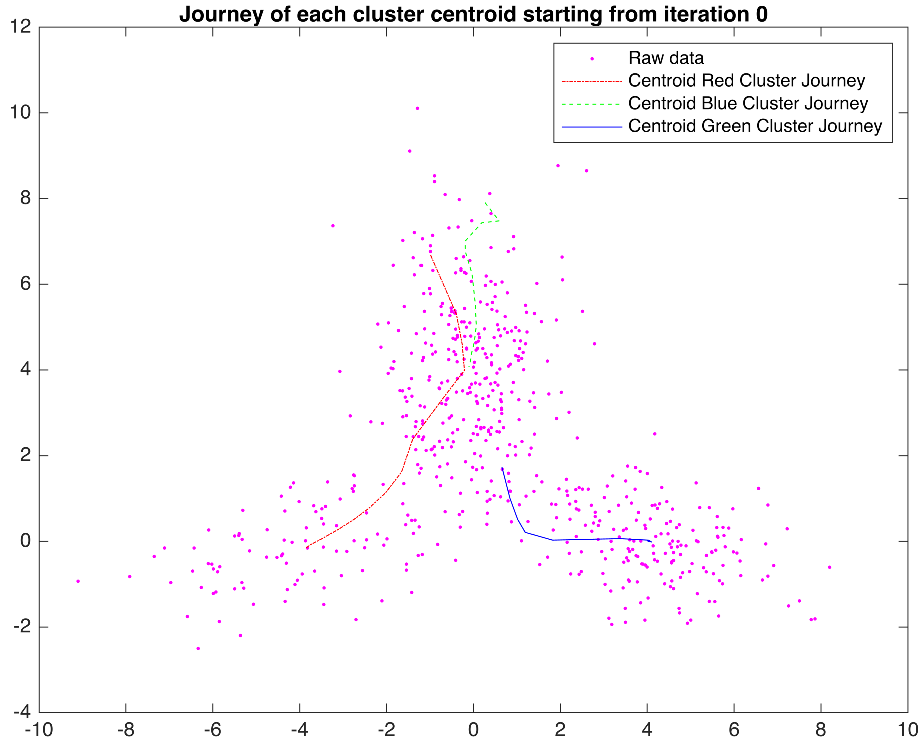
```
        plots_done = plots_done + 1;
    end
end
```



Part (b) Evolution of Representatives

Now, let us see the dynamics of the representatives. Plot the original (raw) data and the trajectories of the cluster representatives in one figure. `squeeze()` might be useful here.

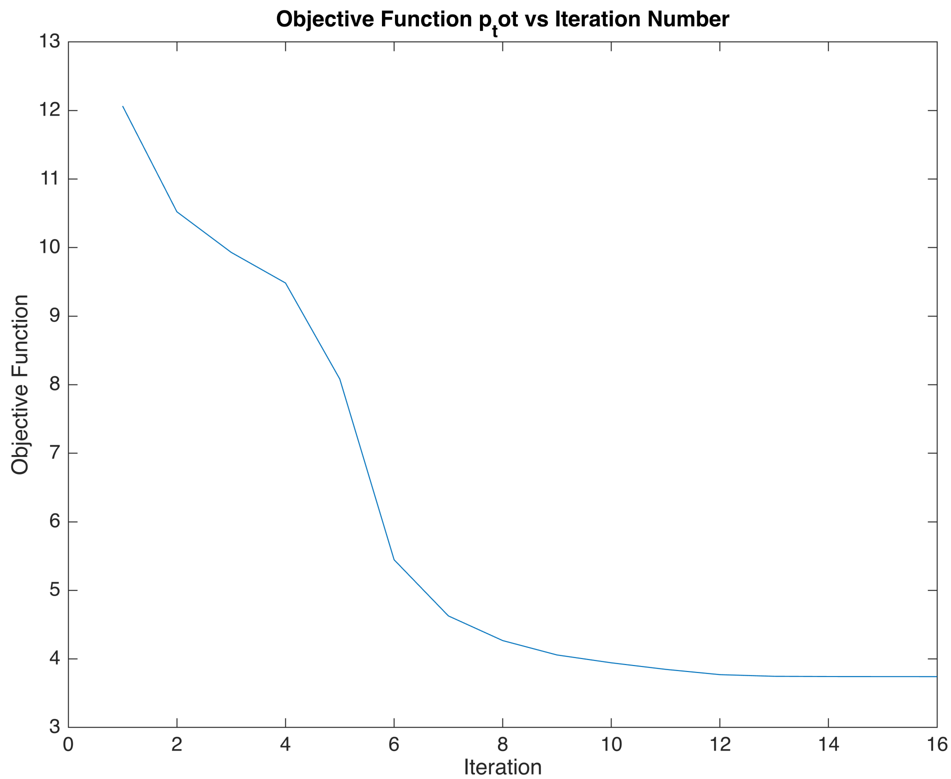
```
figure
plot(x(:, 1), x(:, 2), '.m', journey1(:, 1), journey1(:, 2), '-.r',
journey2(:, 1), journey2(:, 2), '--g', journey3(:, 1), journey3(:, 2), '-b')
legend('Raw data', 'Centroid Red Cluster Journey', 'Centroid Blue Cluster
Journey', 'Centroid Green Cluster Journey');
title('Journey of each cluster centroid starting from iteration 0')
```



Part (c) Objective Function

Plot the values of the objective function against the iteration number. Does the trend match what we expect? Report the minimum value of the objective function using `disp()`.

```
figure;
plot(1:iter, objective_new);
title("Objective Function p_tot vs Iteration Number");
xlabel("Iteration");
ylabel("Objective Function");
```




```
final_list = objective_new(end, :);
disp(strcat('Smallest value of objective function is = ',
num2str(final_list(16))));
```

```
Smallest value of objective function is =3.742
```

Part (d) Comparing with `kmeans()`

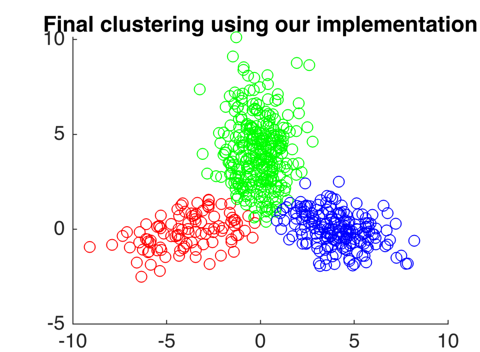
The K-means algorithm is a very popular clustering algorithm and it is often used in various applications. As such, it is implemented in most numerical software packages. In MATLAB, it exists as the built-in function `kmeans()`. In the simplest form, `idx = kmeans(X, k)` performs K-means clustering to partition the rows in the data matrix `X` into `k` clusters, and returns a vector `idx` containing cluster indices for each row. To compare your implementation with the built-in one, plot the clustering your obtained in (a) and the clustering obtained by using `kmeans()`.

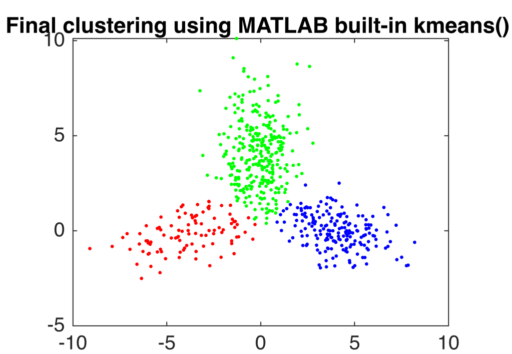
```
plots_done = 1
```

```
plots_done = 1
```

```
subplot(2, 2, plots_done);
hold on;
for k = 1:K
    x_k = x(c == k, :);
    color = colors(k);
    scatter(x_k(:, 1), x_k(:, 2), 30, color);
```

```
end
title('Final clustering using our implementation');
hold off;
builtinplot = subplot(2, 2, 3);
idx = kmeans(x, K);
cluster1points = zeros(0, 2);
cluster2points = zeros(0, 2);
cluster3points = zeros(0, 2);
for i = 1:n
    vector = x(i, :);
    if idx(i) == 1
        cluster1points = [cluster1points; vector];
    end
    if idx(i) == 2
        cluster2points = [cluster2points; vector];
    end
    if idx(i) == 3
        cluster3points = [cluster3points; vector];
    end
end
plot(builtinplot, cluster1points(:, 1), cluster1points(:, 2), '.r',
cluster2points(:, 1), cluster2points(:, 2), '.b', cluster3points(:, 1),
cluster3points(:, 2), '.g');
title('Final clustering using MATLAB built-in kmeans() ');
```





Problem 5 (10 points) Gram Matrices for Text Classification

Gram matrices are used in many different applications. Here is a simple example of application to classification of text documents. Suppose we have a collection of n text documents and we want to measure similarity between them. First, we predefine a list of words (the "dictionary") $w_1, w_2, w_3, \cdots, w_m$ appearing in the documents. Then, for each text i we construct a vector $x_i \in \mathbb{R}^m$ whose j^{th} component is the number of times the word w_j appears in the text. This way, each text is represented as a vector in $\mathbb{R}^m$. This representation is often referred to as the "bag of words" representation.

Let $\langle \cdot, \cdot \rangle$ be the usual dot product in $\mathbb{R}^m$ and $\| \cdot \|$ be the Euclidean norm. Let us normalize all vectors $z_i = \frac{x_i}{\|x_i\|}$ and define:

$$Z = [z_1 \cdots z_n] \in \mathbb{M}_{m \times n}$$

Let G be the Gram matrix associated with $z_1 \cdots z_n$:

$$G = Z^T Z \in \mathbb{M}_{n \times n}$$

Since all z_i are unit vectors:

$$G_{ij} = z_i^T z_j = \|z_i\| \|z_j\| \cos \theta_{ij} = \cos \theta_{ij}$$

where θ_{ij} is the angle between z_i and z_j, which of course is the same as the angle between x_i and x_j. This is widely use as a *measure of similarity* between documents. Hence, the more similar two texts i and j are, the closer G_{ij} is to 1.

Now, let us apply this method to real texts. The task is to compare the text of the US Constitution (found here) with the Wikipedia pages of Bernie Sanders, Hillary Clinton, Donald Trump, Ted Cruz, and John Kasich, and to find the page which is "closest" to the Constitution.

Part (a) Setup

To start, we will define a list of $m = 10$ words. To find the most "relevant words", copy the text of the Constitution and paste it into a word-cloud generator (such as this one). High frequency words appear large and we can form the dictionary from these words. Note that common sense is needed: words like "amendment" and "section" would appear large, but it makes no sense to include them into the dictionary. Also, it makes sense to include "United States" as a single "word", not two separate words, etc. But you are free to chose your own dictionary. In the code box below, please fill out the dictionary with your chosen words.

```
%{
DICTIONARY:
1) president
2) representatives
3) congress
4) members
5) office
6) executive
7) legislative
8) senators
```

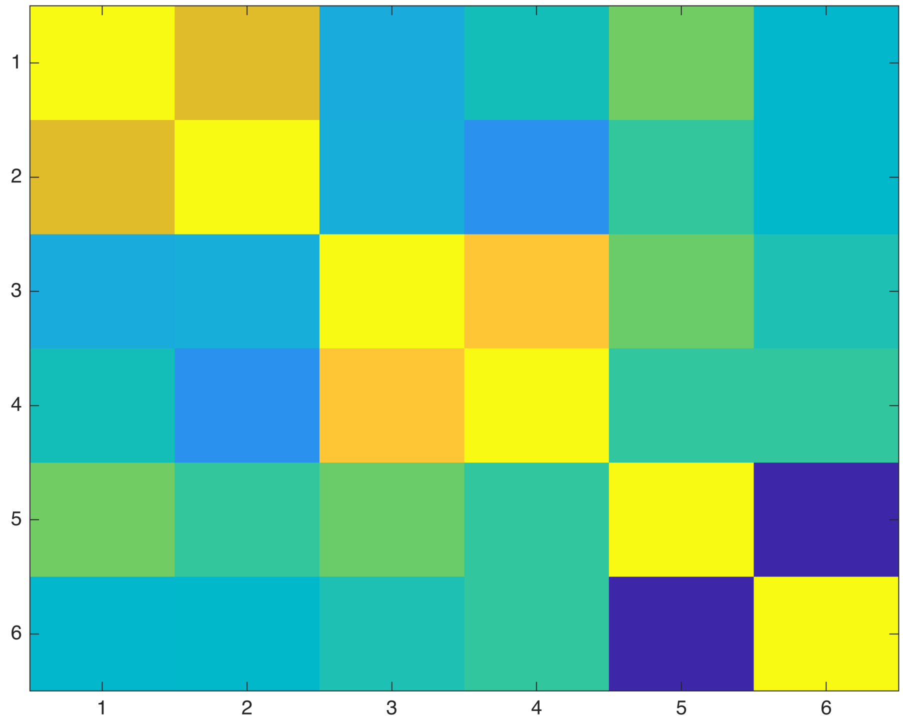
```
9) majority
10) United States
%}
```

Part (b) Fun

With your specified dictionary, let us see which Wikipedia page is closest. First, construct $n = 6$ vectors (as specified above), one for each Wikipedia page and one for the Constitution. No need to get fancy here; just go to the related page and search for the number of occureneces of the words in your dictionary. Put all your results in a column vector for each candidate:

```
%{Creating the dictionaries using ctrl + find to get the occurences
constitution = [ 121 31 62 10 57 16 2 13 14 89 ];
sanders = [ 91 17 61 20 29 0 2 7 3 58 ];
clinton = [ 119 4 25 14 27 0 0 7 3 48 ];
trump = [ 181 7 40 12 50 32 0 0 11 75 ];
cruz = [ 101 1 35 5 21 3 0 18 9 76 ];
kasich = [ 69 20 27 12 23 10 2 7 3 17 ];

%{ Normalising these vectors
constitution = constitution/norm(constitution);
sanders = sanders/norm(sanders);
clinton = clinton/norm(clinton);
trump = trump/norm(trump);
cruz = cruz/norm(cruz);
kasich = kasich/norm(kasich);


%{
Construct the Gram matrix G as discussed
%}
Z = zeros(0, 10);
Z = [Z; constitution];
Z = [Z; sanders];
Z = [Z; clinton];
Z = [Z; trump];
Z = [Z; cruz];
Z = [Z; kasich];
Z = Z';

G = Z' * Z;

maxind = 0;
max = 0;
for i = 2:6
    cos = G(i, 1);
    if cos > max
        max = cos;
        maxind = i;
    end
```

```
end


%{
You can visualize G by uncommenting the following lines of code
%}
figure;
imagesc(G);
```




```



%{
Find the page closest to the Constitution text and report your
answer using disp()
%}
if maxind == 2
    disp('Bernie Sanders');
end
```

```
Bernie Sanders
```

```
if maxind == 3
    disp('Hillary Clinton');
end
if maxind == 4
    disp('Donald Trump');
end
```

```
if maxind == 5
    disp('Ted Cruz');
end
if maxind == 6
    disp('John Kasich');
end
```

Submit the obtained result to the corresponding piazza poll. Remember: this is not about your political preferences, it is about your data analysis :)