ACM - 104, Final

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1) a) Since v is an eigenvector of A and B, if λ_A is the eigenvalue corresponding for A and λ_B for B, then $Av = \lambda_A v$, $Bv = \lambda_B v$. Upon adding,

$$(A+B)v = (\lambda_A + \lambda_B)v,$$

so this is <u>true</u>.

b) Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda = 1$,

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda = 1$,

Then $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, with $\lambda = 2$.

so '1' is not an eigenvalue for $A+B$, this statement is <u>False</u>.

c) Assume for the sake of contradiction that such an A exists. Then, it is representable as $A = QDQ^T$ where D is a diagonalized matrix containing the eigenvalues & Q is an orthogonal matrix containing the eigenvectors. Note then that $Q^T = Q^{-1}$. Finally,

$$A = QDQ^T = QDQ^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \frac{1}{-1}$$

This gives:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \cdot -1$$

$$= \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix} \cdot -1$$

$$= \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$$

This produces a contradiction because A is not symmetric. This statement is False.

d) $A = \begin{bmatrix} 0 & 0 \\ 0 & -2024 \end{bmatrix}$, we know that:

$A^T = \begin{bmatrix} 0 & 0 \\ 0 & -2024 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 0 & 0 \\ 0 & 2024^2 \end{bmatrix}$.

Recall that the eigenvalue of $A^T A$ is precisely the square of the singular value. The $\lambda_1 = 2024^2$, $\lambda_2 = 0$. Now this suggests $\sigma_1 = 2024$, $\sigma_2 = 0$. However 0 is not a singular value because $\nexists\, u \neq 0$ s.t. $A^T A u = 0$. Thus, the statement is True.

e) For a matrix A, let the SVD:

$$A = P \Sigma Q^T,$$

then the pseudo-inverse:

$$A^+ = Q \Sigma^{-1} P^T.$$

Finally, $(A^+)^+ = (P^T)^T (\Sigma^{-1})^{-1} Q^T$

$$= P \Sigma Q^T,$$

by properties of the matrices involved in SVD. Since $P \Sigma Q^T = A$, this statement is True.

2) $V = P^{(1)}$ is a vector space of polynomials $p(x) = ax + b$, $a, b \in \mathbb{R}$. The inner product:

$$\langle p, q \rangle = \int_0^1 x\, p(x)\, q(x)\, dx$$

a) Note if p is orthogonal to $p^* = x$, then

$\langle p, p^* \rangle = \langle p, x \rangle = 0$. (say $p = ax + b$)

$$\Rightarrow \int_0^1 p \cdot x^2 dx = 0 \Rightarrow \int_0^1 (ax+b)x^2 dx$$

$$\Rightarrow \left[\frac{ax^4}{4} + \frac{bx^3}{3}\right]_0^1 = 0 \Rightarrow \frac{a}{4} + \frac{b}{3} = 0 \Rightarrow b = \frac{-3a}{4}$$

All polynomials $p = ax - \frac{3a}{4}$ are orthogonal to $p^* = x$, where $a \in \mathbb{R}$.

b) Say in the previous part that $a = 1$, the $p = x - \frac{3}{4}$, and $p^* = x$. These two are linearly independent, & we only have two free parameters in $P^{(1)}$. Then:

$$\mathcal{U} = \left\{ \frac{x}{\|x\|}, \frac{x - 3/4}{\|x - 3/4\|} \right\},$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\int_0^1 x^3 \, dx} = \sqrt{1/4} = \frac{1}{2},$$

$$\|x - \frac{3}{4}\| = \sqrt{\langle x - \frac{3}{4}, x - \frac{3}{4} \rangle} = \sqrt{\int_0^1 \left(x - \frac{3}{4}\right)^2 x \, dx} = \sqrt{\left[\frac{x^4}{4} - \frac{x^3}{2} + \frac{9x^2}{32}\right]_0^1}$$

$$= \sqrt{\frac{1}{4} - \frac{1}{2} + \frac{9}{32}} = \sqrt{\frac{8}{32} - \frac{16}{32} + \frac{9}{32}} = \sqrt{\frac{1}{32}} = \frac{1}{4\sqrt{2}}$$

Then, $\mathcal{U} = \left\{ 2x, 4\sqrt{2}\,x - 3\sqrt{2} \right\}$

3) First, we find the eigenvalues:

$$|A - \lambda I| = 0 \implies \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = 0$$

$$\implies \lambda^2 + 1 = 0 \implies \lambda = \pm i.$$

Now, we find the corresponding eigenvectors:

when $\lambda_1 = +i$, let $v_1 = \langle x, y \rangle^T$. Then,

$$Av_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$$\lambda_1 v_1 = \begin{pmatrix} xi \\ yi \end{pmatrix}, \quad \begin{pmatrix} xi \\ yi \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$\implies ix = y, \quad iy = -x \implies$ if $x = -i$, $y = 1$,

$v_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$, similarly, when $\lambda_2 = -i$,

$$Av_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

$$\lambda_2 v_2 = \begin{pmatrix} -ix \\ -iy \end{pmatrix}, \quad \begin{pmatrix} -ix \\ -iy \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} \implies \begin{matrix} x = iy \\ y = -ix \end{matrix}$$

when $x = i$, $y = 1$, so $v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$. Finally,

$$A = V \Lambda V^{-1} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}^{-1}$$

4) Let's first find the singular values of this matrix we have:

$$A^T A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Then, for the eigenvalues, we have

$$\det\left[A^T A - \lambda I\right] = \det\begin{pmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 4-\lambda & 0 \\ 0 & 0 & 0 & 9-\lambda \end{pmatrix} = 0$$

$$\Rightarrow -\lambda(1-\lambda)(4-\lambda)(9-\lambda) = 0 \Rightarrow \lambda = 0, 1, 4, 9$$

Then the singular values are $1, 2, 3$.

$$\Sigma = \begin{pmatrix} \sigma_3 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, we find the eigenvectors of $A^T A$.

(Ignoring $\lambda = 0$), when $\lambda = 1$:

$$(A^TA - \lambda I \mid 0) = \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right] \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

when $\lambda = 4$,

$$(A^TA - \lambda I \mid 0) = \left[\begin{array}{cccc|c} -4 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \end{array}\right] \Rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

when $\lambda = 9$,

$$(A^TA - \lambda I \mid 0) = \left[\begin{array}{cccc|c} -9 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

so, $\Phi = [v_3, v_2, v_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\Rightarrow \Phi^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Recall for P that:

$$p_1 = \frac{Av_1}{\sigma_1} = \frac{1}{1} \cdot A\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$p_2 = \frac{Av_2}{\sigma_2} = \frac{1}{2} \cdot A\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$p_3 = \frac{Av_3}{\sigma_3} = \frac{1}{3} \cdot A\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Then $P = [p_3, p_2, p_1]$, and finally:

$$A = P\Sigma Q^T = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}3&0&0\\0&2&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}$$

5) If A is the same as in problem 4, we know that

$$A^T A = \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&4&0\\0&0&0&9\end{bmatrix}$$

Also, $A^T b = \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&2&0&0\\0&0&3&0\end{bmatrix}\begin{bmatrix}0\\0\\0\\2024\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

To solve for: $A^T A x = A^T b$ is akin to:

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array}\right] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, all solutions $\vec{x}_{0_i} = x_{0_i} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, where x_{0_i} is a scalar. Now, to minimize the Euclidian norm, we may simply $x_0 = 0$, then $\vec{x}_0 = x^* = 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

6) We already have the SVD:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The rank-1 truncated SVD:

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the least erroneous rank-1 approx of A when the error is measured with the Frobenius Norm.

7) We are given:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(2-\lambda)^2 - 1\right] + 1\left[(-1)(2-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda)(1-\lambda) - (2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)((3-\lambda)(1-\lambda) - 1) = 0$$

so, when $2-\lambda = 0$, $\lambda_1 = 2 > 0$

when $(3-\lambda)(1-\lambda) - 1 = 0$, $3 - 3\lambda - \lambda + \lambda^2 - 1 = 0$

$$\Rightarrow \lambda^2 - 4\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-8}}{2}, \quad \lambda_2 = \frac{4+2\sqrt{2}}{2} > 0,$$

& $\lambda_3 = \dfrac{4-2\sqrt{2}}{2} > 0$. Hence proved.

8) Recall that if $A = P\Sigma Q^T$ is an SVD of A, then the singular values of A are the diagonal values of Σ.

Then, if $A = P\Sigma Q^T$, we also have:

$$A^T = (P\Sigma Q^T)^T = (Q^T)^T \Sigma^T P^T$$
$$= Q\Sigma^T P^T.$$

Note that this is a SVD of A^T. Now observe that the values on the diagonal of Σ^T are the same as Σ, in fact, $\Sigma = \Sigma^T$, because Σ has $\Sigma_{ij} = 0$, $i \neq j$.

Therefore, the singular values coincide.