ACM 104 Problem Set 2

Amitesh Pandey

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Problem 1: Subspaces

(a) We proceed with a proof by counterexample. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$

Then $\det(A), \det(B) = 0$ but $\det(A+B) = 1 \neq 0$. So $A + B \notin W$, which means W is not closer under addition, so it is not a subspace of V.

(b) (i) See that $\mathbf{0} \in W$ since $\mathrm{tr}\mathbf{0} = 0$. (ii) Now if $A, B \in W$, then

$$\sum_{i=1}^{n} a_{ii} = 0, \sum_{i=1}^{n} b_{ii} = 0 \implies \sum_{i=1}^{n} (a_{ii} + b_{ii}) = 0$$

Then this means $(A + B) \in W$. (iii) Finally also if $A \in W$ then for any $k \in \mathbb{R}$, we have

$$\sum_{i=1}^{n} a_{ii} = 0 \implies \sum_{i=1}^{n} k a_{ii} = 0$$

Since $kA \in W$. So W has an identity, is closer under addition, and scalar multiplication, it is a subspace of V.

(c) We proceed with a proof by counterexample. Let $f_1(x) = 1$, then obviously $f_1(0)f_1(1) = 1$ so $f_1 \in W$. Then for any $k \in \mathbb{R}$, we have $kf_1(x) = k$. So $kf_1(0)kf_1(1) = k^2$. For all $k \in \mathbb{R} \setminus \{-1, 1\}$, this condition fails. Since W is not closed under scalar multiplication, it is not a subspace.

(d) (i) See that if $f(t) = 0$, then

$$f\left(\frac{1}{2}\right) = \int_0^1 f(t) \mathrm{d}t = \int_0^1 0 \mathrm{d}t = 0$$

so $f(t) = 0 \in W$. (ii) Now if $f_1, f_2 \in W$, then

$$(f_1 + f_2)(1/2) = \int_0^1 (f_1 + f_2)(t) \mathrm{d}t = \int_0^1 f_1(t) \mathrm{d}t + \int_0^1 f_2(t) \mathrm{d}t = 0 + 0 = 0$$

So $f_1 + f_2 \in W$. (iii) Now if $f_1 \in W$, then

$$kf_1(1/2) = \int_0^1 kf_1(t) \mathrm{d}t = k \int_0^1 f_1(t) \mathrm{d}t = k \cdot 0 = 0$$

so $kf_1 \in W$. W has an identity, is closed under addition and scalar multiplication, so it's a subspace of V.

(e) (i) See that if $v_z(x, y)$ such that all points (x, y) map to $\mathbf{0}$, then

$$\nabla \cdot v_z = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} = 0 + 0 = 0$$

So $v_z \in W$. (ii) Then for $v, v' \in W$, we have

$$(v + v')(x, y) = [v_1(x, y) + v_1'(x, y), v_2(x, y) + v_2'(x, y)]$$

Then

$$\nabla \cdot (v + v') = \frac{\partial(v_1 + v_1')}{\partial x} + \frac{\partial(v_2 + v_2')}{\partial y} = \frac{\partial v_1}{\partial x} + \frac{\partial v_1'}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_2'}{\partial y} = 0$$

Thus $(v + v') \in W$. (iii) Now if $v \in W$, then for kv, $k \in \mathbb{R}$,

$$\nabla \cdot (kv) = \frac{\partial k v_1}{\partial x} + \frac{\partial k v_2}{\partial y} = k\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right) = k(0) = 0$$

So $kv \in W$. W has an identity, is closed under addition and scalar multiplication, so it's a subspace.

Problem 2: Polynomials

(a) To check whether the quadratics are linearly independent or not, consider the matrix $A = [p_1, p_2, p_3]$, where p_1, p_2, p_3 are the coefficients of the three quadratics. Then

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ -3 & 2 & 1 \end{pmatrix}$$

Now, if $\text{rank}(A) = 3$, then the quadratics are linearly independent. We know rank is invariant of row operations, so apply the following operations: $r_3 := r_3 + 3r_1$, and then $r_3 := r_3 + 2r_2$. Then

$$A' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{pmatrix}$$

Clearly A' is in row-echelon form and its $\text{rank}(A') = 3$, given by the number of non-zero rows it contains. Since $\text{rank}(A') = 3$, we have also that $\text{rank}(A) = 3$, so the three polynomials are linearly independent.

(b) Since we have a collection of three polynomials that are linearly independent and all in $\mathcal{P}^{(2)}$, and the fact that any element in $\mathcal{P}^{(2)}$ can be described by a linear combination of three of its linearly independent elements, we can conclude that the three polynomials span $\mathcal{P}^{(2)}$.

(c) Since p_1, p_2, p_3 are linearly independent and span $\mathcal{P}^{(2)}$, we can conclude that they form a basis for it too. It is easy to check that when $q(x) = 1$, we need coordinates $(-1/8, 1/4, 1/8)$ in this basis to achieve $q(x)$.

Problem 3: Fibonacci Sequences

(a) To show that $\mathcal{F}$ is a vector space, we will first note that $\mathcal{F}$ is a subset of the space of infinitely dimensional real-valued vectors. In other words, $(x_1, x_2, \dots x_n) \in \mathcal{F} \implies (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Now,

1. Identity: When $x_1, x_2 = 0$, $\mathfrak{f}_z = (0, 0, \dots) \in \mathcal{F}$. So $\mathcal{F}$ has an additive identity.
2. Closure under addition: Consider $\mathfrak{f}_1 = (x_1, x_2, \dots)$ and $\mathfrak{f}_2 = (y_1, y_2, \dots)$, both in $\mathcal{F}$, then by definition we have

$$\mathfrak{f}_1 + \mathfrak{f}_2 = \mathfrak{f}_s = (x_1 + y_1, x_2 + y_2, \dots)$$

   Let $\mathfrak{f}_3 = (z_1, z_2, \dots)$. For any $k \geq 3$, it is obvious that $z_k = x_k + y_k$. Also, $z_{k-1} = x_{k-1} + y_{k-1}$ and similarly $z_{k-2} = x_{k-2} + y_{k-2}$. But since $\mathfrak{f}_1, \mathfrak{f}_2 \in \mathcal{F}$, we have $x_k = x_{k-1} + x_{k-2}$, and $y_k = y_{k-1} + y_{k-2}$. Then

$$z_k = x_k + y_k = x_{k-1} + x_{k-2} + y_{k-1} + y_{k-2} = (x_{k-1} + y_{k-1}) + (x_{k-2} + y_{k-2}) = z_{k-1} + z_{k-2}$$

   This is sufficient to show that $\mathfrak{f}_3 = (z_1, z_2, \dots) \in \mathcal{F}$
3. Closure under scalar multiplication: Consider $\alpha \in \mathbb{R}$, and $\mathfrak{f} = (x_1, x_2, \dots) \in \mathcal{F}$, by definintion,

$$\alpha\mathfrak{f} = (z_1, z_2, \dots) = (\alpha x_1, \alpha x_2, \dots)$$

   For a $k \geq 3$, we have

$$z_k = \alpha x_k = \alpha(x_{k-1} + x_{k-2}) = \alpha x_{k-1} + \alpha x_{k-2} = z_{k-1} + z_{k-2}$$

   Thus $\alpha\mathfrak{f} \in \mathcal{F}$. So it is closed under scalar multiplication.

Note that we are permitted to use the distributivity and commutativity of addition because we have already observed that $\mathcal{F}$ is a subset of $\mathbb{R}^n$ for arbitrary n. Since $\mathcal{F}$ obeys (1), (2), and (3), it is a vector space.

(b) Note that any generalized Fibonacci sequence only depends on the first two x_1, x_2, so $\dim(\mathcal{F}) = 2$. To find $\mathcal{F}$'s basis, we need to find exactly two elements $\mathfrak{f}_1, \mathfrak{f}_2 \in \mathcal{F}$ that span $\mathcal{F}$. Note that $\mathfrak{f}_1 = (1, 0, \dots)$ and $\mathfrak{f}_2 = (0, 1, \dots)$ are both linearly independent of each other and span $\mathcal{F}$. Thus the basis for $\mathcal{F}$ is $\langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle$.

(c) It is easy to check that the coordinates of $\mathfrak{f}^*$ in that basis are $(1, 1)$ since $1 \cdot \mathfrak{f}_1 + 1 \cdot \mathfrak{f}_2 = (1, 1, \dots) = \mathfrak{f}^*$.

Problem 4: Fundamental Matrix Subspaces: Low-Dim Example

(a) Recall that the kernel of A, or $\mathrm{Ker}(A)$ is defined as

$$\mathrm{Ker}(A) = \{\mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} = \mathbf{0}\}$$

Essentially, any $\mathbf{x} = \langle x_1, x_2 \rangle \in \mathrm{Ker}(A)$ obeys

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \\ a_{31}x_1 + a_{32}x_2 &= 0 \end{aligned}$$

On substituting from the matrix, we get

$$\begin{aligned} 2x_1 + 0x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \\ 20x_1 + 24x_2 &= 0 \end{aligned}$$

It's easy to see that the solution set to the above equations is unique, thus $\mathrm{Ker}(A) = \{\mathbf{0}\}$, or its $\dim(\mathrm{Ker}(A)) = 0$, since it has no basis. Now for the cokernel of A, consider instead the matrix

$$A^T = \begin{pmatrix} 2 & 2 & 20 \\ 0 & 2 & 24 \end{pmatrix}$$

Then $\mathrm{coKer}(A) = \mathrm{Ker}(A^T)$, which is the solution set of

$$\begin{aligned} 2x_1 + 2x_2 + 20x_3 &= 0 \\ 0x_1 + 2x_2 + 24x_3 &= 0 \end{aligned}$$

Which simplifies to $x_1 = 2x_3$, $x_2 = -12x_3$. This can be reduced to the form

$$\begin{pmatrix} 2 \\ -12 \\ 1 \end{pmatrix} t = \mathbf{x}$$

Since the vector $\langle 2, -12, 1 \rangle^T$ is a basis for $\mathrm{Ker}(A^T)$, we conclude that $\dim(\mathrm{Ker}(A^T)) = \dim(\mathrm{coKer}(A)) = 1$. For the image and coimage of A, note that the reduced row-echelon form of A is given by

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This means that $\mathrm{Im}(A) = \mathrm{span}\{\langle 1, 0, 0 \rangle^T, \langle 0, 1, 0 \rangle^T\}$, since the image is the span of the column vectors and that $\mathrm{coIm}(A) = \mathrm{span}\{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$, since it is the span of the non-zero row vectors. This mean that that

$$\begin{aligned} \dim(\mathrm{Im}(A)) &= 2 \\ \dim(\mathrm{coIm}(A)) &= 2 \end{aligned}$$

(b) Since we did not use the rank-nullity theorems for part (a), we have the bases readily available to us in our solution to (a). On compiling the different bases we calculated, we have no basis for $\mathrm{Ker}(A)$, and $\langle 2, 12, 1\rangle^T$ as the basis for $\mathrm{coKer}(A)$, and $\{\langle 1,0,0\rangle^T, \langle 0,1,0\rangle^T\}$ as the basis for $\mathrm{Im}(A)$, and $\{\langle 1,0\rangle, \langle 0,1\rangle\}$ as the basis for $\mathrm{coIm}(A)$.

Problem 5: Fundamental Matrix Subspaces: High-Dim Example

Let's proceed with the kernel of B. First note that starting at $k = n, n-1, \dots, 2$, we can do: $r_k := r_k - r_{k-1}$.

$$B' = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \vdots & \vdots \\ n & n & \dots & n \end{pmatrix} \xrightarrow[\forall k, 3\le k\le n]{r_k := r_k - r_2} \begin{pmatrix} 1 & 2 & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\forall i, 1\le i\le n]{r_2 := r_2 - r_1} \begin{pmatrix} 1 & 2 & \dots & n \\ 0 & -n & \dots & -n^2+n \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we do $r_2 := r_2/(-n)$, to get

$$B' = \begin{pmatrix} 1 & 2 & \dots & n \\ 0 & 1 & \dots & n-1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 := r_1 - r_2} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & n-1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_1 := r_1 - r_2} \begin{pmatrix} 1 & 0 & \dots & 2-n \\ 0 & 1 & \dots & n-1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced row-echelon form of B. For $\mathrm{Ker}(B)$, we would have $\mathbf{x}$ such that

$$B'\mathbf{x} = 0$$

Let $\mathbf{x} = \langle x_1, x_2, \dots, x_n\rangle$, then

$$x_1 + \sum_{i=3}^{n}(2-i)x_i = 0$$
$$x_2 + \sum_{i=3}^{n}(i-1)x_i = 0$$

This makes it clear that x_1, x_2 are the pivots and we have $n-2$ free variables $x_3, \dots, x_n$ such that we only need

$$x_1 = -\sum_{i=3}^{n}(2-i)x_i$$
$$x_2 = -\sum_{i=3}^{n}(i-1)x_i$$

Now, notice if for each solution $\mathbf{s}_i = \langle s_1, s_2, \dots, s_n\rangle$, $3 \le i \le n$, we set $s_i = 1$ and all other free entries zero, fill s_1, s_2 as per the formula above, we will have produced $n-2$ linearly independent vectors in the kernel that span it. Thus the set $S = \{\mathbf{s}_i \mid 3 \le i \le n\}$ spans $\mathrm{Ker}(B)$ and forms a basis for it. Since the reduced row-echelon form of B^T can be shown to be the same as that of B, the basis for the cokernel of B is the same as the basis of the kernel of B. Now for the image of B, through elementary row operations, we can conclude that the columns $c_1 = [1, n+1, \dots, n^2+n-1]^T$ and $[1, 1, \dots, 1]$ span $\mathrm{rref}(B)$. The vector $[1, 1, \dots, 1]^T$ is obtained trivially by $c_2 - c_1$. So c_1, c_2 are linearly independent column vectors that span $\mathrm{Im}(B)$, thus they form a basis for it. For the coimage of B, note that the rows of B follow the same pattern of being producible as long as the initial element and the difference are provided. So, r_1, r_2, the first two rows of B will span the coimage by the same logic and form a basis for it.

ACM/IDS 104 - Problem Set 2 - MATLAB Problems

Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

Problem 6 (10 points) Fundamental Matrix Subspaces

Your task for this problem is to write a function that takes a matrix `A` as its argument, and outputs four matrices: `K, I, cK` and `cI` where:

- Columns of `K` form a basis of the kernel of `A`. If $\ker A = \{0\}$, then `K` must be a zero vector of the appropriate dimension.
- Columns of `I` form a basis of the image of `A`. If $\operatorname{im} A = \{0\}$, then `I` must be a zero vector of the appropriate dimension.
- Columns of `cK` form a basis of the cokernel of `A`. If $\operatorname{coker} A = \{0\}$, then `cK` must be a zero vector of the appropriate dimension.
- Columns of `cI` form a basis of the coimage of `A`. If $\operatorname{coim} A = \{0\}$, then `cI` must be a zero vector of the appropriate dimension.

Move to the bottom of this livescript to write the function.

Now, let us test our function:

```
A = [2 0; 2 2; 20 24] % feel free to define A as you like
```

```
A = 3×2
     2     0
     2     2
    20    24
```

```
[K, I, cK, cI] = subspacer(A); % this is how you call a MATLAB function
disp(K);
```

```
     0
     0
```

```
disp(I);
```

```
   -0.0409   -0.9856
   -0.0897   -0.1597
   -0.9951    0.0549
```

```
disp(cK);
```

```
   -0.1638
    0.9831
   -0.0819
```

```
disp(cI);
```

```
  -0.6423   -0.7665
  -0.7665    0.6423
```

START HERE by writing the function:

```
function [K, I, cK, cI] = subspacer(A)
%{
This is the MATLAB function syntax.
-> [K, I, cK, cI] are the outputs of the function.
-> "subspacer" is the name of the function. (you can change that if
                            you wish but make sure you change
                            every function call as well!)
-> A is the argument of the function.
%}
[m, n] = size(A);
r = rank (A);
r2 = rank(A');
%{
We start by finding out the dimensions and rank of A.
Let us consider the matrix K. There exist 2 cases:
1) The kernel is trivial i.e. kerA = {0}
2) The kernel is not trivial -> Hint: use null()
Complete the following if/else statement.
%}
if r == n % this condition is done for you
    K = zeros(n,1);
else
    K = null(A);
end
%{
Now, let us consider the matrix cK.
As above, there exist 2 cases. Remember, you can use ' to
transpose a matrix.
Write a similar if/else statement to produce cK.
%}
if r2 == m
    cK = zeros(m,1);
else
    cK = null(A');
end

%{
For the image I and coimage cI, there exists only 1 condition
we must test, and that is if rankA = 0. With this in mind,
complete the following if/else statement.
-> Hint: orth() is useful here.
%}
if r == 0
    I = zeros(m,1);
```

```
    cI = zeros(n,1);

else
    I = orth(A);
    cI = orth(A');

end
end
```