ACM 104 Midterm

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November 2024

Problem 1

(a) *False*. If u is a linear combination of v and w,

$$u = k_1 v + k_2 w$$

for $k_1, k_2 \in \mathbb{Z}$. However, we can solve for v as follows:

$$\frac{1}{k_1}u - \frac{k_2}{k_1}w = v$$

It is obvious that $1/k_1$ and k_2/k_1 need not be integers, thus the statement is false.

(b) *True.* First note that if $\text{Ker}A = \text{Ker}B$, then A and B must have the same number of columns n. This is because for any matrix $T \in M_{m\times n}$ over field $\mathbb{R}$, the kernel of T is defined as

$$\text{Ker}T = \{\mathbf{x} \in \mathbb{R}^n \mid T\mathbf{x} = \mathbf{0}\}$$

It wouldn't be possible for all elements of A and B to have equal length unless the number of columns of the two matrices were equal. Also trivially $\dim(\text{Ker}A) = \dim(\text{Ker}B)$. By the rank-nullity theorem

$$\dim(\text{Ker}A) = n - \text{Rank}A = \dim(\text{Ker}B) = n - \text{Rank}B \implies \text{Rank}A = \text{Rank}B$$

(c) *False.* Consider the counterexample:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Then, for the reduced row-echelon form we have

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

So then we have

$$\text{Ker}A = \{\langle 0, 1\rangle^T k \mid k \in \mathbb{R}\}$$
$$\text{Im}A = \{\langle 1, 1\rangle^T\}$$

Clearly $\text{Ker}A \cap \text{Im}A \neq \{0\}$.

(d) *True*. If A and B are positive definite, we know $x^T Ax, x^T Bx > 0$ for all $x \neq \mathbf{0}$. This then implies

$$\begin{aligned} x^T Ax + x^T Bx &> 0 \\ (x^T)^{-1}(x^T Ax + x^T Bx) &> 0 \\ (A+B)x &> 0 \\ x^T(A+B)x &> 0 \end{aligned}$$

Thus $A+B$ is also positive definite.

(e) *True*. This is because $||u|| \leq ||u+v+(-v)|| = ||u+v|| + ||-v||$ by the Triangle inequality, but $||-v|| = ||v||$ so we finally have $||u|| < ||u+v|| + ||v||$ as desired.

Problem 2

Solution. We need to check for k_1, k_2, k_3, k_4 that

$$k_1 \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + k_2 \begin{pmatrix} e \\ \pi \\ \sqrt{104} \end{pmatrix} + k_3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{pmatrix} + k_4 \begin{pmatrix} x \\ 0 \\ 4x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get the system of equations

$$\begin{aligned} k_1 + k_2 e + k_3/2 + k_4 x &= 0 \\ k_2\pi + k_3/3 &= 0 \\ 4k_1 + \sqrt{104}k_2 + k_3/4 + 4xk_4 &= 0 \end{aligned}$$

But then $k_2 = \frac{k_3}{3\pi}$, then substitute this and multiply the first equation by 4 on each side

$$4k_1 + \frac{4e}{3\pi}k_3 + 2k_3 + 4xk_4 = 0$$

Now subtracting from the third equation, we get

$$\left(\frac{4e}{3\pi} + 2 - \frac{\sqrt{104}}{3\pi} - \frac{1}{4}\right)k_3 = 0 \implies k_3 = 0 \implies k_2 = 0 \implies k_1, k_4 = 0$$

For no value of x can this system be solved, thus these vectors are never linearly independent. Conversely, they are linearly dependent for all $x \in \mathbb{R}$.

Problem 3

Solution. When $x = -1$, we have the equation $a - b + c = 0 \implies c = b - a$, so we have the quadratic polynomials of form $p(x) = ax^2 - bx + b - a = (x^2 - 1)a + (1 - x)b$. Observe that $\{t^2 - 1, 1 - t\}$ spans this vector space, and they are obviously linearly independent, thus they form a basis for this space.

Problem 4

Solution. Let's first find the reduced row-echelon form of A. To do so:

$$\text{rref}(A) = \begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix} \xrightarrow{r_2 := r_2 - 3r_1} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

Let's also find $\text{rref}(A^T)$, which is

$$\text{rref}(A^T) = \begin{pmatrix} 1 & 3 \\ -3 & -9 \end{pmatrix} \xrightarrow{r_2 := r_2 + 3r_1} \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

Now, the kernel of A is simply given by $\mathbf{x} = \langle x_1, x_2 \rangle^T$ such that

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives us $x_1 - 3x_2 = 0$, thus

$$\text{Ker}A = \{\langle 3t, t \rangle^T \mid t \in \mathbb{R}\}$$

The basis for $\text{Ker}A$ is trivially $\{\langle 3, 1 \rangle\}$. For the $\text{CoKer}A$, we have $\mathbf{x} = \langle x_1, x_2 \rangle^T$ such that

$$\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which simply gives us $x_1 = -3x_2$, thus

$$\text{CoKer}A = \{\langle -3t, t \rangle \mid t \in \mathbb{R}\}$$

The basis for $\text{CoKer}(A)$ is trivially then $\{\langle -3, 1 \rangle\}$. For the image of A, $\text{Im}(A)$, it's just the set of columns of the original matrix that correspond to $\text{rref}(A)$'s pivot. Clearly the basis for the image then is the first column of A, $\{\langle 1, 3 \rangle\}$. For the co-image of A, we simply have the image of A^T, once again, the first column of A^T corresponds to $\text{rref}(A^T)$'s pivot, so the basis for the coimage is simply the first column of A^T or $\{1, -3\}$.

Final Answer

These are the final bases compiled:

$$\begin{aligned} \text{Basis of Ker}A &= \{\langle 3, 1 \rangle\} \\ \text{Basis of Im}A &= \{\langle 1, 3 \rangle\} \\ \text{Basis of CoKer}A &= \{\langle -3, 1 \rangle\} \\ \text{Basis of CoIm}A &= \{\langle 1, -3 \rangle\} \end{aligned}$$

Problem 5

Solution. For the angle θ between x and x^2 we have

$$\theta = \cos^{-1}\left(\frac{\langle x, x^2 \rangle}{||\langle x, x \rangle|| \cdot ||\langle x^2, x^2 \rangle||}\right)$$

Now let's first compute

$$\langle x, x^2 \rangle = \int_0^1 (x^3 + 2x)\mathrm{d}x = \left[\frac{x^4}{4} + x^2\right]_0^1 = \frac{5}{4}$$

Now let's compute

$$||\langle x, x \rangle|| = \sqrt{\int_0^1 (x^2 + 1)\mathrm{d}x} = \sqrt{\left[\frac{x^3}{3} + 1\right]_0^1} = \sqrt{\frac{4}{3}}$$

Now let's compute

$$||\langle x^2, x^2 \rangle|| = \sqrt{\int_0^1 (x^4 + 4x^2)\mathrm{d}x} = \sqrt{\left[\frac{x^5}{5} + 4\frac{x^3}{3}\right]_0^1} = \sqrt{\frac{23}{15}}$$

Finally, for θ we have

$$\theta = \cos^{-1}\left(\frac{5/4}{\sqrt{\frac{4}{3}}\sqrt{\frac{23}{15}}}\right) = \cos^{-1}\left(\frac{15\sqrt{5}}{8\sqrt{23}}\right) = \cos^{-1}\left(\frac{33.54}{38.36}\right) \implies \theta = 29^\circ$$

Problem 6

Solution. Notice that maximizing $f(x, y, z)$ is akin to minimizing $g(x, y, z) = -f(x, y, z)$, specifically (with $\mathbf{x} = \langle x, y, z \rangle$) minimizing

$$g(\mathbf{x}) = -z + x + y + x^2 + y^2 + z^2 + xz$$

For some $\mathbf{x} \in \mathbb{R}^n$, when $K = K^T$, $g(\mathbf{x})$ can be expressed as a general quadratic form:

$$g(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathfrak{f} + c$$

Notice that K is simply

$$K = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

To calculate $\mathfrak{f}$, it is easy to multiply $\mathbf{x}^T K \mathbf{x}$, set $c = 0$, and find out that

$$\mathfrak{f} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix}$$

takes care of the 1st order terms of $g(\mathbf{x})$. Now the global minimizer (unique) is to be $x^* = K^{-1}\mathfrak{f}$ if K is positive definite. Let's check through Sylvester's criterion if this is the case.

1. Upper Left 1-1 determinant: $\det\begin{pmatrix} 1 \end{pmatrix} = 1 > 0$
2. Upper Left 2-2 determinant:
$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \times 1 - 0 = 1 > 0$$
3. Upper Left 3-3 determinant:
$$\det\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} = 1 \times 1 \times \frac{3}{4} = \frac{3}{4} > 0$$

Since K is positive definite, solving for $x^* = K^{-1}\mathfrak{f}$ provides us with a unique global minimum for $g(\mathbf{x})$. At this point, we can simply calculate the following:

$$x^* = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 0 & \frac{-2}{3} \\ 0 & 1 & 0 \\ \frac{-2}{3} & 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{-1}{2} \\ 1 \end{pmatrix}$$

Thus g is minimized globally and uniquely at $x = -1, y = -0.5, z = 1$ thus $f(x, y, z)$ is maximized at this point. Specifically, the maximum value is

$$f\left(-1, \frac{-1}{2}, 1\right) = 3.5 - 2.25 = 1.25 = \frac{5}{4}$$

Problem 7

Solution. For this problem, first note the linear system can be represented in the form $A\mathbf{x} = b$, with $\mathbf{x} = \langle x, y \rangle$ as

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{pmatrix} \langle x, y \rangle = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

Notice that the kernel of A contains only the zero vector because $x + 2y = 0, -x + 2y = 0$ can never have a solution where $x \neq 0$, but that implies $y = 0$, which is the only solution. Since $\text{Ker}A = \{\mathbf{0}\}$, the least squares solution to $A\mathbf{x} = b$ is given simply by calculating

$$\begin{aligned} x^* &= (A^T A)^{-1} A^T b \\ &= \left(\begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -1 & 2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 11 & -3 \\ -3 & 9 \end{pmatrix}^{-1} \left(\begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right) \\ &= \begin{pmatrix} \frac{1}{10} & \frac{1}{30} \\ \frac{1}{30} & \frac{11}{90} \end{pmatrix} \begin{pmatrix} -3 \\ 10 \end{pmatrix} = \begin{pmatrix} \frac{1}{30} \\ \frac{101}{90} \end{pmatrix} \end{aligned}$$

Thus $x^* = \langle \frac{1}{30}, \frac{101}{90} \rangle^T$ is the least squares solution

Problem 8

Proof. Recall to show two sets $\text{Ker}(A) = \text{Ker}(A^T A)$, it is sufficient to show for arbitrary $u \in \text{Ker}(A)$, $u \in \text{Ker}(A^T A)$ and for arbitrary $v \in \text{Ker}(A^T A)$, $v \in \text{Ker}(A)$. In the first case, $u \in \text{Ker}(A) \implies A \cdot u = 0 \implies A^T A u = 0 \implies u \in \text{Ker}(A^T A)$. In the second case, $v \in \text{Ker}(A^T A) \implies A^T A v = 0 \implies A^T v^T A v = 0 \implies (Av)^T A v = 0$ but $(Av)^T Av$ induces the standard norm $|| \cdot ||$, thus $||Av||^2 = 0 \implies ||Av|| = 0 \implies Av = 0$ thus $v \in \text{Ker}(A)$. $\square$