ACM 104 Problem Set 4

Amitesh Pandey

November 2024

Problem 1: Least Squares Solution

Solution. The least squares solution to $Ax = b$ is unique if and only if the columns of A are linearly independent. It's easy to see that the columns of A indeed are linearly independent. For invertible $A^T A$, the solution is given by

$$\hat{x} = (A^T A)^{-1} A^T b$$

This means that

$$\begin{aligned}
\hat{x} &= \left(\begin{pmatrix} 1 & 0 & 1 & -3 \\ 2 & -2 & 5 & 1 \\ -1 & 3 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 1 & -3 \\ 2 & -2 & 5 & 1 \\ -1 & 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 6 \\ 8 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 & -3 \\ 2 & -2 & 5 & 1 \\ -1 & 3 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 6 \\ 8 \end{pmatrix} \\
&= \begin{pmatrix} \frac{132}{1171} & \frac{6}{1171} & \frac{61}{1171} \\ \frac{6}{1171} & \frac{107}{2342} & \frac{56}{1171} \\ \frac{61}{1171} & \frac{56}{1171} & \frac{179}{1171} \end{pmatrix} \cdot \begin{pmatrix} -18 \\ 28 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}
\end{aligned}$$

The least squares error is given by $\varepsilon = ||A\hat{x} - b||^2$, but $A\hat{x}$ exactly equals b for us, so the least squares error is 0.

Problem 2: Interpolation for Integration

Solution. (a) Given only two interpolating points x_0, x_1, we will have an interpolating polynomial of degree 1, or $p_1(x)$. Generally, this polynomial for $f(x)$ is given by

$$p_1(x) = f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}x + \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1}$$

Letting $x_0 = a, x_1 = b$, we obtain

$$p_1(x) = \frac{f(b) - f(a)}{b - a}x + \frac{af(b) - bf(a)}{b - a}$$

Then

$$\int_a^b f(x)\mathrm{d}x \approx \int_a^b \left(\frac{f(b) - f(a)}{b - a}x + \frac{af(b) - bf(a)}{b - a} \right) \mathrm{d}x = \left(\frac{b - a}{2} \right) (f(b) + f(a))$$

(b) Now, letting $x_0 = \frac{2}{3}a + \frac{1}{3}b$, $x_1 = \frac{1}{3}a + \frac{2}{3}b$, we have

$$p_1(x) = \frac{f(x_0) - f(x_1)}{b-a}(3x) + 3\left(\frac{x_0 f(x_1) - x_1 f(x_0)}{b-a}\right)$$

$$\int_a^b f(x)\mathrm{d}x \approx \int_a^b \left(\frac{f(x_0) - f(x_1)}{b-a}(3x) + 3\left(\frac{x_0 f(x_1) - x_1 f(x_0)}{b-a}\right)\right)\mathrm{d}x$$

This gets us

$$\frac{3bf(x_0)}{2} + \frac{3af(x_0)}{2} - \frac{3bf(x_1)}{2} - \frac{3af(x_1)}{2} + 3x_0 f(x_1) - 3x_1 f(x_0) = \frac{b-a}{2}\left(f(x_0) + f(x_1)\right)$$

Finally, we have

$$\int_a^b f(x)\mathrm{d}x \approx \left(\frac{b-a}{2}\right)\left(f\left(\frac{2}{3}a + \frac{1}{3}b\right) + f\left(\frac{1}{3}a + \frac{2}{3}b\right)\right)$$

(c) For e^x from 0 to 1

1. Trapezoid Rule:

$$\int_0^1 e^x \mathrm{d}x \approx \frac{1}{2}(e+1) = 1.855$$

2. Open Rule:

$$\int_0^1 e^x \mathrm{d}x \approx \frac{1}{2}(e^{1/3} + e^{2/3}) = 1.671$$

Actual value: $e - 1 = 1.71$, so the open rule is closer. Now for $\sin(x)$ from 0 to π, we have

1. Trapezoid Rule:

$$\int_0^\pi \sin(x)\mathrm{d}x \approx \frac{\pi}{2}(0) = 0$$

2. Open Rule

$$\int_0^\pi \sin(x)\mathrm{d}x \approx \frac{\pi}{2}\left(\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\right) = 0.866$$

Actual value: 2, so open rule is closer.

Problem 3: Least Squares to Data Fitting

Solution. (a) We will have r such that

$$\begin{aligned} r_i &= y^{(i)} - f\left(x_1^{(i)}, x_2^{(i)}\right) \\ &= y^{(i)} - \begin{bmatrix} 1 & x_1^{(i)} & x_2^{(i)} & x_1^{(i)}x_2^{(i)} \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix} \end{aligned}$$

Overall r is as such

$$r = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ \vdots \\ y^{(m)} \end{bmatrix} - \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_1^{(1)}x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_1^{(2)}x_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} & x_1^{(m)}x_2^{(m)} \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix}$$