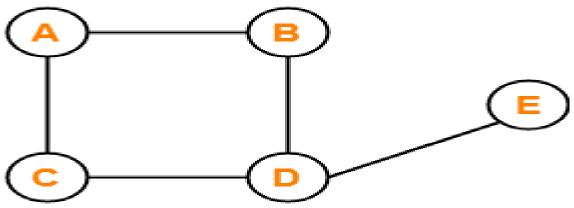
UNIT-6: GRAPH

- A graph is a collection of vertices connected to each other through a set of edges.
- The study of graphs is known as Graph Theory.

Formal Definition: A graph is defined as an ordered pair of a set of vertices and a set of edges. G = (V, E)

Here, V is the set of vertices and E is the set of edges connecting the vertices.

Example-



Example of Graph

In this graph V = { A , B , C , D , E } E = { AB , AC , BD , CD , DE }

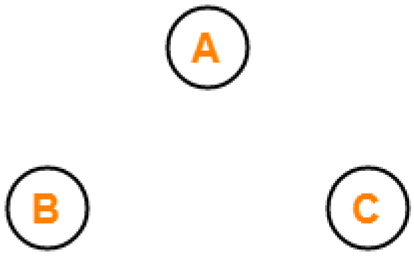
Types of Graphs- Various important types of graphs in graph theory are-

1. Null Graph
2. Trivial Graph
3. Non-directed Graph
4. Directed Graph
5. Connected Graph
6. Disconnected Graph
7. Regular Graph
8. Complete Graph
9. Cycle Graph
10. Cyclic Graph
11. Acyclic Graph
12. Finite Graph
13. Infinite Graph
14. Bipartite Graph
15. Planar Graph
16. Simple Graph
17. Multi Graph
18. Pseudo Graph
19. Euler Graph
20. Hamiltonian Graph

1. Null Graph

- A graph whose edge set is empty is called as a null graph.
- In other words, a null graph does not contain any edges in it.

Example-



Example of Null Graph

Here,

- This graph consists only of the vertices and there are no edges in it.
- Since the edge set is empty, therefore it is a null graph.

2. Trivial Graph

- A graph having only one vertex in it is called as a trivial graph.
- It is the smallest possible graph.

Example-

Example of Trivial Graph

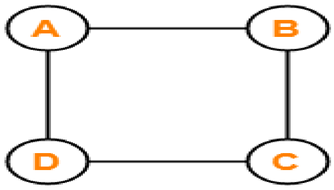
Here,

- This graph consists of only one vertex and there are no edges in it.
- Since only one vertex is present, therefore it is a trivial graph.

3. Non-Directed Graph

- A graph in which all the edges are undirected is called as a non-directed graph.
- In other words, edges of an undirected graph do not contain any direction.

Example-



Example of Non-Directed Graph

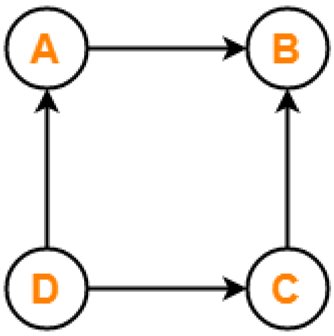
Here,

- This graph consists of four vertices and four undirected edges.
- Since all the edges are undirected, therefore it is a non-directed graph.

4. Directed Graph

- A graph in which all the edges are directed is called as a directed graph.
- In other words, all the edges of a directed graph contain some direction.
- Directed graphs are also called as **digraphs**.

Example-



Example of Directed Graph

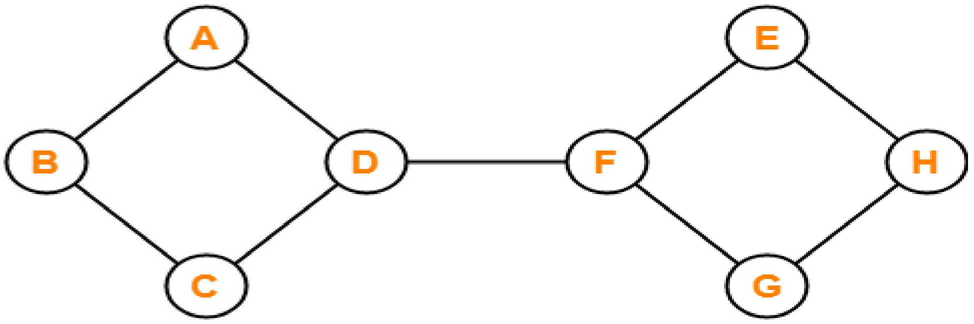
Here,

- This graph consists of four vertices and four directed edges.
- Since all the edges are directed, therefore it is a directed graph.

5. Connected Graph

- A graph in which we can visit from any one vertex to any other vertex is called as a connected graph.
- In connected graph, at least one path exists between every pair of vertices.

Example-



Example of Connected Graph

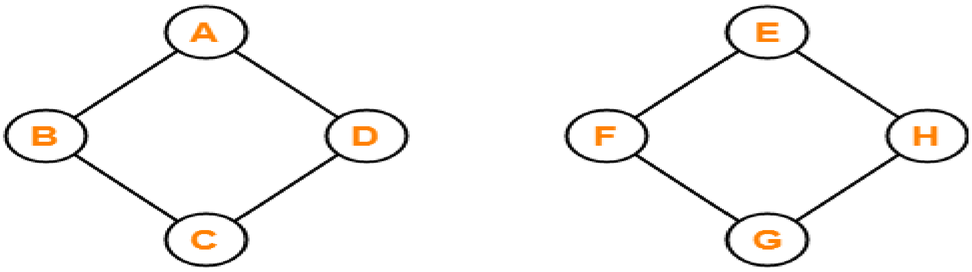
Here,

- In this graph, we can visit from any one vertex to any other vertex.
- There exists at least one path between every pair of vertices.
- Therefore, it is a connected graph.

6. Disconnected Graph

- A graph in which there does not exist any path between at least one pair of vertices is called as a disconnected graph.

Example-



Example of Disconnected Graph

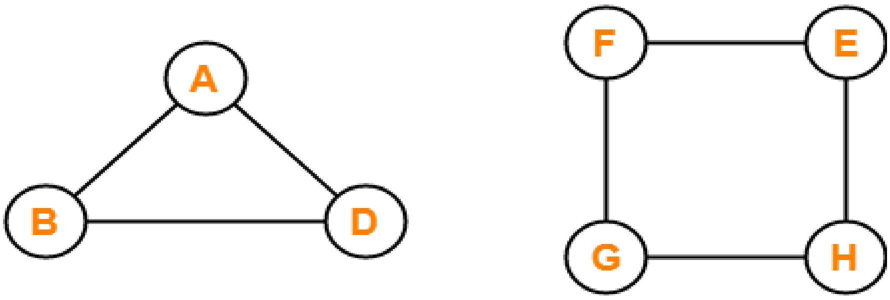
Here,

- This graph consists of two independent components which are disconnected.
- It is not possible to visit from the vertices of one component to the vertices of other component.
- Therefore, it is a disconnected graph.

7. Regular Graph

- A graph in which degree of all the vertices is same is called as a regular graph.
- If all the vertices in a graph are of degree ‘k’, then it is called as a “**k-regular graph** “.

Examples-



Examples of Regular Graph

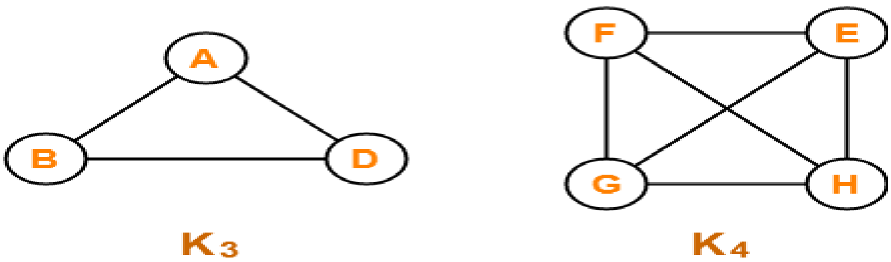
In these graphs,

- All the vertices have degree-2.
- Therefore, they are 2-Regular graphs.

8. Complete Graph

- A graph in which exactly one edge is present between every pair of vertices is called as a complete graph.
- A complete graph of ‘n’ vertices contains exactly nC2 edges.
- A complete graph of ‘n’ vertices is represented as **Kn**.

Examples-



Examples of Complete Graph

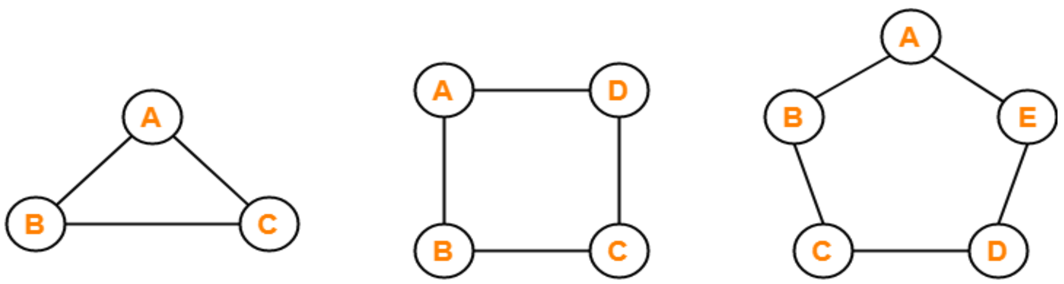
In these graphs,

- Each vertex is connected with all the remaining vertices through exactly one edge.
- Therefore, they are complete graphs.

9. Cycle Graph

- A simple graph of 'n' vertices (n>=3) and n edges forming a cycle of length 'n' is called as a cycle graph.
- In a cycle graph, all the vertices are of degree 2.

Examples-



Examples of Cycle Graph

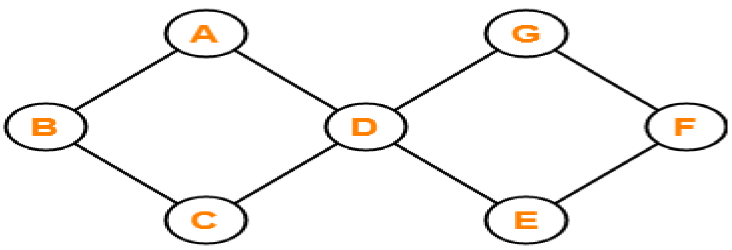
In these graphs,

- Each vertex is having degree 2.
- Therefore, they are cycle graphs.

10. Cyclic Graph-

- A graph containing at least one cycle in it is called as a cyclic graph.

Example-



Example of Cyclic Graph

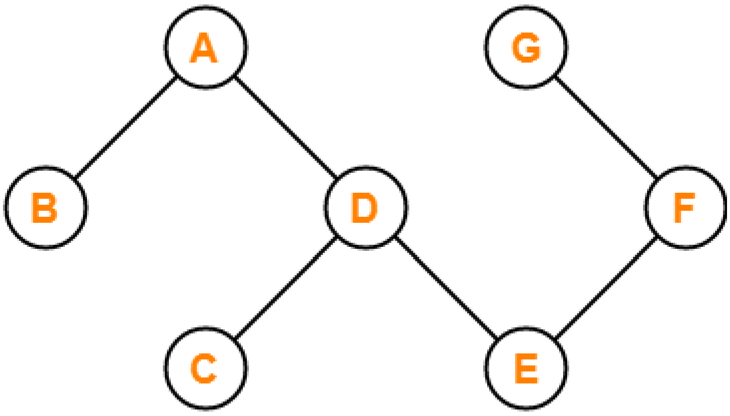
Here,

- This graph contains two cycles in it.
- Therefore, it is a cyclic graph.

11. Acyclic Graph

- A graph not containing any cycle in it is called as an acyclic graph.

Example-



Example of Acyclic Graph

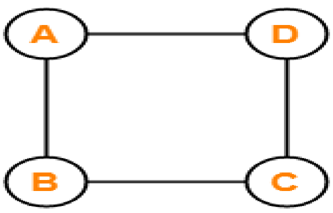
Here,

- This graph does not contain any cycle in it.
- Therefore, it is an acyclic graph.
-

12. Finite Graph

- A graph consisting of finite number of vertices and edges is called as a finite graph.

Example-



Example of Finite Graph

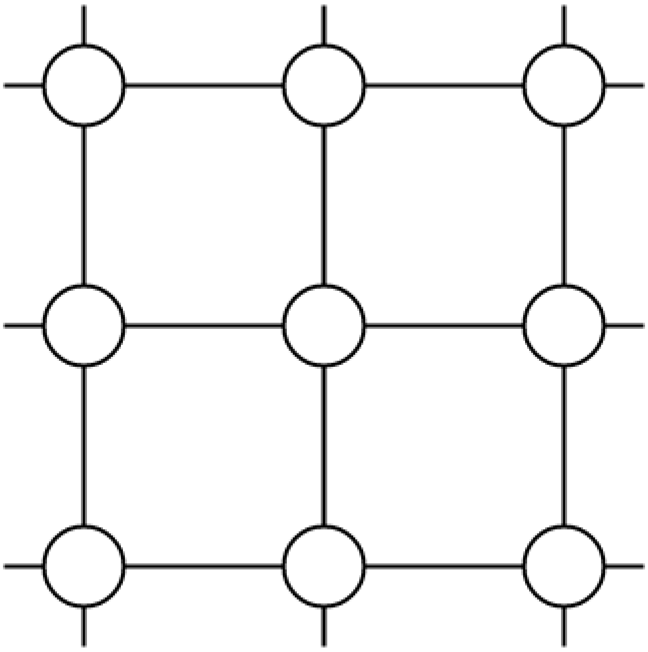
Here,

- This graph consists of finite number of vertices and edges.
- Therefore, it is a finite graph.

13. Infinite Graph

- A graph consisting of infinite number of vertices and edges is called as an infinite graph.

Example-



Example of Infinite Graph

Here,

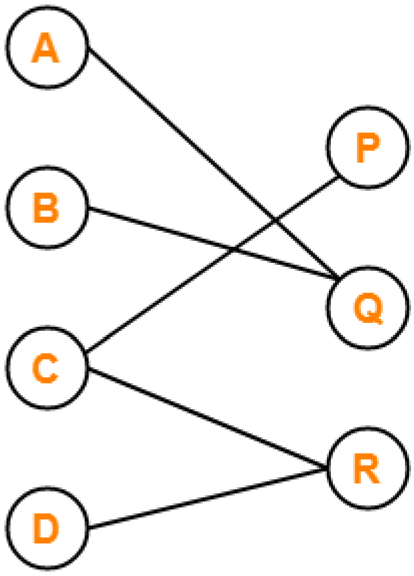
- This graph consists of infinite number of vertices and edges.
- Therefore, it is an infinite graph.

14. Bipartite Graph

A bipartite graph is a graph where-

- Vertices can be divided into two sets X and Y.
- The vertices of set X only join with the vertices of set Y.
- None of the vertices belonging to the same set join each other.

Example-



Example of Bipartite Graph

Complete Bipartite Graph-

A complete bipartite graph may be defined as follows-

A bipartite graph where every vertex of set X is joined to every vertex of set Y

is called as complete bipartite graph.

OR

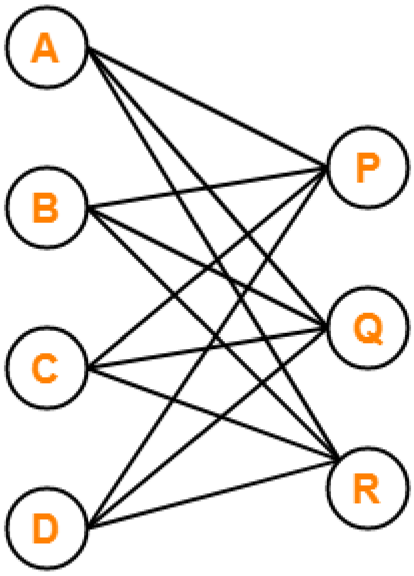
Complete bipartite graph is a bipartite graph which is complete.

OR

Complete bipartite graph is a graph which is bipartite as well as complete.

Complete Bipartite Graph Example-

The following graph is an example of a complete bipartite graph-



Example of Complete Bipartite Graph

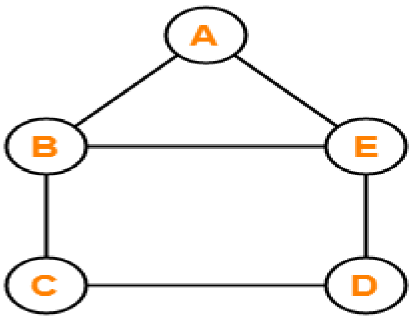
Here,

- This graph is a bipartite graph as well as a complete graph.
- Therefore, it is a complete bipartite graph.
- This graph is called as $K_{4,3}$.

15. Planar Graph

A planar graph is a graph that we can draw in a plane such that no two edges of it cross each other.

Example-



Example of Planar Graph

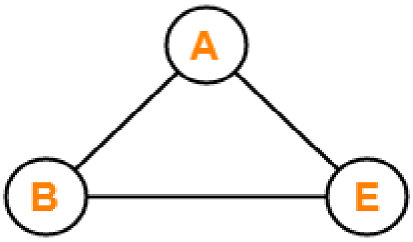
Here,

- This graph can be drawn in a plane without crossing any edges.
- Therefore, it is a planar graph.

16. Simple Graph

- A graph having no self-loops and no parallel edges in it is called as a simple graph.

Example-



Example of Simple Graph

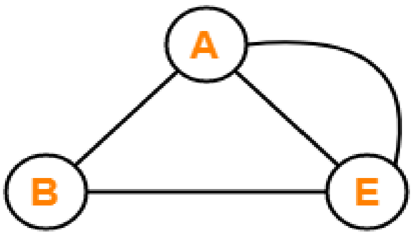
Here,

- This graph consists of three vertices and three edges.
- There are neither self-loops nor parallel edges.
- Therefore, it is a simple graph.

17. Multi Graph

- A graph having no self-loops but having parallel edge(s) in it is called as a multi graph.

Example-



Example of Multi Graph

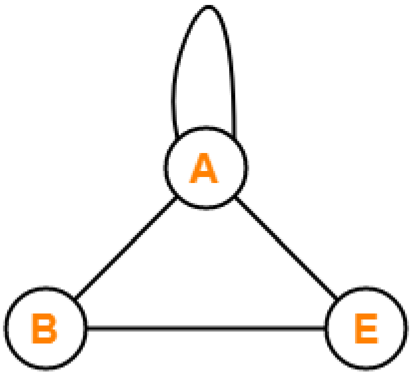
Here,

- This graph consists of three vertices and four edges out of which one edge is a parallel edge.
- There are no self-loops but a parallel edge is present.
- Therefore, it is a multi-graph.

18. Pseudo Graph

- A graph having no parallel edges but having self-loop(s) in it is called as a pseudo graph.

Example-



Example of Pseudo Graph

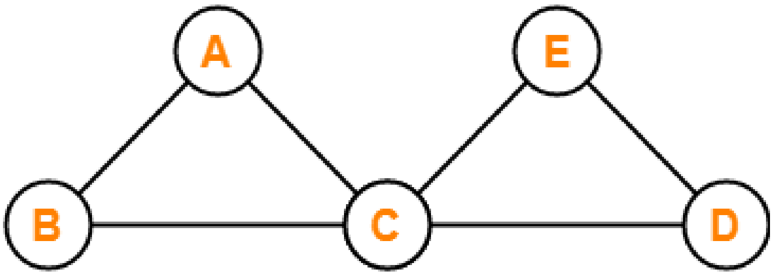
Here,

- This graph consists of three vertices and four edges out of which one edge is a self-loop.
- There are no parallel edges but a self-loop is present.
- Therefore, it is a pseudo graph.

19. Euler Graph

- Euler Graph is a connected graph in which all the vertices are even degree.

Example-



Example of Euler Graph

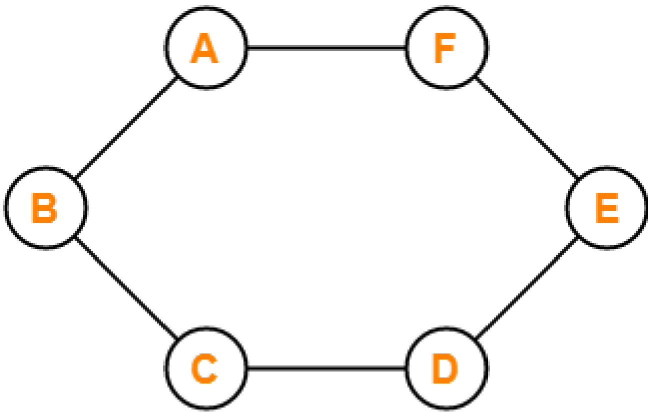
Here,

- This graph is a connected graph.
- The degree of all the vertices is even.
- Therefore, it is an Euler graph.

20. Hamiltonian Graph

- If there exists a closed walk in the connected graph that visits every vertex of the graph exactly once (except starting vertex) without repeating the edges, then such a graph is called as a Hamiltonian graph.

Example-



Example of Hamiltonian Graph

Here,

- This graph contains a closed walk ABCDEFG that visits all the vertices (except starting vertex) exactly once.

- All the vertices are visited without repeating the edges.
- Therefore, it is a Hamiltonian Graph.

Important Points-

- Edge set of a graph can be empty but vertex set of a graph cannot be empty.
- Every polygon is a 2-Regular Graph.
- Every complete graph of 'n' vertices is a (n-1)-regular graph.
- Every regular graph need not be a complete graph.

Remember

The following table is useful to remember different types of graphs-

	Self-Loop(s)	Parallel Edge(s)
Graph	Yes	Yes
Simple Graph	No	No
Multi Graph	No	Yes
Pseudo Graph	Yes	No

Applications of Graph Theory-

Graph theory has its applications in diverse fields of engineering-

1. Electrical Engineering-

- The concepts of graph theory are used extensively in designing circuit connections.
- The types or organization of connections are named as topologies.
- Some examples for topologies are star, bridge, series and parallel topologies.

2. Computer Science-

Graph theory is used for the study of algorithms such as-

- **Kruskal's Algorithm**
- **Prim's Algorithm**
- **Dijkstra's Algorithm**

3. Computer Network-

The relationships among interconnected computers in the network follows the principles of graph theory.

4. Science-

Following structures are represented by graphs-

- Molecular structure of a substance
- Chemical structure of a substance
- DNA structure of an organism etc

5. Linguistics-

The parsing tree of a language and grammar of a language uses graphs.

6. Other Applications-

- Routes between the cities are represented using graphs.
- Hierarchical ordered information such as family tree are represented using special types of graphs called trees

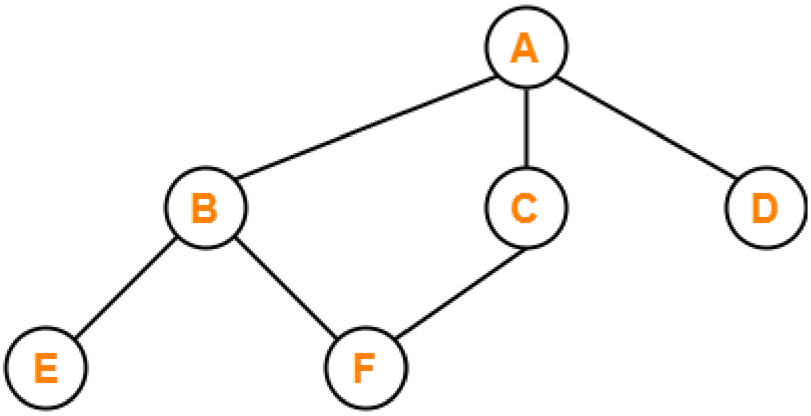
Graph Traversal Techniques

Depth First Search-

- Depth First Search or DFS is a graph traversal algorithm.
- It is used for traversing or searching a graph in a systematic fashion.
- DFS uses a strategy that searches “deeper” in the graph whenever possible.
- Stack data structure is used in the implementation of depth first search.

DFS Example-

Consider the following graph-



Depth First Search Example

The depth first search traversal order of the above graph is-

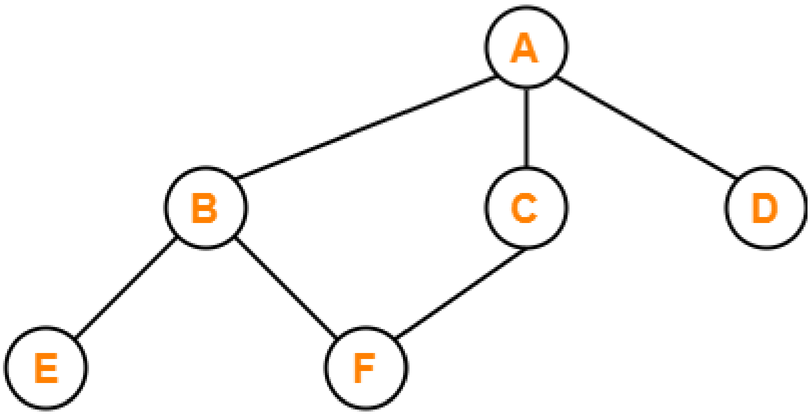
A, B, E, F, C, D

Breadth First Search-

- Breadth First Search or BFS is a graph traversal algorithm.
- It is used for traversing or searching a graph in a systematic fashion.
- BFS uses a strategy that searches in the graph in breadth first manner whenever possible.
- Queue data structure is used in the implementation of breadth first search.

BFS Example-

Consider the following graph-



Breadth First Search Example

The breadth first search traversal order of the above graph is-

A, B, C, D, E, F

Following are the important differences between BFS and DFS.

Sr. No.	Key	BFS	DFS
1	Definition	BFS, stands for Breadth First Search.	DFS, stands for Depth First Search.
2	Data structure	BFS uses Queue to find the shortest path.	DFS uses Stack to find the shortest path.
3	Source	BFS is better when target is closer to Source.	DFS is better when target is far from source.
4	Suitablity for decision tree	As BFS considers all neighbour so it is not suitable for decision tree used in puzzle games.	DFS is more suitable for decision tree. As with one decision, we need to traverse further to augment the decision. If we reach the conclusion, we won.
5	Speed	BFS is slower than DFS.	DFS is faster than BFS.
6	Time Complexity	Time Complexity of BFS = O(V+E) where V is vertices and E is edges.	Time Complexity of DFS is also O(V+E) where V is vertices and E is edges.

Topological Sorting

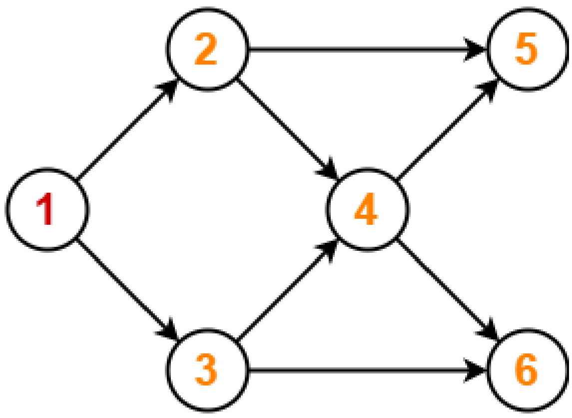
Topological Sort is a linear ordering of the vertices in such a way that if there is an edge in the DAG going from vertex ‘u’ to vertex ‘v’, then ‘u’ comes before ‘v’ in the ordering.

It is important to note that-

1. Topological Sorting is possible if and only if the graph is a Directed Acyclic Graph.
2. There may exist multiple different topological orderings for a given directed acyclic graph.

Topological Sort Example-

Consider the following directed acyclic graph-



Topological Sort Example

For this graph, following 4 different topological orderings are possible-

- **1 2 3 4 5 6**
- **1 2 3 4 6 5**
- **1 3 2 4 5 6**
- **1 3 2 4 6 5**

Applications of Topological Sort- Few important applications of topological sort are-

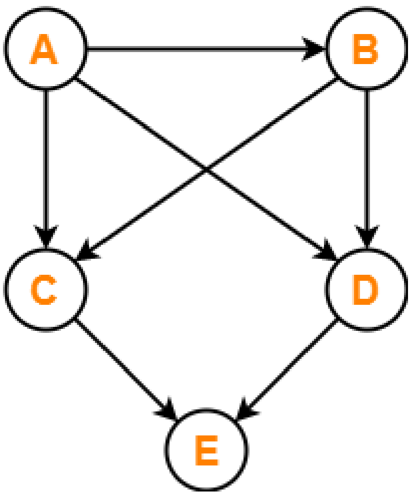
1. Scheduling jobs from the given dependencies among jobs
2. Instruction Scheduling

3. Determining the order of compilation tasks to perform in make files
4. Data Serialization

PRACTICE PROBLEMS BASED ON TOPOLOGICAL SORT-

Problem-01:

Find the number of different topological orderings possible for the given graph-

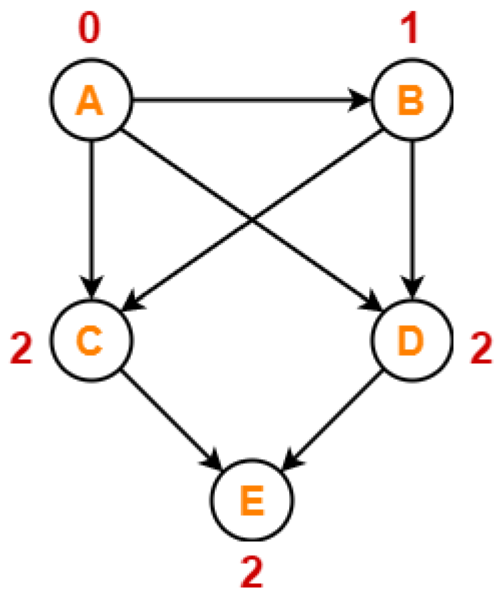


Solution-

The topological orderings of the above graph are found in the following steps-

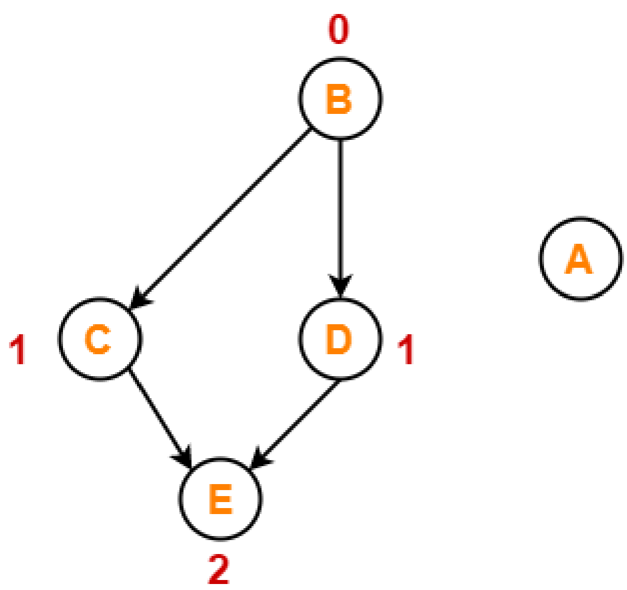
Step-01:

Write in-degree of each vertex-



Step-02:

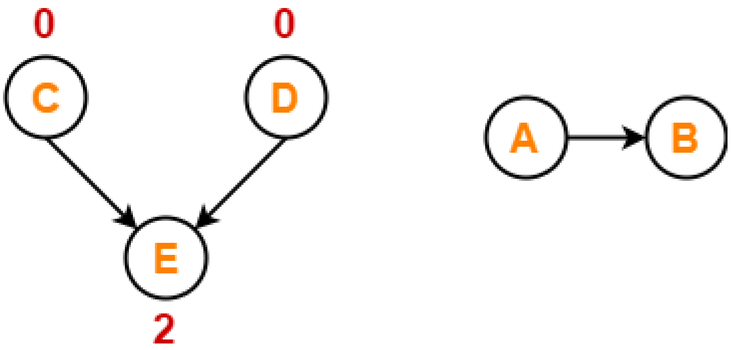
- Vertex-A has the least in-degree.
- So, remove vertex-A and its associated edges.
- Now, update the in-degree of other vertices.



Step-03:

- Vertex-B has the least in-degree.
- So, remove vertex-B and its associated edges.

- Now, update the in-degree of other vertices.



Step-04:

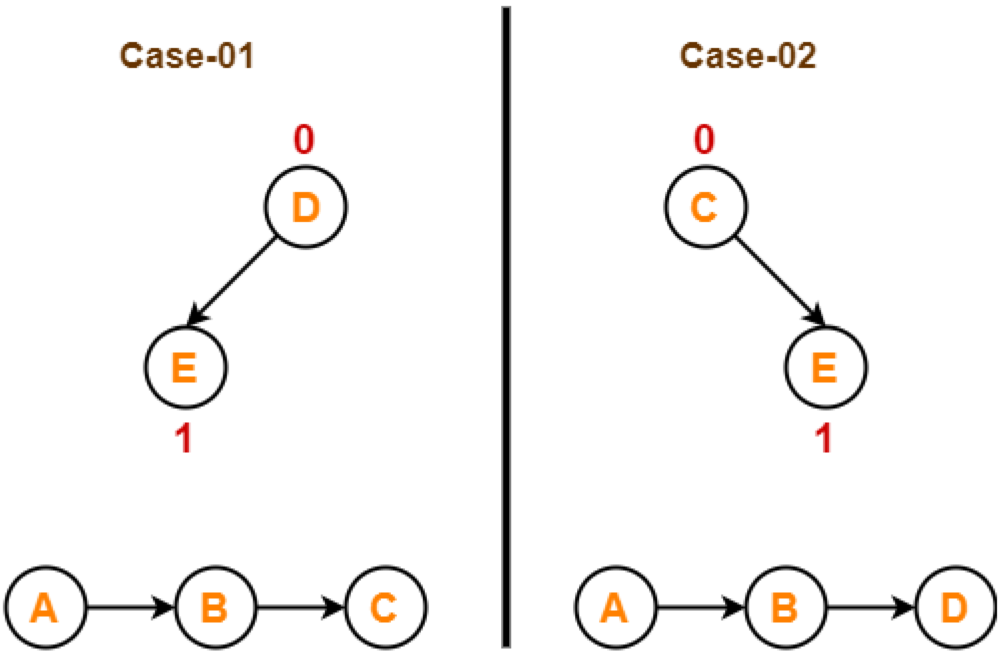
There are two vertices with the least in-degree. So, following 2 cases are possible-

In case-01,

- Remove Vertex-C and its associated edges.
- Then, update the in-degree of other vertices.

In case-02,

- Remove Vertex-D and its associated edges.
- Then, update the in-degree of other vertices.



Step-05:

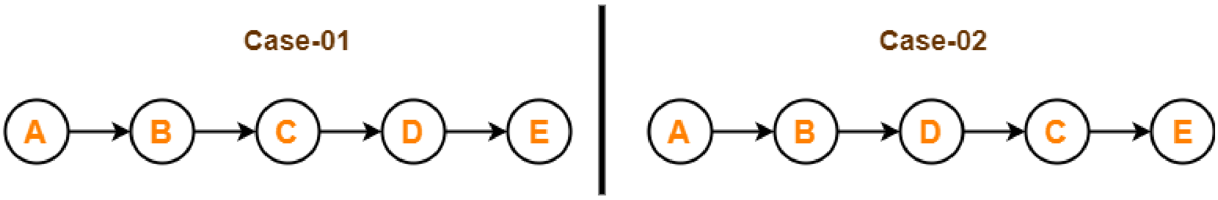
Now, the above two cases are continued separately in the similar manner.

In case-01,

- Remove Vertex-D since it has the least in-degree.
- Then, remove the remaining Vertex-E.

In case-02,

- Remove Vertex-C since it has the least in-degree.
- Then, remove the remaining Vertex-E.



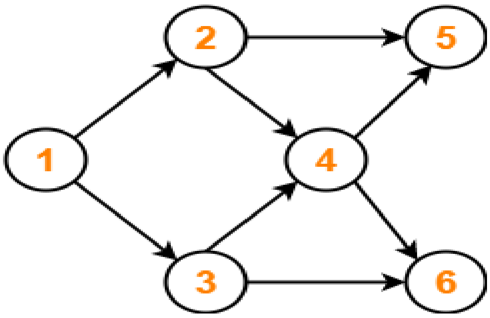
Conclusion-

For the given graph, following **2** different topological orderings are possible-

- **A B C D E**
- **A B D C E**

Problem-02:

Find the number of different topological orderings possible for the given graph-

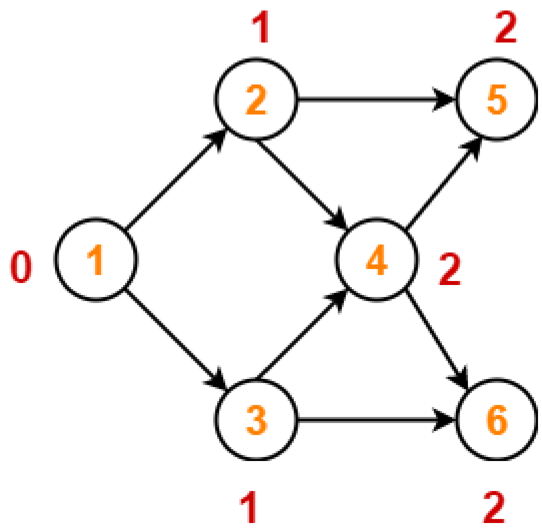


Solution-

The topological orderings of the above graph are found in the following steps-

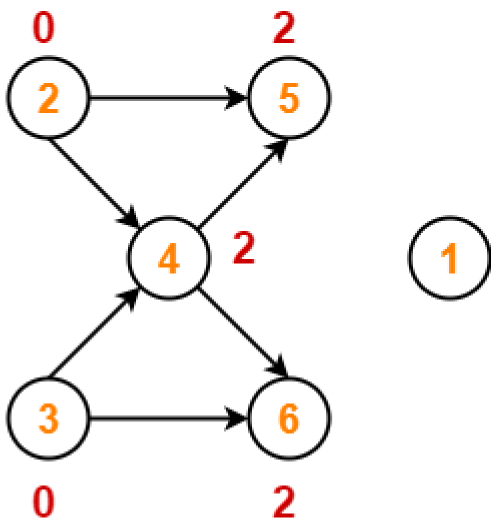
Step-01:

Write in-degree of each vertex-



Step-02:

- Vertex-1 has the least in-degree.
- So, remove vertex-1 and its associated edges.
- Now, update the in-degree of other vertices.



Step-03:

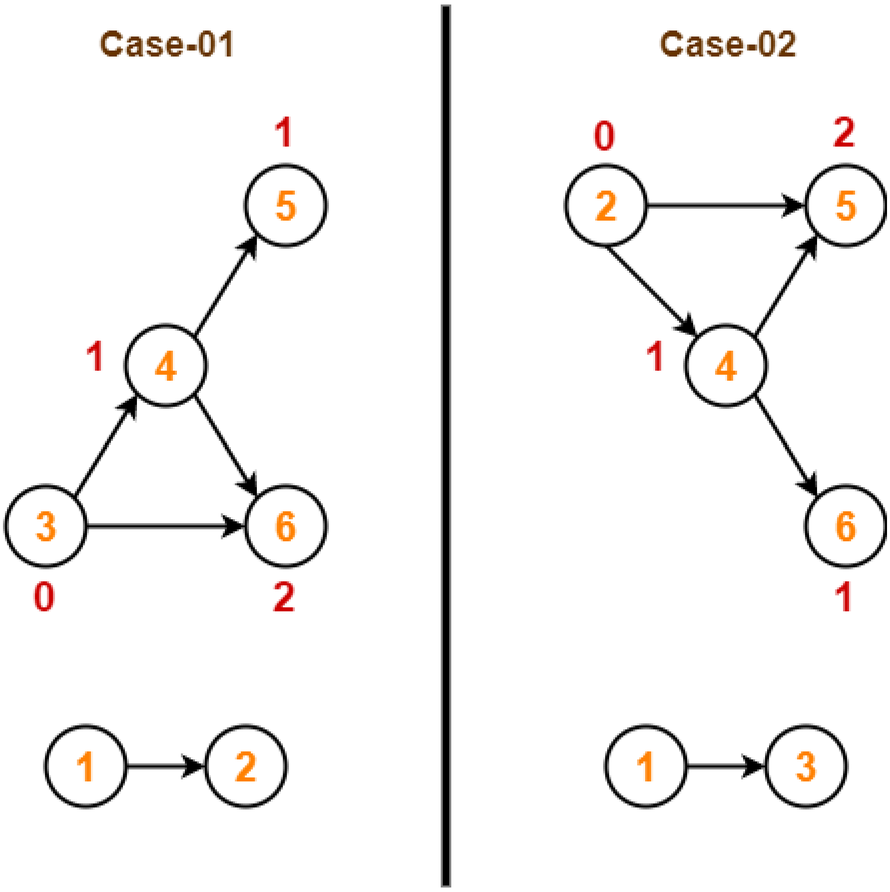
There are two vertices with the least in-degree. So, following 2 cases are possible-

In case-01,

- Remove vertex-2 and its associated edges.
- Then, update the in-degree of other vertices.

In case-02,

- Remove vertex-3 and its associated edges.
- Then, update the in-degree of other vertices.



Step-04:

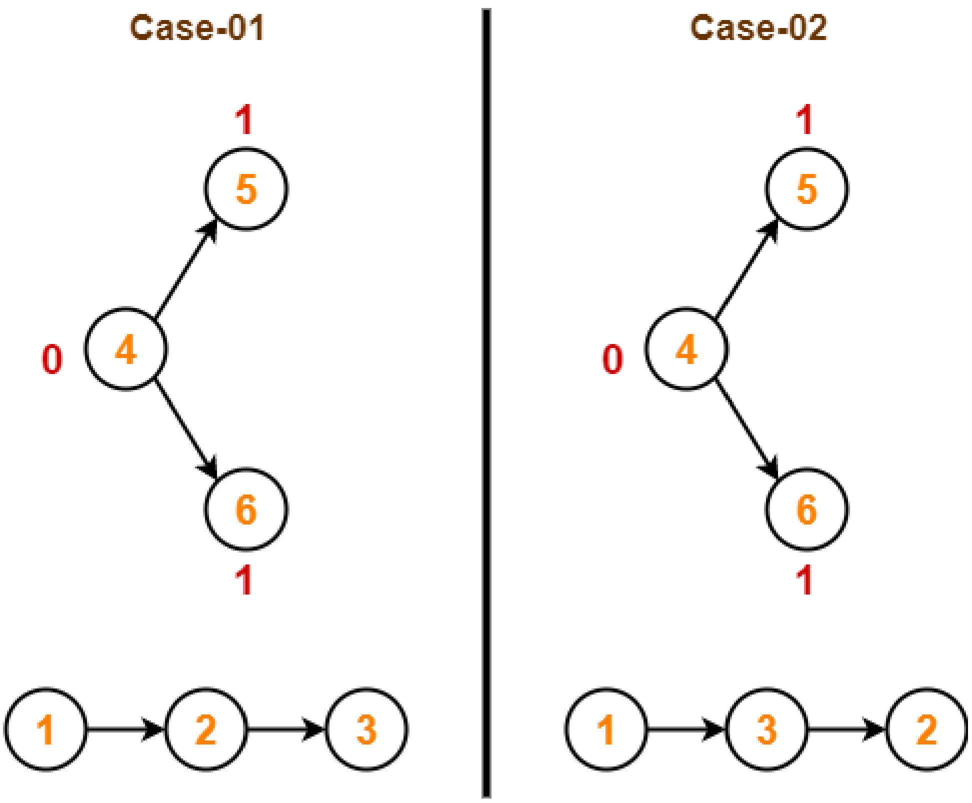
Now, the above two cases are continued separately in the similar manner.

In case-01,

- Remove vertex-3 since it has the least in-degree.
- Then, update the in-degree of other vertices.

In case-02,

- Remove vertex-2 since it has the least in-degree.
- Then, update the in-degree of other vertices.



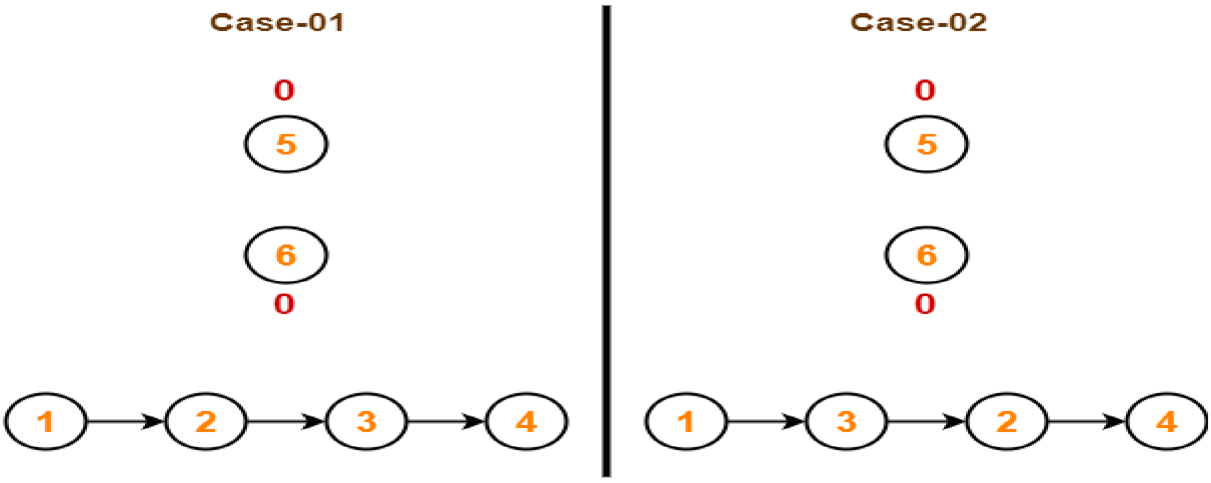
Step-05:

In case-01,

- Remove vertex-4 since it has the least in-degree.
- Then, update the in-degree of other vertices.

In case-02,

- Remove vertex-4 since it has the least in-degree.
- Then, update the in-degree of other vertices.

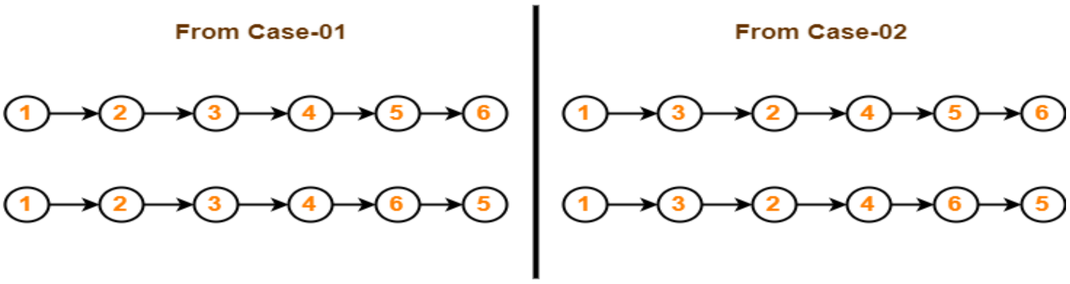


Step-06:

In case-01,

- There are 2 vertices with the least in-degree.
- So, 2 cases are possible.
- Any of the two vertices may be taken first.

Same is with case-02.



Conclusion-

For the given graph, following **4** different topological orderings are possible-

- **1 2 3 4 5 6**
- **1 2 3 4 6 5**
- **1 3 2 4 5 6**
- **1 3 2 4 6 5**

THE END