Problem statement: **Find n^{th} Fibonacci number.**

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Iterative Method:
Algorithm:

```
f1(n):
    first=0;
    second=1;
    for i = 3 to n
    {
        next = first + second;
        first = second;
        second = next;
    }
   return next;
```

Time complexity: O(1) + O(n) + O(1) = O(n)

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Recursive Method:
Algorithm:

```
f2(n):
   if n <= 1
      return n
   return f2(n - 1) + f2(n - 2)
```

Time complexity: ?
If n=0 => F(0) = 0
n=1 => F(1) = 1
n>1 => F(n) = F(n-1) + F(n-2)

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Example-1:

What is the solution of the recurrence relation fn = fn-1 + fn-2 with f0=0 and f1=1?

Solution:

Rewrite recurrence, $f_n - f_{n-1} - f_{n-2} = 0$

Compare recurrence with $a_0 t_n + a_1 t_{n-1} + ... + a_k t_{n-k} = 0$

$K=2$, $a_0=1$, $a_1=-1$, $a_2=-1$

Characteristic eq. is $x^2 - x - 1 = 0$

Quadratic equation

$ax^2+bx+c=0$

a=1, b=-1, c=-1

delta=root(b^2-4ac) = root(1 + 4) =root(5)

root1 = (-b + delta)/2a = (1 +root(5))/2

root2= (-b-delta)/2a = (1 -root(5))/2

Possible roots are r1 = $(1+\sqrt{5})/2$, r2 = $(1-\sqrt{5})/2$

General Solution is $fn = c1r1^n + c2r2^n$

$= c1\left(\frac{1+\sqrt{5}}{2}\right)^n + c2\left(\frac{1-\sqrt{5}}{2}\right)^n$

Use initial condition to find c1 and c2

$n=0 \Rightarrow f0 = c1 + c2 \quad \Rightarrow 0 = c1 + c2 \quad \Rightarrow c1=-c2$

$n=1 \Rightarrow f1 = c1((1+\sqrt{5})/2) + c2((1-\sqrt{5})/2) \quad \Rightarrow 1 = c1((1+\sqrt{5})/2) + c2((1-\sqrt{5})/2)$

Solving above equations, $c1=1/\sqrt{5}$, $c2=-1/\sqrt{5}$

Thus, $fn = (1/\sqrt{5})\,((1+\sqrt{5})/2)^n - (1/\sqrt{5})\,((1-\sqrt{5})/2)^n$

$fn = O(1.6^n) = O(2^n)$

Example-2:

Solve the recurrence:

$$\mathbf{t_n} = \begin{cases} \mathbf{0} & \mathbf{, if\ n = 0} \\ \mathbf{5} & \mathbf{, if\ n = 1} \\ \mathbf{3t_{n-1} + 4t_{n-2}} & \mathbf{, otherwise} \end{cases}$$

Rewrite recurrence, $t_n - 3t_{n-1} - 4t_{n-2} = 0$

Compare recurrence with $a_0 t_n + a_1 t_{n-1} + \ldots + a_k t_{n-k} = 0$

$K=2$, $a_0=1$, $a_1=-3$, $a_2=-4$

Characteristic eq. is $x^2 - 3x - 4 = 0$

Possible roots are $r1 = -1$, $r2 = 4$

General Solution is $tn = c1r1^n + c2r2^n$

$= c1(-1)^n + c2(4)^n$

Use initial condition to find c1 and c2

$n=0 \Rightarrow t0 = c1 + c2 \Rightarrow 0 = c1 + c2 \quad \Rightarrow c1=-c2$

$n=1 \Rightarrow t1 = c1(-1) + c2(4) \quad \Rightarrow 5 = c2 + 4c2 \quad \Rightarrow \quad c2=1$

Solving above equations, $c1=-1$, $c2=1$

Thus, $tn = (-1)(-1)^n + (1)(4)^n$

$tn = O(4^n)$

Example-3:

Solve the recurrence:

$$t_n = \begin{cases} \mathbf{n} & \mathbf{, if\ n = 0, 1, 2} \\ \mathbf{5t_{n-1} - 8t_{n-2} + 4t_{n-3}} & \mathbf{, otherwise} \end{cases}$$

Rewrite recurrence, $t_n - 5t_{n-1} + 8t_{n-2} - 4t_{n-3} = 0$

Compare recurrence with $a_0 t_n + a_1 t_{n-1} + \dots + a_k t_{n-k} = 0$

$K=3$, $a_0=1$, $a_1=-5$, $a_2=8$, $a_3=-4$

Characteristic eq. is $x^3 - 5x^2 + 8x - 4 = 0$

Possible roots are r1 = 1, r2 = 2, r3 = 2 => <u>r1=1, m1=1</u> and <u>r2=2, m2=2</u>

General Solution is $tn = c1r1^n + c2r2^n + c3*n*r2^n$

$$tn = c1\,(1)^n + c2(2)^n + c3n(2)^n \qquad \text{----------------- (a)}$$

Use initial condition to find c1,c2 and c3

n=0 => t0 = c1 + c2 => 0 = c1 + c2 => c1=-c2 --------------- (1)

n=1 => $t1 = c1(1)^1+c2(2)^1+c3(2)^1$ => 1 = c1+2c2 + 2c3 ----------------- (2)

n=2 => $t2 =c1(1)^2+c2(2)^2+c32(2)^2$ => 2 = c1+4c2+8c3 ----------------- (3)

Solving above equations (1), (2) and (3), c1=-2, c2=2, c3=-(1/2)

Put c1, c2 and c3 into equation (a)

$tn = -2(1)^n + 2*2^n - (1/2)*n*2^n$

Thus, $tn = 2*2^n - n*2^{n-1} - 2$

$tn = O(2^n)$