Homework 1 Part 2

This is an individual assignment.

Due: Monday, September 26 @ 11:59pm

In [1]:
```
# import all libraries and magics
%matplotlib inline
import matplotlib.pyplot as plt
import numpy as np
from sklearn.model_selection import KFold, train_test_split
import scipy.stats as stats
```

Problem 1 (22.5 points)

Consider the diabetes data:

In [2]:
```
from sklearn.datasets import load_diabetes

diabetes = load_diabetes(return_X_y=False)
print(diabetes.DESCR)
```

```
.. _diabetes_dataset:

Diabetes dataset
----------------

Ten baseline variables, age, sex, body mass index, average blood
pressure, and six blood serum measurements were obtained for each of n =
442 diabetes patients, as well as the response of interest, a
quantitative measure of disease progression one year after baseline.

**Data Set Characteristics:**

  :Number of Instances: 442

  :Number of Attributes: First 10 columns are numeric predictive values

  :Target: Column 11 is a quantitative measure of disease progression one year after
baseline

  :Attribute Information:
      - age     age in years
      - sex
      - bmi     body mass index
      - bp      average blood pressure
      - s1      tc, total serum cholesterol
      - s2      ldl, low-density lipoproteins
      - s3      hdl, high-density lipoproteins
      - s4      tch, total cholesterol / HDL
      - s5      ltg, possibly log of serum triglycerides level
      - s6      glu, blood sugar level

Note: Each of these 10 feature variables have been mean centered and scaled by the st
andard deviation times `n_samples` (i.e. the sum of squares of each column totals 1).

Source URL:
https://www4.stat.ncsu.edu/~boos/var.select/diabetes.html

For more information see:
Bradley Efron, Trevor Hastie, Iain Johnstone and Robert Tibshirani (2004) "Least Angl
e Regression," Annals of Statistics (with discussion), 407-499.
(https://web.stanford.edu/~hastie/Papers/LARS/LeastAngle_2002.pdf)
```

This dataset is already described in the **feature space**. Each input sample x_i is described as 10-dimensional feature vctor $\phi(x_i)$. The features correspond to: age, sex, bmi, bp, s1, s2, s3, s4, s5 and s6 measurements (read the description above for more details). The target variable corresponds a measure of diabetes disease progression one year after baseline.

Let's load the data as a `pandas` dataframe:

In [3]:
```
import numpy as np
import pandas as pd

df_diabetes = pd.DataFrame(data=np.hstack((diabetes.target[:,np.newaxis],diabetes.data
                  columns=['Target']+diabetes.feature_names)

df_diabetes
```

Out[3]:

	Target	age	sex	bmi	bp	s1	s2	s3	s4	
0	151.0	0.038076	0.050680	0.061696	0.021872	-0.044223	-0.034821	-0.043401	-0.002592	0
1	75.0	-0.001882	-0.044642	-0.051474	-0.026328	-0.008449	-0.019163	0.074412	-0.039493	-0
2	141.0	0.085299	0.050680	0.044451	-0.005671	-0.045599	-0.034194	-0.032356	-0.002592	0
3	206.0	-0.089063	-0.044642	-0.011595	-0.036656	0.012191	0.024991	-0.036038	0.034309	0
4	135.0	0.005383	-0.044642	-0.036385	0.021872	0.003935	0.015596	0.008142	-0.002592	-0
...	...	...	...	...	...	...	...	...	...	...
437	178.0	0.041708	0.050680	0.019662	0.059744	-0.005697	-0.002566	-0.028674	-0.002592	0
438	104.0	-0.005515	0.050680	-0.015906	-0.067642	0.049341	0.079165	-0.028674	0.034309	-0
439	132.0	0.041708	0.050680	-0.015906	0.017282	-0.037344	-0.013840	-0.024993	-0.011080	-0
440	220.0	-0.045472	-0.044642	0.039062	0.001215	0.016318	0.015283	-0.028674	0.026560	0
441	57.0	-0.045472	-0.044642	-0.073030	-0.081414	0.083740	0.027809	0.173816	-0.039493	-0

442 rows × 11 columns

The goal is to fit a linear regression model on the provided features, i.e., the model is of the form:

$$y(x) = w_0 + w_1\phi_1(x) + w_2\phi_2(x) + w_3\phi_3(x) + \cdots + w_{10}\phi_{10}(x)$$

where w_0 is the bias (or intercept) coefficient and all other $w_i, i = 1, \ldots, 10$ correspond to the coefficient associated with feature ϕ_i (age, sex, bmi, bp, etc.).

Questions

1. Randomly partition the data into training (70%) and test sets (30%).
2. Use a 5-fold cross-validation strategy to determine the hyperparameter values to fit a linear regression model with ridge regularization for this dataset. Show and document your work.
3. Evaluate performance in the test set.
4. Determine the final value for the intercept and coefficients of the linear regression model. Plot all 11 values as a `stem` plot.
5. Based on this plot, which input variable (also referred to the independent variable) has the most contribution for predicting the target variable (also referred to the dependent variable)?

In [4]:

```
Train_dataset = np.array(df_diabetes.iloc[:,1:])
Target_dataset = np.array(df_diabetes.iloc[:,0])
```

```
# Split dataset randomly into training 70% and test 30%

X_train, X_test, t_train, t_test = train_test_split(Train_dataset, Target_dataset, tes
print(X_train.shape,X_test.shape,t_train.shape,t_test.shape)
```

```
(309, 10) (133, 10) (309,) (133,)
```

In [5]:
```
def Regression_model(x,t,M,lam):

    #feature matrix
    X = np.zeros((len(x),M+1))
    X[:,0] = 1
    X[:,1:] = np.array([x[:,m] for m in range(M)]).T

    #Compute the solution for the parameter w
    w = np.linalg.inv(X.T@X+lam*np.eye(M+1))@X.T@t

    #Compute model prediction
    y = X@w

    return w,y

def Regression_model_test(x,M,w):

    #feature matrix
    X = np.zeros((len(x),M+1))
    X[:,0] = 1
    X[:,1:] = np.array([x[:,m] for m in range(M)]).T

    #Prediction for test set
    y = X@w

    return y
```

In [6]:
```
k = 5 #number of folds
kf= KFold(n_splits=k,shuffle=True)

M_vals = range(1,11) #model orders
lam_vals = np.arange(0.1,1.1,0.1) #set of values for lambda

MSE_train_avg_array = []
MSE_val_avg_array = []


for M in M_vals:
    for lam in lam_vals:

        print('M value: ',M)
        print('lambda value: ',lam)

        #for each training/validation split
        f = 1

        #Intializing performance measures
        MSE_train_avg = 0
        MSE_val_avg = 0


        for train_index, val_index in kf.split(X_train):
            print('\nFold',f)
```

```
            # Selecting training set using indeces from kf.split
            X_train2,X_validation = X_train[train_index],X_train[val_index]

            # Select validation set using indeced from kf.split
            T_train2, T_validation = t_train[train_index],t_train[val_index]

            #train model with training set

            w,y_train = Regression_model(X_train2,T_train2,M,lam)

            #evaluate trained model in validation set

            y_val = Regression_model_test(X_validation,M,w)

            #Perfromance Measure
            MSE_train = np.mean((T_train2 - y_train)**2)
            MSE_val = np.mean((T_validation - y_val)**2)

            # Average performance measure
            MSE_train_avg = MSE_train_avg+MSE_train
            MSE_val_avg = MSE_val_avg+MSE_val
            print('MSE Training = ', MSE_train)
            print('MSE Validation = ', MSE_val)
            f+=1

    print('\nAverage Performance in Training = ', MSE_train_avg/k)
    print('Average Performance in Validation = ', MSE_val_avg/k)
    print('-----------------------------------------------------\n')
    MSE_train_avg_array.append([MSE_train_avg/k,M,lam])
    MSE_val_avg_array.append([MSE_val_avg/k,M,lam])
```

```
M value:  1
lambda value:  0.1

Fold 1
MSE Training =  6152.051412265145
MSE Validation =  5713.122226120677

Fold 2
MSE Training =  6071.160263951632
MSE Validation =  5963.600653208849

Fold 3
MSE Training =  5945.373895088402
MSE Validation =  6444.078128746192

Fold 4
MSE Training =  6174.976111474362
MSE Validation =  5541.53500567769

Fold 5
MSE Training =  5868.177292052579
MSE Validation =  6764.625960720862
M value:  1
lambda value:  0.2

Fold 1
MSE Training =  6143.095128219603
MSE Validation =  5720.48482636294

Fold 2
MSE Training =  6085.286953972211
MSE Validation =  6151.89156041568

Fold 3
MSE Training =  5872.4068233767375
MSE Validation =  6942.853837144927

Fold 4
MSE Training =  5917.395180378026
MSE Validation =  6573.161013089096

Fold 5
MSE Training =  6188.446111399414
MSE Validation =  5635.6810632107
M value:  1
lambda value:  0.30000000000000004

Fold 1
MSE Training =  5945.722162443371
MSE Validation =  6570.439566625934

Fold 2
MSE Training =  6120.370245646056
MSE Validation =  5809.6139736734585

Fold 3
MSE Training =  6220.414744861911
MSE Validation =  5548.106411615158

Fold 4
```

```
MSE Training =  5981.248615866121
MSE Validation =  6696.649374058058

Fold 5
MSE Training =  6038.3071628347725
MSE Validation =  6124.320951336031
M value:  1
lambda value:  0.4

Fold 1
MSE Training =  6083.128038479375
MSE Validation =  6113.3427930037315

Fold 2
MSE Training =  6169.56266472148
MSE Validation =  5717.544787529738

Fold 3
MSE Training =  6111.604370009088
MSE Validation =  6110.26569870542

Fold 4
MSE Training =  6011.239112484945
MSE Validation =  6390.280076467404

Fold 5
MSE Training =  6037.065643375913
MSE Validation =  6244.003327642613
M value:  1
lambda value:  0.5

Fold 1
MSE Training =  6270.198550465532
MSE Validation =  5341.711236132983

Fold 2
MSE Training =  5933.090989996709
MSE Validation =  6845.7944547682755

Fold 3
MSE Training =  6189.64599641532
MSE Validation =  5734.069510796127

Fold 4
MSE Training =  5992.810997618172
MSE Validation =  6542.762511547762

Fold 5
MSE Training =  6094.450499626295
MSE Validation =  6160.860956236371
M value:  1
lambda value:  0.6

Fold 1
MSE Training =  6257.371773171242
MSE Validation =  5516.105939653641

Fold 2
MSE Training =  5953.855019306057
MSE Validation =  6730.3290635826625
```

```
Fold 3
MSE Training =  6095.098826434869
MSE Validation =  6195.247290976566

Fold 4
MSE Training =  6312.954957155948
MSE Validation =  5338.464708764139

Fold 5
MSE Training =  5913.278574393073
MSE Validation =  6990.983315129243
M value:  1
lambda value:  0.7000000000000001

Fold 1
MSE Training =  6037.555217294795
MSE Validation =  6397.499916907068

Fold 2
MSE Training =  6128.5550539248625
MSE Validation =  6116.71357113051

Fold 3
MSE Training =  6352.236261124099
MSE Validation =  5181.434011999953

Fold 4
MSE Training =  5960.3492798165325
MSE Validation =  6848.150048086857

Fold 5
MSE Training =  6110.933381439557
MSE Validation =  6219.172428595051
M value:  1
lambda value:  0.8

Fold 1
MSE Training =  6252.0540956782825
MSE Validation =  5805.329196465452

Fold 2
MSE Training =  6105.483298733786
MSE Validation =  6165.86833987771

Fold 3
MSE Training =  6228.2863672244575
MSE Validation =  5627.411826767239

Fold 4
MSE Training =  5851.589422143666
MSE Validation =  7348.588959560751

Fold 5
MSE Training =  6192.944417645333
MSE Validation =  5870.943519070649
M value:  1
lambda value:  0.9

Fold 1
```

```
MSE Training =  6290.885039768927
MSE Validation =  5582.137258690717

Fold 2
MSE Training =  6040.895727919905
MSE Validation =  6732.9087039220185

Fold 3
MSE Training =  5935.889436940269
MSE Validation =  6891.835640918219

Fold 4
MSE Training =  6255.688879540993
MSE Validation =  5713.751683977547

Fold 5
MSE Training =  6148.871788640471
MSE Validation =  5954.978318711588
M value:  1
lambda value:  1.0

Fold 1
MSE Training =  5757.177308444131
MSE Validation =  7652.543965680001

Fold 2
MSE Training =  6047.863164547148
MSE Validation =  6477.171539246743

Fold 3
MSE Training =  6235.065461888178
MSE Validation =  6002.895762599285

Fold 4
MSE Training =  6361.613728259803
MSE Validation =  5382.27617685072

Fold 5
MSE Training =  6286.051570166659
MSE Validation =  5617.797191173006

Average Performance in Training =  6137.554246661184
Average Performance in Validation =  6226.536927109951
-----------------------------------------------------

M value:  2
lambda value:  0.1

Fold 1
MSE Training =  6000.037713251653
MSE Validation =  6323.8076202629445

Fold 2
MSE Training =  6115.847930655456
MSE Validation =  5841.75494901887

Fold 3
MSE Training =  5942.109528728643
MSE Validation =  6518.937376539196
```

```
Fold 4
MSE Training =  5828.0440085675045
MSE Validation =  6940.168570637024

Fold 5
MSE Training =  6287.092653076583
MSE Validation =  5057.260332769655
M value:  2
lambda value:  0.2

Fold 1
MSE Training =  6205.303484628701
MSE Validation =  5531.096637166248

Fold 2
MSE Training =  6058.39905724163
MSE Validation =  6101.538646543808

Fold 3
MSE Training =  5949.481940851897
MSE Validation =  6494.83856080105

Fold 4
MSE Training =  5878.125821480973
MSE Validation =  6822.531574195233

Fold 5
MSE Training =  6162.773902819755
MSE Validation =  5612.513144772962
M value:  2
lambda value:  0.30000000000000004

Fold 1
MSE Training =  6410.698426139918
MSE Validation =  4804.214348500889

Fold 2
MSE Training =  6254.000197228246
MSE Validation =  5414.270954190005

Fold 3
MSE Training =  5900.382194060778
MSE Validation =  6808.565281737965

Fold 4
MSE Training =  5702.35815229081
MSE Validation =  7635.518123519538

Fold 5
MSE Training =  6006.995811460667
MSE Validation =  6275.442562142789
M value:  2
lambda value:  0.4

Fold 1
MSE Training =  6121.3531002058
MSE Validation =  5926.563836370269

Fold 2
MSE Training =  5917.569329560278
```

```
MSE Validation =  6720.499634342103

Fold 3
MSE Training =  6043.19275756898
MSE Validation =  6323.774963154893

Fold 4
MSE Training =  6187.752283800116
MSE Validation =  5752.218220826036

Fold 5
MSE Training =  6109.180938024684
MSE Validation =  6004.907886028892
M value:  2
lambda value:  0.5

Fold 1
MSE Training =  6650.311923815658
MSE Validation =  3974.9109088284767

Fold 2
MSE Training =  6025.083263032623
MSE Validation =  6416.421678898421

Fold 3
MSE Training =  6014.151178161921
MSE Validation =  6512.400312040216

Fold 4
MSE Training =  5579.783212876243
MSE Validation =  8351.464064192352

Fold 5
MSE Training =  6144.6614967806
MSE Validation =  5879.460915934334
M value:  2
lambda value:  0.6

Fold 1
MSE Training =  6081.839801037552
MSE Validation =  6334.021444662015

Fold 2
MSE Training =  6137.336043213498
MSE Validation =  6047.347826287533

Fold 3
MSE Training =  6129.1828390970295
MSE Validation =  6106.902826636647

Fold 4
MSE Training =  6144.571639882507
MSE Validation =  5935.108706468661

Fold 5
MSE Training =  6008.189396371469
MSE Validation =  6498.039376017456
M value:  2
lambda value:  0.7000000000000001
```

```
Fold 1
MSE Training =  5680.378652794995
MSE Validation =  7866.415711638609

Fold 2
MSE Training =  6095.985339608117
MSE Validation =  6387.830167093819

Fold 3
MSE Training =  6187.6775208657855
MSE Validation =  5846.133937333771

Fold 4
MSE Training =  6236.36901827641
MSE Validation =  5659.038272356196

Fold 5
MSE Training =  6372.181700879788
MSE Validation =  5046.596153147903
M value:  2
lambda value:  0.8

Fold 1
MSE Training =  5993.953439658598
MSE Validation =  6669.178999001775

Fold 2
MSE Training =  6151.826506018929
MSE Validation =  6056.438123528338

Fold 3
MSE Training =  5997.319006230135
MSE Validation =  6641.225479859925

Fold 4
MSE Training =  6160.266439525275
MSE Validation =  6111.964537510218

Fold 5
MSE Training =  6317.345107631368
MSE Validation =  5346.501083560023
M value:  2
lambda value:  0.9

Fold 1
MSE Training =  6101.87134262225
MSE Validation =  6178.675620413933

Fold 2
MSE Training =  5948.217263028127
MSE Validation =  6889.644321752713

Fold 3
MSE Training =  6192.939897129246
MSE Validation =  6167.044890209538

Fold 4
MSE Training =  6263.803326272696
MSE Validation =  5603.465994342685
```

```
Fold 5
MSE Training =  6107.91020799391
MSE Validation =  6312.446191111544
M value:  2
lambda value:  1.0

Fold 1
MSE Training =  6082.007827585816
MSE Validation =  6380.669686850559

Fold 2
MSE Training =  6232.286711204069
MSE Validation =  5786.176099683459

Fold 3
MSE Training =  6255.097067924915
MSE Validation =  5758.375884911206

Fold 4
MSE Training =  5922.133276184686
MSE Validation =  7062.34653037614

Fold 5
MSE Training =  6218.33562290913
MSE Validation =  5940.7042723475

Average Performance in Training =  6141.972101161724
Average Performance in Validation =  6185.654494833772
----------------------------------------------------

M value:  3
lambda value:  0.1

Fold 1
MSE Training =  3779.9205747395636
MSE Validation =  4147.349680766893

Fold 2
MSE Training =  3645.195157948238
MSE Validation =  4748.639911856269

Fold 3
MSE Training =  3822.8324888784077
MSE Validation =  3815.1327083748106

Fold 4
MSE Training =  3924.278758005458
MSE Validation =  3451.140429797554

Fold 5
MSE Training =  3923.9519516629025
MSE Validation =  3427.206290990641
M value:  3
lambda value:  0.2

Fold 1
MSE Training =  3664.0995529131797
MSE Validation =  4724.716553868697

Fold 2
```

```
MSE Training =  4090.1687289215465
MSE Validation =  3618.9437459533474

Fold 3
MSE Training =  3919.6286908125426
MSE Validation =  3925.706564627749

Fold 4
MSE Training =  3967.2024142471687
MSE Validation =  3921.487298516922

Fold 5
MSE Training =  3979.7070820445933
MSE Validation =  3954.2767787421185
M value:  3
lambda value:  0.30000000000000004

Fold 1
MSE Training =  4049.2101631939063
MSE Validation =  4458.1449483001215

Fold 2
MSE Training =  3984.879433388452
MSE Validation =  4475.387492739204

Fold 3
MSE Training =  4112.591873117303
MSE Validation =  3728.9295799248453

Fold 4
MSE Training =  3947.016823246367
MSE Validation =  4231.28047810967

Fold 5
MSE Training =  4125.93838818141
MSE Validation =  3923.9161723269585
M value:  3
lambda value:  0.4

Fold 1
MSE Training =  4152.935198584011
MSE Validation =  4152.2067313906

Fold 2
MSE Training =  4346.959261803704
MSE Validation =  3253.8960100149393

Fold 3
MSE Training =  4123.0448266809935
MSE Validation =  4789.968661916458

Fold 4
MSE Training =  4152.443955321741
MSE Validation =  4180.50302730655

Fold 5
MSE Training =  4072.695632998569
MSE Validation =  4781.513055416485
M value:  3
lambda value:  0.5
```

```
Fold 1
MSE Training =  4173.194263290284
MSE Validation =  5387.892958670635

Fold 2
MSE Training =  4344.869013490411
MSE Validation =  4255.61412770952

Fold 3
MSE Training =  4260.647313527048
MSE Validation =  4148.341246074005

Fold 4
MSE Training =  4362.099943584376
MSE Validation =  3811.8456306644375

Fold 5
MSE Training =  4259.916850096493
MSE Validation =  4332.0654186975025
M value:  3
lambda value:  0.6

Fold 1
MSE Training =  4319.833647915843
MSE Validation =  4509.945763954134

Fold 2
MSE Training =  4480.12726180344
MSE Validation =  4077.3085001252634

Fold 3
MSE Training =  4202.643283269922
MSE Validation =  5245.256827383136

Fold 4
MSE Training =  4425.87360523711
MSE Validation =  4312.228589239544

Fold 5
MSE Training =  4544.021784839084
MSE Validation =  4008.509823892413
M value:  3
lambda value:  0.7000000000000001

Fold 1
MSE Training =  4404.111083122063
MSE Validation =  5036.042864342069

Fold 2
MSE Training =  4453.740456350182
MSE Validation =  4721.969772643983

Fold 3
MSE Training =  4612.543080412246
MSE Validation =  4267.263613621901

Fold 4
MSE Training =  4361.987017149486
MSE Validation =  4919.449349606376
```

```
Fold 5
MSE Training =  4640.771736712629
MSE Validation =  3633.1041160886584
M value:  3
lambda value:  0.8

Fold 1
MSE Training =  4578.530549704723
MSE Validation =  4716.654628817551

Fold 2
MSE Training =  4543.403717191946
MSE Validation =  5021.261714989034

Fold 3
MSE Training =  4628.353312620983
MSE Validation =  4188.589332913628

Fold 4
MSE Training =  4751.366999692491
MSE Validation =  3722.6861026716883

Fold 5
MSE Training =  4398.571646414181
MSE Validation =  5647.034048253197
M value:  3
lambda value:  0.9

Fold 1
MSE Training =  4635.9628214388185
MSE Validation =  5693.3550518599495

Fold 2
MSE Training =  4590.895208217794
MSE Validation =  4448.145047111744

Fold 3
MSE Training =  4601.671434599606
MSE Validation =  4950.439303557892

Fold 4
MSE Training =  4587.014176758973
MSE Validation =  4950.228655888442

Fold 5
MSE Training =  4867.605667905262
MSE Validation =  3882.1016386420406
M value:  3
lambda value:  1.0

Fold 1
MSE Training =  4705.11478018381
MSE Validation =  5359.180628342652

Fold 2
MSE Training =  4660.156409835085
MSE Validation =  4997.987581342166

Fold 3
```

```
MSE Training =  4792.987211172959
MSE Validation =  4338.3839442308035

Fold 4
MSE Training =  4726.97984528368
MSE Validation =  4679.637139119151

Fold 5
MSE Training =  4825.496186139526
MSE Validation =  4588.496806501456

Average Performance in Training =  4742.146886523013
Average Performance in Validation =  4792.737219907246
----------------------------------------------------

M value:  4
lambda value:  0.1

Fold 1
MSE Training =  3591.834807686103
MSE Validation =  3375.3527592171467

Fold 2
MSE Training =  3605.4772492613583
MSE Validation =  3476.8704437871306

Fold 3
MSE Training =  3338.5483473554536
MSE Validation =  4419.166683486613

Fold 4
MSE Training =  3533.6675498493382
MSE Validation =  3822.288002368723

Fold 5
MSE Training =  3547.449708919779
MSE Validation =  3404.05102910886
M value:  4
lambda value:  0.2

Fold 1
MSE Training =  3570.860351001523
MSE Validation =  4028.1566991160767

Fold 2
MSE Training =  3775.6260823363314
MSE Validation =  3223.3060022446493

Fold 3
MSE Training =  3527.438075703829
MSE Validation =  3888.6093202120965

Fold 4
MSE Training =  3515.159753558023
MSE Validation =  4046.42984808551

Fold 5
MSE Training =  3735.127877783325
MSE Validation =  3238.312776093454
M value:  4
```

```
lambda value:  0.30000000000000004

Fold 1
MSE Training =  3699.901182622269
MSE Validation =  3806.971079337425

Fold 2
MSE Training =  3758.7459538481953
MSE Validation =  3794.2314765936844

Fold 3
MSE Training =  3680.785801790322
MSE Validation =  3947.0503610916726

Fold 4
MSE Training =  3893.347592469243
MSE Validation =  2994.1491246873293

Fold 5
MSE Training =  3603.267470592846
MSE Validation =  4458.419126521155
M value:  4
lambda value:  0.4

Fold 1
MSE Training =  3717.177462339912
MSE Validation =  4354.925554951963

Fold 2
MSE Training =  3731.1167115594444
MSE Validation =  4161.569445334344

Fold 3
MSE Training =  3829.2295018600043
MSE Validation =  3857.1096263388927

Fold 4
MSE Training =  4131.020350701385
MSE Validation =  2667.5859487643534

Fold 5
MSE Training =  3776.414712374362
MSE Validation =  4507.257474556054
M value:  4
lambda value:  0.5

Fold 1
MSE Training =  4071.984879704202
MSE Validation =  3620.535762087757

Fold 2
MSE Training =  3851.880773908982
MSE Validation =  4430.863552774414

Fold 3
MSE Training =  3983.3977240740683
MSE Validation =  3644.3763277774506

Fold 4
MSE Training =  3958.177758646862
```

```
MSE Validation =  3839.7588591350477

Fold 5
MSE Training =  3852.786933601192
MSE Validation =  4419.4522797831305
M value:  4
lambda value:  0.6

Fold 1
MSE Training =  4102.1967366691
MSE Validation =  3758.032310816942

Fold 2
MSE Training =  3930.955122685792
MSE Validation =  4437.669340415528

Fold 3
MSE Training =  4142.04643231396
MSE Validation =  3911.107237144595

Fold 4
MSE Training =  4085.883679984737
MSE Validation =  4074.5947123113447

Fold 5
MSE Training =  3927.929340844412
MSE Validation =  4571.297155704696
M value:  4
lambda value:  0.7000000000000001

Fold 1
MSE Training =  4098.565426092227
MSE Validation =  4595.902109488664

Fold 2
MSE Training =  3947.6047853927057
MSE Validation =  4605.640916033914

Fold 3
MSE Training =  4334.979228976488
MSE Validation =  3159.928970156169

Fold 4
MSE Training =  3984.898789477293
MSE Validation =  5051.165433813598

Fold 5
MSE Training =  4307.5456516395425
MSE Validation =  3694.6491360900463
M value:  4
lambda value:  0.8

Fold 1
MSE Training =  4033.9741142546977
MSE Validation =  5203.869193981128

Fold 2
MSE Training =  4312.203002322701
MSE Validation =  4218.01958392486
```

```
Fold 3
MSE Training =  4253.560242985612
MSE Validation =  3812.365666191217

Fold 4
MSE Training =  4415.059066380404
MSE Validation =  3344.703053378199

Fold 5
MSE Training =  4121.06308279027
MSE Validation =  4915.222645564708
M value:  4
lambda value:  0.9

Fold 1
MSE Training =  4281.931995555223
MSE Validation =  4702.425234917681

Fold 2
MSE Training =  4206.1837446273785
MSE Validation =  5004.851607302154

Fold 3
MSE Training =  4322.259397105636
MSE Validation =  4368.625766538803

Fold 4
MSE Training =  4297.91634396259
MSE Validation =  4262.300595291599

Fold 5
MSE Training =  4410.726393798347
MSE Validation =  3938.276712840686
M value:  4
lambda value:  1.0

Fold 1
MSE Training =  4344.524812121543
MSE Validation =  4964.944295269451

Fold 2
MSE Training =  4319.806772414242
MSE Validation =  4797.395282189718

Fold 3
MSE Training =  4361.189260607141
MSE Validation =  4631.58910770622

Fold 4
MSE Training =  4532.448206028328
MSE Validation =  3796.508174699659

Fold 5
MSE Training =  4352.350196006247
MSE Validation =  4284.207640946548

Average Performance in Training =  4382.0638494355
Average Performance in Validation =  4494.928900162319
----------------------------------------------------------
```

```
M value:  5
lambda value:  0.1

Fold 1
MSE Training =  3476.1371810457513
MSE Validation =  3928.8360074350035

Fold 2
MSE Training =  3569.885332030878
MSE Validation =  3558.6683293850115

Fold 3
MSE Training =  3523.8276533203257
MSE Validation =  3645.716881229561

Fold 4
MSE Training =  3613.9367912280222
MSE Validation =  3209.5471045245595

Fold 5
MSE Training =  3432.368143398021
MSE Validation =  3955.2754818647886
M value:  5
lambda value:  0.2

Fold 1
MSE Training =  3548.005985650053
MSE Validation =  3813.629801027527

Fold 2
MSE Training =  3583.220716518767
MSE Validation =  3743.289339329786

Fold 3
MSE Training =  3698.323385415369
MSE Validation =  3379.946286535862

Fold 4
MSE Training =  3527.483844626296
MSE Validation =  3873.507393637434

Fold 5
MSE Training =  3714.6638159208605
MSE Validation =  3763.91871507902
M value:  5
lambda value:  0.30000000000000004

Fold 1
MSE Training =  3682.530498066765
MSE Validation =  3659.6381210811783

Fold 2
MSE Training =  3720.1229405365534
MSE Validation =  3737.8898158085644

Fold 3
MSE Training =  3723.9779966612887
MSE Validation =  4004.6810308042204

Fold 4
```

```
MSE Training =  3830.575390454228
MSE Validation =  3477.2598720020083

Fold 5
MSE Training =  3611.5191762178943
MSE Validation =  4203.738366153523
M value:  5
lambda value:  0.4

Fold 1
MSE Training =  3936.0983715341254
MSE Validation =  3337.09929581175

Fold 2
MSE Training =  3759.748771730267
MSE Validation =  3958.6001575284463

Fold 3
MSE Training =  3697.573378216038
MSE Validation =  4457.6127496040635

Fold 4
MSE Training =  3797.0172774903754
MSE Validation =  4068.958104449286

Fold 5
MSE Training =  3914.010104174644
MSE Validation =  3723.6156657427464
M value:  5
lambda value:  0.5

Fold 1
MSE Training =  3769.0004332394633
MSE Validation =  4675.80961523552

Fold 2
MSE Training =  4031.5553772167837
MSE Validation =  3179.2754655980325

Fold 3
MSE Training =  3934.315845870359
MSE Validation =  3825.3851047991416

Fold 4
MSE Training =  3941.130878142053
MSE Validation =  4141.953494494258

Fold 5
MSE Training =  3932.5107338913595
MSE Validation =  4393.74104367218
M value:  5
lambda value:  0.6

Fold 1
MSE Training =  4262.918697483198
MSE Validation =  3088.457859377586

Fold 2
MSE Training =  3991.529041641404
MSE Validation =  4331.620566358621
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```
Fold 3
MSE Training =  3955.6835132872693
MSE Validation =  4476.327644081359

Fold 4
MSE Training =  3905.0417201940745
MSE Validation =  4426.9976628237555

Fold 5
MSE Training =  3957.4091356180757
MSE Validation =  4541.0878493537975
M value:  5
lambda value:  0.7000000000000001

Fold 1
MSE Training =  4172.554004366556
MSE Validation =  3899.30143256157

Fold 2
MSE Training =  4077.7620096079504
MSE Validation =  4229.483792965285

Fold 3
MSE Training =  4164.94202156346
MSE Validation =  3948.1088437035187

Fold 4
MSE Training =  4234.441466202632
MSE Validation =  3747.289242114807

Fold 5
MSE Training =  3914.9268549460676
MSE Validation =  5299.296111758314
M value:  5
lambda value:  0.8

Fold 1
MSE Training =  4282.285296673166
MSE Validation =  3802.019779221485

Fold 2
MSE Training =  4253.388184694969
MSE Validation =  3964.4476366896247

Fold 3
MSE Training =  4120.114978345413
MSE Validation =  4618.683839505182

Fold 4
MSE Training =  4189.1085634904175
MSE Validation =  4457.491089446723

Fold 5
MSE Training =  4184.822867532006
MSE Validation =  4458.295581704913
M value:  5
lambda value:  0.9

Fold 1
```

```
MSE Training =  4417.197067394922
MSE Validation =  4050.1122260550483

Fold 2
MSE Training =  4227.2285748758
MSE Validation =  4293.610940922006

Fold 3
MSE Training =  4372.842744849458
MSE Validation =  3959.0164449779077

Fold 4
MSE Training =  4288.793257780934
MSE Validation =  4126.94781882209

Fold 5
MSE Training =  4093.9005939772896
MSE Validation =  5592.294734840444
M value:  5
lambda value:  1.0

Fold 1
MSE Training =  4366.035486443334
MSE Validation =  4072.33285124634

Fold 2
MSE Training =  4317.007826763526
MSE Validation =  4675.883535190873

Fold 3
MSE Training =  4090.1317659821966
MSE Validation =  5709.060890014934

Fold 4
MSE Training =  4443.618385848444
MSE Validation =  4229.518388038346

Fold 5
MSE Training =  4561.806340636248
MSE Validation =  3589.4681781348763

Average Performance in Training =  4355.7199611347505
Average Performance in Validation =  4455.252768525073
----------------------------------------------------

M value:  6
lambda value:  0.1

Fold 1
MSE Training =  3478.113624687009
MSE Validation =  3852.595607537566

Fold 2
MSE Training =  3552.237809445786
MSE Validation =  3557.942897892351

Fold 3
MSE Training =  3381.6332903798802
MSE Validation =  4248.620647610925
```

```
Fold 4
MSE Training =  3447.995841780248
MSE Validation =  3971.7255782146626

Fold 5
MSE Training =  3666.6931939223055
MSE Validation =  2887.1744536868073
M value:  6
lambda value:  0.2

Fold 1
MSE Training =  3568.3566794534277
MSE Validation =  3960.941207765838

Fold 2
MSE Training =  3624.7232739866804
MSE Validation =  3479.7978980080293

Fold 3
MSE Training =  3591.99096390176
MSE Validation =  3887.8808025424933

Fold 4
MSE Training =  3528.2279857733392
MSE Validation =  3943.018125016772

Fold 5
MSE Training =  3705.1709142765126
MSE Validation =  3343.2777178588335
M value:  6
lambda value:  0.30000000000000004

Fold 1
MSE Training =  3767.4742265309924
MSE Validation =  4088.8476732033455

Fold 2
MSE Training =  3733.706153670107
MSE Validation =  3729.0028102944507

Fold 3
MSE Training =  3549.6982769857873
MSE Validation =  4413.234340724686

Fold 4
MSE Training =  3919.6328312433657
MSE Validation =  2800.178882144028

Fold 5
MSE Training =  3582.1804259453643
MSE Validation =  4185.431649129676
M value:  6
lambda value:  0.4

Fold 1
MSE Training =  3979.086056177261
MSE Validation =  3317.6305640547266

Fold 2
MSE Training =  3850.2823058876947
```

```
MSE Validation =  3659.071948917503

Fold 3
MSE Training =  3553.6889494081643
MSE Validation =  5062.592372317366

Fold 4
MSE Training =  3683.8347060108504
MSE Validation =  4579.199788124999

Fold 5
MSE Training =  4006.2130469096087
MSE Validation =  3197.7454238838945
M value:  6
lambda value:  0.5

Fold 1
MSE Training =  3665.5421633165556
MSE Validation =  5892.640680775711

Fold 2
MSE Training =  3850.6391590062876
MSE Validation =  4374.837298194206

Fold 3
MSE Training =  3918.4078040704953
MSE Validation =  3666.130301096415

Fold 4
MSE Training =  4078.3329999164625
MSE Validation =  3274.0830118209183

Fold 5
MSE Training =  4024.8865863855667
MSE Validation =  3695.6647805461985
M value:  6
lambda value:  0.6

Fold 1
MSE Training =  3938.1864170516346
MSE Validation =  4410.498406221133

Fold 2
MSE Training =  3943.1232784625186
MSE Validation =  4210.5461662805565

Fold 3
MSE Training =  4440.395097131351
MSE Validation =  2549.7176513423656

Fold 4
MSE Training =  3834.226677287798
MSE Validation =  5092.268969100969

Fold 5
MSE Training =  3911.4906458318874
MSE Validation =  4738.158831988825
M value:  6
lambda value:  0.7000000000000001
```

```
Fold 1
MSE Training =  4198.474734164792
MSE Validation =  3835.4544611675515

Fold 2
MSE Training =  4122.748242855435
MSE Validation =  4367.49683322303

Fold 3
MSE Training =  4131.055346166856
MSE Validation =  4005.373346156757

Fold 4
MSE Training =  4201.027660940154
MSE Validation =  3710.103455724484

Fold 5
MSE Training =  3937.5517355778684
MSE Validation =  5054.076369157327
M value:  6
lambda value:  0.8

Fold 1
MSE Training =  4266.3455563548805
MSE Validation =  3973.8935879738906

Fold 2
MSE Training =  4225.62841912908
MSE Validation =  4037.5793670823705

Fold 3
MSE Training =  4111.403373806617
MSE Validation =  4881.376830096782

Fold 4
MSE Training =  4323.946349298058
MSE Validation =  3933.392717681389

Fold 5
MSE Training =  4081.9301288933184
MSE Validation =  4706.169299815092
M value:  6
lambda value:  0.9

Fold 1
MSE Training =  4397.922651572036
MSE Validation =  4094.574574796434

Fold 2
MSE Training =  4240.157932605551
MSE Validation =  4747.398711478335

Fold 3
MSE Training =  4163.613168762003
MSE Validation =  4504.69726141203

Fold 4
MSE Training =  4238.324923692741
MSE Validation =  4628.034552932979
```

```
Fold 5
MSE Training =  4379.491431399889
MSE Validation =  3909.579456701484
M value:  6
lambda value:  1.0

Fold 1
MSE Training =  4401.713558962399
MSE Validation =  4029.6159462934943

Fold 2
MSE Training =  4355.331211548237
MSE Validation =  4895.784255147845

Fold 3
MSE Training =  4497.041441856336
MSE Validation =  3900.182114906973

Fold 4
MSE Training =  4261.616577885619
MSE Validation =  4883.289798578417

Fold 5
MSE Training =  4303.456742194254
MSE Validation =  4485.7227232199675

Average Performance in Training =  4363.8319064893685
Average Performance in Validation =  4438.918967629339
----------------------------------------------------

M value:  7
lambda value:  0.1

Fold 1
MSE Training =  3154.6591970833333
MSE Validation =  3432.698765301651

Fold 2
MSE Training =  3336.9800761638594
MSE Validation =  2641.7202983462334

Fold 3
MSE Training =  3175.5473212023226
MSE Validation =  3184.222858237993

Fold 4
MSE Training =  3076.7078475147964
MSE Validation =  3553.440601172943

Fold 5
MSE Training =  3020.3414019010406
MSE Validation =  3781.330416458186
M value:  7
lambda value:  0.2

Fold 1
MSE Training =  3170.442649746544
MSE Validation =  3947.038250758495

Fold 2
```

```
MSE Training =  3325.6666495922573
MSE Validation =  3212.275260389532

Fold 3
MSE Training =  3307.0753849651746
MSE Validation =  3370.779049346922

Fold 4
MSE Training =  3129.8133815411916
MSE Validation =  3913.5777945569625

Fold 5
MSE Training =  3477.062224733678
MSE Validation =  2625.824542626725
M value:  7
lambda value:  0.30000000000000004

Fold 1
MSE Training =  3429.1797477991477
MSE Validation =  3320.469460695759

Fold 2
MSE Training =  3608.9634380740995
MSE Validation =  2745.8623513706775

Fold 3
MSE Training =  3243.079695171848
MSE Validation =  4004.369425827072

Fold 4
MSE Training =  3420.315525896469
MSE Validation =  3380.2752103915973

Fold 5
MSE Training =  3260.7687753495065
MSE Validation =  4381.3415943539
M value:  7
lambda value:  0.4

Fold 1
MSE Training =  3497.0712059810976
MSE Validation =  3613.862346393934

Fold 2
MSE Training =  3417.8040461081814
MSE Validation =  3969.3955713241035

Fold 3
MSE Training =  3508.0612191243563
MSE Validation =  3854.8436659143786

Fold 4
MSE Training =  3530.1714328367007
MSE Validation =  3466.9434470799106

Fold 5
MSE Training =  3585.09709226717
MSE Validation =  3407.2885939236076
M value:  7
lambda value:  0.5
```

```
Fold 1
MSE Training =  3662.0813976638246
MSE Validation =  3576.7377111501887

Fold 2
MSE Training =  3468.8493539829906
MSE Validation =  4440.248603841819

Fold 3
MSE Training =  3590.0556311203536
MSE Validation =  3798.075312366703

Fold 4
MSE Training =  3703.4694174712877
MSE Validation =  3282.3234009270195

Fold 5
MSE Training =  3674.710295424839
MSE Validation =  3554.5640823033086
M value:  7
lambda value:  0.6

Fold 1
MSE Training =  3410.3014481544537
MSE Validation =  4788.903335429404

Fold 2
MSE Training =  3916.023400250003
MSE Validation =  3249.9562695610434

Fold 3
MSE Training =  3802.526916967196
MSE Validation =  3795.2362592001136

Fold 4
MSE Training =  3652.280859230254
MSE Validation =  4082.060036428971

Fold 5
MSE Training =  3851.3142880447226
MSE Validation =  3089.9850074699057
M value:  7
lambda value:  0.7000000000000001

Fold 1
MSE Training =  3722.563405391416
MSE Validation =  4137.5774877080585

Fold 2
MSE Training =  3784.7568478237713
MSE Validation =  3886.575510869209

Fold 3
MSE Training =  3789.3478908403163
MSE Validation =  4278.561576606169

Fold 4
MSE Training =  3869.4149582098553
MSE Validation =  3718.5089261341786
```

```
Fold 5
MSE Training =  3939.8057843694273
MSE Validation =  3639.703068105698
M value:  7
lambda value:  0.8

Fold 1
MSE Training =  3865.6443288492646
MSE Validation =  3763.24381583285

Fold 2
MSE Training =  3964.6990589210805
MSE Validation =  3737.9887456772526

Fold 3
MSE Training =  4146.923701940046
MSE Validation =  3104.9489468330853

Fold 4
MSE Training =  3875.1041307654627
MSE Validation =  4724.523546783477

Fold 5
MSE Training =  3716.873166874779
MSE Validation =  4649.043558875212
M value:  7
lambda value:  0.9

Fold 1
MSE Training =  3841.4924494872917
MSE Validation =  4526.731807391127

Fold 2
MSE Training =  4145.429637163524
MSE Validation =  3443.4080506371333

Fold 3
MSE Training =  3841.898619745414
MSE Validation =  4874.784051842094

Fold 4
MSE Training =  4059.856371874342
MSE Validation =  3973.638846306916

Fold 5
MSE Training =  4090.8577536542953
MSE Validation =  3617.2028557601525
M value:  7
lambda value:  1.0

Fold 1
MSE Training =  4232.25093206098
MSE Validation =  3669.312547777824

Fold 2
MSE Training =  4041.8761297496508
MSE Validation =  4278.327822440268

Fold 3
```

```
MSE Training =  4123.187018649139
MSE Validation =  4082.5286562437273

Fold 4
MSE Training =  3948.0972621650753
MSE Validation =  4465.612712918129

Fold 5
MSE Training =  4019.0887804848994
MSE Validation =  4446.888618079842

Average Performance in Training =  4072.900024621949
Average Performance in Validation =  4188.5340714919585
----------------------------------------------------

M value:  8
lambda value:  0.1

Fold 1
MSE Training =  3016.654975010347
MSE Validation =  3531.5629402177296

Fold 2
MSE Training =  3106.119333058891
MSE Validation =  3575.7059929803568

Fold 3
MSE Training =  3085.128086360375
MSE Validation =  3325.5036791823973

Fold 4
MSE Training =  3222.530126156294
MSE Validation =  2773.7813747222062

Fold 5
MSE Training =  3116.3020075118066
MSE Validation =  3187.945083895455
M value:  8
lambda value:  0.2

Fold 1
MSE Training =  3181.372408671558
MSE Validation =  3456.9811843487364

Fold 2
MSE Training =  3113.3030447206197
MSE Validation =  3675.682564261974

Fold 3
MSE Training =  3138.842513185364
MSE Validation =  3763.410587783956

Fold 4
MSE Training =  3569.152553159605
MSE Validation =  1985.1669164162279

Fold 5
MSE Training =  3151.013289783843
MSE Validation =  3631.7908883330415
M value:  8
```

```
lambda value:  0.30000000000000004

Fold 1
MSE Training =  3256.119073791597
MSE Validation =  3734.602109463332

Fold 2
MSE Training =  3354.394042884337
MSE Validation =  3173.054520918033

Fold 3
MSE Training =  3328.9457189823684
MSE Validation =  3777.222988574326

Fold 4
MSE Training =  3391.4211492092068
MSE Validation =  3212.33210304146

Fold 5
MSE Training =  3294.6579452356964
MSE Validation =  3640.8986432342863
M value:  8
lambda value:  0.4

Fold 1
MSE Training =  3341.7072695822094
MSE Validation =  3888.9775542097627

Fold 2
MSE Training =  3576.465930446317
MSE Validation =  3096.1708362739287

Fold 3
MSE Training =  3502.730426406257
MSE Validation =  3409.839275454949

Fold 4
MSE Training =  3329.9637538806132
MSE Validation =  3853.916520200187

Fold 5
MSE Training =  3447.661697685585
MSE Validation =  3389.6961532035966
M value:  8
lambda value:  0.5

Fold 1
MSE Training =  3605.1866566937547
MSE Validation =  3651.8862771238764

Fold 2
MSE Training =  3517.3581664796066
MSE Validation =  3524.368008103808

Fold 3
MSE Training =  3391.616184331207
MSE Validation =  4313.499912454613

Fold 4
MSE Training =  3683.314226842076
```

```
MSE Validation =  3157.2055932337157

Fold 5
MSE Training =  3454.4182886945264
MSE Validation =  3853.935743702068
M value:  8
lambda value:  0.6

Fold 1
MSE Training =  3752.832149791684
MSE Validation =  3090.430003855811

Fold 2
MSE Training =  3553.80573321051
MSE Validation =  4031.0407917204584

Fold 3
MSE Training =  3689.7572688161677
MSE Validation =  3758.863700209914

Fold 4
MSE Training =  3618.4944666616248
MSE Validation =  3626.3866134116683

Fold 5
MSE Training =  3571.9417212766084
MSE Validation =  4053.487957367935
M value:  8
lambda value:  0.7000000000000001

Fold 1
MSE Training =  3848.9123451320834
MSE Validation =  3374.1454004560483

Fold 2
MSE Training =  3712.8915752906905
MSE Validation =  3987.916304109604

Fold 3
MSE Training =  3638.261995709423
MSE Validation =  3601.3604260085503

Fold 4
MSE Training =  3788.1806297264693
MSE Validation =  3705.108073210774

Fold 5
MSE Training =  3629.7240342021846
MSE Validation =  4500.239147453897
M value:  8
lambda value:  0.8

Fold 1
MSE Training =  3938.243409681561
MSE Validation =  3592.110514446598

Fold 2
MSE Training =  4039.196615066893
MSE Validation =  2955.531563741374
```

```
Fold 3
MSE Training =  3796.929968680076
MSE Validation =  4219.776594890689

Fold 4
MSE Training =  3560.2722604790947
MSE Validation =  4698.3179055399805

Fold 5
MSE Training =  3692.159070372392
MSE Validation =  4117.215080649524
M value:  8
lambda value:  0.9

Fold 1
MSE Training =  3873.425127960385
MSE Validation =  3604.997499566892

Fold 2
MSE Training =  3924.8962124967343
MSE Validation =  3594.010305595386

Fold 3
MSE Training =  3805.568418728283
MSE Validation =  4934.452318471293

Fold 4
MSE Training =  3926.935575120678
MSE Validation =  4130.908296833809

Fold 5
MSE Training =  3858.792165023238
MSE Validation =  4032.598036836786
M value:  8
lambda value:  1.0

Fold 1
MSE Training =  3861.8800782565722
MSE Validation =  4264.653703797514

Fold 2
MSE Training =  3934.095625449709
MSE Validation =  4283.8829289122095

Fold 3
MSE Training =  4023.8984932348103
MSE Validation =  3792.748122646461

Fold 4
MSE Training =  3968.0042416532638
MSE Validation =  3909.1269616508002

Fold 5
MSE Training =  4022.4829735148096
MSE Validation =  4017.2431241596064

Average Performance in Training =  3962.072282421833
Average Performance in Validation =  4053.530968233318
----------------------------------------------------
```

```
M value:  9
lambda value:  0.1

Fold 1
MSE Training =  2782.882714054933
MSE Validation =  3574.1324455430567

Fold 2
MSE Training =  3004.4281741309005
MSE Validation =  2708.1357256181304

Fold 3
MSE Training =  2900.243597341595
MSE Validation =  3294.731518310971

Fold 4
MSE Training =  3032.6386564244026
MSE Validation =  2521.875579952691

Fold 5
MSE Training =  2878.2485501354254
MSE Validation =  3301.050943303641
M value:  9
lambda value:  0.2

Fold 1
MSE Training =  2927.9588094015767
MSE Validation =  3374.4929568625093

Fold 2
MSE Training =  2956.0109773174663
MSE Validation =  3243.488885697796

Fold 3
MSE Training =  3062.51175110338
MSE Validation =  2875.852877234398

Fold 4
MSE Training =  2999.5366411999016
MSE Validation =  3152.953343424016

Fold 5
MSE Training =  3004.050112671217
MSE Validation =  3030.769675108145
M value:  9
lambda value:  0.30000000000000004

Fold 1
MSE Training =  3106.1174117388896
MSE Validation =  3213.2673914356933

Fold 2
MSE Training =  3172.160309828983
MSE Validation =  2945.2615901201807

Fold 3
MSE Training =  3181.676081535911
MSE Validation =  3007.7231771680954

Fold 4
```

```
MSE Training =  3015.670827676724
MSE Validation =  3276.041282362795

Fold 5
MSE Training =  2870.656452735719
MSE Validation =  3920.7754340870592
M value:  9
lambda value:  0.4

Fold 1
MSE Training =  3256.8020135857637
MSE Validation =  2976.056585197594

Fold 2
MSE Training =  3139.2166642804796
MSE Validation =  3122.853670528278

Fold 3
MSE Training =  3079.1469223554054
MSE Validation =  3477.770795912405

Fold 4
MSE Training =  3070.4416046460947
MSE Validation =  3943.0066017473378

Fold 5
MSE Training =  3274.494308135017
MSE Validation =  2852.1294687505115
M value:  9
lambda value:  0.5

Fold 1
MSE Training =  3364.9533095995585
MSE Validation =  2906.91188191596

Fold 2
MSE Training =  3095.5156465818204
MSE Validation =  4022.3932616818124

Fold 3
MSE Training =  3378.0376811293268
MSE Validation =  2543.649387445113

Fold 4
MSE Training =  3340.06147302699
MSE Validation =  3355.850065541847

Fold 5
MSE Training =  3100.7603771303197
MSE Validation =  3889.297586186205
M value:  9
lambda value:  0.6

Fold 1
MSE Training =  3474.0393919343433
MSE Validation =  2565.1082413425406

Fold 2
MSE Training =  3390.8850650830814
MSE Validation =  3524.6013645122553
```

```
Fold 3
MSE Training =  3348.2597734896863
MSE Validation =  3589.098987442993

Fold 4
MSE Training =  3173.4661085972457
MSE Validation =  4140.292571191868

Fold 5
MSE Training =  3246.836580375433
MSE Validation =  3824.5982855633138
M value:  9
lambda value:  0.7000000000000001

Fold 1
MSE Training =  3446.5827870732146
MSE Validation =  3342.3997885851113

Fold 2
MSE Training =  3579.1434261042655
MSE Validation =  2807.3582064950256

Fold 3
MSE Training =  3315.8294098997544
MSE Validation =  4085.3645152913687

Fold 4
MSE Training =  3538.712738899321
MSE Validation =  3027.333997965069

Fold 5
MSE Training =  3232.1687536332925
MSE Validation =  4228.780917630006
M value:  9
lambda value:  0.8

Fold 1
MSE Training =  3468.9867243279045
MSE Validation =  3914.193757427571

Fold 2
MSE Training =  3514.553279684414
MSE Validation =  3354.311651269549

Fold 3
MSE Training =  3435.2345228132535
MSE Validation =  3767.9578213234063

Fold 4
MSE Training =  3391.3639824640127
MSE Validation =  4052.875330207666

Fold 5
MSE Training =  3684.1364816872124
MSE Validation =  2840.360878699693
M value:  9
lambda value:  0.9

Fold 1
```

```
MSE Training =  3605.424466666952
MSE Validation =  3755.3076220105745

Fold 2
MSE Training =  3526.5965992588094
MSE Validation =  3742.0272581242925

Fold 3
MSE Training =  3610.809332112565
MSE Validation =  3151.6914601920776

Fold 4
MSE Training =  3566.8825109956756
MSE Validation =  4070.1834127743823

Fold 5
MSE Training =  3543.9611525213663
MSE Validation =  3774.246012072601
M value:  9
lambda value:  1.0

Fold 1
MSE Training =  3881.7822032523454
MSE Validation =  2860.741712313515

Fold 2
MSE Training =  3776.8835603872963
MSE Validation =  3172.871434097054

Fold 3
MSE Training =  3576.1916977563333
MSE Validation =  4032.7441980018853

Fold 4
MSE Training =  3593.1803133280105
MSE Validation =  3810.681452815018

Fold 5
MSE Training =  3372.8055781314124
MSE Validation =  5101.281487121605

Average Performance in Training =  3640.1686705710795
Average Performance in Validation =  3795.664056869816
----------------------------------------------------

M value:  10
lambda value:  0.1

Fold 1
MSE Training =  2890.8465910230457
MSE Validation =  3057.6338325238935

Fold 2
MSE Training =  3031.5303507547615
MSE Validation =  2677.5391219860117

Fold 3
MSE Training =  2915.3987041957835
MSE Validation =  3110.0507777887105
```

```
Fold 4
MSE Training =  2793.751784152223
MSE Validation =  3604.5228964142966

Fold 5
MSE Training =  2965.516874861407
MSE Validation =  2769.6168551436363
M value:  10
lambda value:  0.2

Fold 1
MSE Training =  3052.7057541208123
MSE Validation =  2844.159093496954

Fold 2
MSE Training =  2830.074424702828
MSE Validation =  3579.8216910309206

Fold 3
MSE Training =  3157.049807753546
MSE Validation =  2472.653016932104

Fold 4
MSE Training =  2999.86692930784
MSE Validation =  3241.9707959907737

Fold 5
MSE Training =  2878.3356536109322
MSE Validation =  3591.038343419208
M value:  10
lambda value:  0.30000000000000004

Fold 1
MSE Training =  3135.5025246791893
MSE Validation =  2921.92770423566

Fold 2
MSE Training =  2991.2059171547216
MSE Validation =  3503.5133743490333

Fold 3
MSE Training =  3072.667619133961
MSE Validation =  3274.755256291814

Fold 4
MSE Training =  3049.6312603126553
MSE Validation =  3098.1174907013155

Fold 5
MSE Training =  3077.8877062978763
MSE Validation =  3231.4468419367
M value:  10
lambda value:  0.4

Fold 1
MSE Training =  3137.7605059467423
MSE Validation =  3531.185421718986

Fold 2
MSE Training =  3099.750636811731
```

```
MSE Validation =  3392.63364495106

Fold 3
MSE Training =  3269.1261878445403
MSE Validation =  2647.303604087468

Fold 4
MSE Training =  3050.3123355265916
MSE Validation =  3446.49379968495

Fold 5
MSE Training =  3198.103588410193
MSE Validation =  3219.335544379923
M value:  10
lambda value:  0.5

Fold 1
MSE Training =  3206.5852388692633
MSE Validation =  3325.945622238295

Fold 2
MSE Training =  3175.231571552482
MSE Validation =  3615.564969266002

Fold 3
MSE Training =  3247.265661182411
MSE Validation =  3233.051866148005

Fold 4
MSE Training =  3097.6354409958376
MSE Validation =  3866.807187828966

Fold 5
MSE Training =  3411.83314778149
MSE Validation =  2751.432505736065
M value:  10
lambda value:  0.6

Fold 1
MSE Training =  3183.9003166182915
MSE Validation =  3957.8353154142183

Fold 2
MSE Training =  3346.335065920692
MSE Validation =  3281.143977481606

Fold 3
MSE Training =  3456.892330044022
MSE Validation =  3028.6138481214907

Fold 4
MSE Training =  3362.6148632099007
MSE Validation =  3241.748390538682

Fold 5
MSE Training =  3199.6191583772757
MSE Validation =  3654.8340090086463
M value:  10
lambda value:  0.7000000000000001
```

```
Fold 1
MSE Training =  3544.7832022758575
MSE Validation =  3043.4384828926572

Fold 2
MSE Training =  3286.8758518471946
MSE Validation =  4102.924099618606

Fold 3
MSE Training =  3295.934719643069
MSE Validation =  4017.7908243182405

Fold 4
MSE Training =  3438.603263503666
MSE Validation =  3287.9229200892614

Fold 5
MSE Training =  3364.1014850858305
MSE Validation =  3052.356748929791
M value:  10
lambda value:  0.8

Fold 1
MSE Training =  3479.045352741143
MSE Validation =  3307.311651034017

Fold 2
MSE Training =  3412.4004179190947
MSE Validation =  3697.5985479476167

Fold 3
MSE Training =  3530.1907586332823
MSE Validation =  3132.8394325047093

Fold 4
MSE Training =  3355.303004142222
MSE Validation =  4154.092974825889

Fold 5
MSE Training =  3519.06758807592
MSE Validation =  3632.4198210421373
M value:  10
lambda value:  0.9

Fold 1
MSE Training =  3685.4315204560635
MSE Validation =  3046.5656064200907

Fold 2
MSE Training =  3348.036532769551
MSE Validation =  4310.491463334961

Fold 3
MSE Training =  3667.514186078087
MSE Validation =  2933.538716279578

Fold 4
MSE Training =  3468.4851329222142
MSE Validation =  4137.7856653902745
```

```
Fold 5
MSE Training =  3490.2296392303842
MSE Validation =  3761.861996860441
M value:  10
lambda value:  1.0

Fold 1
MSE Training =  3630.317332044433
MSE Validation =  3701.873386911112

Fold 2
MSE Training =  3615.742603822985
MSE Validation =  3400.6432538593267

Fold 3
MSE Training =  3352.3058832282295
MSE Validation =  4939.644298350945

Fold 4
MSE Training =  3719.0034081901845
MSE Validation =  3205.724494373102

Fold 5
MSE Training =  3679.343563716755
MSE Validation =  3215.7622756635806

Average Performance in Training =  3599.3425582005175
Average Performance in Validation =  3692.7295418316126
----------------------------------------------------------
```

In [7]:

```python
MSE_train_avg_min = 0
MSE_val_avg_min = 0
for i in range (1, len(MSE_train_avg_array)):
    if MSE_train_avg_array[i][0] + MSE_val_avg_array[i][0] < MSE_train_avg_array[i-1][
        MSE_train_avg_min = MSE_train_avg_array[i]
        MSE_val_avg_min = MSE_val_avg_array[i]
    else:
        MSE_train_avg_min = MSE_train_avg_array[i]
        MSE_val_avg_min = MSE_val_avg_array[i]

print(MSE_train_avg_min)
print(MSE_val_avg_min)

Best_M = MSE_train_avg_min[1]
Best_lam = MSE_train_avg_min[2]

print(Best_M , Best_lam)
```

```
[3599.3425582005175, 10, 1.0]
[3692.7295418316126, 10, 1.0]
10 1.0
```

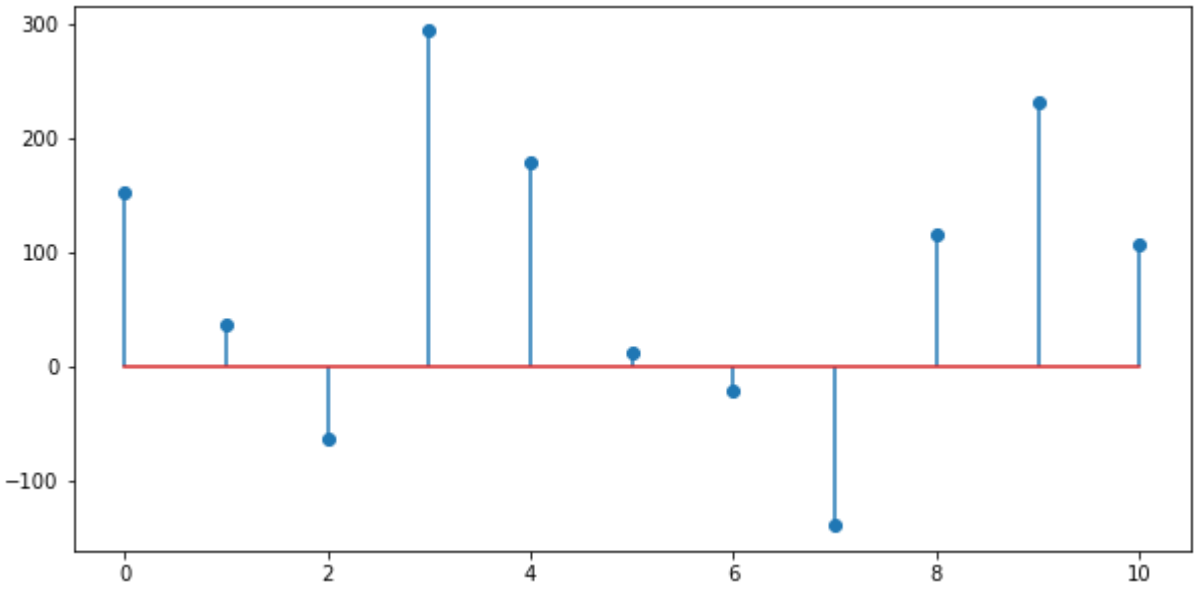
In [8]:

```python
# Training the full model with best values of M and lam obtained from K-fold Cross val
w,y_final = Regression_model(X_train,t_train,Best_M,Best_lam)
print(w)

# value of y from the trained model
y_val_final = Regression_model_test(X_test,Best_M,w)
```

```
MSE = np.mean((t_test - y_val_final)**2)
print(MSE)

intercept = w[0];
coefficient = w[1:];

print("intercept: ", intercept)
print("coefficiant: ", coefficient)

# plotting
m = [i for i in range(M+1)]
fig = plt.figure(figsize = (10,5))
plt.stem(m,w)
```

```
[ 151.83795214   37.25359537  -62.81959403  294.37256604  179.73613433
   11.99678664  -21.20478753 -139.95973386  115.77914634  232.12731557
  107.00626591]
3069.1673738881323
intercept:  151.83795213900083
coefficiant:  [  37.25359537  -62.81959403  294.37256604  179.73613433   11.99678664
  -21.20478753 -139.95973386  115.77914634  232.12731557  107.00626591]
<StemContainer object of 3 artists>
```

Out[8]:



Answer 1.5

Based on the stem plot from above the input variable at the index 3 has the most contribution in the output variable. In the provided Diabetes dataset the index 3 belongs to bmi . Thus, BMI input variable contributes most in the output variable.

Problem 2 (25 points)

Suppose that a taxi company wants to estimate the average number of trips per hour for the upcoming weekend in Downtown Gainesville. The company is working under the assumption

that the number of passengers can be modeled with a Poisson random variable (RV) with parameter λ ($\lambda > 0$).

- The poisson RV with parameter λ has the following probability mass function (PMF): $p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$.

The company's engineers decide to use a Gamma RV with parameters $\alpha = 5$ and $\beta = 0.5$ as the prior probability for the unknown parameter λ.

- The Gamma RV with parameters α and β ($\alpha, \beta > 0$) has the following probability density function: $f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$.

Suppose that you have a set of data with 10 samples:

$$x = [12, 5, 10, 10, 7, 17, 6, 11, 9, 9]$$

Answer the following questions:

1. Compute the MLE estimate for λ. Show all your work.
2. Compute the MAP estimate for λ. Show all your work.
3. Does Poisson-Gamma form a prior conjugate relationship? Provide the pseudo-code for online updated of the prior parameters.
4. Suppose the dataset `hourly_trips.npy` is the dataset coming in hourly (one sample at a time). Use this data to perform online update of the prior parameters. Start with an initial guess of $\alpha = 3$ and $\beta = 1$. Include a plot showing the estimated value for λ (using MLE and MAP) as data samples are received. (The true value is $\lambda = 10$.)

Question-01 :-

Compute MLE estimator for λ

$$\Rightarrow p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \longrightarrow (PMF)$$

Thus data likelihood.

$$\mathcal{L}^0 = P(X_1 \cap X_2 \cap \dots \cap X_N | \lambda)$$

$$= P(X_1|\lambda) \cdot P(X_2|\lambda) \cdots P(X_N|\lambda)$$

↳ X_i's are conditionally independent

$$= \prod_{i=1}^{N} P(X_i|\lambda)$$

↳ Identically distributed.

$$= \prod_{i=1}^{N} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$\therefore \arg\max_{\lambda} \mathcal{L}^0 = \arg\max_{\lambda} \left[\prod_{i=1}^{N} \frac{\lambda^{x_i} \cdot e^{-\lambda}}{x_i!} \right]$$

$$\mathcal{L} = \ln(\mathcal{L}^0)$$

$$= \ln \left[\prod_{i=1}^{N} \frac{\lambda^{x_i} \cdot e^{-\lambda}}{x_i!} \right]$$

$$= \sum_{i=1}^{N} \ln \left(\frac{\lambda^{x_i} \cdot e^{-\lambda}}{x_i!} \right)$$

$$= \sum_{i=1}^{N} \left(\left(\ln \lambda^{x_i} \right) + \ln\left(e^{-\lambda}\right) - \ln(x_i!) \right)$$

$$= \sum_{i=1}^{N} \left(x_i \ln(\lambda) - \lambda \ln(e) - \ln(x_i!) \right)$$

$$\therefore \mathcal{L} = \sum_{i=1}^{N} \left(x_i \ln(\lambda) - \lambda \ln(e) - \ln(x_i!) \right)$$

Solving for λ: $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$

$$\Leftrightarrow \sum_{i=1}^{N} \left[x_i \cdot \frac{1}{\lambda} - \ln(e) - 0 \right] = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \left[\frac{x_i}{\lambda} - 1 \right] = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \frac{x_i}{\lambda} - \sum_{i=1}^{N} 1 = 0$$

$$\Leftrightarrow \frac{1}{\lambda} \sum_{i=1}^{N} x_i - N = 0$$

$$\Leftrightarrow \lambda = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\therefore \boxed{\lambda_{MLE} = \frac{1}{N} \sum_{i=1}^{N} x_i} \longrightarrow \underline{\text{Ans}}$$

Question - 02 :-

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$\& \quad P(\lambda|\alpha,\beta)$$

$$\mathcal{L}_0 = P(x|\lambda) \times P(\lambda|\alpha,\beta)$$

$$= \left(\prod_{i=1}^{N} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}\right) \cdot \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}\right)$$

$$\mathcal{L} = \ln(\mathcal{L}_0)$$

$$= \sum_{i=1}^{N} (x_i \ln(\lambda) - \lambda \ln(e) - \ln(x_i!)) + \alpha \ln\beta - \ln(\Gamma\alpha) + \alpha - 1 \ln(\lambda) - \beta\lambda \ln(e)$$

$$= \sum_{i=1}^{N} x_i \ln(\lambda) - \sum_{i=1}^{N} \lambda \cdots - \sum_{i=1}^{N} \ln(x_i!) + \alpha \ln\beta + \ln(\Gamma(\alpha)) + (\alpha-1)\ln(\lambda) - \beta\lambda .$$

Thus after removing the constant values with respect to λ we get

$$\alpha \sum_{i=1}^{N} x_i \ln(\lambda) - N\lambda + (\alpha-1)\ln\lambda - \beta\lambda$$

$$\Rightarrow \left(\sum_{i=1}^{N} x_i^{\circ} + \alpha - 1\right) \ln(\lambda) - \lambda(\beta + N).$$

$$\frac{\partial L}{\partial \lambda} = 0$$

$$\Leftrightarrow \left(\sum_{i=1}^{N} x_i^{\circ} + \alpha - 1\right) \frac{1}{\lambda} - (\beta + N) = 0$$

$$\Leftarrow \sum_{i=1}^{N} x_i^{\circ} + \alpha - 1 = \lambda(\beta + N)$$

$$\therefore \lambda = \frac{1}{(\beta + N)} \cdot \left(\sum_{i=1}^{N} x_i^{\circ} + \alpha - 1\right)$$

$$\therefore \boxed{\lambda_{MAP} = \frac{\left(\sum_{i=1}^{N} x_i^{\circ} + \alpha - 1\right)}{(\beta + N)}} \Rightarrow \underline{\text{Ans}}.$$

Question-03 :-

In MAP :

Prior : $P(\lambda | \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha - 1} \cdot e^{-\beta\lambda}$

Posterior $P(\lambda|x)$

$$\propto \lambda^{\sum_{i=1}^{N} x_i^{\circ} + \alpha - 1} \cdot e^{-\lambda(N+\beta)}$$

When the prior and the posterior have the same parametric form after ignoring the constants they are said to have a conjugate Relationship.

Thus from above we can say that their is a conjugate Prior relationship between poisson - Gamma forms.

Pseudo - code for online update :-

$t = 0$ (iteration)

① Initialize the parameters of prior. $\alpha^{(t)}, \beta^{(t)}$.

② As we received data, d:

(2.1) Estimate posterior :

$$\lambda^{\sum_{i=1}^{N} x_i^{\circ} + \alpha^{(t)} - 1} \cdot e^{-\lambda(N+\beta)^{(t)}}.$$

(2.2) Compute estimate for u using MAP.

$$\Rightarrow \lambda_{MAP}^{(t)} = \sum_{i=1}^{N} x_i^{\circ} + \alpha^{(t)} - 1$$

$N + \beta^{(t)}$

(2.3) update the prior parameters with those from posterior:

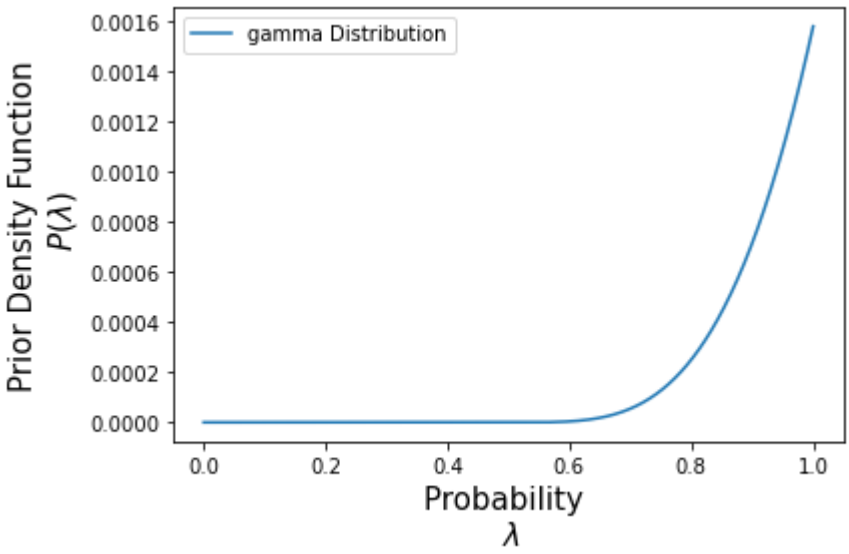
$$\alpha^{(t+1)} \leftarrow \alpha^{(t)} + \sum_{i=1}^{N} x_i$$

$$\beta^{(t+1)} \leftarrow \beta^{(t)} + N$$

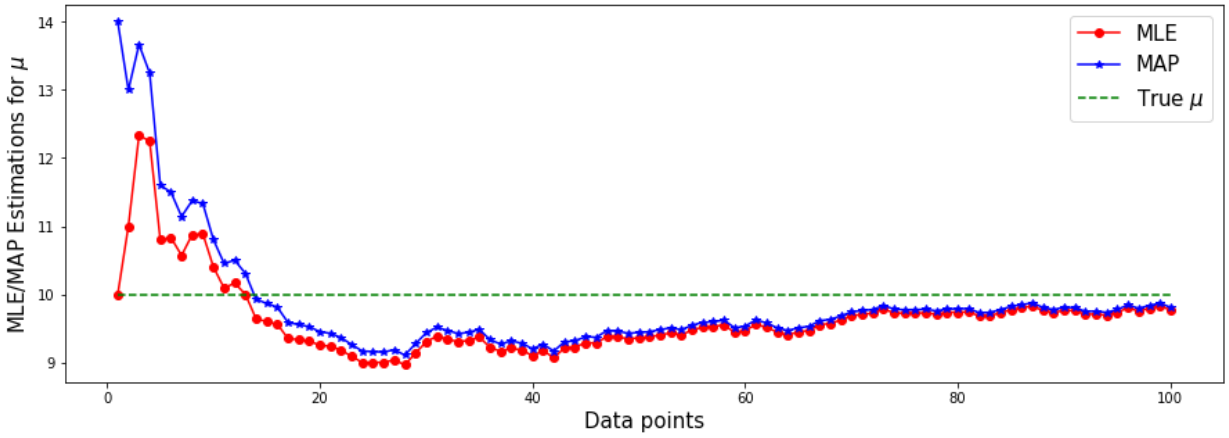
(2.4) $t \leftarrow t+1$

In [9]:
```
a = 5
b = 0.5

gamma =stats.gamma(a,b)
x = np.linspace(0,1,1000)
```

```
plt.plot(x, gamma.pdf(x), label='gamma Distribution')
plt.legend(loc='best')
plt.xlabel('Probability \n $\lambda$',fontsize=15)
plt.ylabel('Prior Density Function\n $P(\lambda)$',fontsize=15);
```

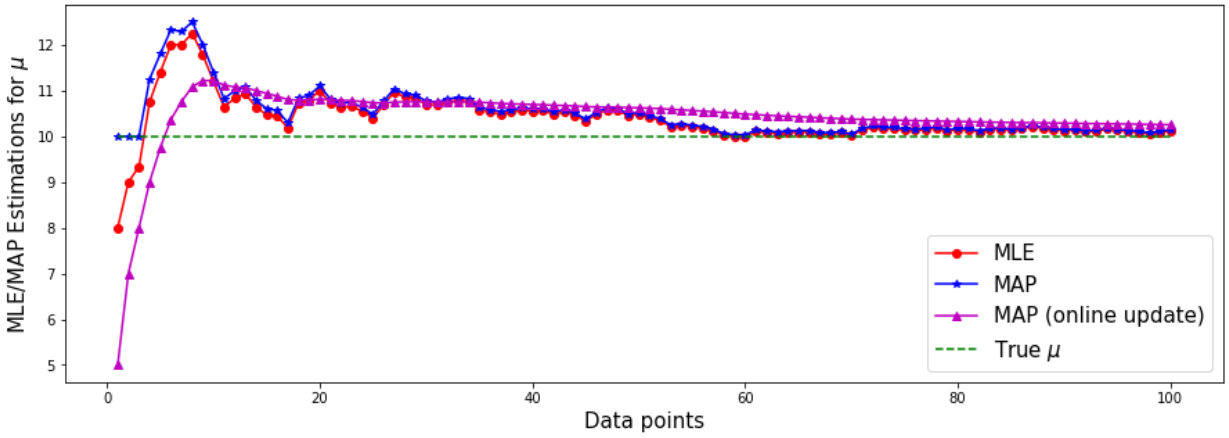


In [10]:
```
# True value of the unknown parameter
truelem = 10 # 0.5 for a fair coin

# Prior Initial Parameters
a=5 # alpha
b=0.5 # beta

# Sampling Training Data
Nattempts = 100
Outcomes = stats.poisson(truelem).rvs(Nattempts)

# Computing MLE and MAP estimates as data is being collected
lem_MLE = []
lem_MAP = []
for i in range(1,Nattempts+1):
    lem_MLE += [np.sum(Outcomes[:i])/len(Outcomes[:i])]
    lem_MAP += [(np.sum(Outcomes[:i])+a-1)/(len(Outcomes[:i]+b))]

# Plotting estimates
plt.figure(figsize=(15,5))
plt.plot(range(1,Nattempts+1), lem_MLE, '-or', label='MLE')
plt.plot(range(1,Nattempts+1), lem_MAP, '-*b', label='MAP')
plt.plot(range(1,Nattempts+1), [truelem]*Nattempts, '--g', label='True $\mu$')
plt.xlabel('Data points',size=15)
plt.ylabel('MLE/MAP Estimations for $\mu$',size=15)
plt.legend(fontsize=15);
```



In [11]:

```
# True value of the unknown parameter
truelem = 10

# Prior Initial Parameters
a=3; a_init = a
b=1; b_init = b

# Sampling Training Data
Nattempts = 100
Outcomes = stats.poisson(truelem).rvs(Nattempts)

# Computing MLE and MAP estimates as data is being collected
lem_MLE = []
lem_MAP = []
lem_MAP_update = []
for i in range(1,Nattempts+1):
    lem_MLE += [np.sum(Outcomes[:i])/len(Outcomes[:i])]
    lem_MAP += [(np.sum(Outcomes[:i])+a_init-1)/(len(Outcomes[:i]+b_init))]
    lem_MAP_update += [(np.sum(Outcomes[:i])+a-1)/(len(Outcomes[:i])+b)]
    a += np.sum(Outcomes[:i])
    b += len(Outcomes[:i])

# Plotting estimates
plt.figure(figsize=(15,5))
plt.plot(range(1,Nattempts+1), lem_MLE, '-or', label='MLE')
plt.plot(range(1,Nattempts+1), lem_MAP, '-*b', label='MAP')
plt.plot(range(1,Nattempts+1), lem_MAP_update, '-^m', label='MAP (online update)')
plt.plot(range(1,Nattempts+1), [truelem]*Nattempts, '--g', label='True $\mu$')
plt.xlabel('Data points',size=15)
plt.ylabel('MLE/MAP Estimations for $\mu$',size=15)
plt.legend(fontsize=15);
```



In [ ]:

Submit Your Solution

Confirm that you've successfully completed the assignment.

Along with the Notebook, include a PDF of the notebook with your solutions.

`add` and `commit` the final version of your work, and `push` your code to your GitHub repository.

Submit the URL of your GitHub Repository as your assignment submission on Canvas.
