CS60055: Ubiquitous Computing

Contactless Sensing

Part I: Sensing with RF: Basic Principles

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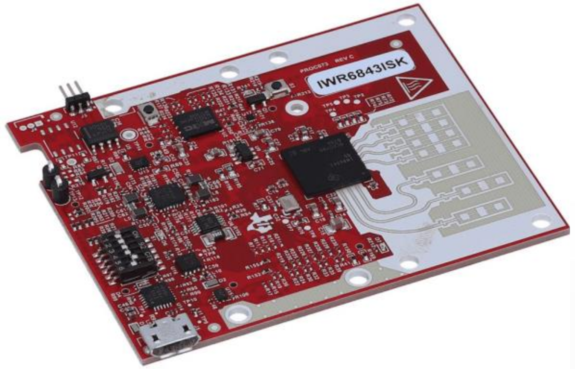
Contactless Sensing

- IMU-based sensing needs the subject to wear some devices: smartwatch, earable, smart glass, smart ring, …
    - Might not be very convient all the times (ex. Sleep monitoring)

- Contactless sensing are useful when the user does not want to attach the sensors with their body
    - Useful for continuous and passive monitoring of human activities
    - Widely used to monitor objects (materials, liquids, structures, ...)

Sensing Modalities



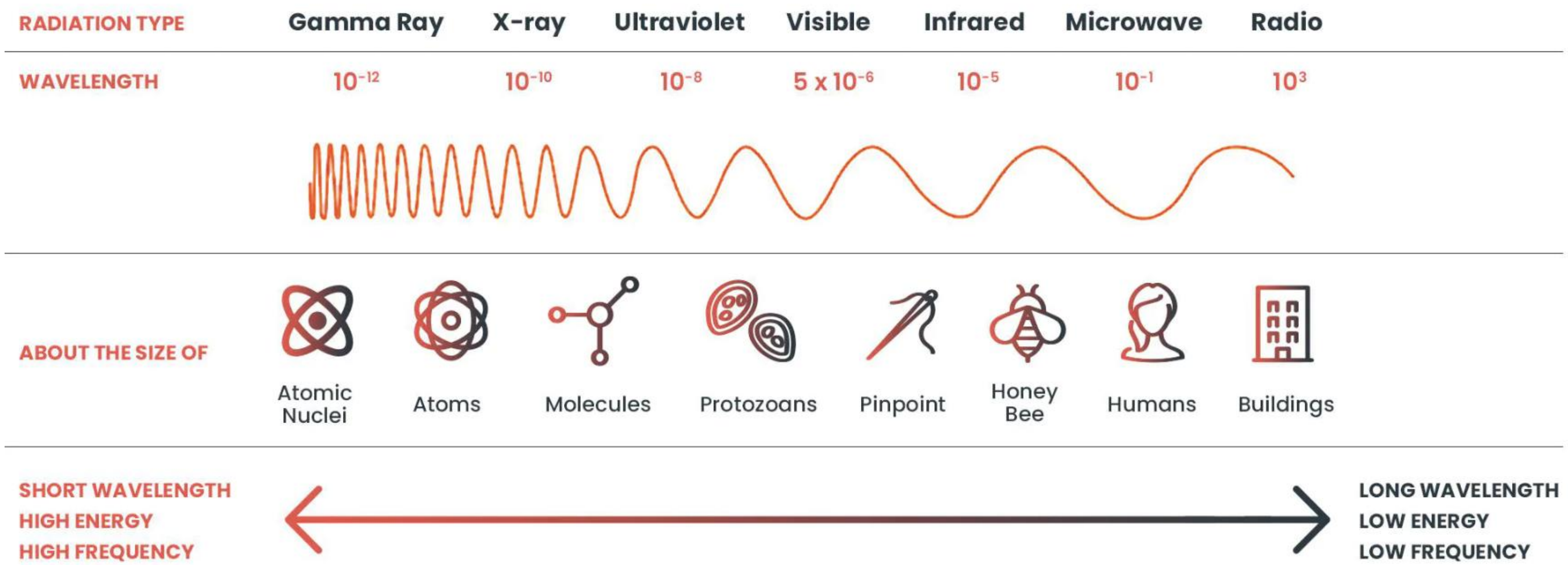
Sensing Modalities

- Primarily two modalities
  - Electromagnetic waves (majority of the sensing devices work on some EM waves)
  - Mechanical waves (acoustic-based sensing)

- The basic operating principles vary depending on the type and the frequency of the waves being used in the sensing applications

Fundamentals of EM Waves

- The existence of EM was predicted by James Clerk Maxwell in 1864
  - Heinrich Hertz confirmed the same in 1887
  - Hertz also demonstrated that EM waves are affected and reflected by solid objects
- The frequency and the wavelength characterizes the waves



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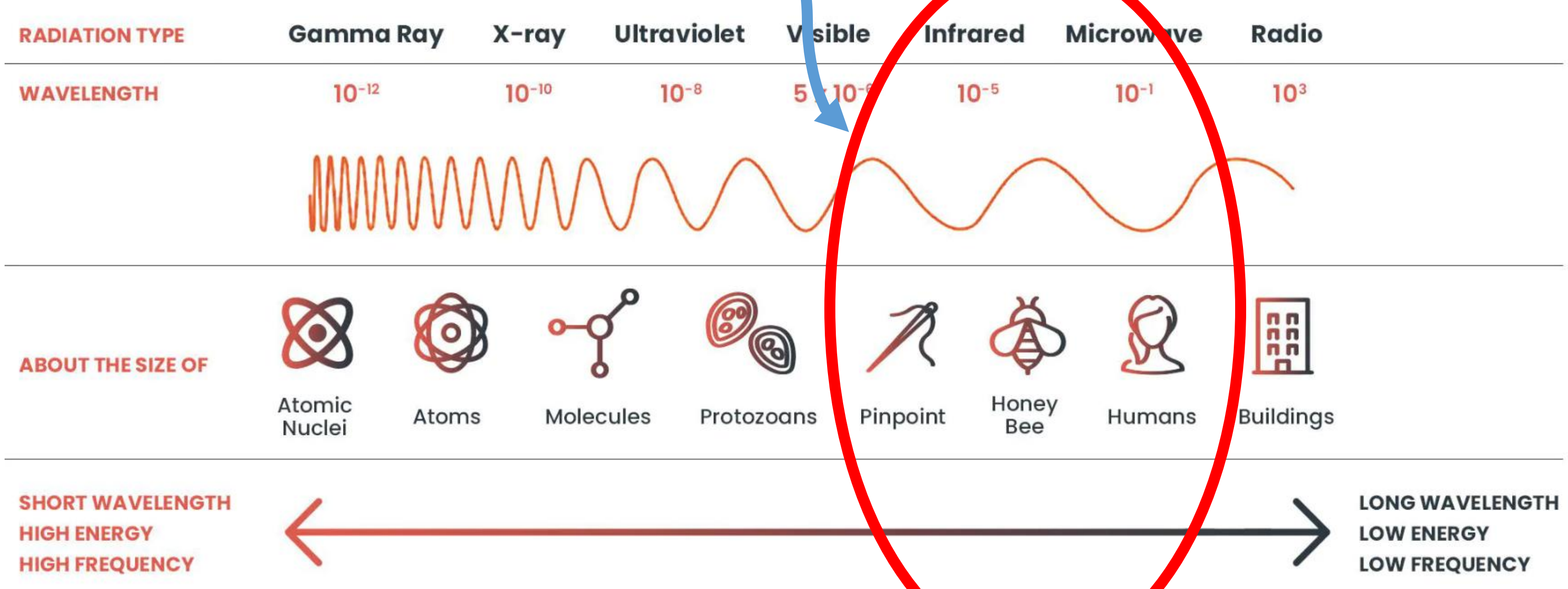
The objects we sense



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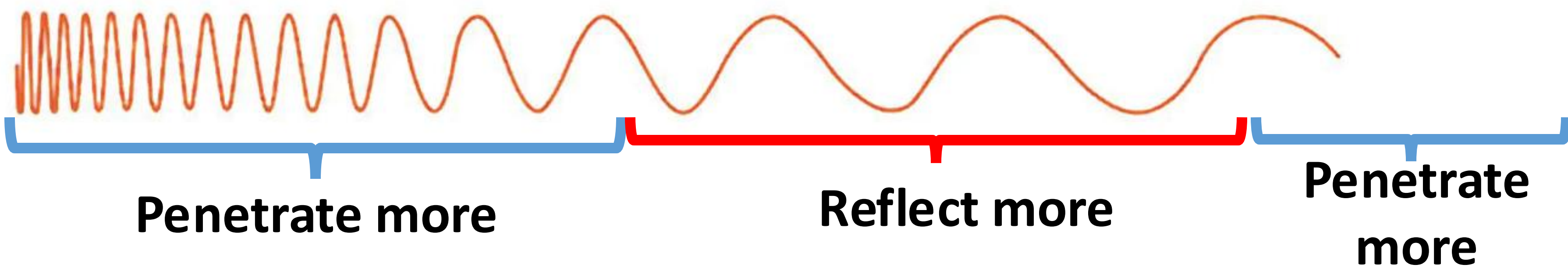
  - **Long wavelength, low frequency:** Penetrate more through physical objects
  - **Medium wavelength, medium frequency:** Reflect more from physical objects
  - **Short wavelength, high frequency**: Penetrate more through the objects

RADIATION TYPE	Gamma Ray	X-ray	Ultraviolet	Visible	Infrared	Microwave	Radio
WAVELENGTH	10^{-12}	10^{-10}	10^{-8}	5×10^{-6}	10^{-5}	10^{-1}	10^{3}



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  - **Long v** jects
  - **Mediu** objects
  - **Short** s

This is just a general idea; the penetration/reflection capability of the signal also depends on its bandwidth and other channel parameters

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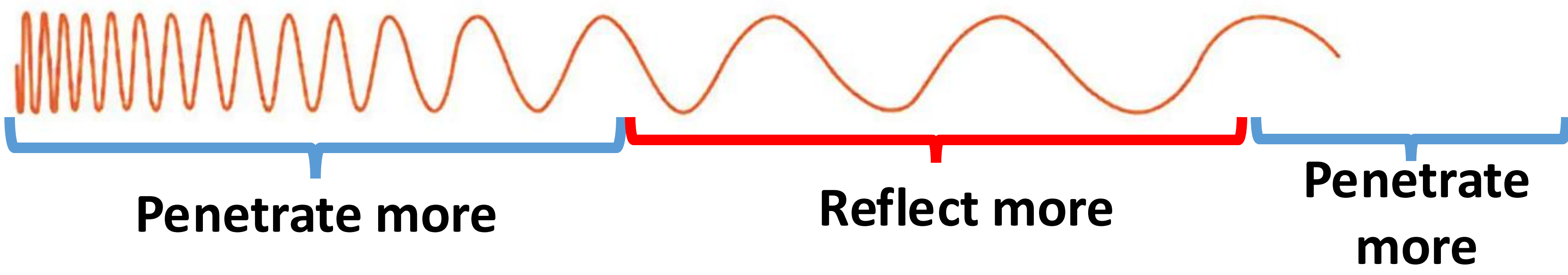

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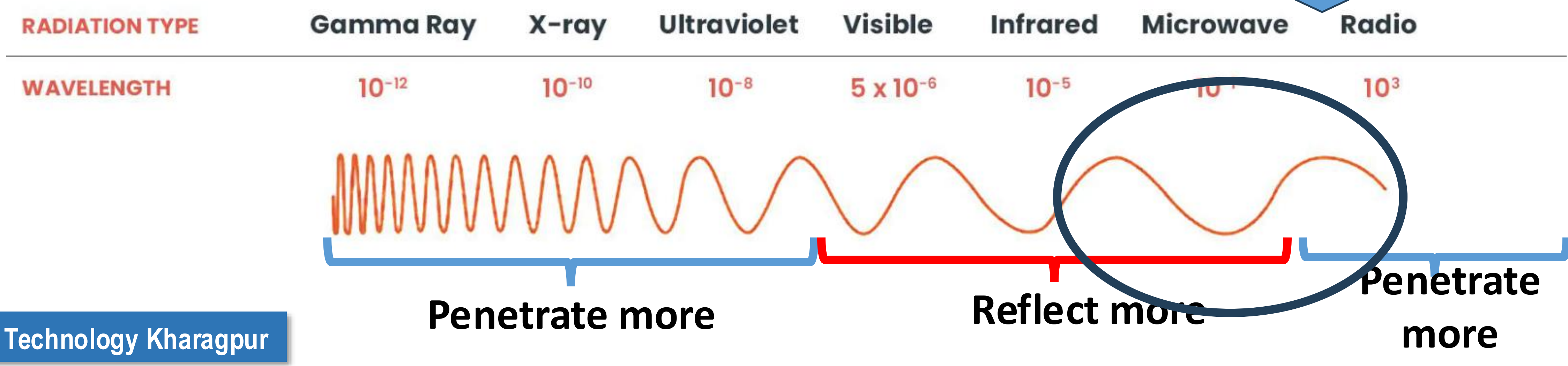
Useful for passive contactless sensing



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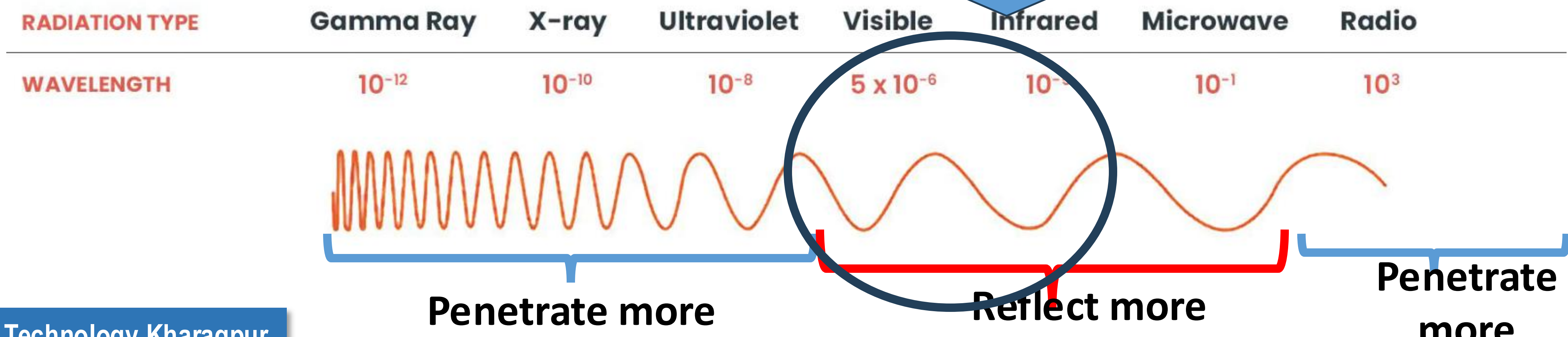
Used widely for sensing, but privacy is a concern (the vision domain)



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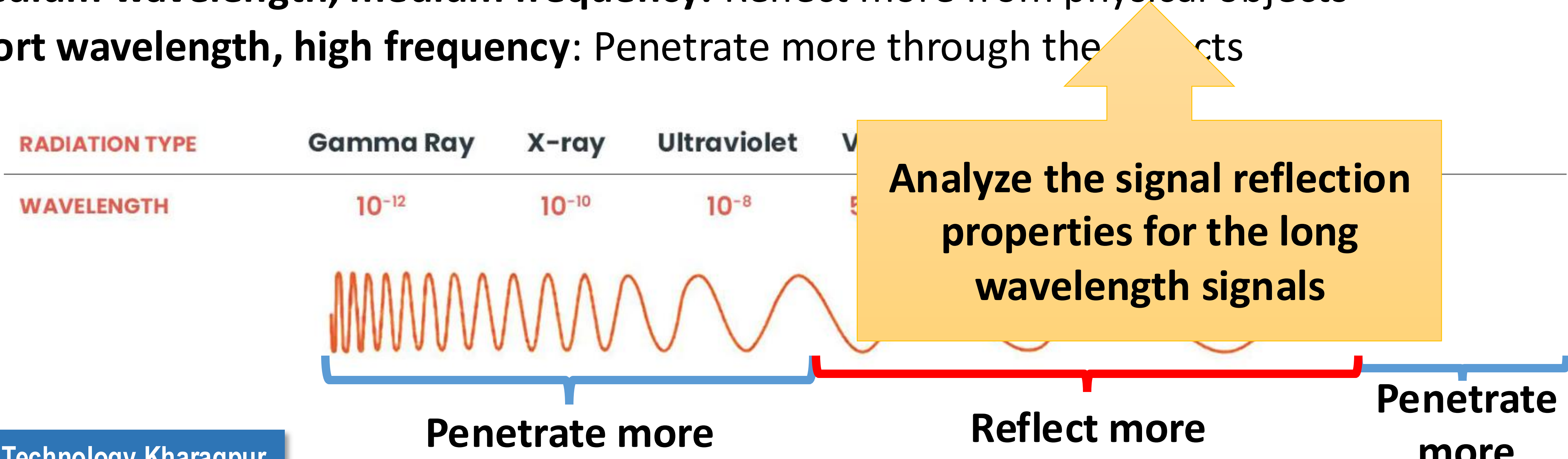

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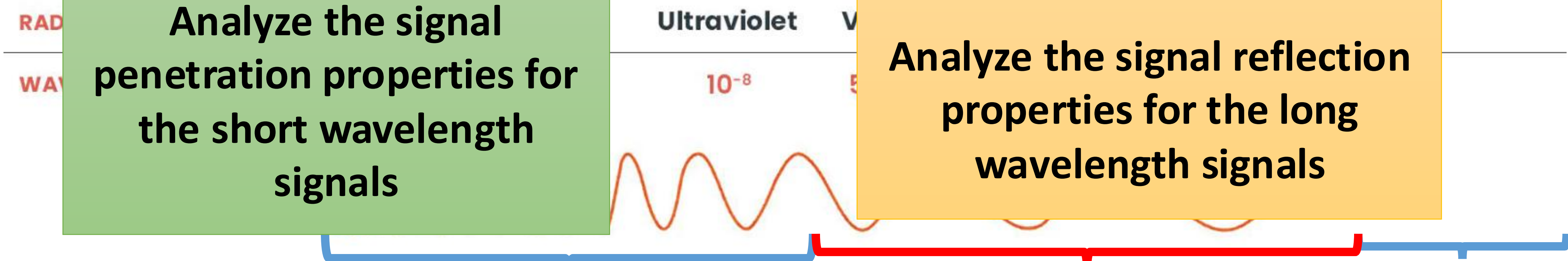
Analyze the signal penetration properties for the short wavelength signals

Analyze the signal reflection properties for the long wavelength signals



Penetrate more

Reflect more

Penetrate more

Radio Waves

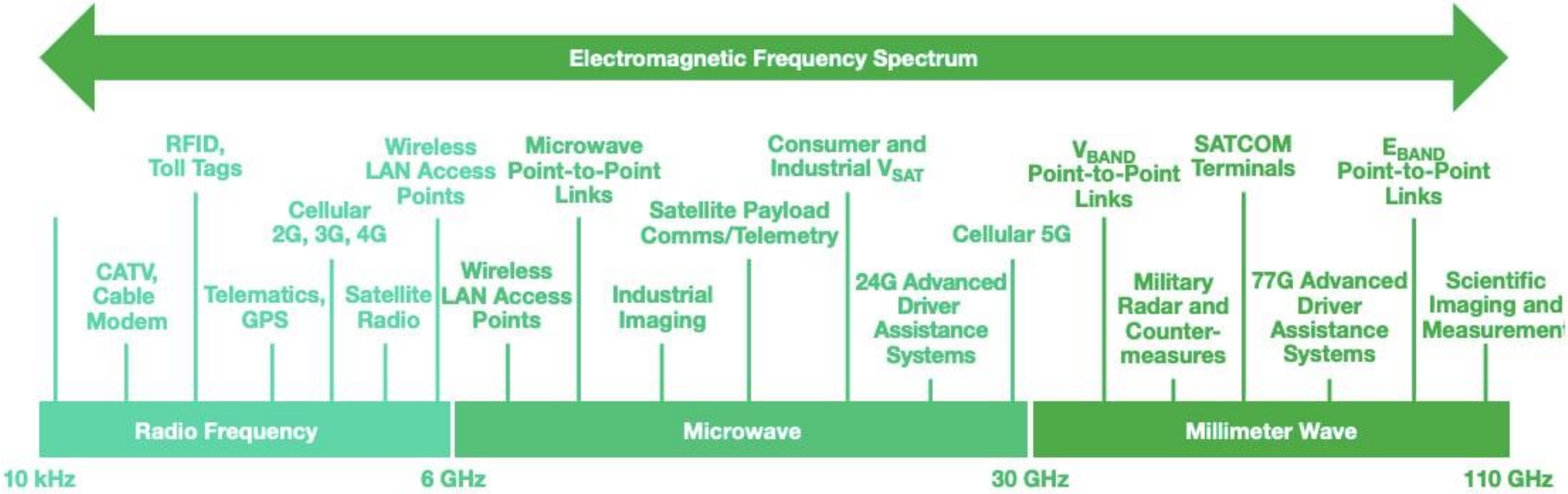
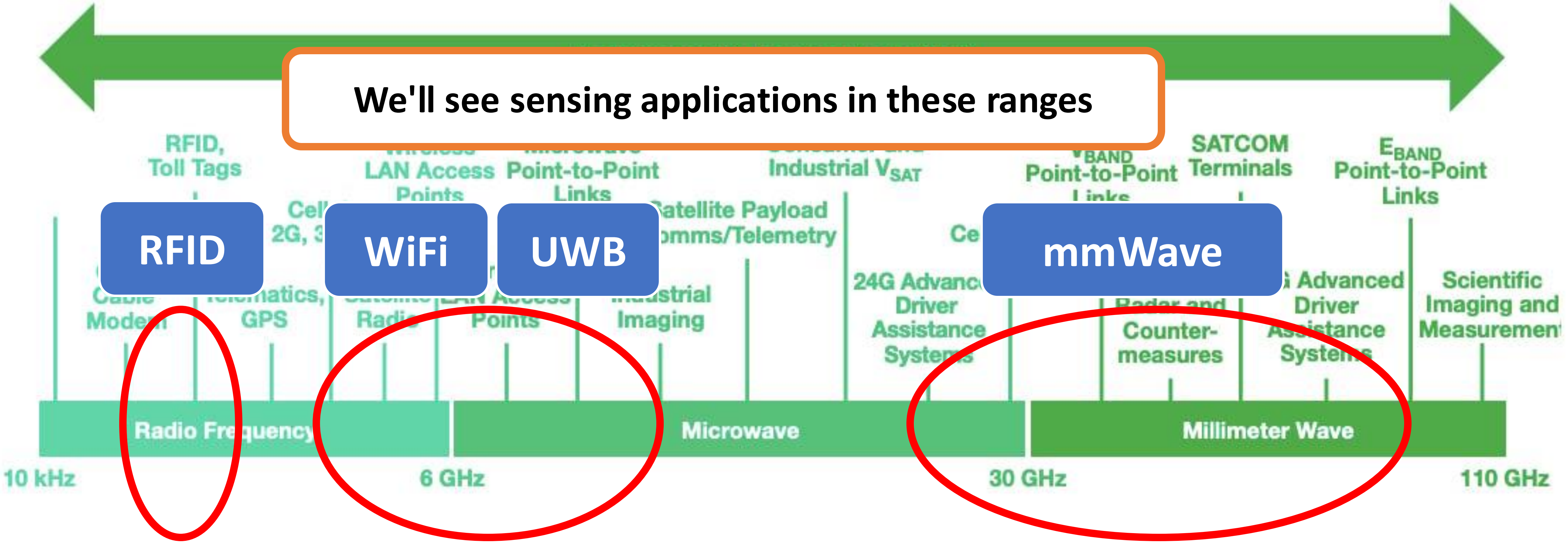


Image Source: https://www.allaboutcircuits.com/technical-articles/basics-of-millimeter-wave-mmwave-technology/

Radio Waves



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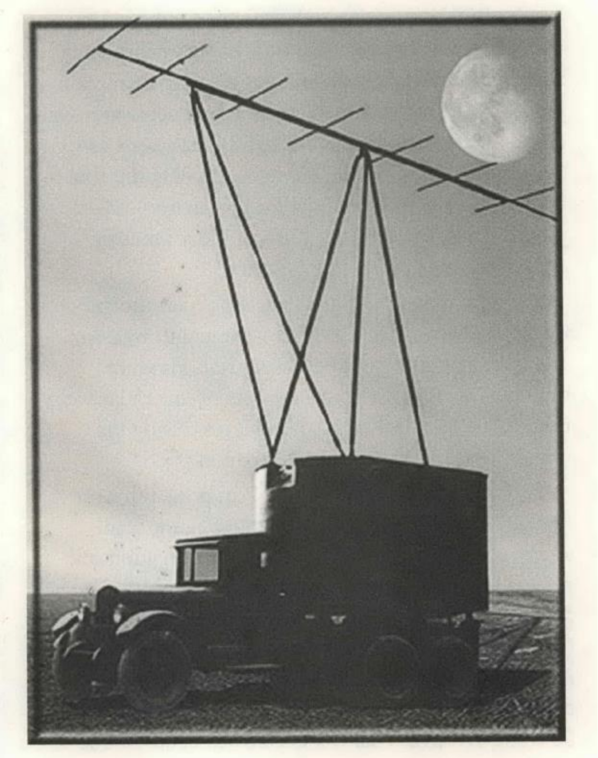
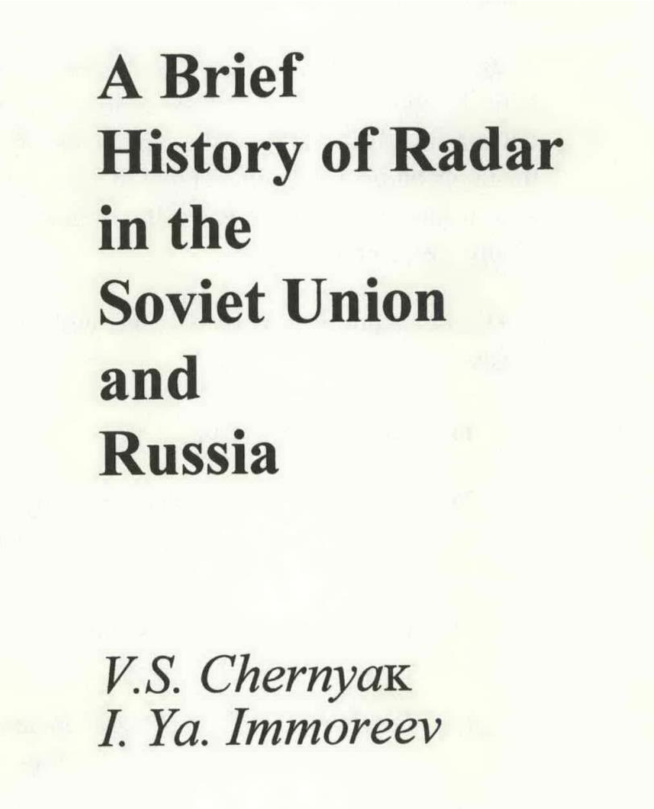
Signal Interference to Detect Motion: The Concept of Radar

- 1897: Alexander Popov, a Russian Imperial Navi physicist, was testing an early version of wireless communication between two ships in Baltic Sea
  - He observed an interference wave pattern caused by a third ship
  - Popov proposed the idea that this might be used to detect moving objects

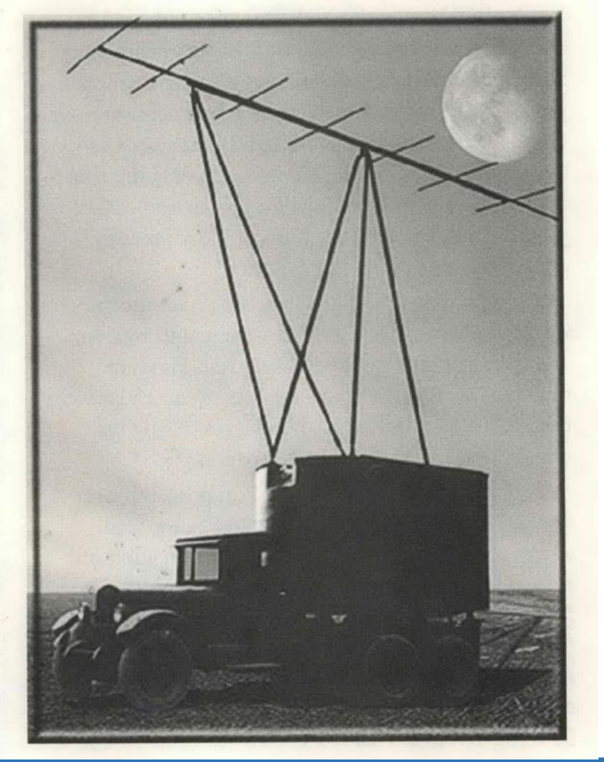


https://ieeexplore.ieee.org/abstract/document/5282288



Signal Interference to Detect Motion: The Concept of Radar

- 1932: Radar as a technical equipment was proposed by a military engineer Piotr Oshchepkov
  - "RUS-1": The first industrial radar (1939): 4m wavelength, transmitter and receiver seperated by 35 km



IEEE MILESTONE IN ELECTRICAL ENGINEERING AND COMPUTING

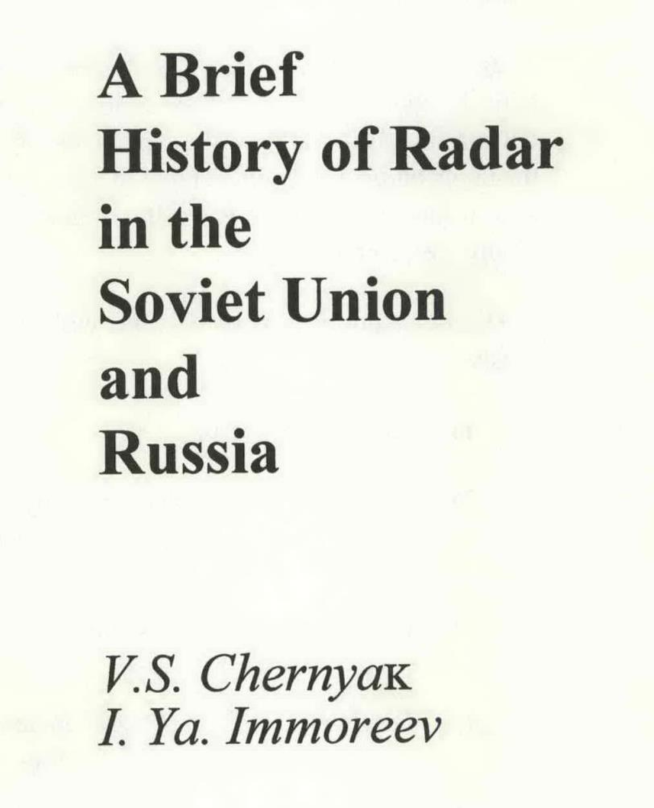
POPOV'S CONTRIBUTION TO THE DEVELOPMENT OF WIRELESS COMMUNICATION, 1895

On 7 May 1895, A. S. Popov demonstrated the possibility of transmitting and receiving short, continuous signals over a distance up to 64 meters by means of electromagnetic waves with the help of a special portable device responding to electrical oscillation which was a significant contribution to the development of wireless communication.

May 2005

INSTITUTE OF ELECTRICAL AND ELECTRONICS ENGINEERS

https://ieeexplore.ieee.org/abstract/document/5282288



Basic Principle: The Doppler Effect

- Change in the frequency of the wave in relation to an observer who is moving relative to the wave source
  - **Wave source moves towards the observer:** Each successive wave cycle is emitted from a position closer to the observer
    - **Time between cycles is reduced; frequency is increased**
  - **Wave source moves away from the observer:** Each successive wave cycle is emitted from a position farther from the observer
    - **Time between cycles is increased; frequency is decreased**

Image Source: Wikipedia

The Doppler Effect

- Let,
  - f_0 is the emitted frequency
  - f is the observed frequency
  - c is the propagation speed of the wave in the medium
  - v_r is the speed of the receiver, v_s is the speed of the source

$$f = \left(\frac{c \pm v_{\mathrm{r}}}{c \mp v_{\mathrm{s}}} \right) f_0$$

- v_r is added to c if the receiver is moving towards the source, subtracted if the receiver is moving away from the source (opposite for v_s)

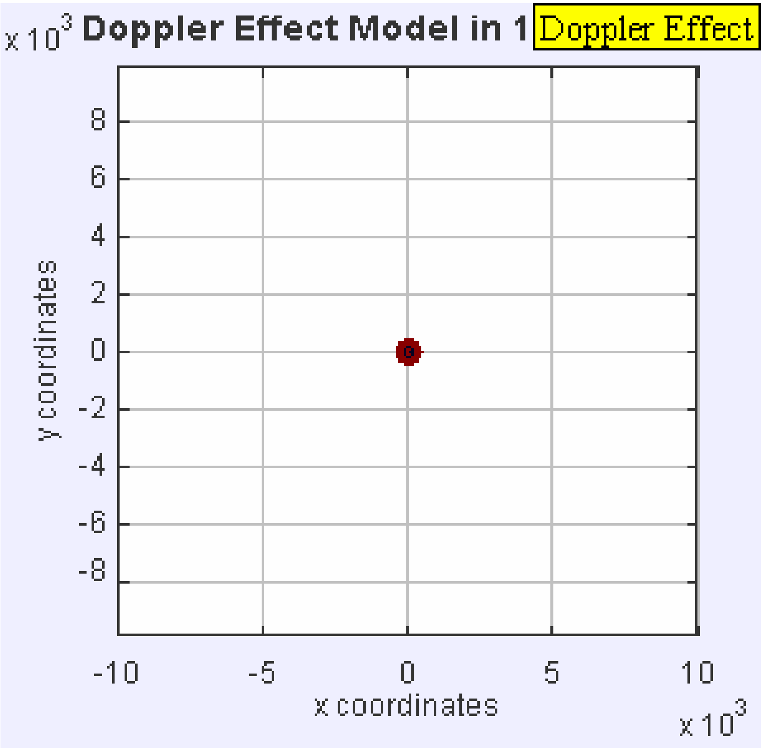
The Doppler Effect

- Equivalently, if the source is directly approaching or receding from the observer,

$$\frac{f}{v_{wr}} = \frac{f_0}{v_{ws}} = \frac{1}{\lambda}$$

  - v_{wr} is the wave speed related to the receiver, v_{ws} is the wave speed related to the source
  - λ is the wavelength

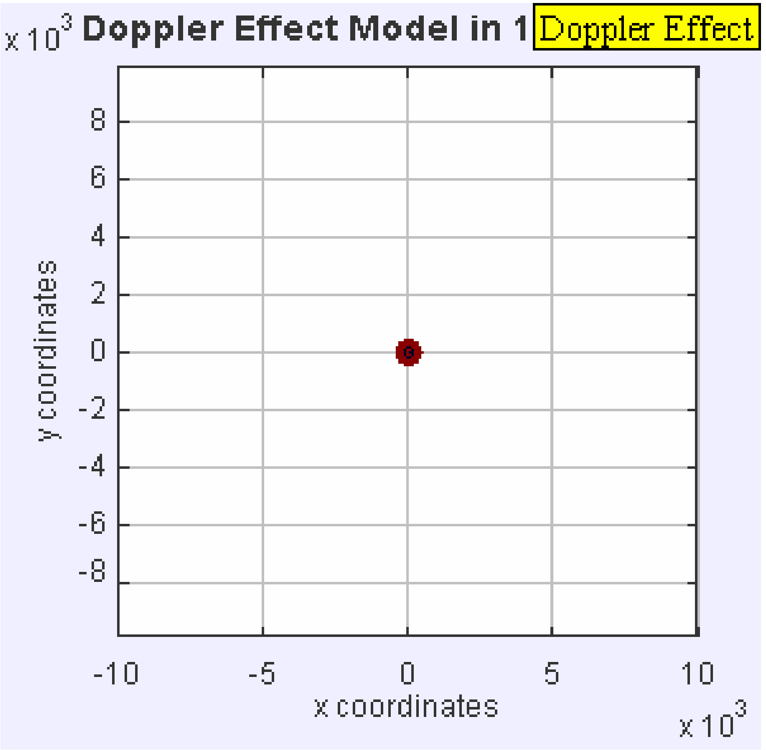
Doppler Effect for the Sound Sources



Stationary sound source

Image Source: Wikipedia

Doppler Effect for the Sound Sources



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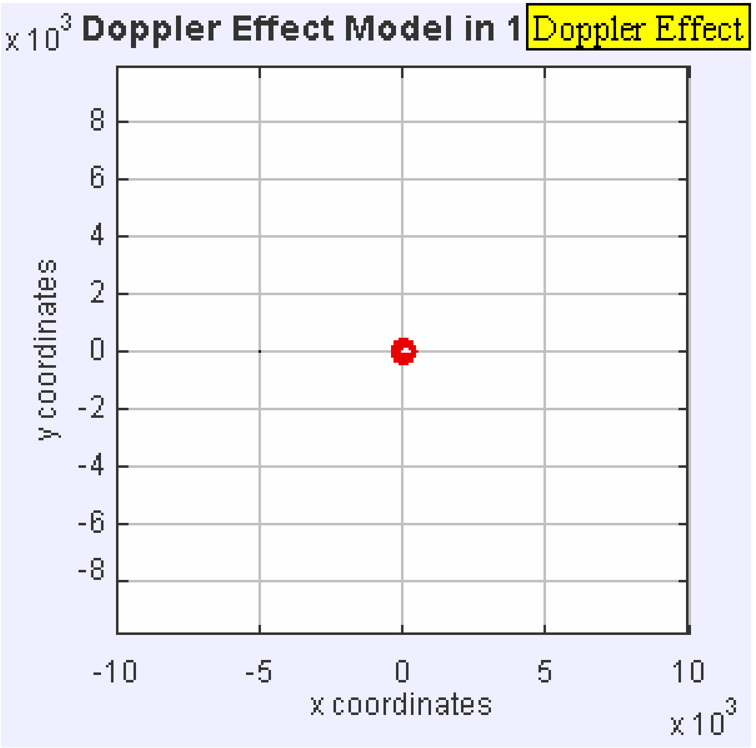
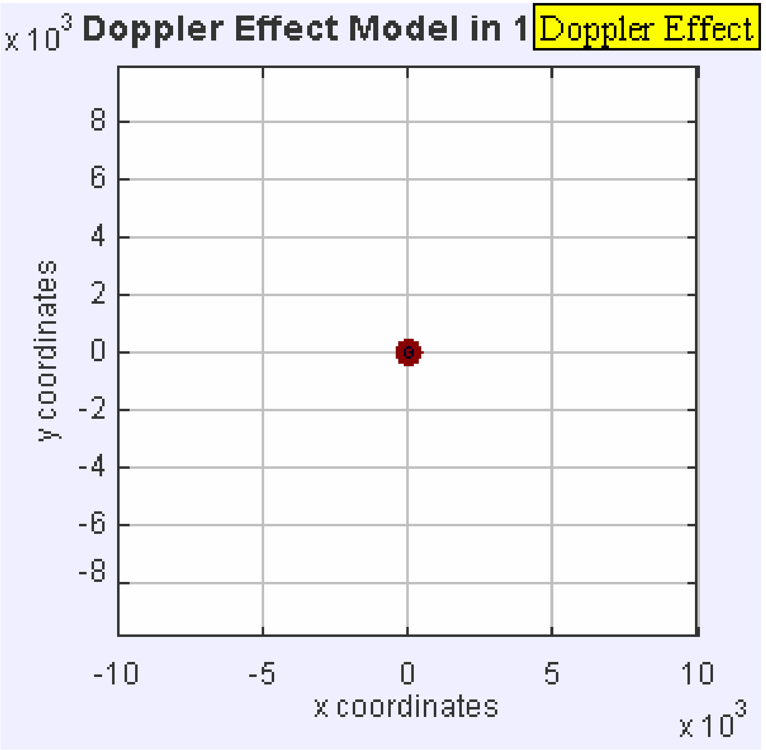


Source moves at a speed 0.7c

Image Source: Wikipedia

Doppler Effect for the Sound Sources



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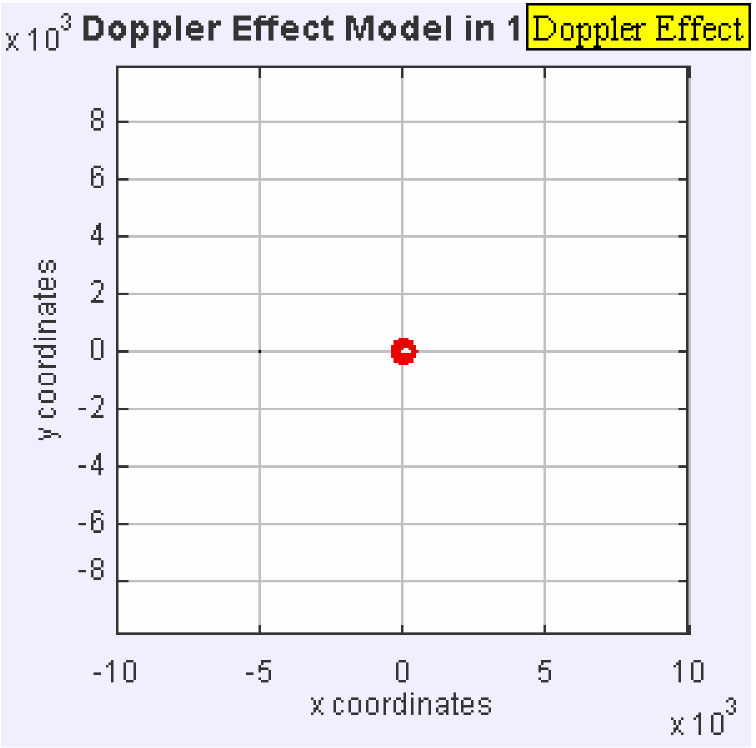

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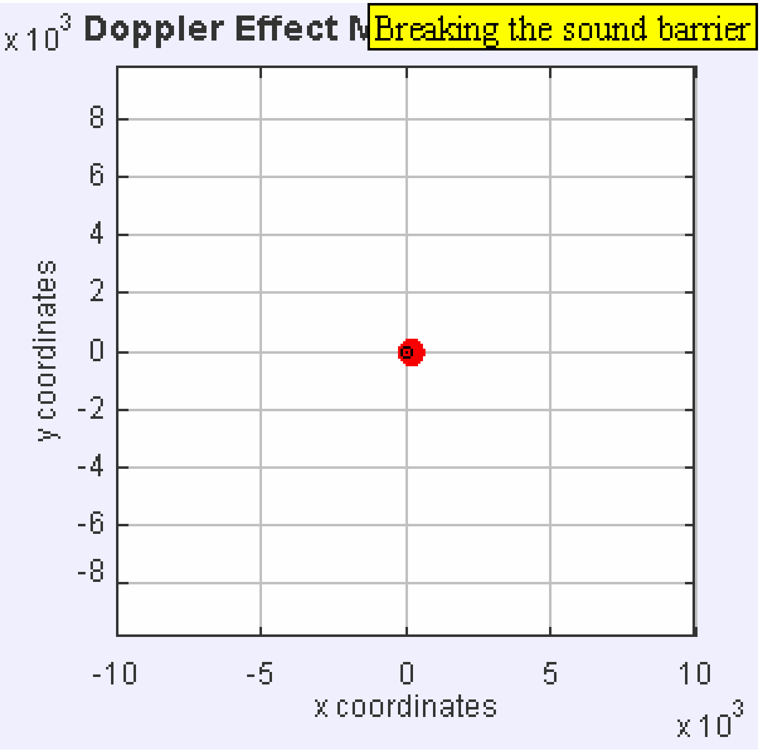

Source moves at a speed c

Image Source: Wikipedia

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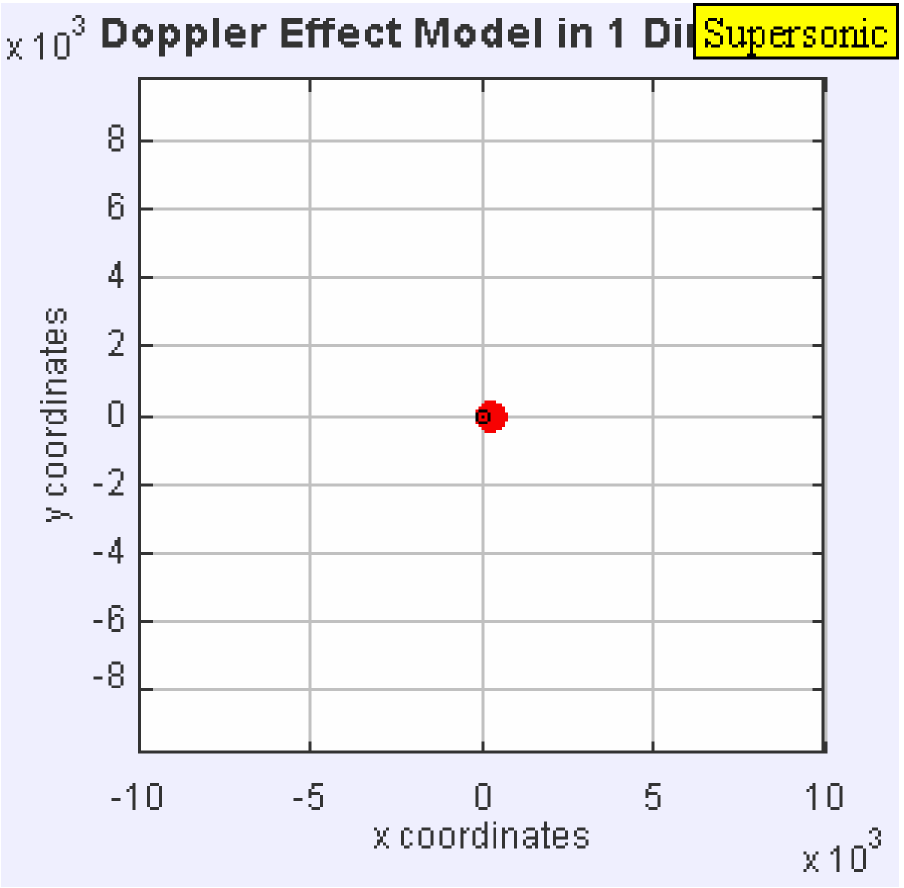

Source moves at a speed 1.4c

Advancing wavefront

Creates a shock wave and consequently the sonic boom

Image Source: Wikipedia

The Wave: Concept of Bandwidth

- Any arbitrary wave signal can be decomposed into a set of sinusoidal wave of multiple frequencies
  - Called the frequency components of the wave signal (or simply, the signal)

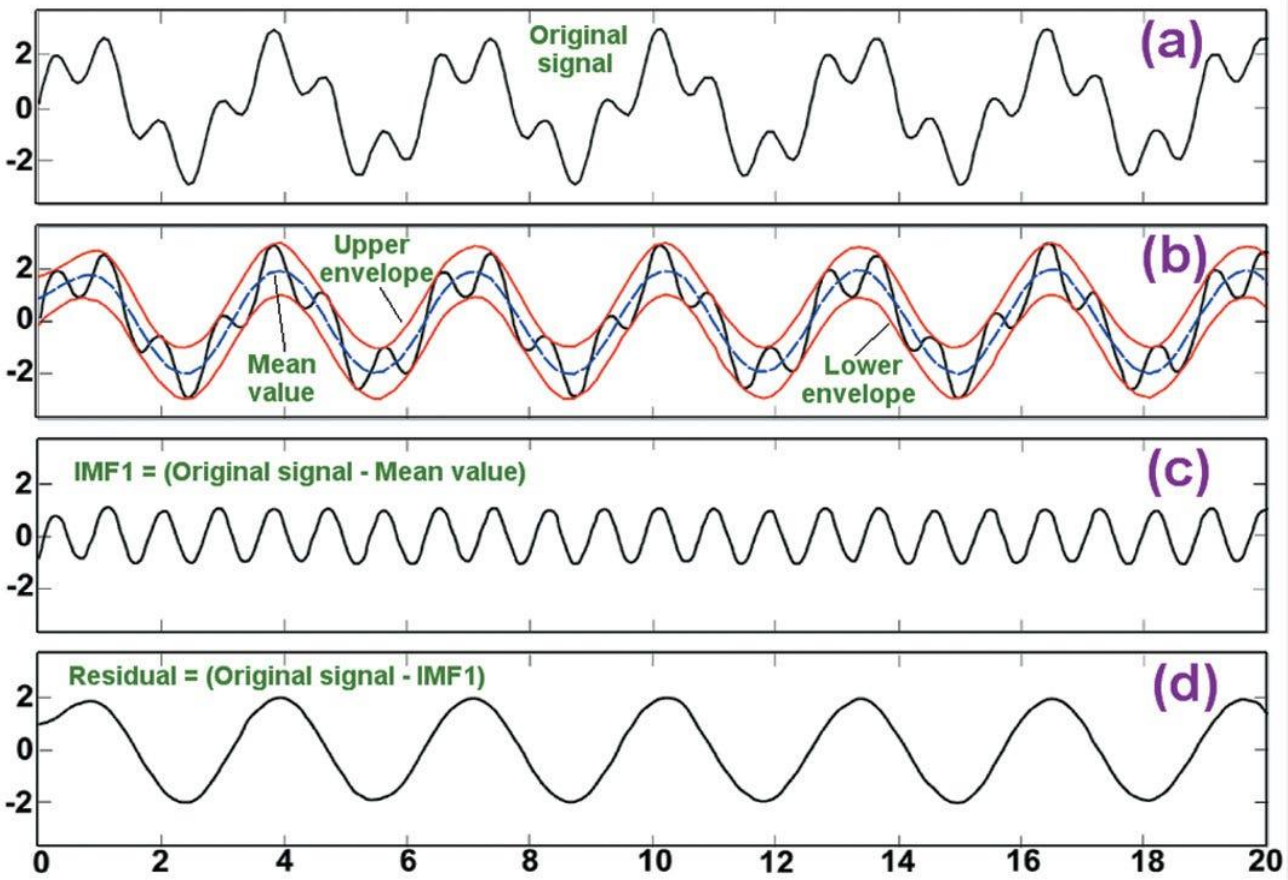

Image Source:
https://towardsdatascience.com/decomposing-signal-using-empirical-mode-decomposition-algorithm-explanation-for-dummy-93a93304c541

The Wave: Concept of Bandwidth

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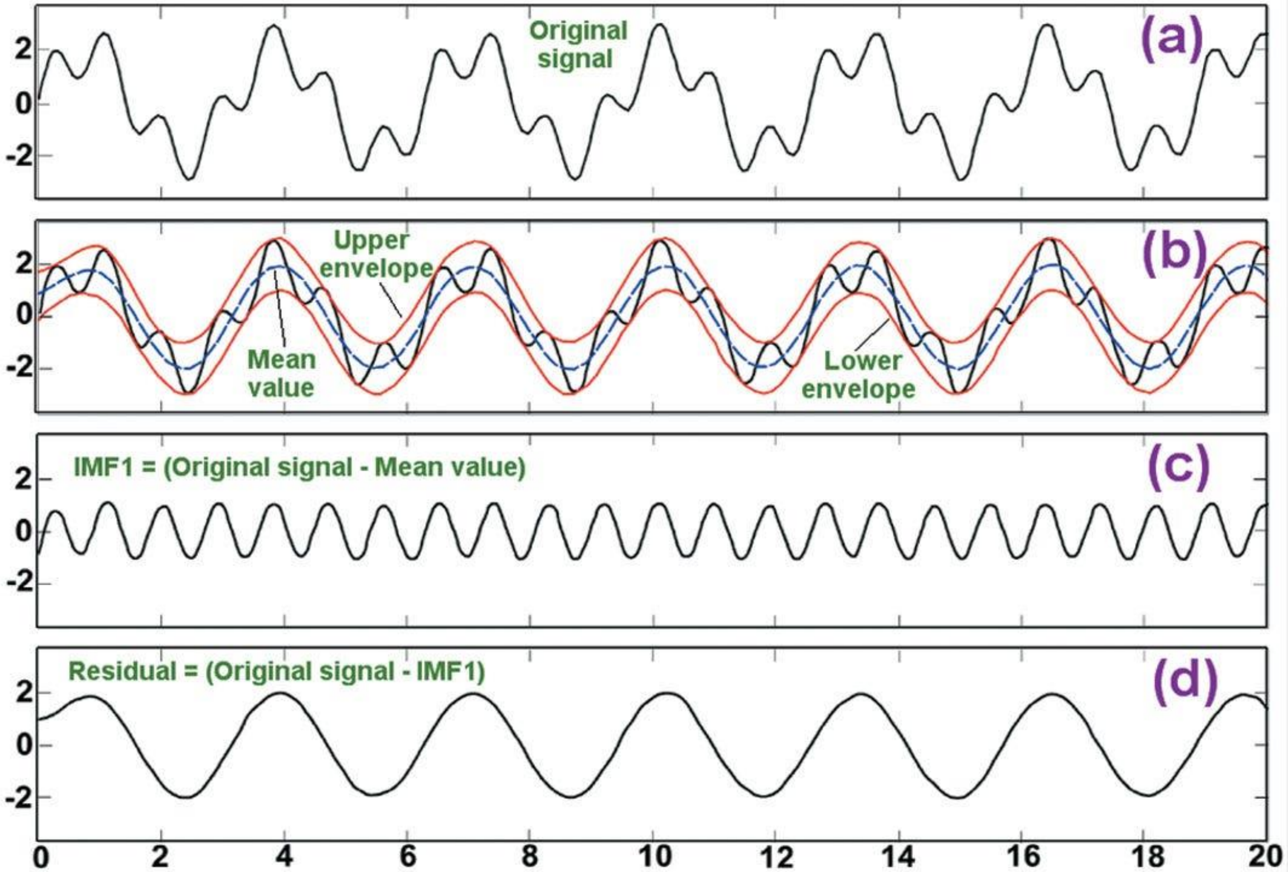
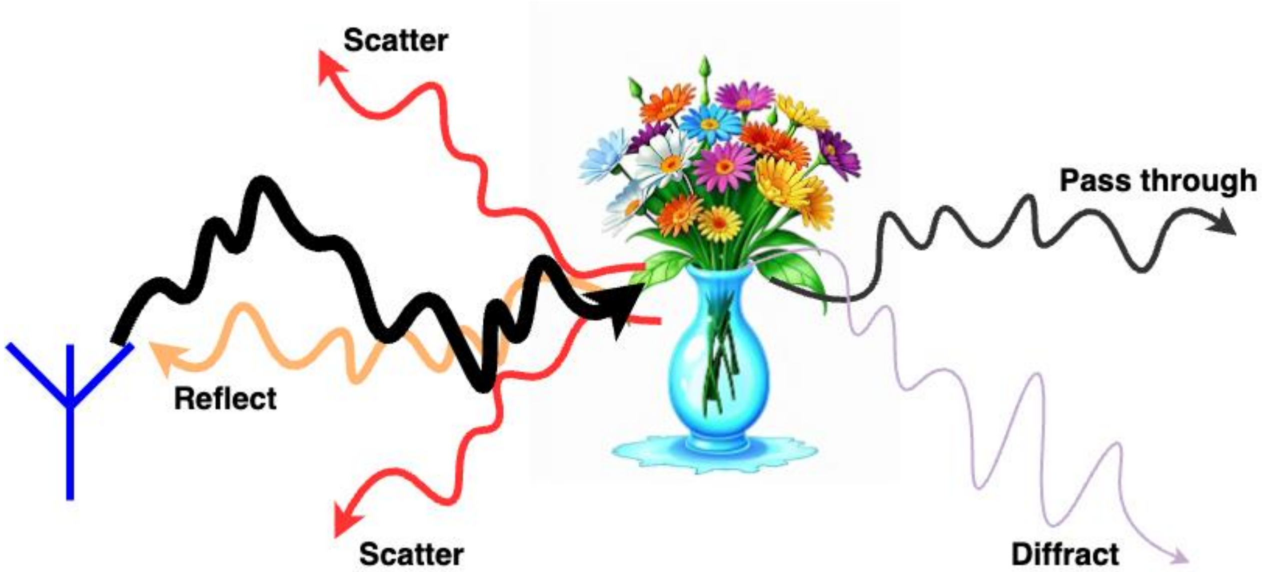
- The bandwidth of a signal is the difference between the highest and the lowest frequency components of that signal
  - We assume that any of the frequency components between the highest and the lowest can be present in the signal



Impact of Bandwidth on Sensing

- Higher bandwidth signal means it is likely to have more number of signal components
  - Therefore, scattering, diffraction, reflection, etc., are likely to be more on high-bandwidth signals



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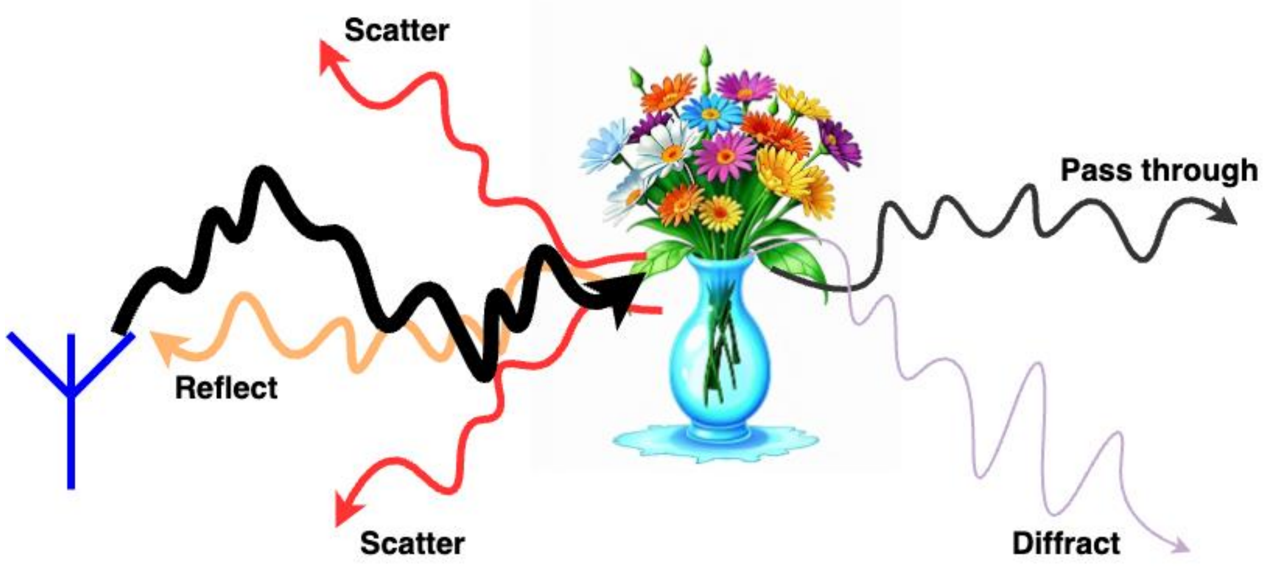

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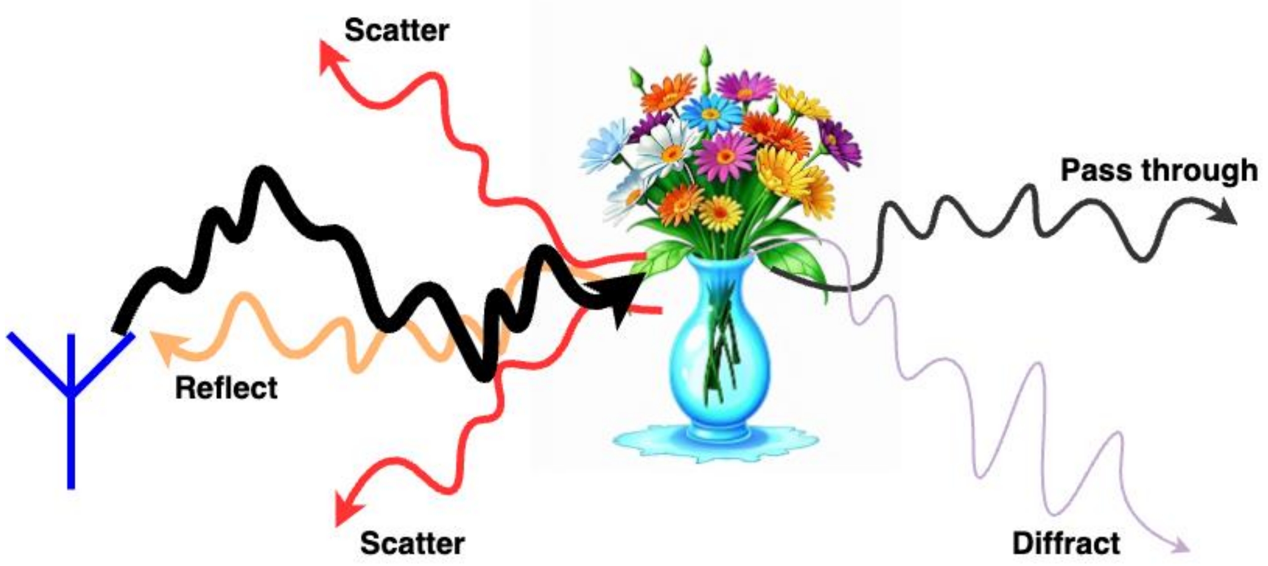
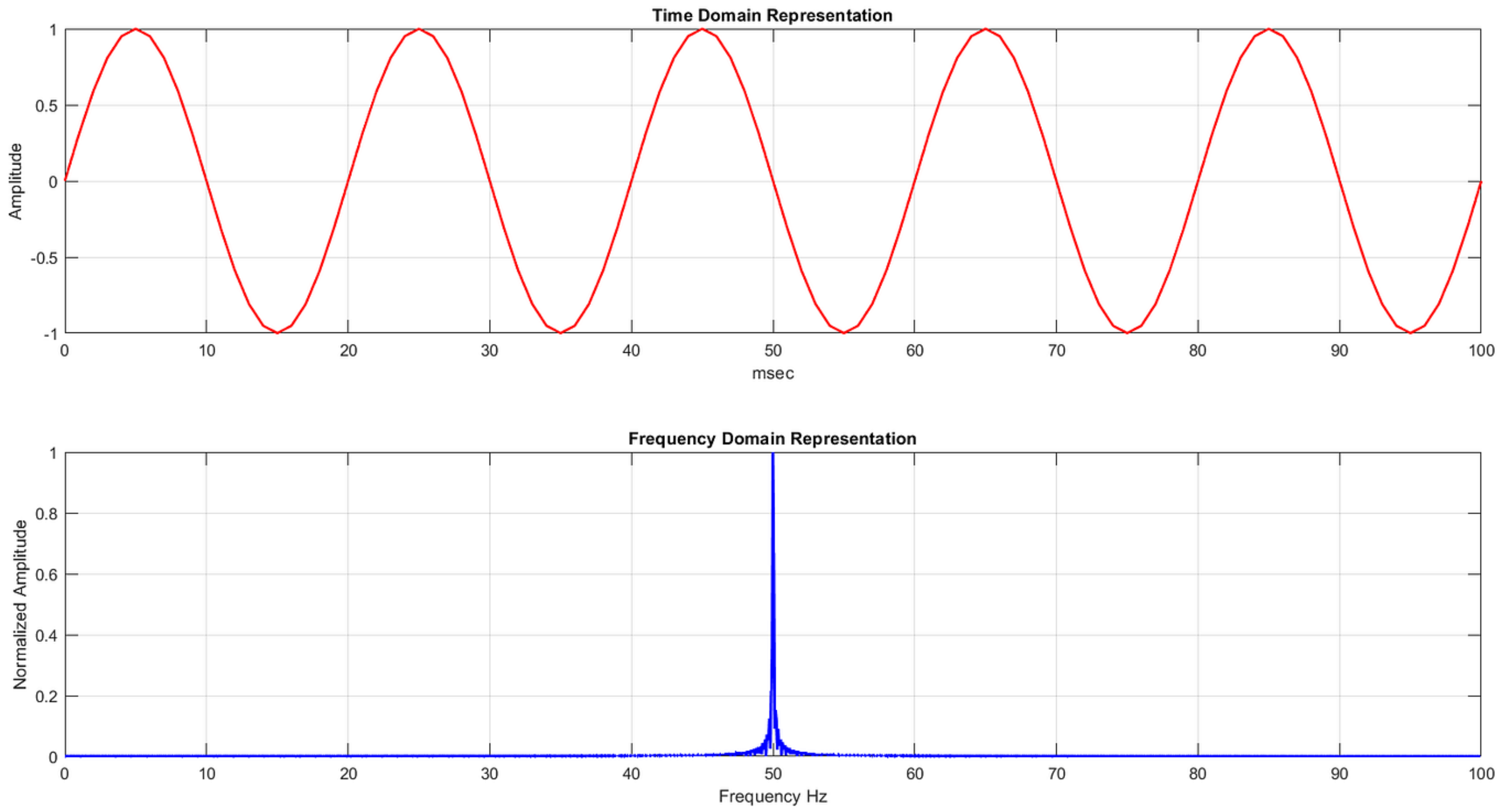
How do we get the various frequency components of a signal?



Fourier Analysis

- Extracts the frequency components of a signal



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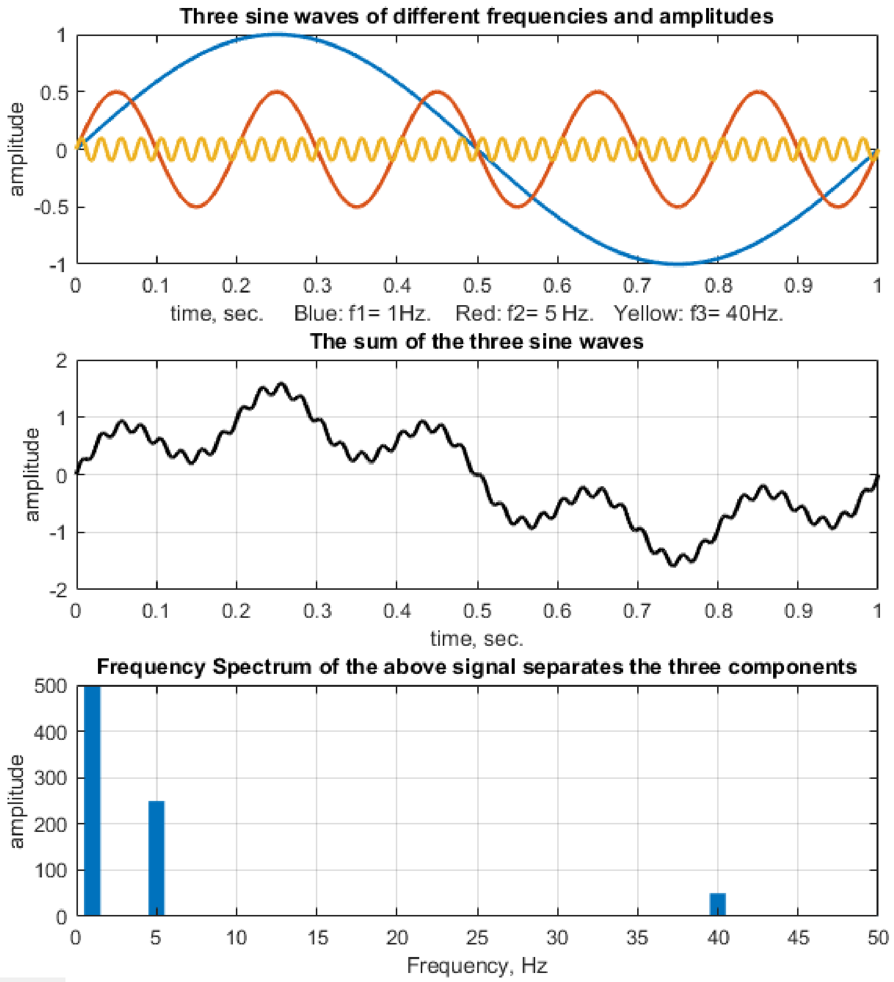

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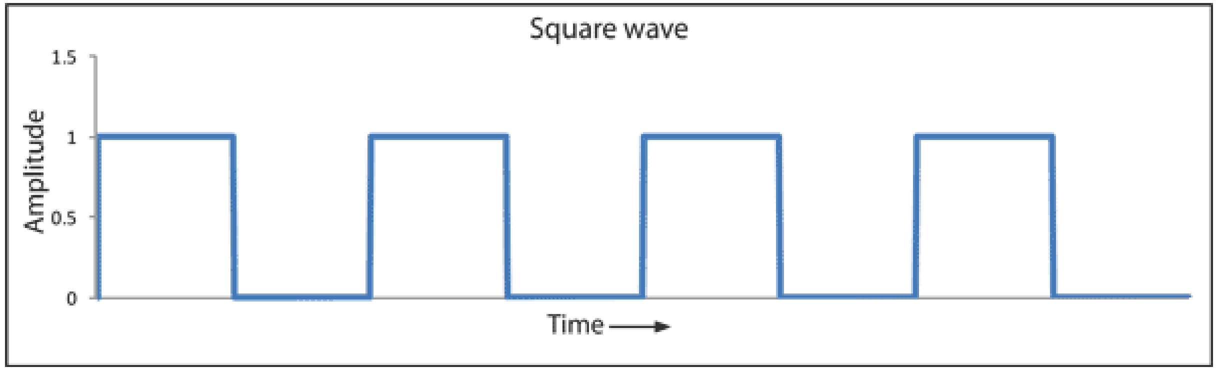

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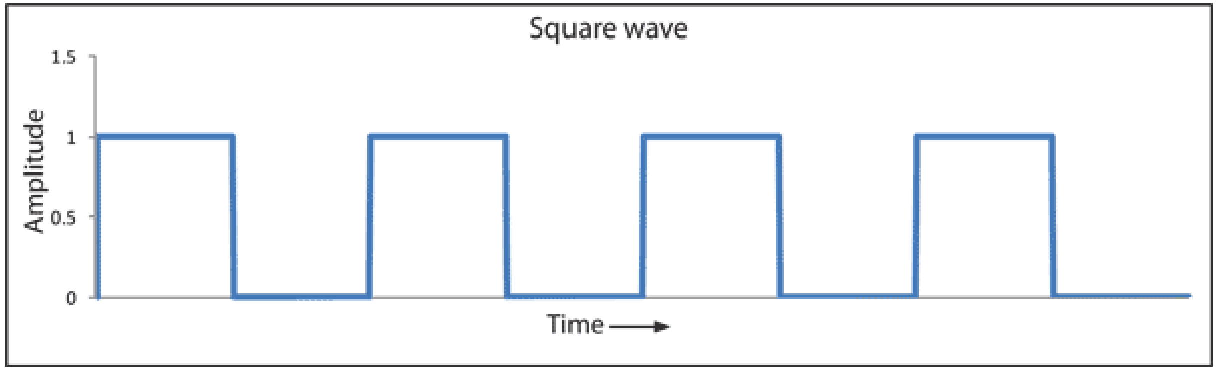

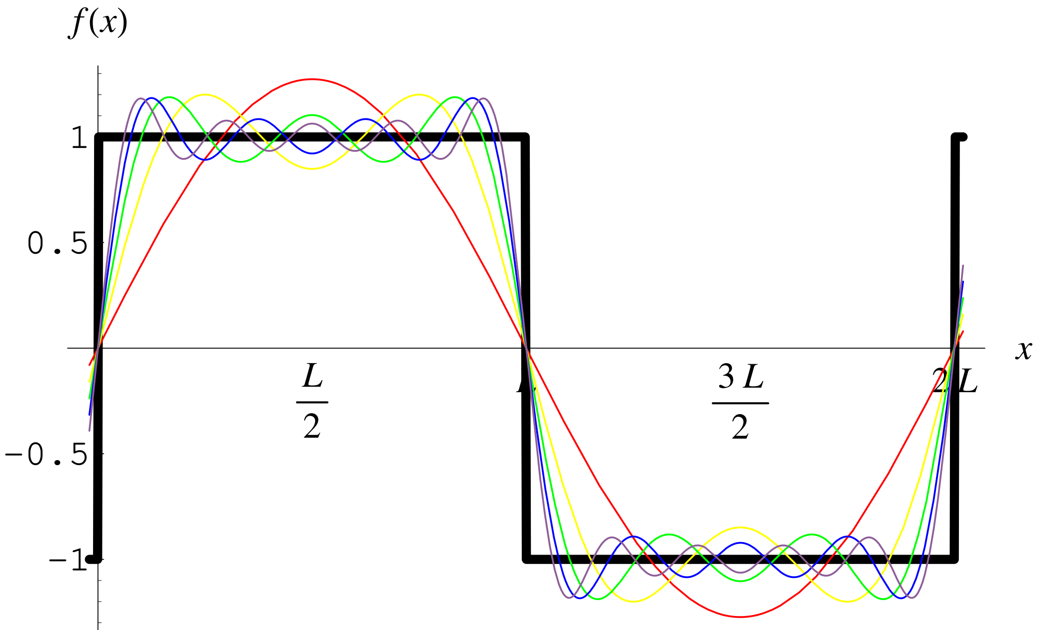

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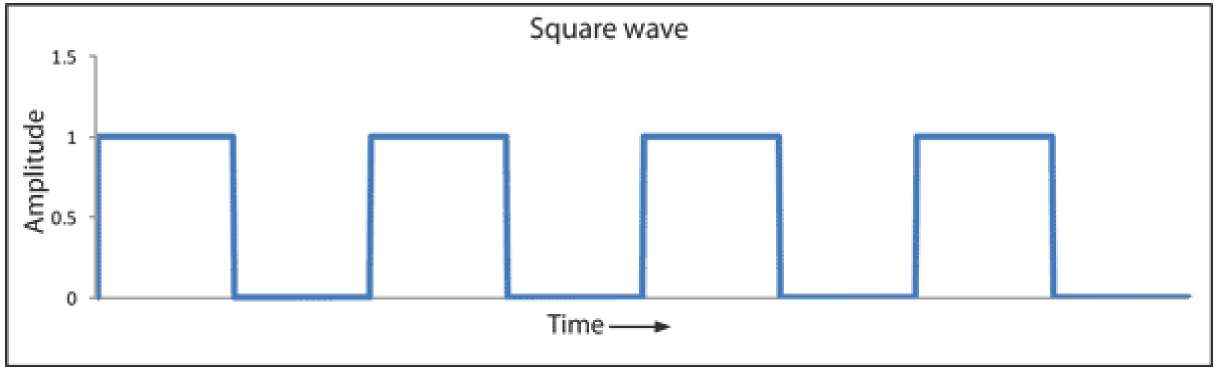

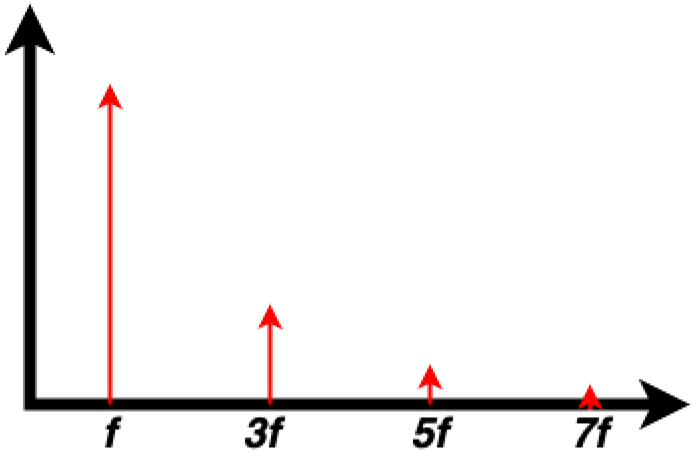

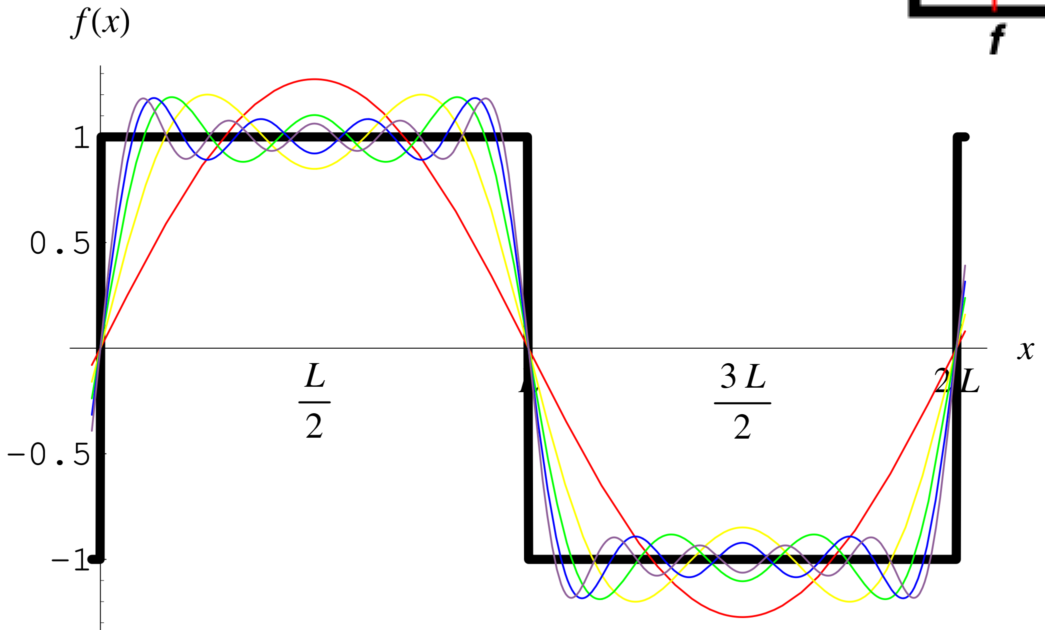

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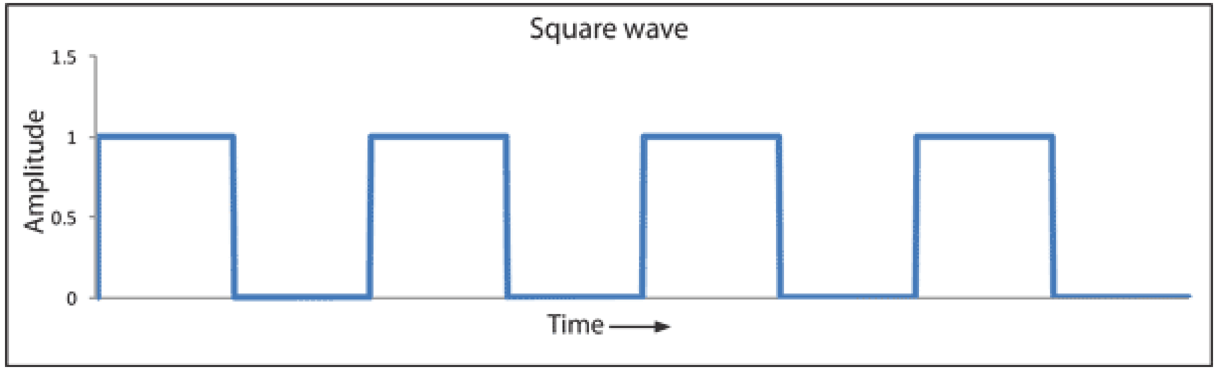

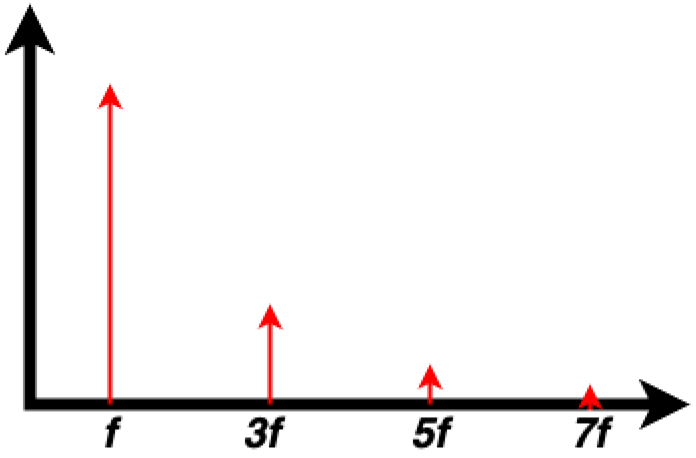

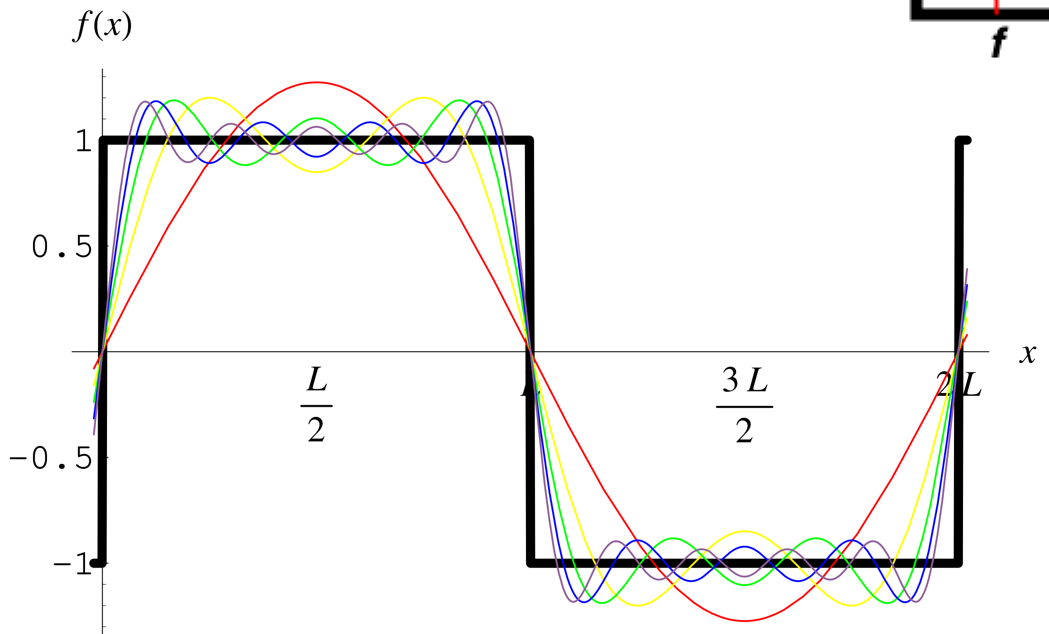

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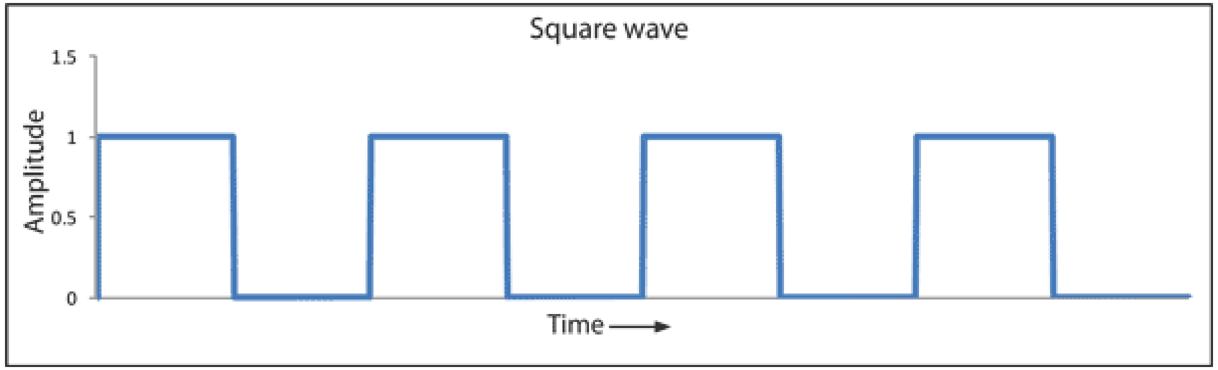
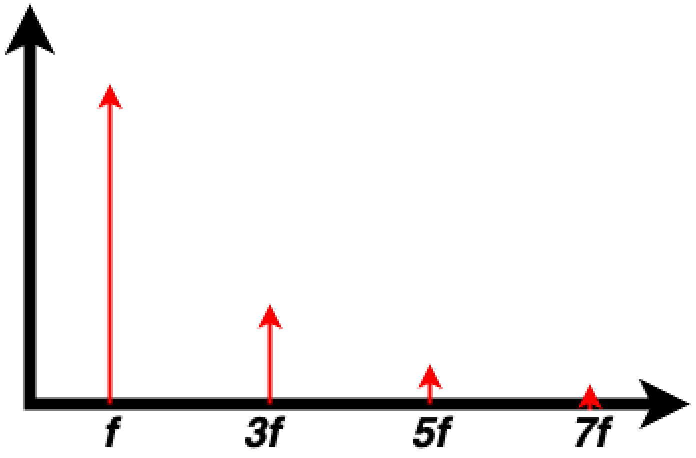
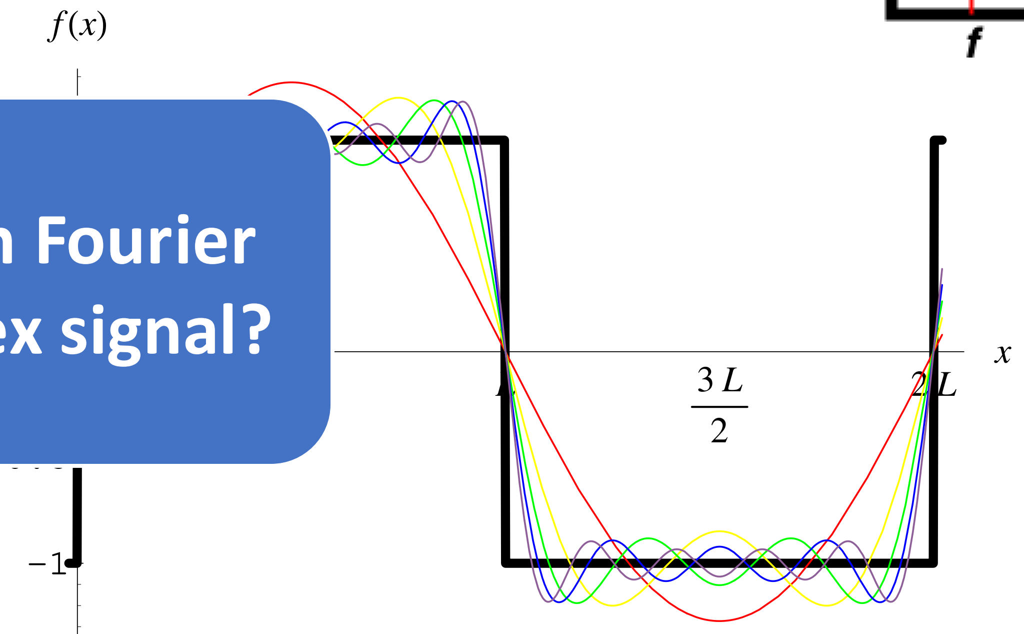
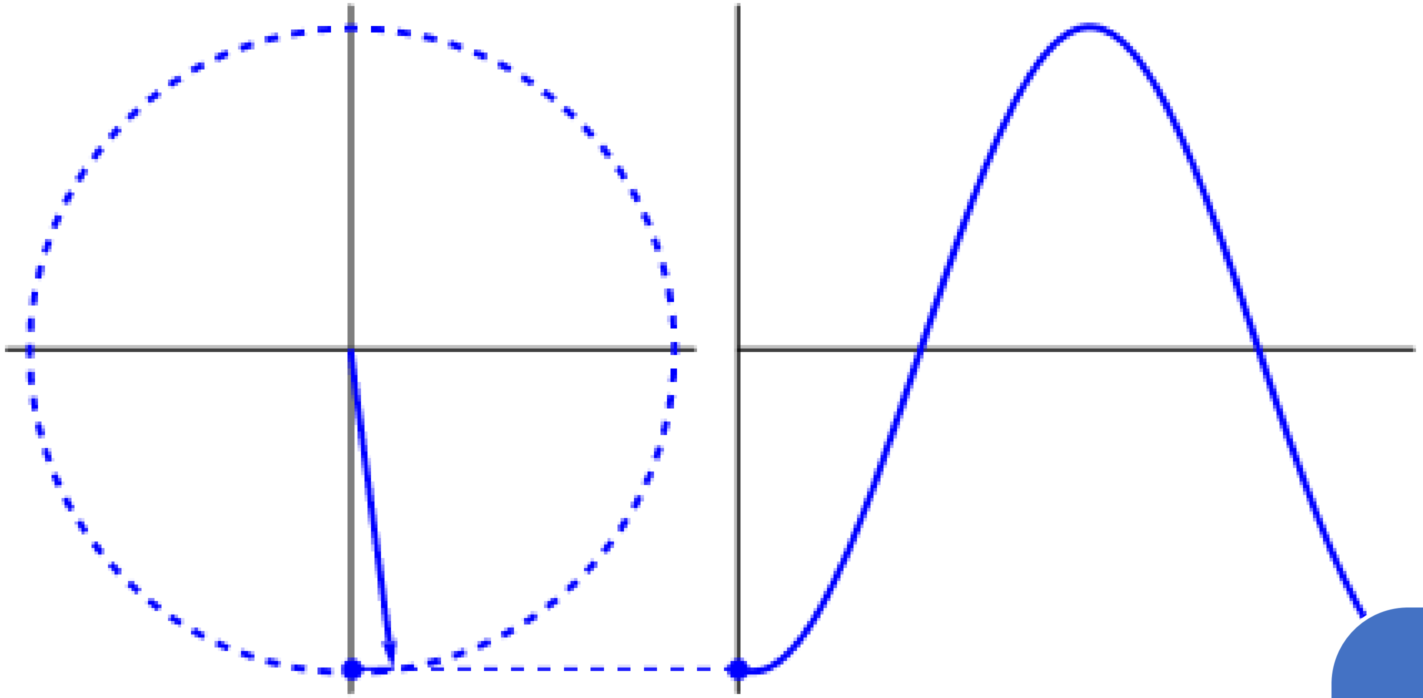






How do we perform Fourier Analysis of a complex signal?

Representation of a Signal



$$y(t) = A \sin(\omega t + \varphi)$$

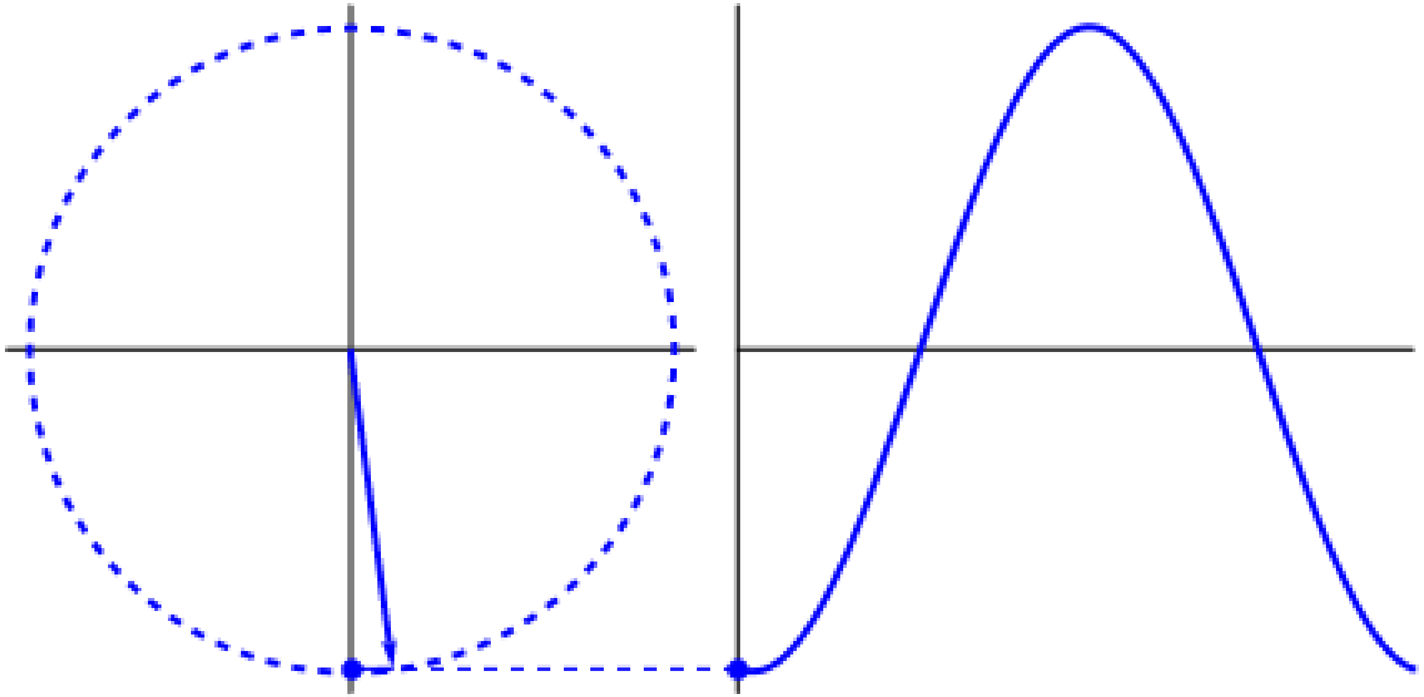
A: ***Amplitude***

ω: Angular frequency

φ: Phase

What is the significance of frequency of this signal?

Fourier Representation in Complex Domain



$y(t) = A \sin(2\pi f t + \varphi)$

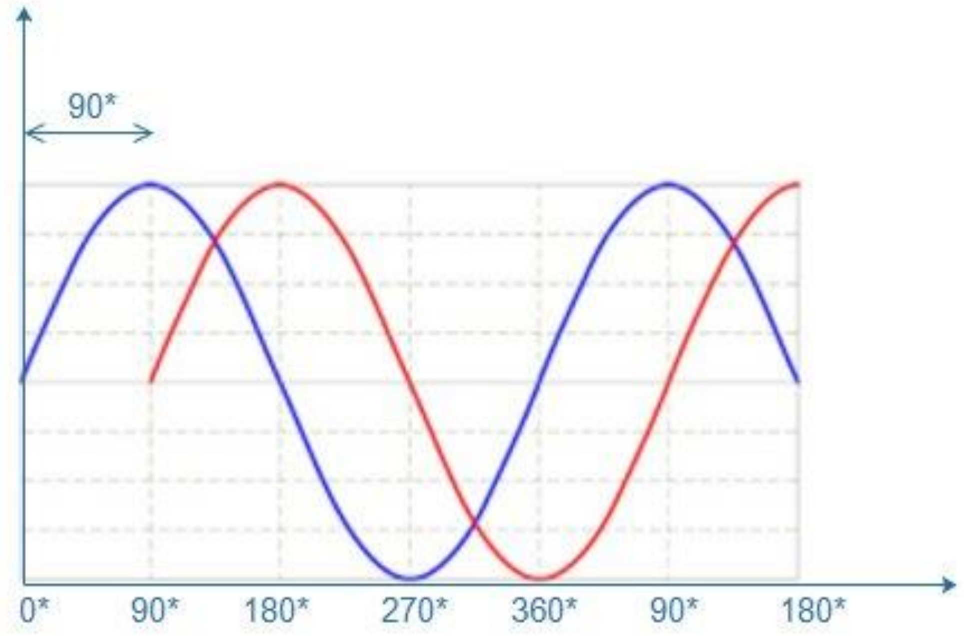
A: ***Amplitude***

f: Ordinary frequency

φ: Phase

The Notion of Phase

- The fraction of the cycle covered upto some time instance *t*
- An important metric for sensing
  - The phase of the reflected/scattered signal depends on the type of the material where the transmitted signal hitted
  - Introduces a shift in the reflected/scattered signal



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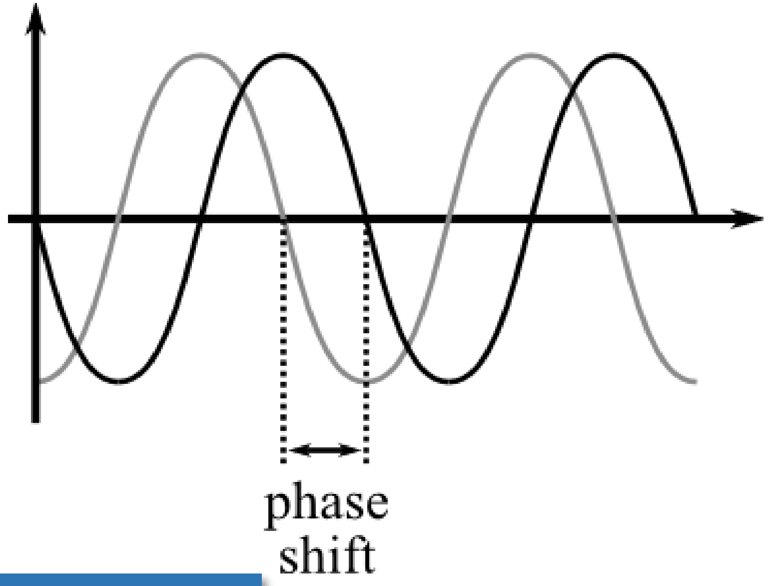
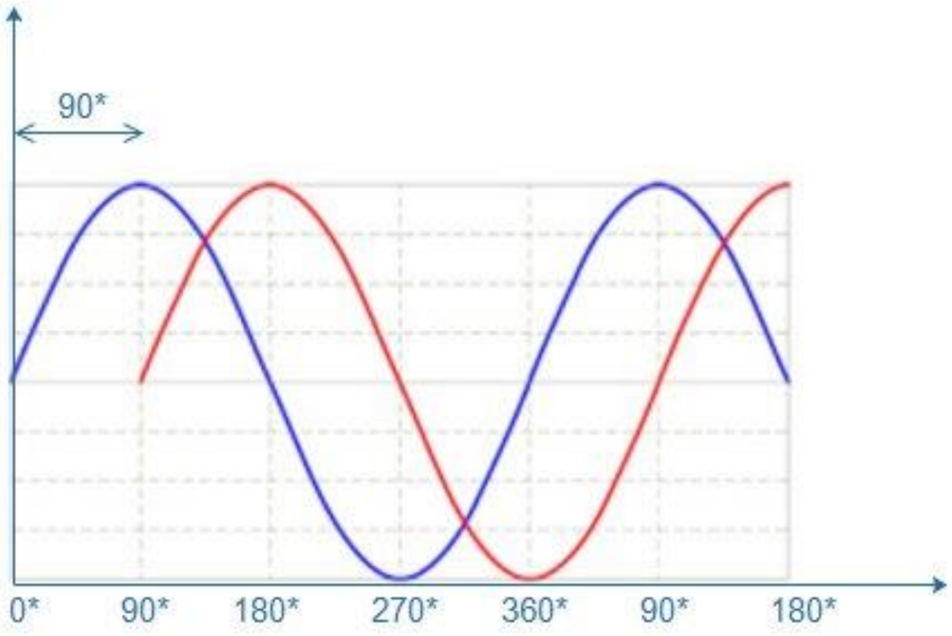
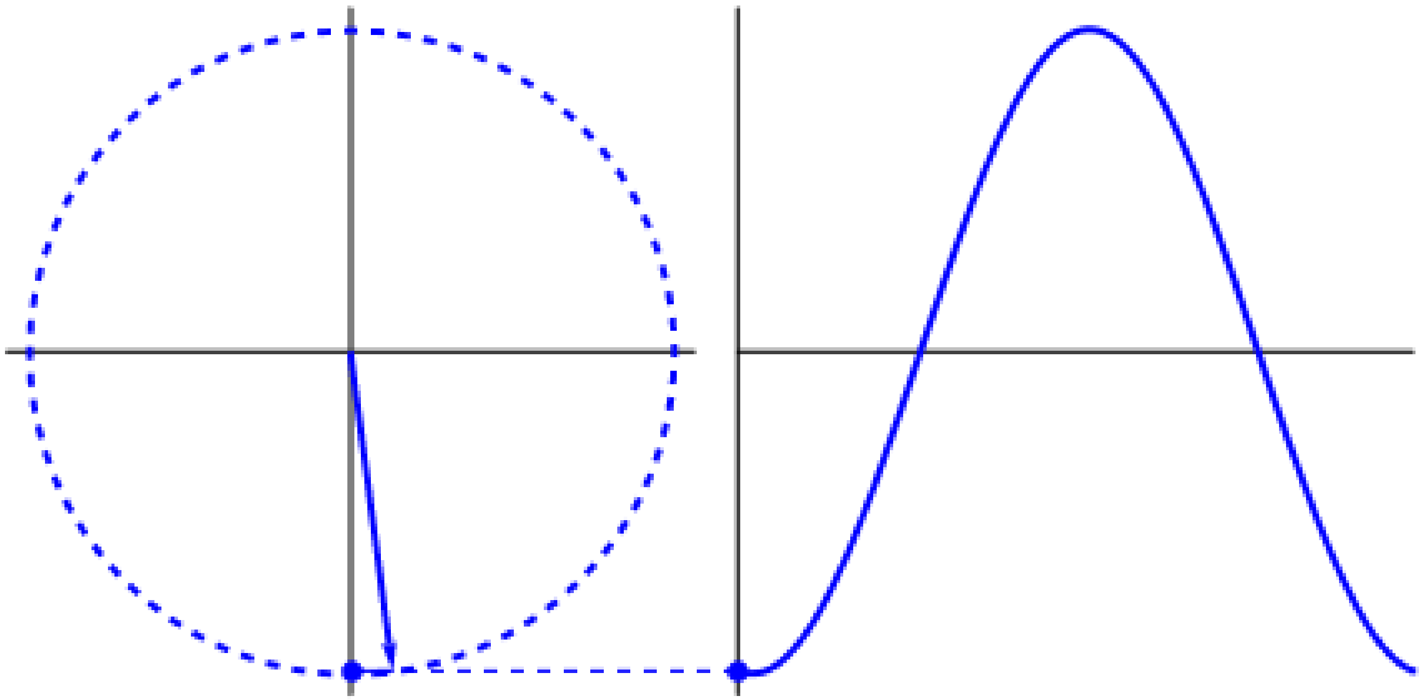
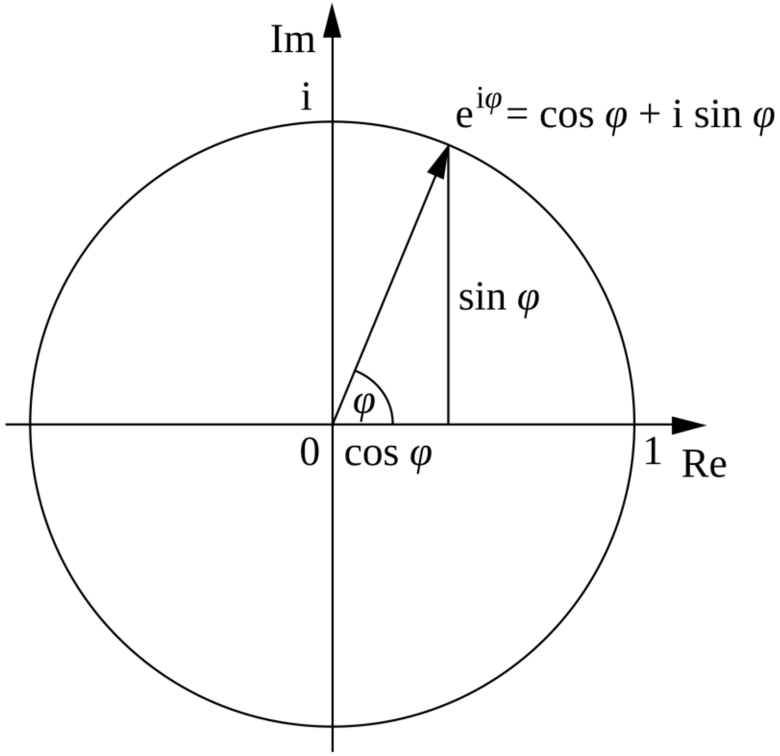
- **Phase difference/phase shift:**





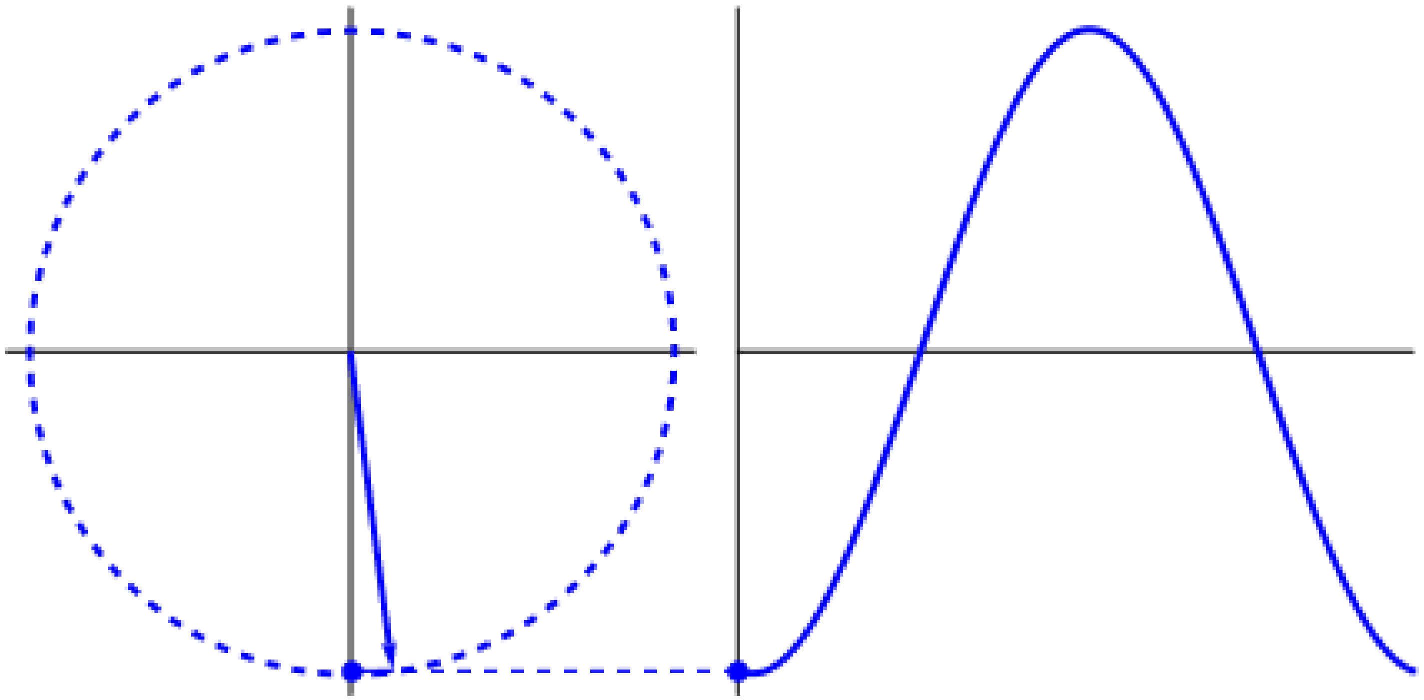
Fourier Representation in Complex Domain





$$S(f) = \int_{-\infty}^{\infty} s(t) \cdot e^{-i2\pi ft}\, dt.$$

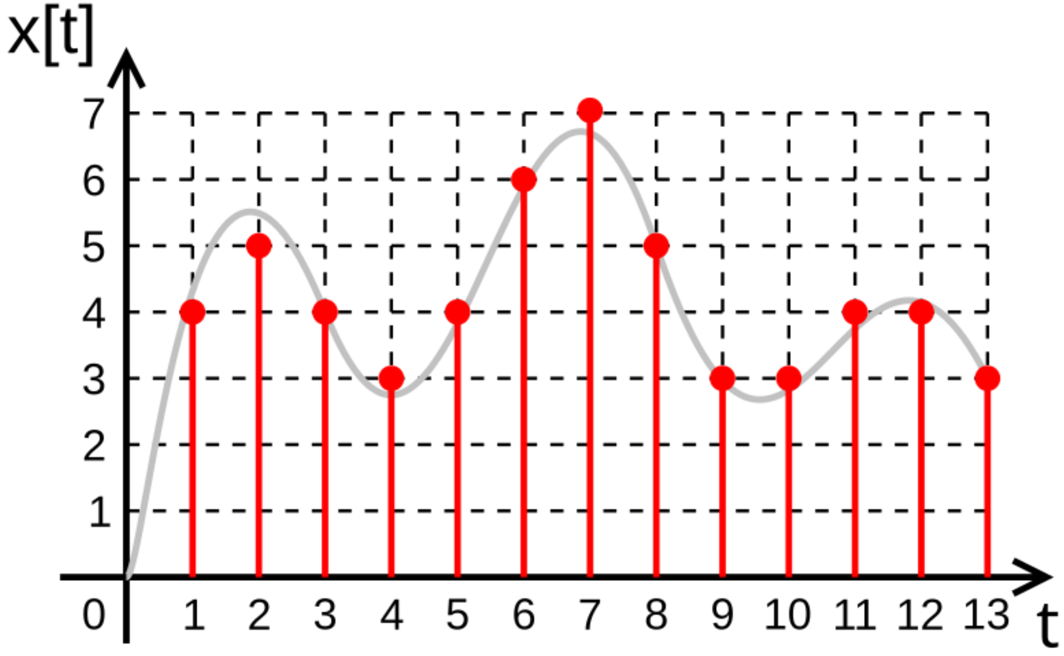
Fourier Representation in Complex Domain



Check this video for a nice explanation of Fourier transformation

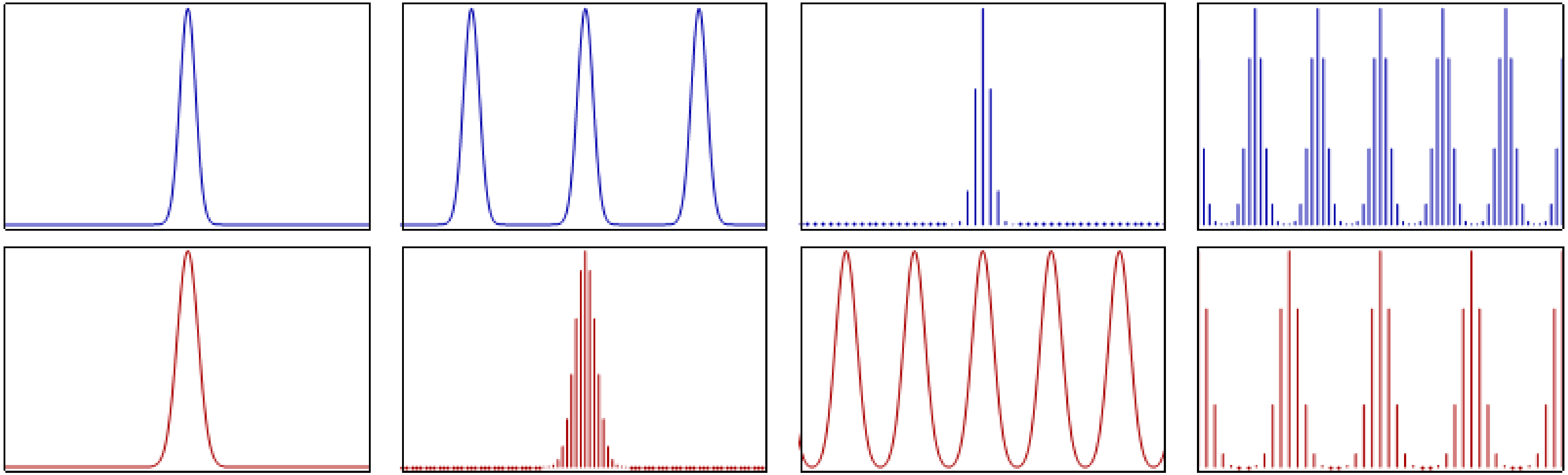
Discrete Fourier Transformation

- In practice, we record the samples of a continuous signal
  - Discrete representation of the signal



- So, we need to perform the Fourier transformation on the discrete samples of the signals
  - Discrete Time Fourier Transform (DTFT)
- However, the input and the output of the samples are also finite
  - Discrete Fourier Transform (DFT): Works on the finite time series data

DTFT and DFT



A Continuous signal and its Fourier transform	Periodic summation of the original signal and its Fourier transform	Original signal discretized and its Fourier transform (DTFT)	Periodic summation of the discrete signal, DFT computes discrete samples of the continuous DTFT

Image Source: Wikipedia

Discrete Fourier Transformation (DFT)

- Let $x_0, \ldots, x_{n-1}$ be complex numbers. The DFT is defined by the formula,

$$X_k = \sum_{m=0}^{n-1} x_m e^{-i2\pi km/n} \qquad k = 0, \ldots, n-1,$$

  - Where $e^{i2\pi/n}$ is the primitive n^{th} root of 1

- Evaluating the above equation needs O(n^2) operations

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DFT Matrix

- An N-point DFT is represented as $X = Wx$
  - x is the original input signal
  - W is *NxN* DFT matrix

$$W = \left(\frac{\omega^{jk}}{\sqrt{N}}\right)_{j,k=0,\ldots,N-1}$$

  - ω is a primitive N^{th} root of unity

$$\omega = e^{-2\pi i/N}$$

- For different values of *j*, *W* can be represented as a matrix

$$W = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

Example: 8-point DFT

$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -i & -i\omega & -1 & -\omega & i & i\omega \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\omega & i & \omega & -1 & i\omega & -i & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & -i & i\omega & -1 & \omega & i & -i\omega \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\omega & i & -\omega & -1 & -i\omega & -i & \omega \end{bmatrix}$$

Example: 8-point DFT

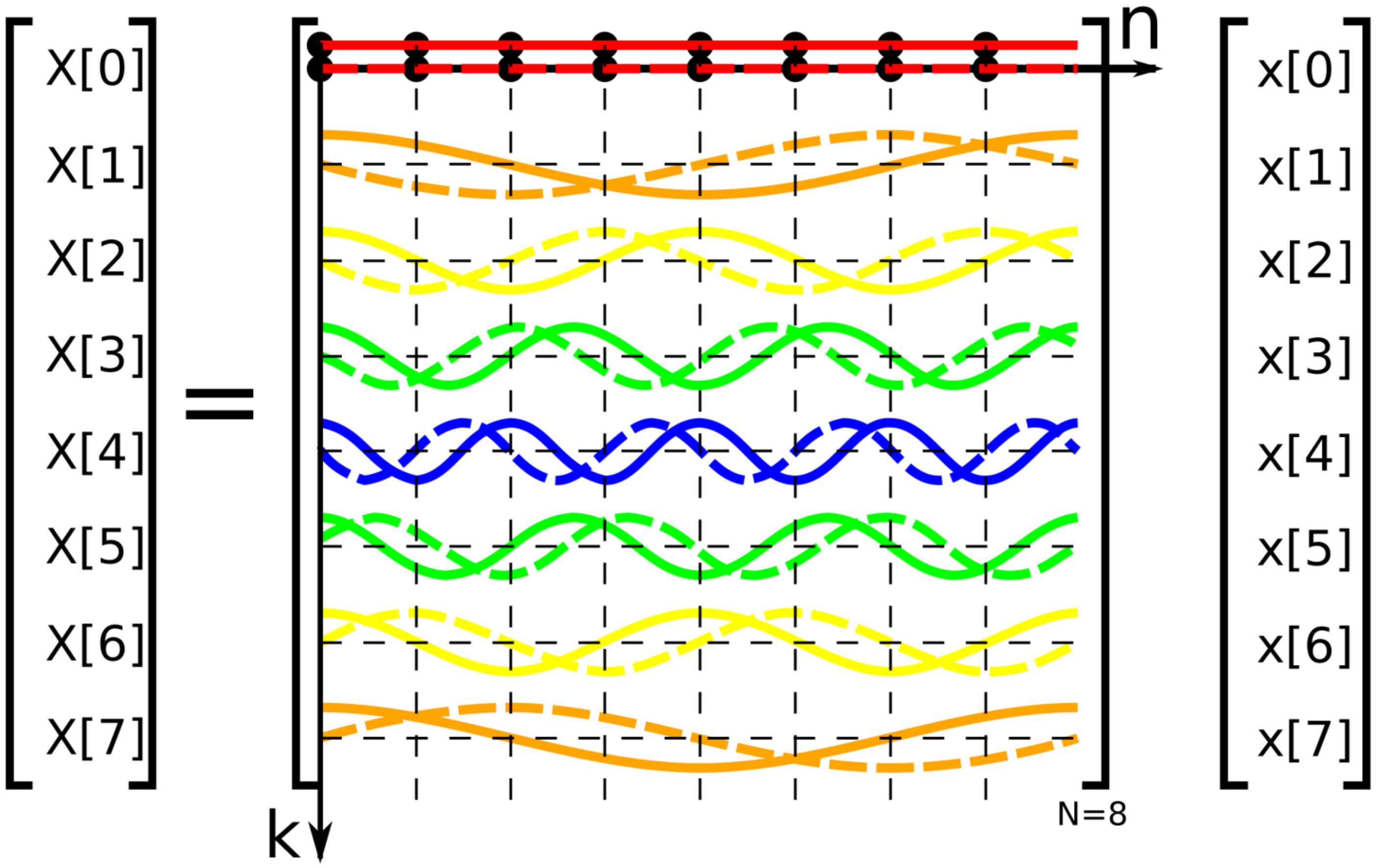
$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -i & -i\omega & -1 & -\omega & i & i\omega \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -i\omega & i & \omega & -1 & i\omega & -i & -\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & -i & i\omega & -1 & \omega & i & -i\omega \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\omega & i & -\omega & -1 & -i\omega & -i & \omega \end{bmatrix}$$

$$\omega = e^{-\frac{2\pi i}{8}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

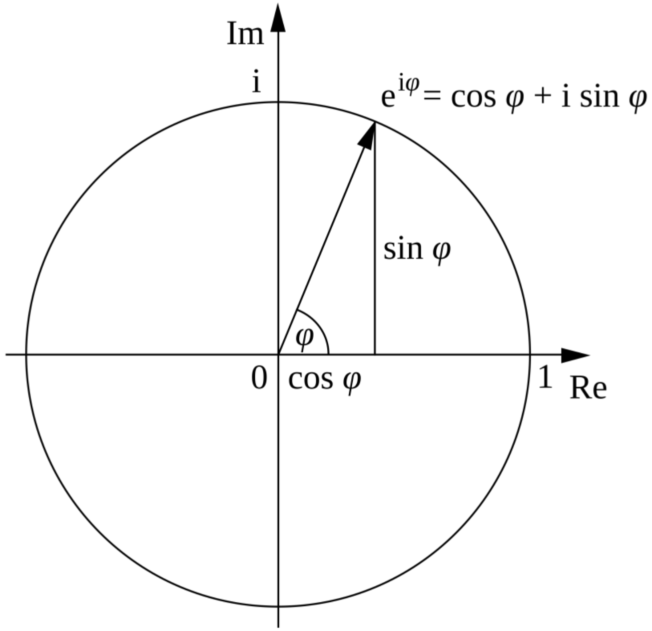
Example: 8-point DFT

$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

Example: 8-point DFT: Pictorial Representation

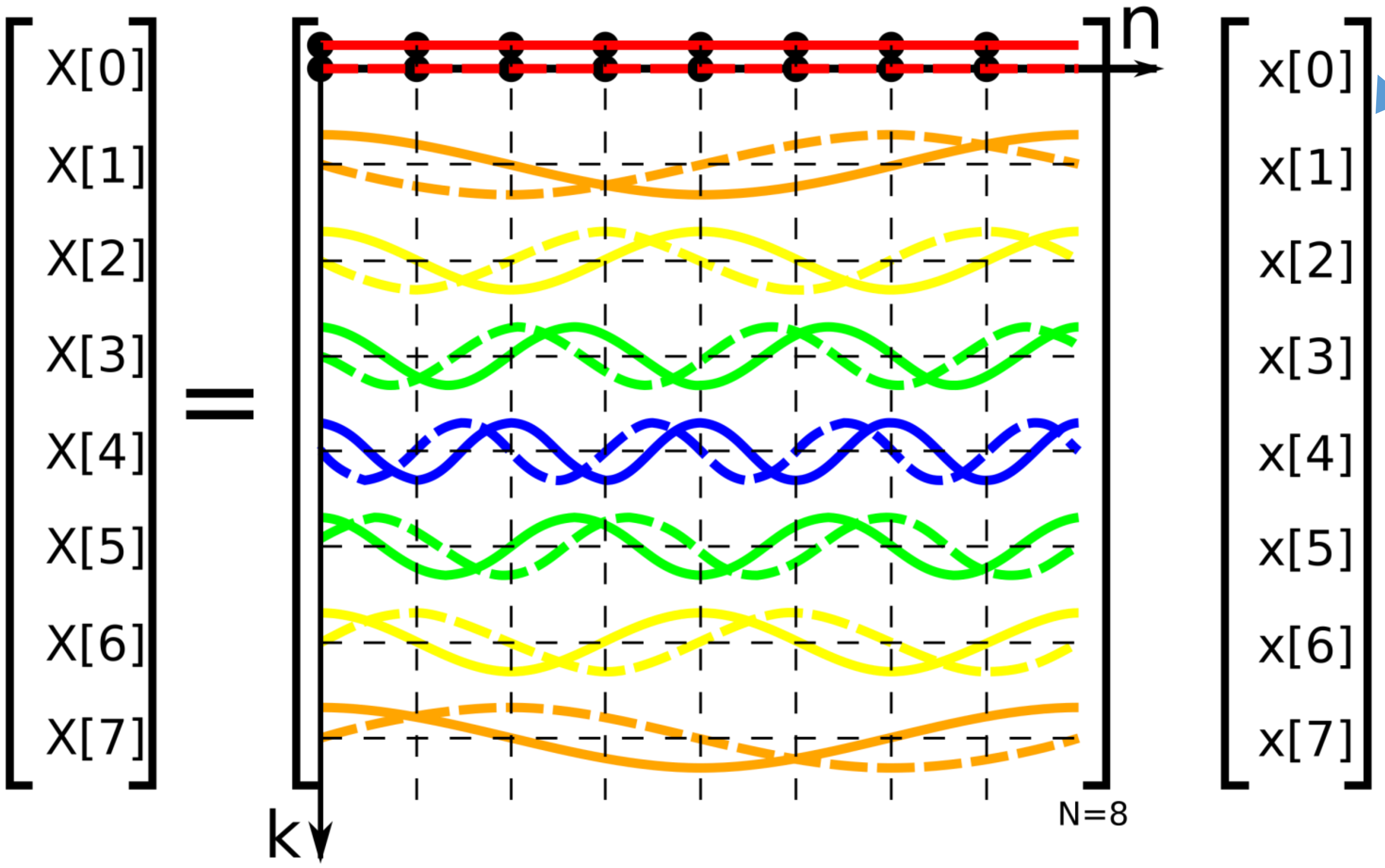


$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$



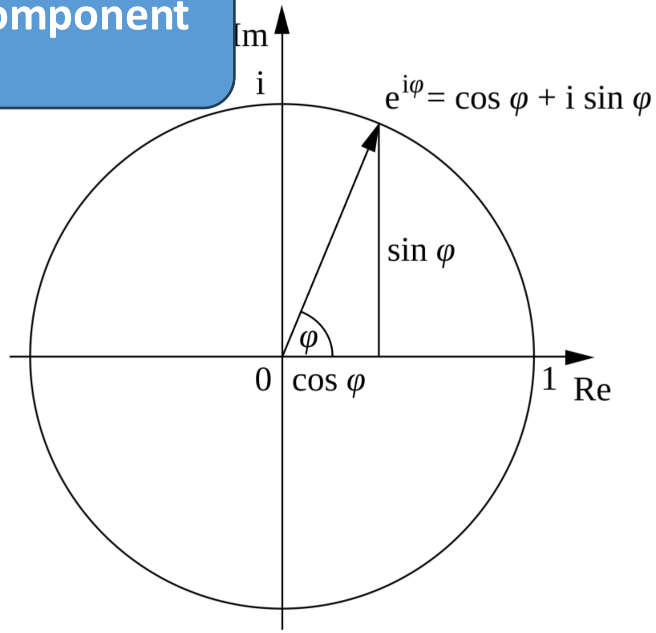
Cosine wave: Solid Line
Sine wave: Dashed line

Example: 8-point DFT: Pictorial Representation



$$W = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

DC Component



Cosine wave: Solid Line
Sine wave: Dashed line

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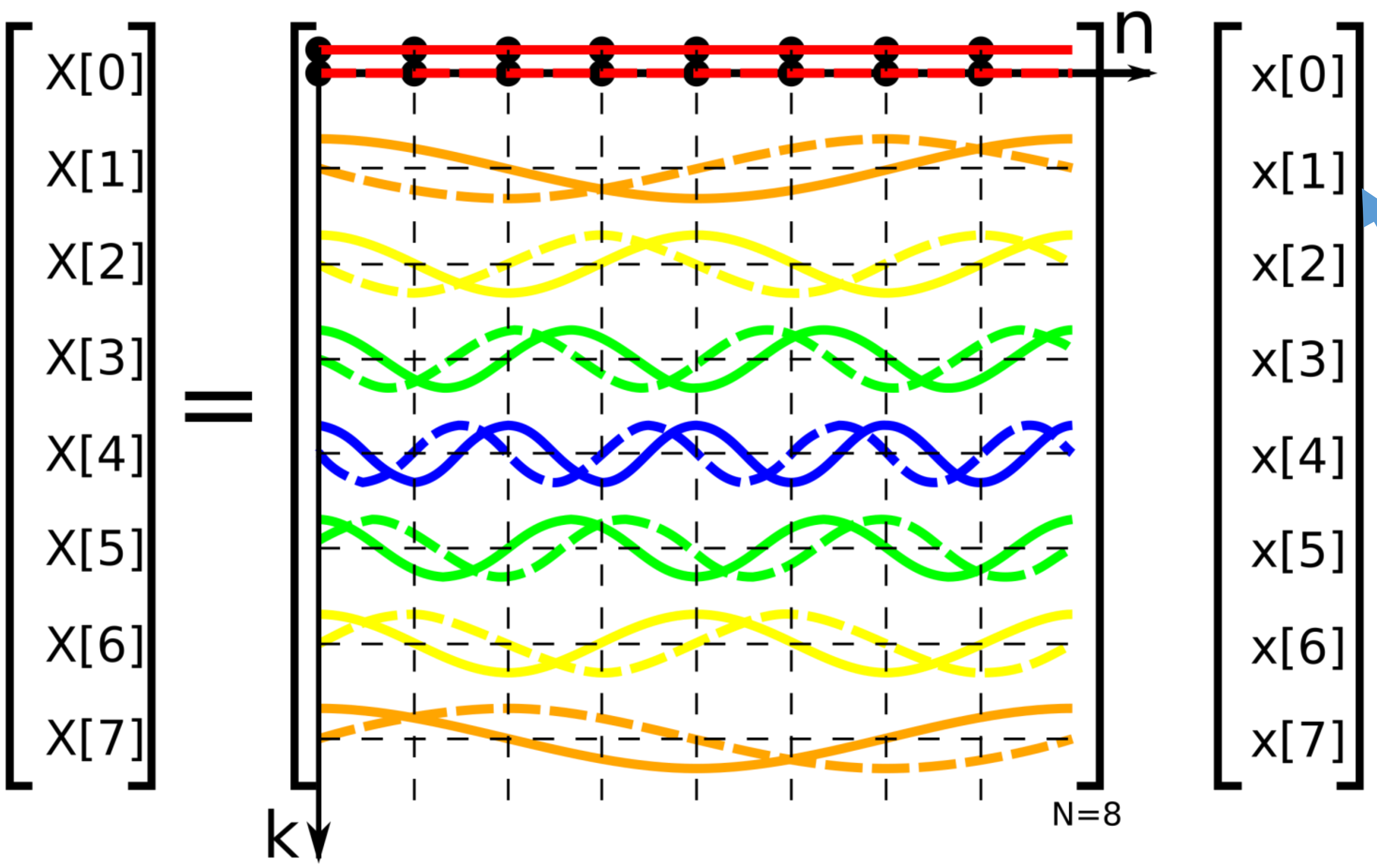

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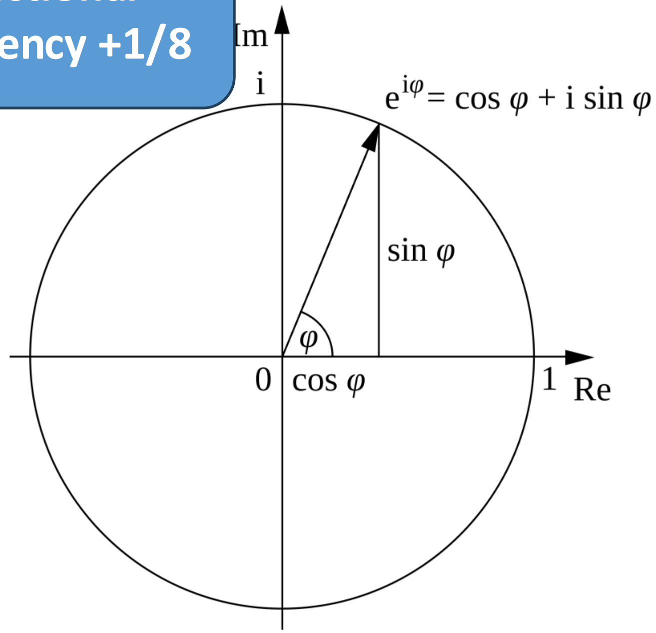
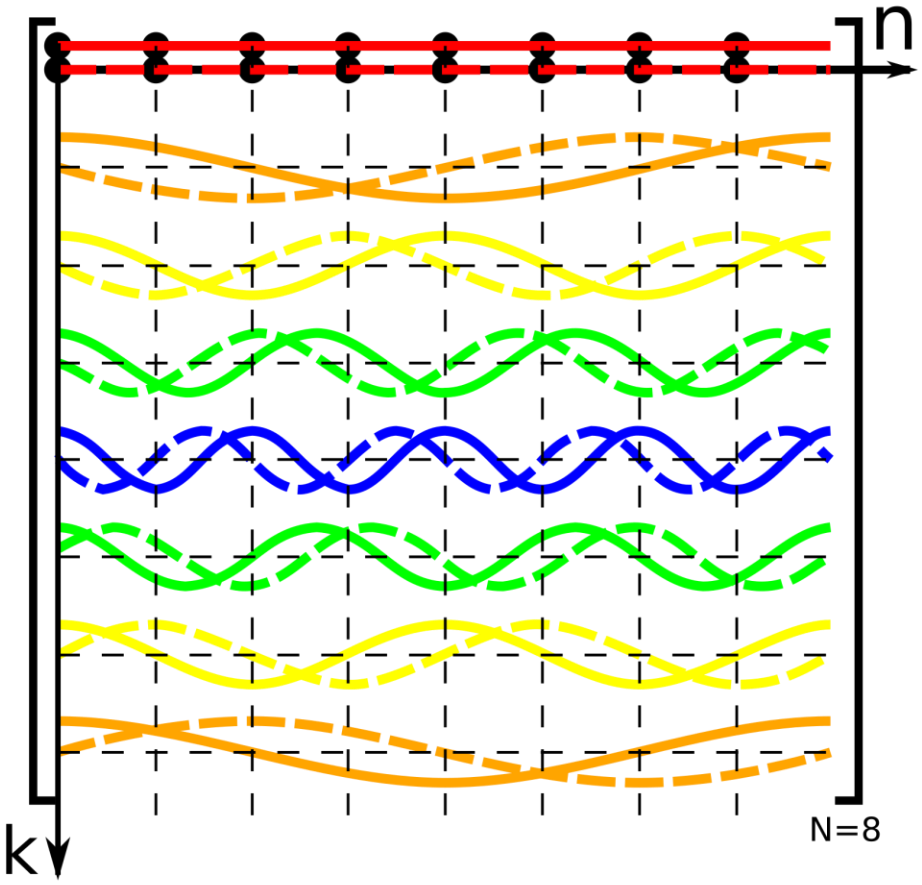
Fractional frequency +1/8



Cosine wave: Solid Line
Sine wave: Dashed line

Example: 8-point DFT: Pictorial Representation

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} =$$



$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

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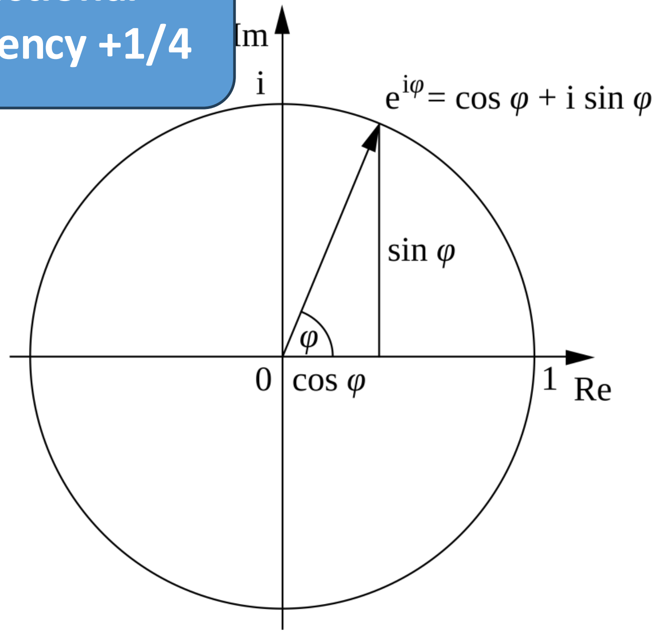
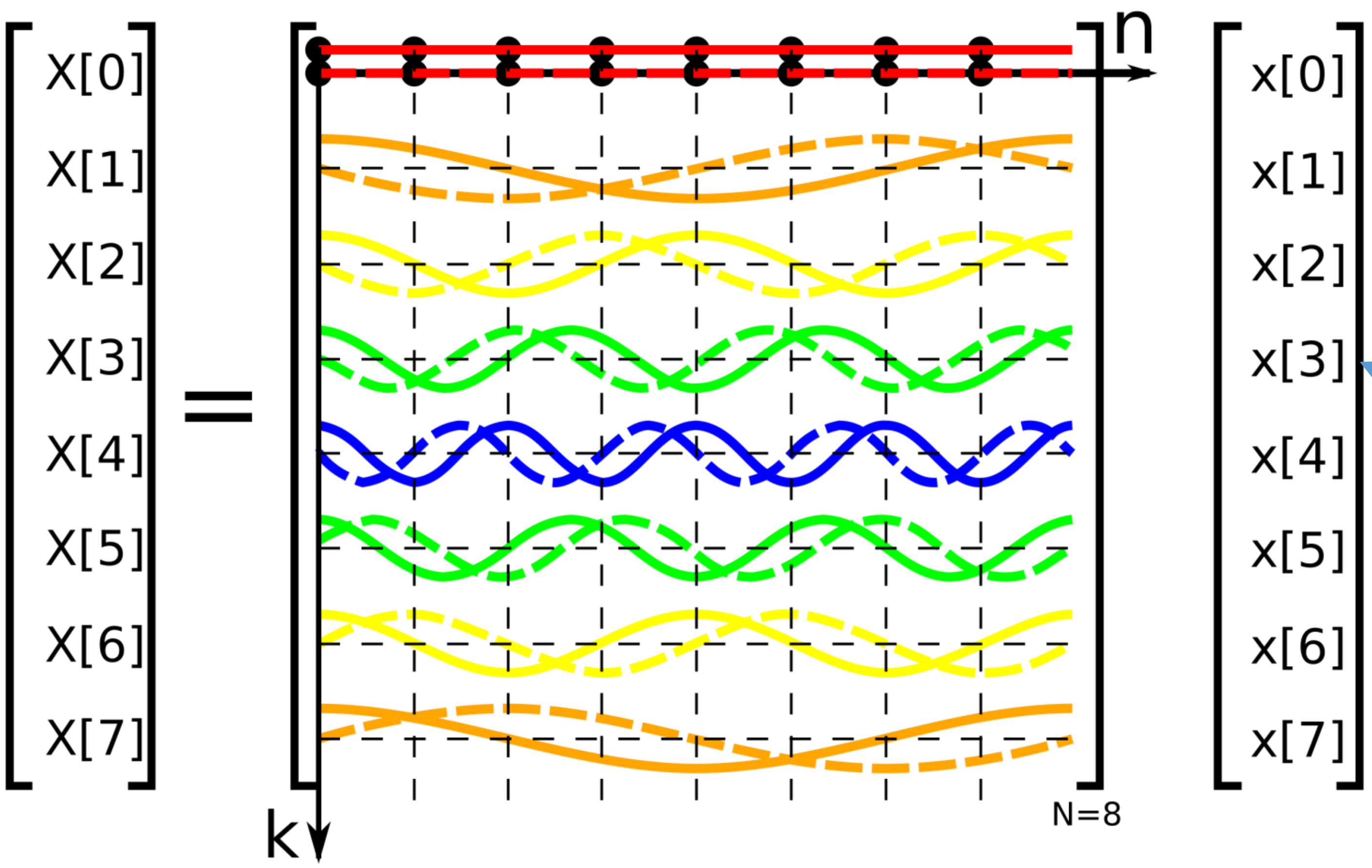
Fractional frequency +1/4



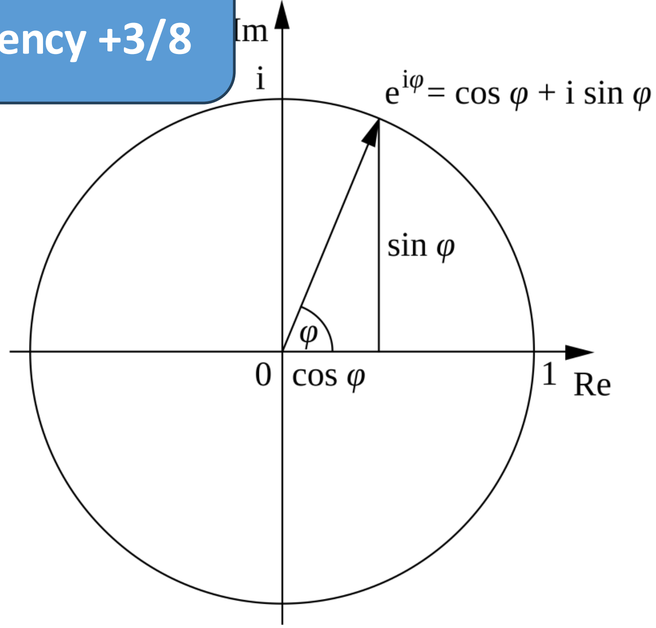
Cosine wave: Solid Line
Sine wave: Dashed line

Example: 8-point DFT: Pictorial Representation



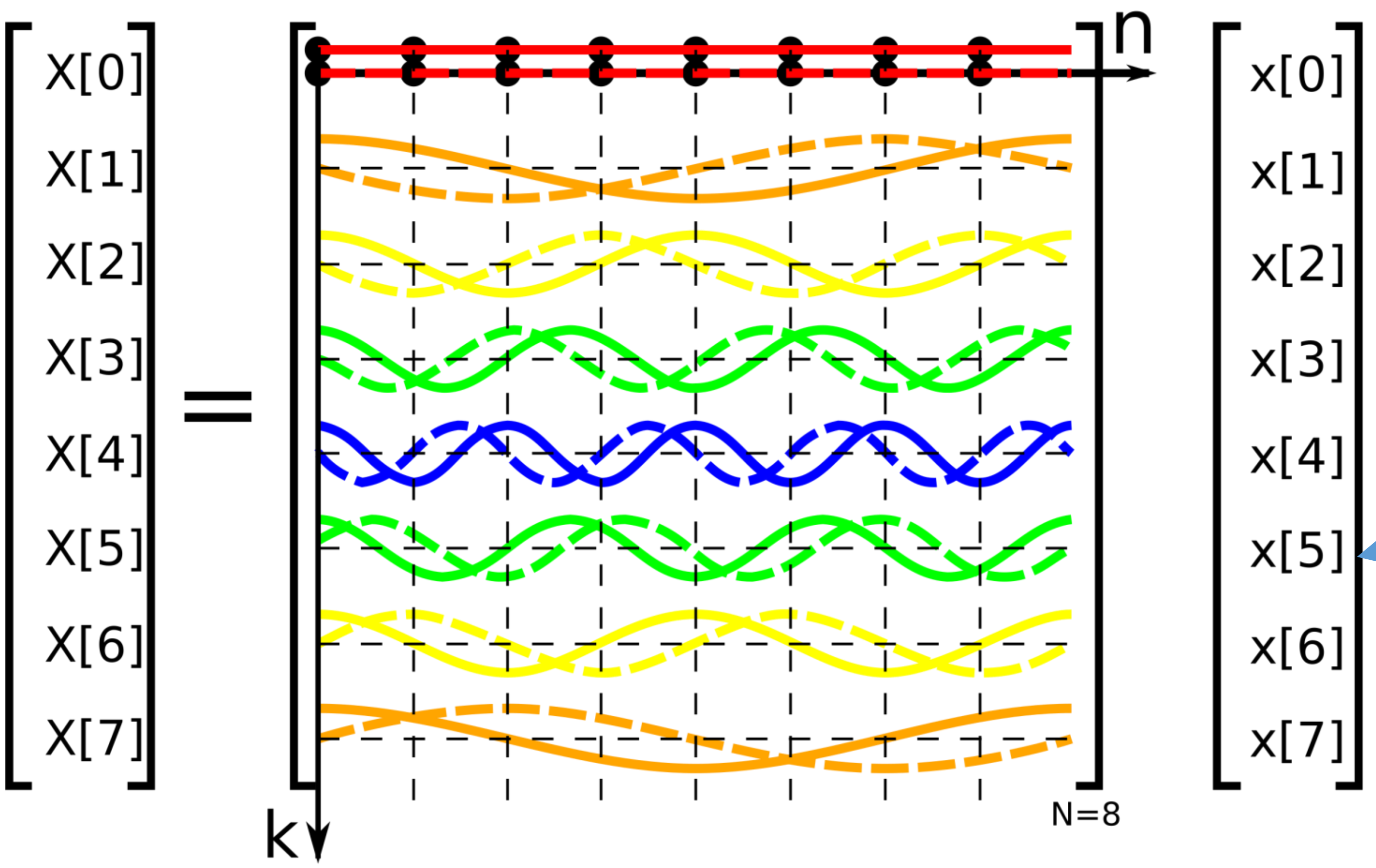
$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

Fractional frequency +3/8



Cosine wave: Solid Line
Sine wave: Dashed line

Example: 8-point DFT: Pictorial Representation



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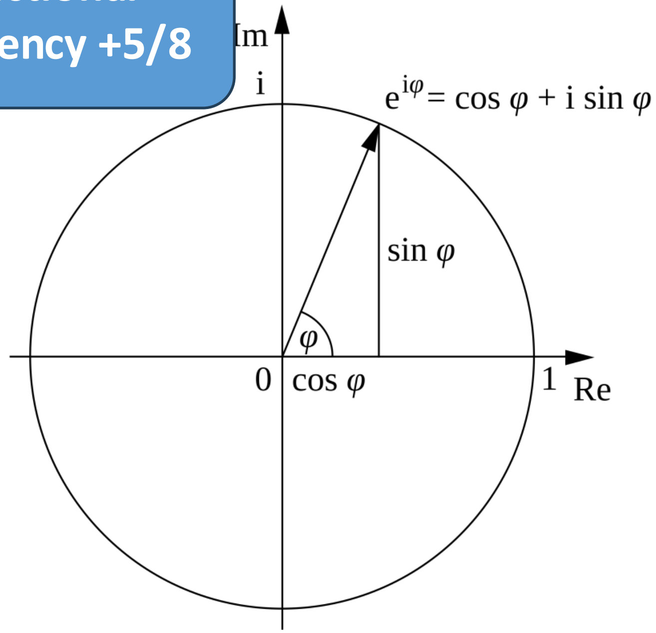
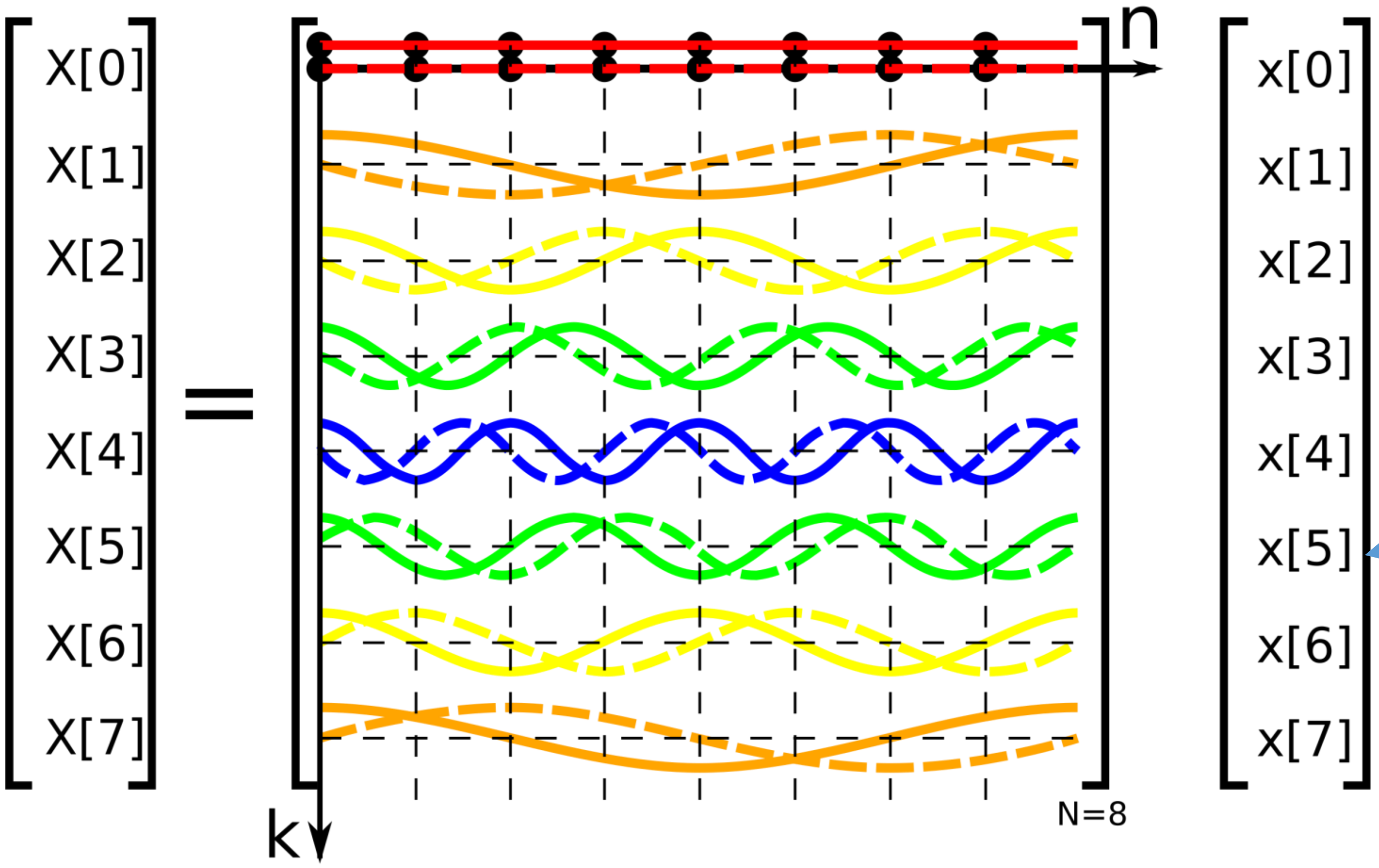
Fractional frequency +5/8



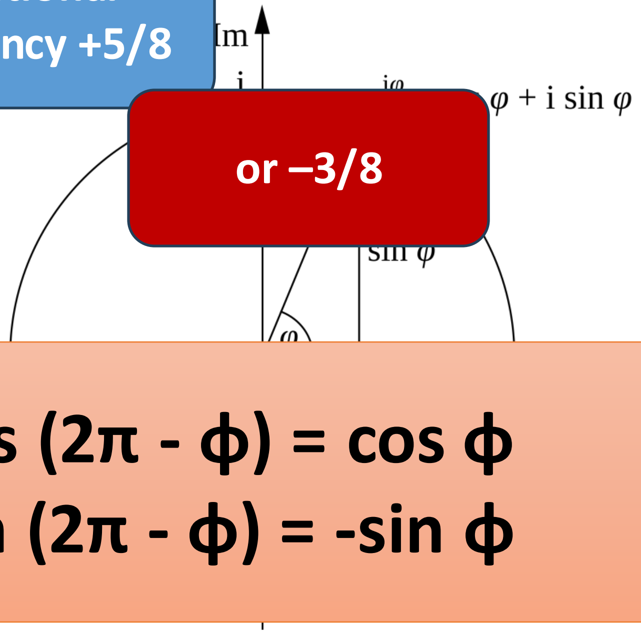
Cosine wave: Solid Line
Sine wave: Dashed line

Example: 8-point DFT: Pictorial Representation



$$W = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

Fractional frequency +5/8

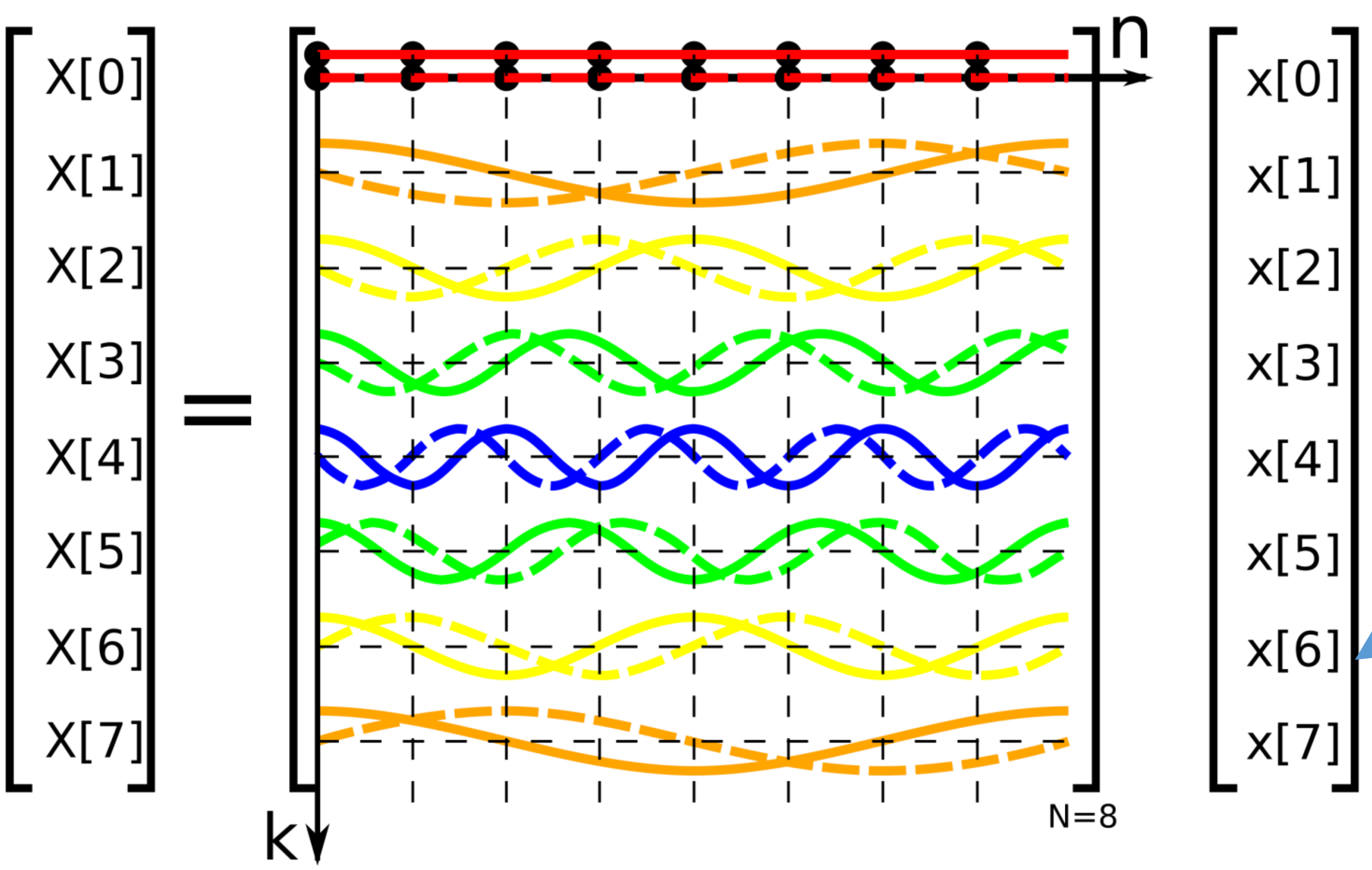


or –3/8

Cosine wave: Solid Line
Sine wave: Dashed line

$\cos(2\pi - \phi) = \cos \phi$
$\sin(2\pi - \phi) = -\sin \phi$

Example: 8-point DFT: Pictorial Representation



Cosine wave: Solid Line
Sine wave: Dashed line

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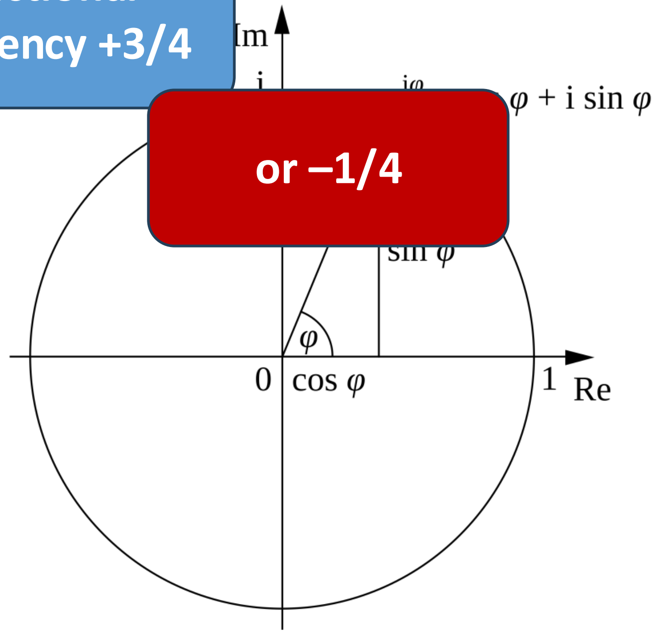
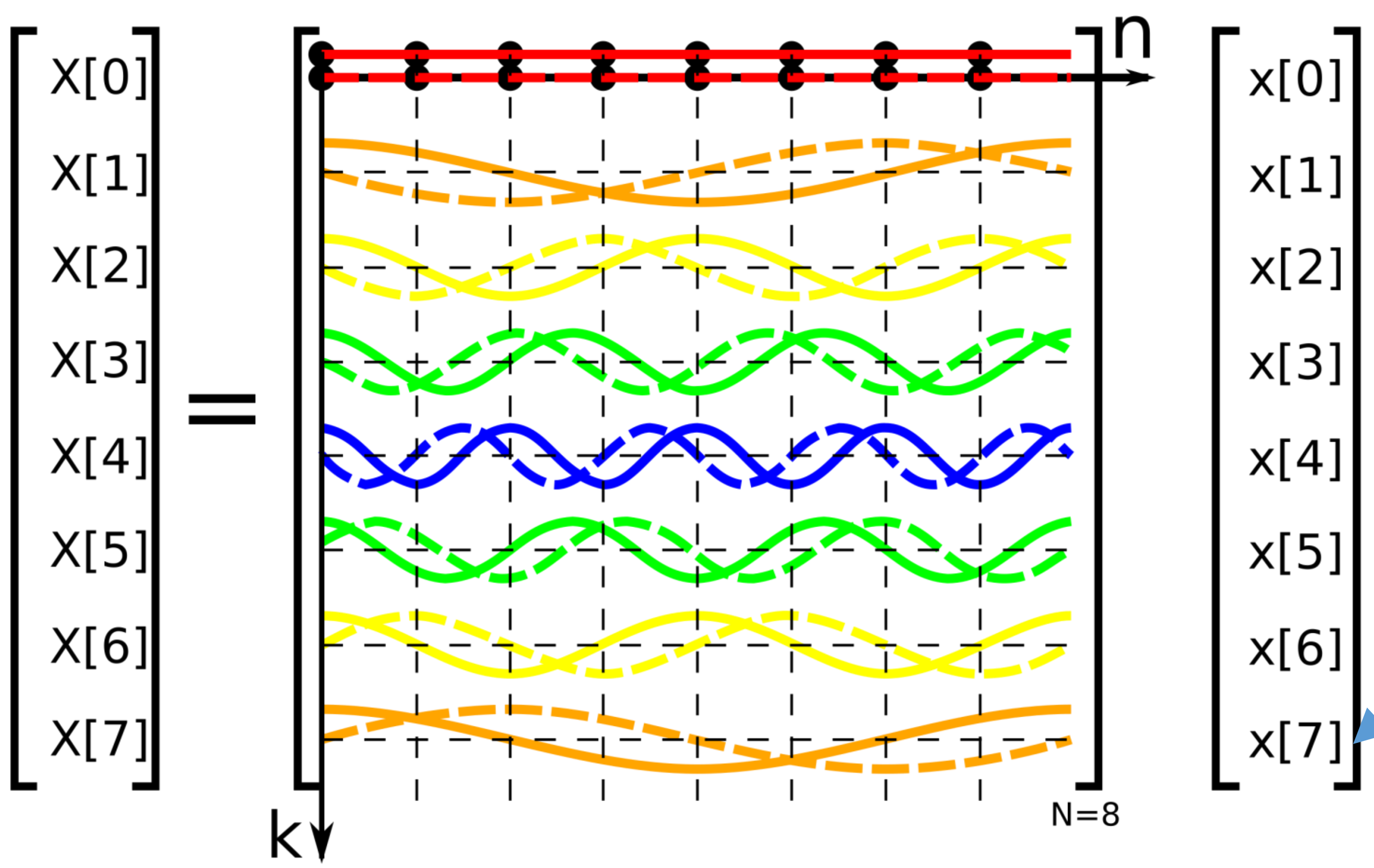
Fractional frequency +3/4

or –1/4



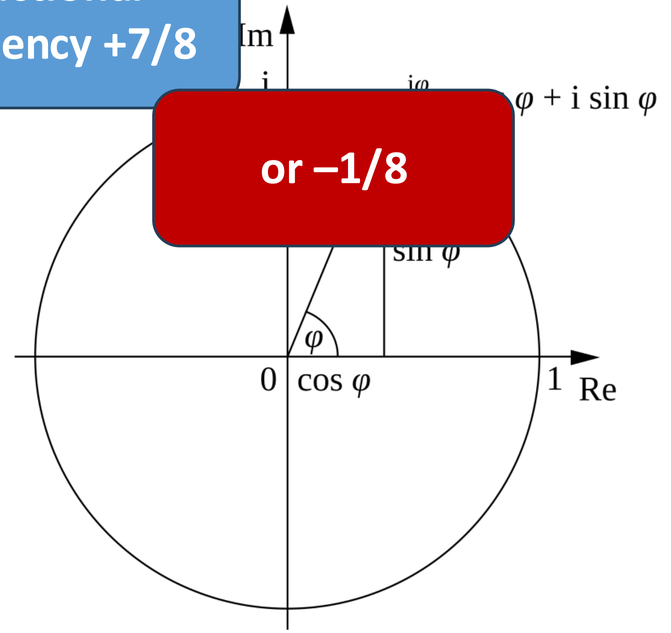
Example: 8-point DFT: Pictorial Representation



$$W = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

Fractional frequency +7/8

or –1/8



Cosine wave: Solid Line
Sine wave: Dashed line

Discrete Fourier Transformation (DFT)

- Let $x_0, \ldots, x_{n-1}$ be complex numbers. The DFT is defined by the formula,

$$X_k = \sum_{m=0}^{n-1} x_m e^{-i2\pi km/n} \qquad k = 0, \ldots, n-1,$$

  - Where $e^{i2\pi/n}$ is the primitive n^{th} root of 1

- Evaluating the above equation needs $O(n^2)$ operations

Can we reduce the complexity?

Fast Fourier Transform (FFT)

- Divide the DFT matrix recursively into smaller DFTs and then combine them
  - Based on the multiplications on complex root of unity
- **Radix-2 decimation-in-time (DIT) FFT**
  - Divide between odd and even inputs

$$X_k = \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}(2m)k} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}$$

Fast Fourier Transform (FFT)

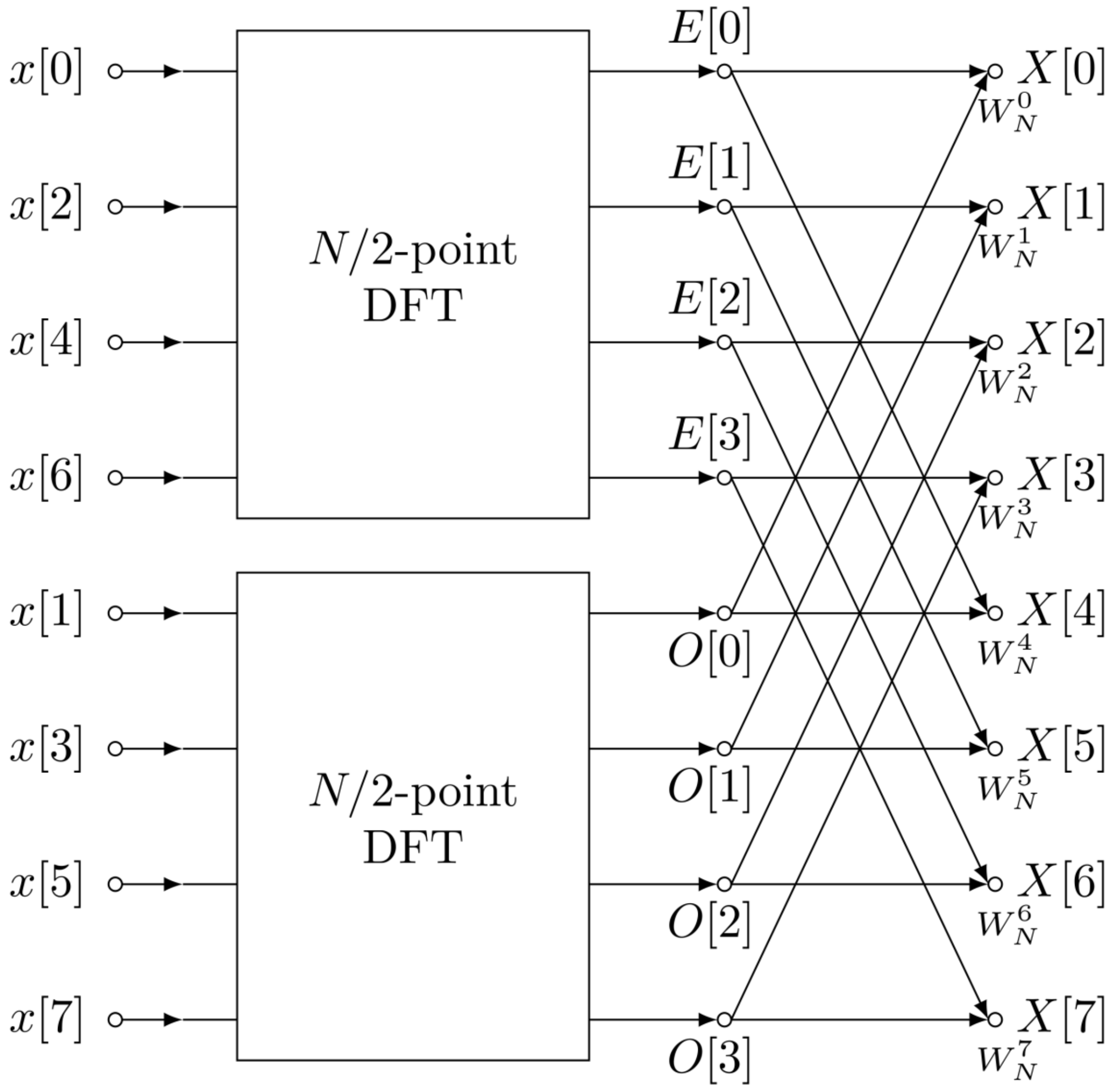
- Divide the DFT matrix recursively into smaller DFTs and then combine them
  - Based on the multiplications on complex root of unity
- **Radix-2 decimation-in-time (DIT) FFT**
  - Divide between odd and even inputs

$$X_k = \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}(2m)k} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}$$

  - By rearranging,

$$X_k = \underbrace{\sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2}mk}}_{\text{DFT of even-indexed part of } x_n} + e^{-\frac{2\pi i}{N}k} \underbrace{\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2}mk}}_{\text{DFT of odd-indexed part of } x_n} = E_k + e^{-\frac{2\pi i}{N}k} O_k \qquad \text{for } k = 0, \dots, \frac{N}{2} - 1.$$

FFT for N = 8

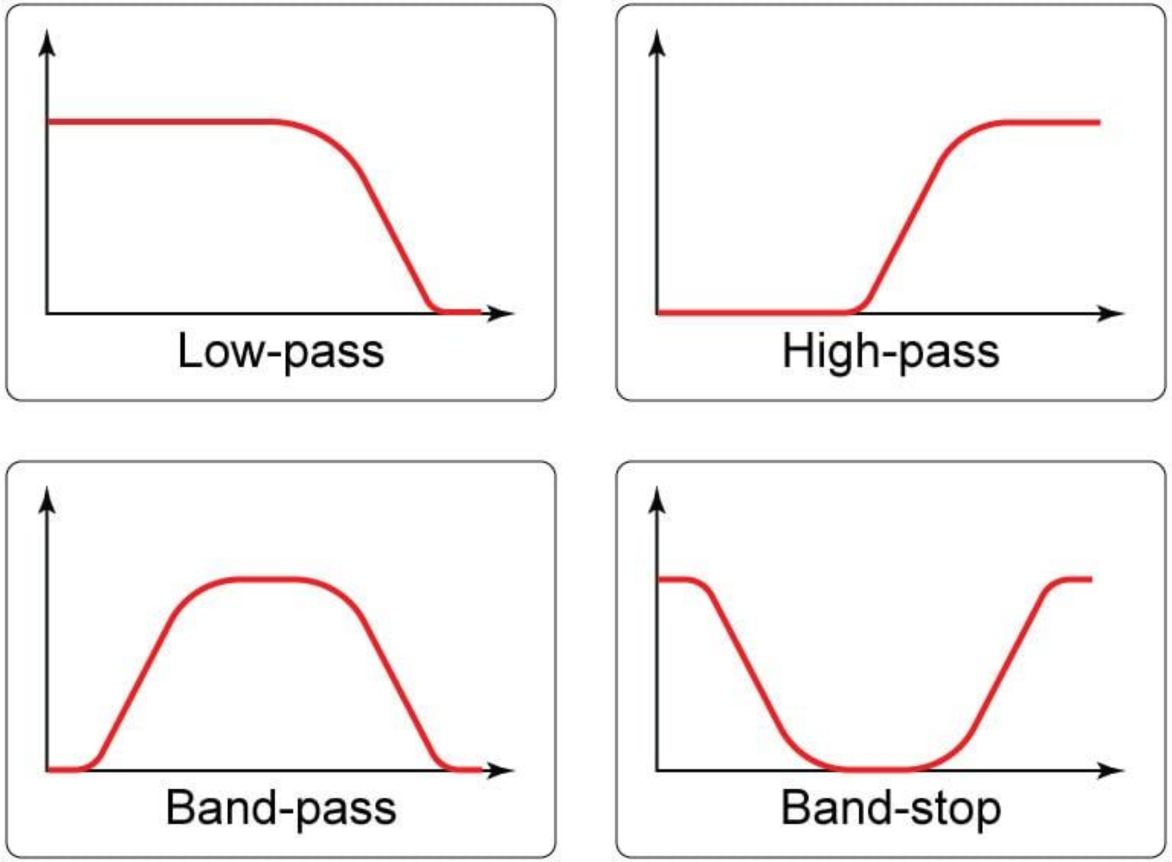


$x[0]$
$x[2]$
$x[4]$
$x[6]$
$x[1]$
$x[3]$
$x[5]$
$x[7]$
$N/2$-point DFT
$N/2$-point DFT
$E[0]$
$E[1]$
$E[2]$
$E[3]$
$O[0]$
$O[1]$
$O[2]$
$O[3]$
$X[0]$ W_N^0
$X[1]$ W_N^1
$X[2]$ W_N^2
$X[3]$ W_N^3
$X[4]$ W_N^4
$X[5]$ W_N^5
$X[6]$ W_N^6
$X[7]$ W_N^7

Application of FFT in Sensing -- Filters

- Extract a portion of the signal components

- Example: Acoustic data processing
  - You want to analyze the acoustic chirp sent from your smartphone
  - However, the sound emitted may get mixed with other environmental noises
  - Pass the received signal through a band-pass filter to extract only the components of the target frequency band



In Summary

- Signal propagation, distortion, reflection, etc., can help us sense the environment
  - Doppler analysis helps in identifying moving objects or movement patters
  - The reflection patterns (frequency components, time of flight, etc.) can be used to compute the distance of the object from the transmitter, angle of arrival, etc.
  - Phase shift can be used to identify material properties

- Fourier transform helps us to identify the frequency components in the signal
  - Helpful for signal analysis (we'll see the details later)
  - Useful for preprocessing the signals – removing unwanted signal components – low-pass/ high-pass/ bandpass filters

Happy Learning!

Some resources related to this topic