Let k, n be natural numbers with k odd. Prove that the sum $1^k + 2^k + 3^k + \dots + n^k$ is divisible by $1+2+3+4+5+\dots n$?

Offcourse ... $1+2+\dots n = \frac{n(n+1)}{2}$.

\# we need to prove that

$$1^k + 2^k + 3^k + \dots n^k \equiv 0 \pmod{\frac{n(n+1)}{2}}$$

This sign shows that the left expression has remainder as zero when $\frac{n(n+1)}{2}$ is divisor.

\# clearly if $\frac{n(n+1)}{2}$ divides the expression then $n(n+1)$ also divides it.

so, we are trying to prove that :

(factor 1) × (factor 2) | (expression)

$\because$ both the factor (n) & $(n+1)$ are coprimes; it is equivalent to proving both of;

(factor 1) | (expression) & (factor2) | (expression)

seperately.

To deal with that we will break up into cases, depending on whether n is even or odd.

Both cases are similar, I will do when n is even;

let $n = 2m$

Replace it in our previous eq^{n}s & we get;

$$1^k + 2^k + 3^k + \dots + (2m)^k \equiv 0 \pmod{m(2m+1)}$$

$\because$ m & $2m+1$ are coprimes, it's equivalent proving that:

$$1^k + 2^k + \dots + (2m)^k \equiv 0 \pmod{m} \quad \text{—(1)}$$

&

$$1^k + 2^k + \dots + (2m)^k \equiv 0 \pmod{2m+1} \quad \text{—(2)}$$

(2) $\rightarrow$ $\because$ k is odd & $2m \equiv -1 \pmod{2m+1}$

$2m-1 \equiv -2 \pmod{2m+1}$

$\vdots$

& so on

$$\therefore 1^k + 2^k + \dots (2m)^k \equiv 1^k + 2^k + \dots + m^k + (-m)^k + \dots + (-2)^k + (-1)^k \pmod{2m+1}$$

$$\Rightarrow 1^k + 2^k + \dots (2m)^k \equiv \cancel{1} + \cancel{2} + \dots \cancel{m} + (\cancel{-m}) + \dots (\cancel{-2}) + (\cancel{-1}) \equiv 0 \pmod{2m+1}.$$

$\because$ k is odd.

So, eq^n (2) is proved.

#(1) $\rightarrow$ $1^k + 2^k + 3^k + \dots + (m-1)^k + (m)^k + (m+1)^k + \dots$

$+ (2m-1)^k + (2m)^k \equiv 0 \pmod{m}$

$\because$ $\left.\begin{array}{l} m \equiv 0 \\ 2m \equiv 0 \\ m+1 \equiv 1 \\ m+2 \equiv 2 \\ \quad \vdots \\ \text{\& so on} \end{array}\right(\text{modulo } m \left.\right)$

$\therefore$ it's equivalent to ~~prove~~ write that :

$\rightarrow$ $1^k + 2^k + 3^k + \dots + (m-1)^k + (0)^k + 1^k + \dots (m-1)^k + 0^k \pmod{m}$

$\rightarrow$ $2(1^k + 2^k + 3^k + \dots + (m-1)^k) \pmod{m}$

Now we can almost do the same thing as for the (mod 2m+1) case, except there is a small hitch when m is even.

* if m is odd then we can reformulate above expression as:—

$\rightarrow$ $2(1^k + 2^k + \dots ((\frac{m-1}{2}))^k + (-\frac{(m-1)}{2})^k + \dots + (-2)^k + (-1)^k \pmod{m}$.

$\equiv 0 \pmod{m}$.

* if m is even ; let say $m = 2p$ then reformulate expression as:—

$\rightarrow$ $2(1^k + 2^k + 3^k + \dots + (m-1)^k) \pmod{m}$

$\equiv 2(1^k + 2^k + \dots + p^k + \dots + (2p-1)^k) \pmod{2p}$.

$\equiv 2(1^k + 2^k + \dots + (p-1)^k) + 2p^k + 2((p+1)^k + \dots (2p-1)^k)$

$\pmod{2p}$

$\equiv 2(1^k + 2^k + \dots + (p-1)^k) + 0 + 2(-(p-1)^k + \dots - (1)^k)$

$\pmod{2p}$

$\equiv 0 \pmod{(2p)}$

Regardless of whether n is odd or even,
we have proved that :
$1^k + 2^k + \cdots + n^k$ is divisible by $\frac{n(n+1)}{2}$ if n is even.

Similar is the proof, when n is odd.