Advanced Practical 2022/2023
Operations Research Case

Lecture: Steady-State Simulation
For an Infinite Horizon

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Steady-State Definition

- Suppose that you describe the system state at time t by X_t;
- You observe the system at time 0 ('now');
- The system is acting already a long time (from time $-\infty$);
- You might be interested in
  (i). What is the CDF $F_0(x)$ of X_0?
  (ii). What is the expectation $\mathbb{E}[X_0]$?
  (iii). More generally, suppose that $h(x)$ is a function acting on the state variable, what is $\mathbb{E}\big[h\big(X_0\big)\big]$?
- F_0 is the *steady-state distribution* of the system;
- The expectations are *steady-state performance measures*.

Looking to the Future

- You might reason as follows;
- The system starts at time 0;
- The system will act forever in the future;
- The system will be in steady-state at time $+\infty$;
- Now we are interested in
  - *(i).* The CDF $F_\infty(x)$ of X_∞;
  - *(ii).* The expectation $\mathbb{E}\left[h\left(X_\infty\right)\right]$.
- Question: what do we mean by random variable X_∞?

Limit Distribution

- Assume that $F_t(x) = \mathbb{P}(X_t \leq x)$ is a CDF for all $t \geq 0$;
- Equivalently, X_t is a proper random variable;

Definition

Assume

(a). The limit

$$F_\infty(x) = \lim_{t \to \infty} F_t(x)$$

exists for all $x \in \mathbb{R}$ (we say *pointwise*);

(b). $F_\infty(\cdot)$ is a CDF (recall the definition of a CDF!)

Then we say that the sequence of random variables $\{X_t, t \geq 0\}$ converges *weakly* or *in distribution* to a (proper!) random variable X_∞; and we denote

$$X_t \xrightarrow{\mathcal{D}} X_\infty$$

Convergence in Mean

Definition

Assume that $X_t \xrightarrow{\mathcal{D}} X_\infty$. If

$$\mathbb{E}[X_\infty] = \lim_{t\to\infty} \mathbb{E}[X_t]$$

we say that the sequence of random variables $\{X_t, t \geq 0\}$ converges *in mean* to X_∞; and we denote

$$X_t \xrightarrow{\mathcal{L}_1} X_\infty$$

Generally you need to check certain sufficient conditions to ensure connvergence in distribution and/or in mean (see the counterexamples next slides).

Counterexample I

- Define random variable X_t for $t \geq 0$ by

$$\mathbb{P}(X_t = 0) = 0.5; \quad \mathbb{P}(X_t = t) = 0.5.$$

- Hence,

$$F_t(x) = \begin{cases} 0, & x < 0; \\ 0.5, & 0 \leq x < t; \\ 1, & x \geq t. \end{cases}$$

- Easy to see that $F_\infty(x) = \lim_{t\to\infty} F_t(x)$ exists (pointwise), with

$$F_\infty(x) = \begin{cases} 0, & x < 0; \\ 0.5, & x \geq 0. \end{cases}$$

- No convergence in distribution.

Counterexample II

- Define random variable X_t for $t \geq 1$ by
$$\mathbb{P}(X_t = 0) = (t-1)/t; \quad \mathbb{P}(X_t = t) = 1/t.$$
- Hence,
$$F_t(x) = \begin{cases} 0, & x < 0; \\ (t-1)/t, & 0 \leq x < t; \\ 1, & x \geq t. \end{cases}$$
- Easy to see that $F_\infty(x) = \lim_{t\to\infty} F_t(x)$ exists (pointwise), with
$$F_\infty(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases}$$
- Conclude $X_t \xrightarrow{\mathcal{D}} X_\infty$, where $\mathbb{P}(X_\infty = 0) = 1$;
- However
$$\lim_{t\to\infty} \mathbb{E}[X_t] = \lim_{t\to\infty} 1 = 1 \neq 0 = \mathbb{E}[X_\infty].$$

Steady-State Simulation

- Suppose that in your system both (i) $X_t \xrightarrow{\mathcal{D}} X_\infty$ and (ii) $h(X_t) \xrightarrow{\mathcal{L}_1} h(X_\infty)$;
- NB: you may assume that also $h(X_t) \xrightarrow{\mathcal{D}} h(X_\infty)$
- The performance measure of interest is the steady-state expectation

$$J = \mathbb{E}\big[h(X_\infty)\big];$$

  How would you estimate J by simulation?
- Notice:
  - –> From (i): $X_\infty \overset{\mathcal{D}}{\approx} X_{\tau_0}$ for large τ_0;
  - –> From (ii): $J = \mathbb{E}\big[h(X_\infty)\big] \approx \mathbb{E}\big[h(X_{\tau_0})\big]$ for large τ_0;

A Steady-State Simulation Run

- Hence,
  - *(i).* you start simulating a run from an arbitrary initial state, for instance the empty state;
  - *(ii).* continue long enough, say until τ_0, at which the system is about in steady-state;
  - *(iii).* continue simulating until $\tau_0 + \tau$;
- Note that for all $t \geq \tau_0$ it holds that $X_t \overset{\mathcal{D}}{\approx} X_\infty$ and $\mathbb{E}\big[h(X_t)\big] \approx J$;
- Thus, define as output of the run

$$Y = \frac{1}{\tau} \int_{\tau_0}^{\tau_0+\tau} h(X_t)\, dt$$

- NB: similarly for discrete processes in which you take sums.

Replication-Deletion

- The output satisfies

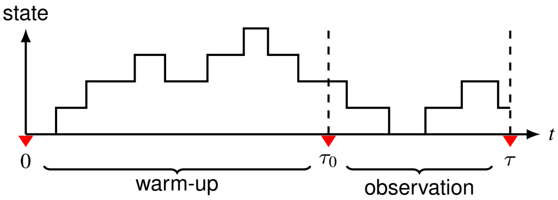
$$\mathbb{E}[Y] = \mathbb{E}\Big[\frac{1}{\tau}\int_{\tau_0}^{\tau_0+\tau} h(X_t)\,dt\Big] = \frac{1}{\tau}\int_{\tau_0}^{\tau_0+\tau} \mathbb{E}\big[h(X_t)\big]\,dt \approx J;$$

- Repeating n times independently, you get an (approximately) unbiased estimator of J by

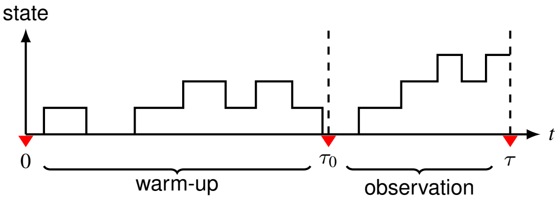
$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

- The rest (SLLN; CLT; confidence interval; accuracy; ...) is the same as in static Monte Carlo simulations.

Illustration



run 1: compute output Y_1 from state process in $[\tau_0, \tau_0 + \tau]$.



run 2: compute output Y_2 from state process in $[\tau_0, \tau_0 + \tau]$.

Discussion

- A simple method for estimating a *steady-state expectation*;
- The initial long period of a run until steady-state is called *warm-up period*, and is not used for computing output;
- The output Y of a run is computed from the *observation period*;
- Pro: iid observations which makes the statistics easy;
- Con: inefficient because each run needs the warm-up period.

More Good News

- Suppose the performance measure is a *long-run fraction or average* quantity; e.g.
  - –> the long-run fraction of time the server is available;
  - –> the long-run average cost per unit of time;
  - –> the long-run fraction of time the machine is down;
  - –> the long-run average number of particles per unit of time;
  - –> etc.
- These can be modelled by

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t h(X_s)\, ds.$$

Almost Sure Limits

- If the system state process $\{X_t, t \geq 0\}$ is *regenerative* these long-run average limits converge almost surely to the steady-state expectation (this is known as an *ergodic theorem*):

$$\lim_{t\to\infty} \underbrace{\frac{1}{t}\int_0^t h(X_s)\,ds}_{\text{random variable}} = \mathbb{E}\big[h(X_\infty)\big] \quad \text{a.s.}$$

- See your textbook on Markov chains;
- Corollary: you estimate long-run averages/fractions similarly as estimating $\mathbb{E}\big[h(X_\infty)\big]$.

The Difficulty of Steady-State Simulation

- How large should τ_0 (length of warm-up period) be before starting the observations?
- This is the *initial transient problem*;
- There are statistical techniques, but in practice you just simulate and experiment;

Steady-State Simulation Strategies

- The replication-deletion method, described in the slides above, is just one of many methods for steady-state simulation.
- The most important ones are
  - *A.* Stationary process (single run)
  - *B.* Replication-deletion method (multiple runs)
  - *C.* Batch means method (single run)
  - *D.* Regenerative method (multiple cycles)
- A. and D. are more involved obtaining variance estimates for the construction of the confidence intervals; not in this course.
- C. is an efficient alternative for B.

C. The Batch Means Method

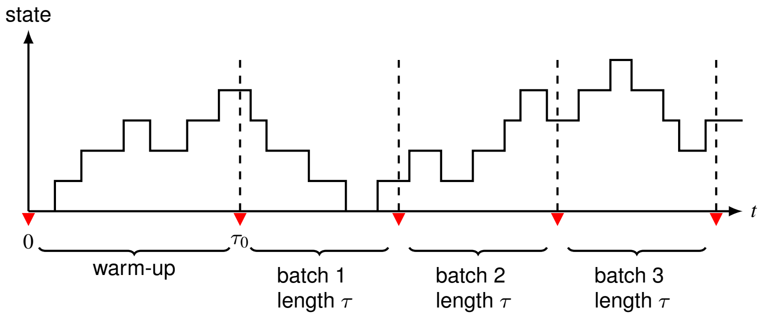
- Remedy for the inefficiency of the replication-deletion method;
  - *(i).* simulate *one long run* of length T;
  - *(ii).* first a warm-up period of length τ_0;
  - *(iii).* then split the remaining time $T - \tau_0$ in n sections (called *batches*) of length $\tau = (T - \tau_0)/n$.
- Compute the sample mean of the i-th batch ($i = 1, \ldots, n$)

$$Y_i = \frac{1}{\tau} \int_0^{\tau} h(X_{\tau_0 + (i-1)\tau + t})\, dt$$

- Final estimator is the average of these *batch means*

$$\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

Illustration



Discussion

- Typically, number of batches n is small.
- The batch length (or size) τ should be large, so that
  1. the correlations between the batch means $Y_i, i = 1, 2, \ldots$ are neglegible;
  2. each batch mean Y_i is approximately Gaussian.
- Thus the Y_i's may be considered as independent Gaussian, with mean ℓ and variance σ^2;

Confidence Interval

- The sample variance to estimate σ^2:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}_n\right)^2.$$

- Since n is small:

$$\sqrt{n}\frac{\overline{Y}_n - \ell}{S} \overset{\mathcal{D}}{\approx} t_{n-1},$$

  where t_{n-1} is the Student-t variable with $n-1$ df.

- CI: $\left(\overline{Y}_n - t_{n-1,1-\alpha/2}S/\sqrt{n},\ \overline{Y}_n + t_{n-1,1-\alpha/2}S/\sqrt{n}\right)$.