# Exercises and Solutions: week 3

## Financial Econometrics

2024-2025

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# CHAPTER 6: Multivariate GARCH models

1. Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$
Apply the VECH operation to the following matrices: $A$, $B$, $A^\top$ and $B^\top$.

**Solution:**
For the matrix $A$, we obtain that
$$\text{vech}(A) = \begin{bmatrix} a \\ c \\ d \end{bmatrix},$$
and
$$\text{vech}(A^\top) = \begin{bmatrix} a \\ b \\ d \end{bmatrix}.$$
The result for $B$ can be trivially obtained as well.

2. Let
$$A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$
Calculate $A \odot B$, $AB$, $A \odot C$ and $AC$.

**Solution:**
We obtain
$$A \odot B = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 \times 1 & 1 \times 1 \\ 0 \times 1 & 1 \times 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}.$$
$$A \odot C = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 \times 1 & 1 \times 0 \\ 0 \times 0 & 1 \times 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

3. We know that the conditional distribution of $\boldsymbol{y}_t = (y_{1t}, y_{2t})^\top$ given the past $Y^{t-1} = \{\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \dots, \}$ is $N_2(\mathbf{0}_2, \mathbf{\Sigma}_t)$, where
$$\mathbf{\Sigma}_t = \begin{bmatrix} 1.2 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$
What is the conditional expectation of $y_{1,t}$ given $y_{2,t}$, i.e. $\mathbb{E}(y_{1,t}|y_{2,t}, Y^{t-1})$.

**Solution:**
First, we revisit some properties of the bivariate normal distribution. Let two random variables $X$ and $Y$ have a joint bivariate normal distribution of the form
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \right)$$
Then, the conditional expectation and variance of $X$ given $Y$ are
$$\mathbb{E}(X|Y) = \mu_x + \rho_{xy}\frac{\sigma_x}{\sigma_y}(Y - \mu_y) = \mu_x + \frac{\sigma_{xy}}{\sigma_y^2}(Y - \mu_y),$$
$$\mathbb{V}ar(X|Y) = \sigma_x^2(1 - \rho_{xy}^2),$$
where $\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$. Therefore, we can use these properties of the bivariate normal to calculate the desired conditional expectation. In particular, we have that
$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} |Y^{t-1} \sim N\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \right),$$
and therefore we obtain
$$\mathbb{E}(y_{1,t}|y_{2,t}, Y^{t-1}) = \frac{0.5}{1.5}y_{2,t} = \frac{1}{3}y_{2,t}.$$

4. An econometrician has obtained the following estimate of the conditional covariance matrix of a DVECH model

$$\begin{bmatrix} \sigma_{1,t}^2 & \sigma_{21,t} \\ \sigma_{21,t} & \sigma_{2,t}^2 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.6 \end{bmatrix} + \begin{bmatrix} 0.7 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} \odot \begin{bmatrix} \sigma_{1,t-1}^2 & \sigma_{21,t-1} \\ \sigma_{21,t-1} & \sigma_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} 0.2 & 0.1 \\ 0.05 & 0.2 \end{bmatrix} \odot \begin{bmatrix} y_{1,t-1}^2 & y_{1,t-1}y_{2,t-1} \\ y_{1,t-1}y_{2,t-1} & y_{2,t-1}^2 \end{bmatrix}.$$

Can this result be correct? explain why.

**Solution:**
The result cannot be correct because the conditional covariance matrix $\mathbf{\Sigma}_t$ must be a symmetric matrix. This is indeed not the case for the estimation results of the DVECH model in the exercise.

5. The conditional covariance matrix $\mathbf{\Sigma}_t$ of a DVECH model is given by

$$\begin{bmatrix} \sigma_{1,t}^2 & \sigma_{21,t} \\ \sigma_{21,t} & \sigma_{2,t}^2 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.2 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.4 \end{bmatrix} \odot \begin{bmatrix} y_{1,t-1}^2 & y_{1,t-1}y_{2,t-1} \\ y_{1,t-1}y_{2,t-1} & y_{2,t-1}^2 \end{bmatrix}$$

   (a) Show that this model is a special case of a bivariate VECH model. Hint: write the model in VECH form.

   (b) Find the unconditional covariance matrix of $\boldsymbol{y}_t = (y_{1,t}, y_{2,t})^\top$.

**Solution:**

(a) A bivariate VECH(0,1) is given by

$$\begin{bmatrix} \sigma_{1,t}^2 \\ \sigma_{21,t} \\ \sigma_{2,t}^2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} y_{1,t-1}^2 \\ y_{1,t-1}y_{2,t-1} \\ y_{2,t-1}^2 \end{bmatrix}.$$

Therefore, the DVECH model in the exercise can be expressed as a bivariate VECH(0,1) with the following parameter values

$$\begin{bmatrix} \sigma_{1,t}^2 \\ \sigma_{21,t} \\ \sigma_{2,t}^2 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.2 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} y_{1,t-1}^2 \\ y_{1,t-1}y_{2,t-1} \\ y_{2,t-1}^2 \end{bmatrix}.$$

(b) We know that for a VECH(0,1) of the form

$$\text{vech}(\mathbf{\Sigma}_t) = W + A_1 + \text{vech}(\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}^\top),$$

the unconditional covariance matrix $\mathbb{V}ar(\boldsymbol{y}_t) = \mathbf{\Sigma}$ is

$$\begin{aligned} \text{vech}(\mathbf{\Sigma}) &= (I_3 - A_1)^{-1}W \\ &= \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0.9 \\ 0.2 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 0.9 \\ 0.2 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 1.43 & 0 & 0 \\ 0 & 1.11 & 0 \\ 0 & 0 & 1.67 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.2 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 1.29 \\ 0.22 \\ 0.83 \end{bmatrix}. \end{aligned}$$

6. For the DVECH model in Exercise 5, find the conditional covariance and the conditional correlation between $y_{1,t}$ and $y_{2,t}$ having observed $y_{1,t-1} = -0.5$ and $y_{2,t-1} = -1.0$.

   **Solution:**
   The conditional covariance between $y_{1,t}$ and $y_{2,t}$ is given by

   $$\sigma_{12,t} = 0.2 + 0.1 y_{1,t-1} y_{2,t-1} = 0.2 + 0.1 \times (-0.5) \times (-1.0) = 0.25.$$

   Whereas the conditional correlation is

   $$\rho_{12,t} = \frac{\sigma_{12,t}}{\sigma_{1,t}\sigma_{2,t}} = \frac{0.25}{\sqrt{0.9 \times 0.975}} = 0.27,$$

   where the conditional variances $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$ are obtained as

   $$\begin{aligned}\sigma^2_{1,t} &= 0.9 + 0.3 y^2_{1,t-1} = 0.9 + 0.3 \times (-0.5)^2 = 0.975,\\ \sigma^2_{2,t} &= 0.5 + 0.4 y^2_{2,t-1} = 0.5 + 0.4 \times (-1.0)^2 = 0.9.\end{aligned}$$

7. The conditional covariance matrix $\mathbf{\Sigma}_t$ of a BEKK model is given by

   $$\begin{bmatrix}\sigma^2_{1,t} & \sigma_{21,t}\\ \sigma_{21,t} & \sigma^2_{2,t}\end{bmatrix} = \begin{bmatrix}0.5 & 0\\ 0.1 & 0.6\end{bmatrix}\begin{bmatrix}0.5 & 0.1\\ 0 & 0.6\end{bmatrix} + \begin{bmatrix}0.5 & 0\\ 0 & 0.6\end{bmatrix}\begin{bmatrix}y^2_{1,t-1} & y_{1,t-1}y_{2,t-1}\\ y_{1,t-1}y_{2,t-1} & y^2_{2,t-1}\end{bmatrix}\begin{bmatrix}0.5 & 0\\ 0 & 0.6\end{bmatrix}$$

   Show that this model is a bivariate DVECH model. Write the model in DVECH form using the Hadamard product $\odot$ notation.

   **Solution:**
   We can proceed as follows

   $$\begin{aligned}\begin{bmatrix}\sigma^2_{1,t} & \sigma_{21,t}\\ \sigma_{21,t} & \sigma^2_{2,t}\end{bmatrix} &= \begin{bmatrix}0.5^2 & 0.1 \times 0.5\\ 0.1 \times 0.5 & 0.1^2 + 0.6^2\end{bmatrix} + \begin{bmatrix}0.5 y^2_{1,t-1} & 0.5 y_{1,t-1}y_{2,t-1}\\ 0.6 y_{1,t-1}y_{2,t-1} & 0.6 y^2_{2,t-1}\end{bmatrix}\begin{bmatrix}0.5 & 0\\ 0 & 0.6\end{bmatrix}\\ &= \begin{bmatrix}0.25 & 0.05\\ 0.05 & 0.37\end{bmatrix} + \begin{bmatrix}0.5^2 y^2_{1,t-1} & 0.5 \times 0.6 y_{1,t-1}y_{2,t-1}\\ 0.5 \times 0.6 y_{1,t-1}y_{2,t-1} & 0.6^2 y^2_{2,t-1}\end{bmatrix}\\ &= \begin{bmatrix}0.25 & 0.05\\ 0.05 & 0.37\end{bmatrix} + \begin{bmatrix}0.25 y^2_{1,t-1} & 0.3 y_{1,t-1}y_{2,t-1}\\ 0.3 y_{1,t-1}y_{2,t-1} & 0.36 y^2_{2,t-1}\end{bmatrix}\\ &= \begin{bmatrix}0.25 & 0.05\\ 0.05 & 0.37\end{bmatrix} + \begin{bmatrix}0.25 & 0.3\\ 0.3 & 0.36\end{bmatrix} \odot \begin{bmatrix}y^2_{1,t-1} & y_{1,t-1}y_{2,t-1}\\ y_{1,t-1}y_{2,t-1} & y^2_{2,t-1}\end{bmatrix}\end{aligned}$$

   This immediately shows that the model is a bivariate DVECH(0,1) and also gives the Hadamard product representation $\mathbf{\Sigma}_t = \boldsymbol{W} + \boldsymbol{A}_1 \odot \boldsymbol{y}_{t-1}\boldsymbol{y}^\top_{t-1}$.

8. For the BEKK model in Exercise 7, find the conditional covariance and the conditional correlation between $y_{1,t}$ and $y_{2,t}$ having observed $y_{1,t-1} = -0.5$ and $y_{2,t-1} = -1.0$.

   **Solution:**
   The conditional covariance between $y_{1,t}$ and $y_{2,t}$ is given by

   $$\sigma_{12,t} = 0.05 + 0.3 y_{1,t-1} y_{2,t-1} = 0.05 + 0.3 \times (-0.5) \times (-1.0) = 0.20.$$

   Whereas the conditional correlation is

   $$\rho_{12,t} = \frac{\sigma_{12,t}}{\sigma_{1,t}\sigma_{2,t}} = \frac{0.20}{\sqrt{0.312 \times 0.730}} = 0.419,$$

   where the conditional variances $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$ are obtained as

   $$\begin{aligned}\sigma^2_{1,t} &= 0.25 + 0.25 y^2_{1,t-1} = 0.25 + 0.25 \times (-0.5)^2 = 0.312,\\ \sigma^2_{2,t} &= 0.37 + 0.36 y^2_{2,t-1} = 0.37 + 0.36 \times (-1.0)^2 = 0.730.\end{aligned}$$

# CHAPTER 8: Econometric analysis with multivariate GARCH

1. Let $\boldsymbol{y}_t = (y_{1,t}, y_{2,t})^\top$ be generated by a DVECH model with conditional covariance matrix $\boldsymbol{\Sigma}_t$ given by

$$\begin{bmatrix} \sigma^2_{1,t} & \sigma_{21,t} \\ \sigma_{21,t} & \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.2 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.4 \end{bmatrix} \odot \begin{bmatrix} y^2_{1,t-1} & y_{1,t-1}y_{2,t-1} \\ y_{1,t-1}y_{2,t-1} & y^2_{2,t-1} \end{bmatrix}$$

Consider the portfolio $y_{pt} = 0.5y_{1,t} + 0.5y_{2,t}$. Find

(a) The conditional variance of the portfolio $y_{p,t}$, i.e. $\sigma^2_{p,t}$.

(b) The conditional variance of the portfolio at time $T+1$, having observed $y_{1,T} = 0.9$ and $y_{2,T} = 1.1$.

(c) The conditional $\alpha$-VaR of the portfolio at time $T+1$ for $\alpha = 0.05$.

(d) The conditional probability that $y_{p,T+1} > 0.6$.

**Solution:**

(a) The conditional variance of the portfolio is

$$\begin{aligned} \mathbb{V}ar(y_{p,t}|Y^{t-1}) &= 0.5^2\mathbb{V}ar(y_{1,t}|Y^{t-1}) + 0.5^2\mathbb{V}ar(y_{2,t}|Y^{t-1}) + 2 \times 0.5 \times 0.5\mathbb{C}ov(y_{1,t}, y_{2,t}|Y^{t-1}) \\ &= 0.25\sigma^2_{1,t} + 0.25\sigma^2_{2,t} + 0.5\sigma_{12,t}. \end{aligned}$$

(b) The conditional variance of the portfolio at time $T+1$ depends on $\sigma^2_{1,T+1}$, $\sigma^2_{2,T+1}$ and $\sigma_{12,T+1}$. Therefore, we fist obtain these quantities

$$\begin{aligned} \sigma^2_{1,T+1} &= 0.9 + 0.3y^2_{1,T} = 1.14, \\ \sigma^2_{2,T+1} &= 0.5 + 0.4y^2_{2,T} = 0.98, \\ \sigma_{12,T+1} &= 0.2 + 0.1y_{1,T}y_{2,T} = 0.3. \end{aligned}$$

Finally, the conditional variance is

$$\sigma^2_{p,T+1} = 0.25\sigma^2_{1,T+1} + 0.25\sigma^2_{2,T+1} + 0.5\sigma_{12,T+1} = 0.68.$$

(c) The $\alpha$-VaR$_{T+1}$ of level $\alpha = 0.05$ is

$$\alpha\text{-VaR}_{T+1} = z_\alpha \sigma_{p,T+1} = -1.64 \times \sqrt{0.68} = -1.35.$$

(d) The desired probability is

$$\begin{aligned} P(y_{p,T+1} > 0.6|Y^T) &= P\left(\frac{y_{p,T+1}}{\sigma_{p,T+1}} > \frac{0.6}{\sigma_{p,T+1}}|Y^T\right) \\ &= 1 - \Phi\left(\frac{0.6}{\sigma_{p,T+1}}\right) \\ &= 1 - \Phi\left(\frac{0.6}{\sqrt{0.68}}\right) \\ &= 1 - \Phi(0.73). \end{aligned}$$

2. Repeat points (a), (b), (c) and (d) of the previous exercise but now assuming that $\boldsymbol{y}_t = (y_{1t}, y_{2t})^\top$ is generated by a bivariate CCC model with $\rho_{12} = 0.5$ and the following conditional variances

$$\sigma^2_{1,t} = 0.1 + 0.3y^2_{1,t-1} + 0.2\sigma^2_{1,t-1},$$

$$\sigma^2_{2,t} = 0.1 + 0.2y^2_{2,t-1} + 0.3\sigma^2_{2,t-1}.$$

For points (b), (c) and (d) consider $\sigma^2_{1,T} = 0.8$ and $\sigma^2_{2,T} = 0.7$.

**Solution:**

(a) The expression of the conditional variance of the portfolio is equivalent to the one of the previous exercise, i.e. $\sigma^2_{p,t} = 0.25\sigma^2_{1,t} + 0.25\sigma^2_{2,t} + 0.5\sigma_{12,t}$.

(b) The conditional variance of the portfolio at time $T+1$ is given by

$$\sigma^2_{p,T+1} = 0.25\sigma^2_{1,T+1} + 0.25\sigma^2_{2,T+1} + 0.5\sigma_{12,T+1}.$$

Therefore we first need to obtain $\sigma^2_{1,T+1}$, $\sigma^2_{2,T+1}$ and $\sigma_{12,T+1}$. In particular, the conditional variances are obtained as

$$\sigma^2_{1,T+1} = 0.1 + 0.3y^2_{1,T} + 0.2\sigma^2_{1,T} = 0.1 + 0.3 \times 0.9^2 + 0.2 \times 0.8 = 0.503,$$
$$\sigma^2_{2,T+1} = 0.1 + 0.2y^2_{2,T} + 0.3\sigma^2_{2,T} = 0.1 + 0.2 \times 1.1^2 + 0.3 \times 0.7 = 0.552,$$

whereas the conditional covariance of the CCC model is obtained as

$$\sigma^2_{12,T+1} = \rho_{12}\sigma_{1,T+1}\sigma_{2,T+1} = 0.5 \times \sqrt{0.503 \times 0.552} = 0.263.$$

As a result we obtain that the conditional variance of the portfolio is given by

$$\sigma^2_{p,T+1} = 0.25 \times 0.503 + 0.25 \times 0.552 + 0.5 \times 0.263 = 0.395.$$

(c) The 5% level conditional VaR at time $T+1$ is

$$\alpha\text{-VaR}_{p,T+1} = z_{0.05}\sigma_{p,T+1} = (-1.64) \times \sqrt{0.395} = -1.03$$

(d) The conditional probability is given by

$$P\left(y_{p,T+1} > 0.6|Y^T\right) = P\left(\frac{y_{p,T+1}}{\sigma_{p,T+1}} > \frac{0.6}{\sigma_{p,T+1}}|Y^T\right) = 1-\Phi\left(\frac{0.6}{\sigma_{p,T+1}}\right) = 1-\Phi\left(\frac{0.6}{\sqrt{0.395}}\right) = 1-\Phi\left(0.955\right),$$

where $\Phi(\cdot)$ is the cdf of a standard normal distribution.

3. We have 2 stocks with log-returns denoted by $\boldsymbol{y}_t = (y_{1t}, y_{2t})^\top$. An appropriate multivariate GARCH model is estimated and we obtain the following conditional covariance matrix $\boldsymbol{\Sigma}_{T+1}$ at time $T+1$

$$\begin{bmatrix} \sigma^2_{1,T+1} & \sigma_{21,T+1} \\ \sigma_{21,T+1} & \sigma^2_{2,T+1} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}.$$

Furthermore, the expected log-returns are $\boldsymbol{\mu}_{T+1} = (\mu_1, \mu_2)^\top = (0.01, 0.01)^\top$. Two econometricians (a) and (b) propose the following weights for the portfolio

(a) The first econometrician proposes $k_{1,T+1} = k_{2,T+1} = 0.5$;

(b) The second econometrician proposes $k_{1,T+1} = 0.4$ and $k_{2,T+1} = 0.6$;

Which of them is proposing the best portfolio at time $T+1$ in terms of Sharpe Ratio? Obtain the Sharpe Ratio for each of the two portfolios.

**Solution:**

(a) The return of the first portfolio is $y_{p,T+1} = 0.5y_{1,T+1} + 0.5y_{2,T+1}$. The conditional expectation and variance are

$$\mu_{p,T+1} = 0.5\mu_{1,T+1} + 0.5\mu_{2,T+1} = 0.5 \times 0.01 + 0.5 \times 0.01 = 0.01.$$
$$\sigma^2_{p,T+1} = 0.25\sigma^2_{1,T+1} + 0.25\sigma^2_{2,T+1} + 0.5\sigma_{12,T+1} = 0.175.$$

Therefore, the Sharpe Ratio is

$$S_{p,T+1} = \frac{\mu_{p,T+1}}{\sigma_{p,T+1}} = \frac{0.01}{\sqrt{0.175}} = 0.0239.$$

(b) The return of the second portfolio is $y_{p,T+1} = 0.4y_{1,T+1} + 0.6y_{2,T+1}$. The conditional expectation and variance are

$$\mu_{p,T+1} = 0.4\mu_{1,T+1} + 0.6\mu_{2,T+1} = 0.4 \times 0.01 + 0.6 \times 0.01 = 0.01.$$
$$\sigma^2_{p,T+1} = 0.16\sigma^2_{1,T+1} + 0.36\sigma^2_{2,T+1} + 0.48\sigma_{12,T+1} = 0.168.$$

Therefore, the Sharpe Ratio is

$$S_{p,T+1} = \frac{\mu_{p,T+1}}{\sigma_{p,T+1}} = \frac{0.01}{\sqrt{0.168}} = 0.0244.$$

We conclude that the second portfolio is better in terms of Sharpe Ratio.