Financial Econometrics

- Week 1, Lecture 2 -

ARCH and GARCH Models

VU Econometrics and Data Science

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Today's class

ARCH models

ARCH models

Objective: study class of **models capable of describing financial returns**, i.e. time-series that:

1. Exhibit time-varying conditional volatility (volatility clustering);
2. Have heavy tails in the unconditional distribution.

ARCH: Autoregressive Conditional Heteroskedasticity

Heteroskedasticity: refers to the variance not being constant

Homoeskedasticity: constant fixed variance

ARCH(1) model: definition (i)

Let $\{y_1, y_2, y_3, \dots\}$ be a sequence of financial returns

Definition: The *Autoregressive Conditional Heteroschedaticity model of order 1*, or **ARCH(1)** model, is given by

$$y_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \omega + \alpha_1 y_{t-1}^2$$

- σ_t^2 is the conditional volatility at time t (σ_t^2 is not observed);
- ε_t are iid Gaussian innovations: $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{NID}(0,1)$;
- $\omega > 0$ and $\alpha_1 \geq 0$ are unknown parameters that determine the behavior of the conditional volatility;
- If $\omega > 0$ and $\alpha_1 \geq 0$, then $\sigma_t^2 > 0$ for all t;

ARCH(1) model: definition (ii)

Definition: The **ARCH(1)** model is

$$y_t = \sigma_t \varepsilon_t \tag{1}$$

$$\sigma_t^2 = \omega + \alpha_1 y_{t-1}^2 \tag{2}$$

- Equation (1) is called the *observation-equation*;
- Equation (2) is called the *updating-equation*;
- The ARCH(1) model is *observation-driven*:
  - $\Rightarrow$ past observations are used to update the values of σ_t^2;
- The ARCH(1) captures time variation in the variance and describes *volatility clustering* typical of stock returns;

ARCH(1) model: definition (iii)

Definition: The **ARCH(1)** model is

$$y_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \omega + \alpha_1 y_{t-1}^2$$

Intuition for specification of σ_t^2: y_{t-1}^2 can be seen as an estimate of the variance at time $t-1$:

- When y_{t-1}^2 is large, then σ_t^2 also tends to be large ($\alpha_1 > 0$);
- When σ_t^2 is large, then y_t^2 is more likely to be large;
- Large y_{t-1}^2 produce large y_t^2 (volatility clustering).

Example: simulated ARCH(1) returns

- α_1 determines the impact of y_{t-1}^2 on the conditional variance σ_t^2.

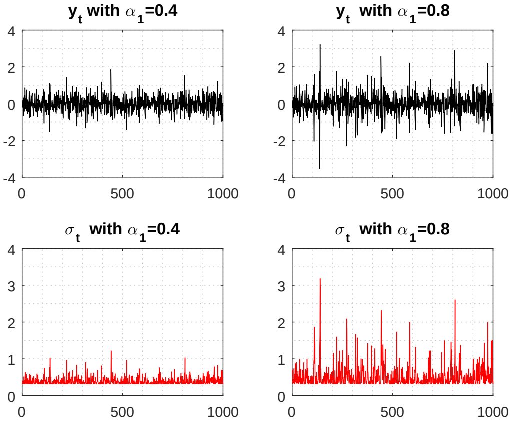
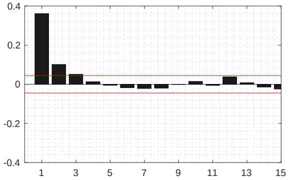


Figure: $(\omega, \alpha_1) = (0.1, 0.4)$ (left) and $(\omega, \alpha_1) = (0.1, 0.8)$ (right).

Example: ACF of simulated ARCH(1)

Important:

- The larger α_1 the stronger the autocorrelation in squared returns
- Returns are uncorrelated regardless of α_1



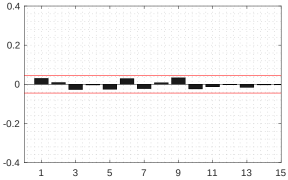


Figure: Sample ACF for squared returns (left) and returns (right). Data simulated with $T = 2000$ and $(\omega, \alpha_1) = (0.1, 0.4)$.

Conditional distribution

Question:

What is the conditional distribution of returns generated by an ARCH(1) model?

Theorem (conditional distribution)

The conditional distribution of y_t *given the past* $Y^{t-1} = \{y_{t-1}, y_{t-2}, \dots\}$ *is normal with mean* $\mathbb{E}(y_t|Y^{t-1}) = 0$ *and variance* $\mathbb{V}ar(y_t|Y^{t-1}) = \sigma_t^2$, *namely* $\boldsymbol{y_t|Y^{t-1} \sim N(0, \sigma_t^2)}$.

Important remark: time-varying *conditional* variance does not imply time-varying *unconditional* variance

Conditional distribution: proof

Proof: The conditional mean is obtained as

$$\mathbb{E}(y_t|Y^{t-1}) = \mathbb{E}(\sigma_t\varepsilon_t|Y^{t-1}) = \sigma_t\mathbb{E}(\varepsilon_t|Y^{t-1}) = \sigma_t\mathbb{E}(\varepsilon_t) = \sigma_t \cdot 0 = 0$$

- second equality holds as σ_t is constant conditional on Y^{t-1}
- third equality holds since ε_t is independent of Y^{t-1}

The conditional variance is obtained as

$$\begin{aligned}\mathbb{V}ar(y_t|Y^{t-1}) &= \mathbb{V}ar(\sigma_t\epsilon_t|Y^{t-1}) = \mathbb{E}(\sigma_t^2\epsilon_t^2|Y^{t-1}) \\ &= \sigma_t^2\mathbb{E}(\epsilon_t^2|Y^{t-1}) = \sigma_t^2\mathbb{E}(\epsilon_t^2) = \sigma_t^2 \cdot 1 = \sigma_t^2\end{aligned}$$

- third equality holds as σ_t^2 is constant conditional on Y^{t-1}
- fourth equality holds since ε_t is independent of Y^{t-1}

Conditional distribution: proof (continued)

Proof: (continued)

The conditional distribution of $y_t = \sigma_t \varepsilon_t$ given Y^{t-1} is Gaussian

$$y_t | Y^{t-1} \sim \sigma_t \varepsilon_t | Y^{t-1} \sim N(0, \sigma_t^2)$$

- conditional on Y^{t-1}, the factor σ_t is a constant and $\varepsilon_t \sim N(0,1)$.
- product of normal with constant is normal:

$$c + d \times N(a, b) = N\Big(c + da \ , \ d^2 b\Big)$$

END OF PROOF.

Conditional distribution: application

Conditional distribution: allows us to calculate the probability of extreme events (risk) conditional on recent stock behavior

Example: Suppose that you have \$1000 in IBM stocks. Let the **log returns** of IBM stocks satisfy the following ARCH(1) dynamics:

$$y_t = \sigma_t \epsilon_t \quad , \quad \sigma_t^2 = 0.01 + 0.54 y_{t-1}^2 .$$

Question: Given that $y_{t-1} = 0.03$, what is the probability that you'll lose more than \$100 at time t? what is the probability that you'll gain more than \$200? What if $y_{t-1} = -0.21$?

Answer: $y_t | y_{t-1} = 0.03 \sim N(0, 0.01 + 0.54 \times 0.03^2) \sim N(0, 0.0105)$

Hence, $P(y_t < -0.1 | y_{t-1} = 0.03) \approx 0.1635 \quad (\approx 0.293 \text{ for } y_{t-1} = -0.21)$

and $P(y_t > 0.2 | y_{t-1} = 0.03) \approx 0.0249 \quad (\approx 0.138 \text{ for } y_{t-1} = -0.21)$

Stochastic properties of ARCH(1)

In the following, we will see that the **ARCH(1) model can describe several empirical features of log-returns**

If the log-returns $\{y_t\}_{t\in\mathbb{Z}}$ are generated by an ARCH(1) model, then

- Unconditional mean of y_t is zero;
- Log-returns $\{y_t\}_{t\in\mathbb{Z}}$ are uncorrelated;
- Unconditional variance of y_t is constant (if $\alpha_1 < 1$);
- $\{y_t\}_{t\in\mathbb{Z}}$ is *white noise*;
- Squared log-returns $\{y_t^2\}_{t\in\mathbb{Z}}$ are autocorrelated;
- The unconditional distribution of y_t has fat tails.

Stochastic properties: unconditional mean

Theorem (unconditional mean)

The returns $\{y_t\}_{t\in\mathbb{Z}}$ generated by an ARCH(1) model have unconditional mean zero, namely $\mathbb{E}(y_t) = 0$.

Proof: We know that the conditional mean $\mathbb{E}(y_t|Y^{t-1})$ is equal to zero. Therefore we obtain that

$$\mathbb{E}(y_t) = \mathbb{E}(\mathbb{E}(y_t|Y^{t-1})) = \mathbb{E}(0) = 0,$$

by an application of the law of total expectation.

END OF PROOF.

Stochastic properties: uncorrelated returns

Theorem (uncorrelated returns)

Returns $\{y_t\}_{t\in\mathbb{Z}}$ generated by an ARCH(1) model have zero autocovariance at any lag $\mathbb{C}ov(y_t, y_{t-l}) = 0$; i.e. they are uncorrelated.

Proof: The autocovariance function for any $l > 0$ is given by

$$\begin{aligned}\mathbb{C}ov(y_t, y_{t-l}) &= \mathbb{E}(y_t y_{t-l}) = \mathbb{E}(\mathbb{E}(y_t y_{t-l}|Y^{t-1})) \\ &= \mathbb{E}(y_{t-l}\mathbb{E}(y_t|Y^{t-1})) = \mathbb{E}(y_{t-l} \cdot 0) = 0\end{aligned}$$

- The second equality holds by the law of total expectation;
- The third equality holds since y_{t-l} is constant for any $l > 0$ conditional on Y^{t-1};
- Zero autocovariance implies zero autocorrelation at any lag.

END OF PROOF.

Stochastic properties: AR representation (i)

Theorem (AR representation)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an ARCH(1) model. Then $\{y_t^2\}_{t\in\mathbb{Z}}$ follows an AR(1) model

$$y_t^2 = \omega + \alpha_1 y_{t-1}^2 + \eta_t$$

where $\{\eta_t\}_{t\in\mathbb{Z}}$ is a white noise sequence.

Proof: Define new error term η_t as $\eta_t = y_t^2 - \sigma_t^2 \sim WN$.
Substitute $\sigma_t^2 = y_t^2 - \eta_t$ in the updating equation:

$$y_t^2 - \eta_t = \omega + \alpha_1 y_{t-1}^2 \quad \Leftrightarrow \quad y_t^2 = \omega + \alpha_1 y_{t-1}^2 + \eta_t.$$

Conclude that $\{y_t^2\}_{t\in\mathbb{Z}}$ follows an AR(1) process.

END OF PROOF.

Stochastic properties: AR representation (ii)

The AR representation tells us that **squared log-returns** $\{y_t^2\}_{t\in\mathbb{Z}}$ generated from an ARCH(1) **are autocorrelated**.

Question: *why is the AR representation useful?*

Answer: you can use your knowledge of time-series econometrics:

1. For ACF-based model selection
2. For obtaining stationarity conditions
3. For obtaining the unconditional variance

Stochastic properties: unconditional variance

Theorem (unconditional variance)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an ARCH(1) model. If $\alpha_1 < 1$, then the unconditional variance of y_t is time-invariant and, in particular, given by $\mathbb{V}ar(y_t) = \omega/(1-\alpha_1)$.

Proof: First, we note that $\mathbb{V}\text{ar}(y_t) = \mathbb{E}(y_t^2)$.

Second, unfold the AR(1) representation

$$y_t^2 = \sum_{i=0}^{\infty} \alpha_1^i \omega + \sum_{i=0}^{\infty} \alpha_1^i \eta_{t-i} = \omega/(1-\alpha_1) + \sum_{i=0}^{\infty} \alpha_1^i \eta_{t-i}.$$

Finally, since $\mathbb{E}(\eta_t) = 0$ for any t we can conclude that (if $\alpha_1 < 1$)

$$\mathbb{E}(y_t^2) = \omega/(1-\alpha_1) + \sum_{i=0}^{\infty} \alpha_1^i \mathbb{E}(\eta_{t-i}) = \omega/(1-\alpha_1).$$

END OF PROOF.

Stochastic properties: stationarity

Until now: we established that returns generated by an ARCH(1) model:

1. have zero unconditional mean,
2. have fixed unconditional variance (if $\alpha_1 < 1$),
3. are uncorrelated at any lag.

Conclusion: $\{y_t\}_{t \in \mathbb{Z}}$ is a *weakly stationary* ***white noise*** sequence

Corollary (stationary white noise)

Let $\{y_t\}_{t \in \mathbb{Z}}$ be generated by an ARCH(1) model with $\alpha_1 < 1$. Then, $\{y_t\}_{t \in \mathbb{Z}}$ is a weakly stationary white noise sequence.

Stochastic properties: fat tails (i)

Note: ARCH(1) model can also generate the fat tails (e.g. large kurtosis) observed in stock returns!

Important: conditional distribution $y_t|Y^{t-1}$ is Gaussian

However: unconditional distribution of ARCH(1) returns is non-Gaussian

Fat tails: unconditional distribution of ARCH(1) returns has kurtosis > 3

$$\text{Kurtosis}(y_t) = \frac{\mathbb{E}(y_t^4)}{\mathbb{E}(y_t^2)^2}$$

Stochastic properties: fat tails (ii)

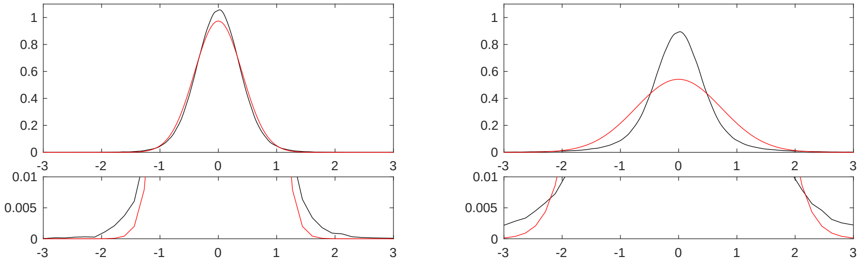


Figure: **Black curve**: unconditional density simulated from ARCH model with $(\omega, \alpha) = (0.1, 0.4)$ (left graph) and $(\omega, \alpha) = (0.1, 0.8)$ (right graph). **Red curve**: Gaussian density. The two bottom figures provide a 'zoom in' on the tails of each density.

Stochastic properties: fat tails (iii)

Note:

- No closed form expression for the unconditional pdf of y_t.
- However the kurtosis can be derived!

Theorem

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an ARCH(1) model with $\alpha_1 < \frac{1}{\sqrt{3}}$. Then the kurtosis of y_t is given by

$$k_u = \frac{\mathbb{E}(y_t^4)}{\mathbb{E}(y_t^2)^2} = \frac{3(1-\alpha_1^2)}{1-3\alpha_1^2} > 3.$$

The Kurtosis of ARCH(1) is larger than 3 (heavy tails)!

The ARCH(q) model (i)

Problem: Often, the conditional variance of stock returns shows strong persistence over time.

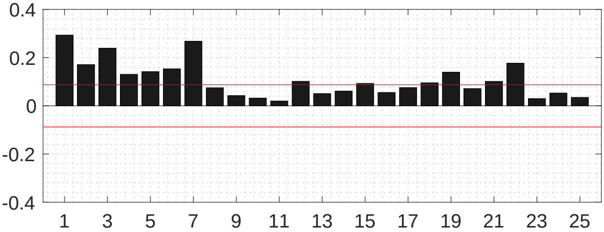


Figure: ACF of daily squared log-returns of the S&P 500 stock index

Note: ACF of squared log-returns does not decay exponentially.

Hence: ARCH(1) is not appropriate!

The ARCH(q) model (ii)

Solution: include more lags of y_t^2 in the updating equation!

Definition: the ***ARCH(2)*** *model* is given by

$$y_t = \sigma_t \epsilon_t \ , \qquad \{\epsilon_t\}_{t \in \mathbb{Z}} \sim \text{NID}(0,1) \ ,$$
$$\sigma_t^2 = \omega + \alpha_1 y_{t-1}^2 + \alpha_2 y_{t-2}^2,$$

where $\omega > 0$, $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ ensures $\sigma_t^2 > 0$.

Definition: the ***ARCH(q)*** *model* is given by

$$y_t = \sigma_t \epsilon_t \ , \qquad \{\epsilon_t\}_{t \in \mathbb{Z}} \sim \text{NID}(0,1) \ ,$$
$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2,$$

where $\omega > 0$, $\alpha_1 \geq 0, \ldots, \alpha_p \geq 0$ ensures $\sigma_t^2 > 0$.

Stochastic properties

Again: we can establish several interesting stochastic properties for the ARCH(q) model

Lemma (some stochastic properties)

ARCH(q) returns have the following properties:

- $\mathbb{E}(y_t|Y^{t-1}) = 0;$
- $\mathbb{V}ar(y_t|Y^{t-1}) = \sigma_t^2;$
- $y_t|Y^{t-1} \sim N(0, \sigma_t^2);$
- *have zero unconditional mean* $\mathbb{E}(y_t) = 0.$
- *are uncorrelated over time* $\mathbb{C}ov(y_t, y_{t-l}) = 0$ *for* $l \neq 0.$

Proof: the same as for ARCH(1) model!

Stochastic properties: AR representation

Note: squared returns of ARCH(q) process admit an AR(q) representation

Theorem

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an ARCH(q) model. Then $\{y_t^2\}_{t\in\mathbb{Z}}$ follows an AR(q) model

$$y_t^2 = \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \eta_t$$

where $\{\eta_t\}_{t\in\mathbb{Z}}$ is a white noise sequence.

Proof: The proof of this theorem is left as an exercise.

Conclusion: ARCH(q) is capable of generating arbitrary dependence structure for first q lags

Stochastic properties: unconditional variance

Theorem (unconditional variance)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an ARCH(q) model. If $\sum_{i=1}^{q}\alpha_i < 1$, then the unconditional variance of y_t is time-invariant and, in particular, given by $\mathbb{V}ar(y_t) = \omega/(1-\sum_{i=1}^{q}\alpha_i)$.

Proof: First note that $\mathbb{V}\text{ar}(y_t) = \mathbb{E}(y_t^2)$.

Next, use the AR(q) representation to conclude that if $\sum_{i=1}^{q}\alpha_i < 1$, then

$$\mathbb{E}(y_t^2) = \omega/(1-\sum_{i=1}^{q}\alpha_i).$$

END OF PROOF.

Stochastic properties: stationarity

Until now: we established that returns generated by an ARCH(q) model:

1. have zero unconditional mean,
2. have fixed unconditional variance (if $\sum_{i=1}^{q} \alpha_i < 1$),
3. are uncorrelated at any lag.

Conclusion: $\{y_t\}_{t\in\mathbb{Z}}$ is a *weakly stationary* ***white noise*** sequence

Corollary (stationary white noise)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an ARCH(q) model with $\sum_{i=1}^{q} \alpha_i < 1$. Then, $\{y_t\}_{t\in\mathbb{Z}}$ is a weakly stationary white noise sequence.

GARCH models

GARCH models

Important: ARCH models with several lags are useful!

Problem: large q leads to many parameters to estimate

Solution: add lags of σ_t^2 in updating equation

Advantage: strong dependence can be described parsimoniously!

GARCH(1,1) model

Definition: the *generalized autoregressive conditional heteroskedasticity model* of order (1,1), or **GARCH(1,1)** is given by

$$y_t = \sigma_t \epsilon_t \ ,$$
$$\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 y_{t-1}^2$$

- $\{\epsilon_t\}_{t \in \mathbb{Z}}$ is an $NID(0,1)$ sequence
- $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ ensure that $\sigma_t^2 > 0$ for all t

Important: The GARCH(1,1) model can easily generate clusters of volatility!

Example: simulated GARCH(1,1)

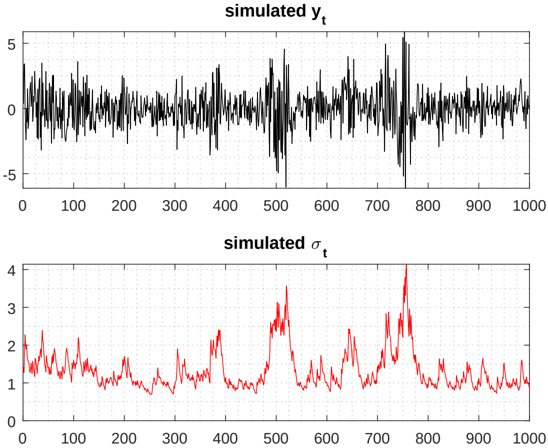


Figure: GARCH(1,1) returns generated with $(\omega, \beta_1, \alpha_1) = (0.1, 0.75, 0.2)$.

Example: simulated GARCH(1,1) ACF

Note: GARCH(1,1) can generate very strong dependence in squared returns when β is large

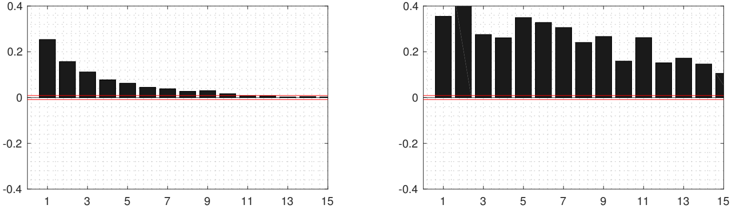


Figure: Sample ACF of squared returns y_t^2 generated by a GARCH(1,1) model with parameters $(\omega, \alpha_1, \beta_1) = (0.1, 0.2, 0.5)$ [left figure] and $(\omega, \alpha_1, \beta_1) = (0.1, 0.2, 0.78)$ [right figure].

Stochastic properties

Again: we can establish several interesting stochastic properties for the GARCH(1,1) model

Lemma (some stochastic properties)

GARCH(1,1) returns have the following properties:

- $\mathbb{E}(y_t|Y^{t-1}) = 0$;
- $\mathbb{V}ar(y_t|Y^{t-1}) = \sigma_t^2$;
- $y_t|Y^{t-1} \sim N(0, \sigma_t^2)$;
- *have zero unconditional mean* $\mathbb{E}(y_t) = 0$.
- *are uncorrelated over time* $\mathbb{C}ov(y_t, y_{t-l}) = 0$ *for* $l \neq 0$.

Proof: the same as for ARCH(1) model!

Stochastic properties: ARMA representation

Lemma (ARMA representation)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by a GARCH(1,1) model. Then $\{y_t^2\}_{t\in\mathbb{Z}}$ admits an ARMA(1,1) representation

$$y_t^2 = \omega + (\alpha_1 + \beta_1)y_{t-1}^2 + \eta_t - \beta_1\eta_{t-1}$$

where $\{\eta_t\}_{t\in\mathbb{Z}}$ is a white noise process.

Proof:

Define $\eta_t = y_t^2 - \sigma_t^2 \sim WN$

Plug in $\sigma_t^2 = y_t^2 - \eta_t$ and $\sigma_{t-1}^2 = y_{t-1}^2 - \eta_{t-1}$ in updating equation

$$y_t^2 = \omega + (\alpha_1 + \beta_1)y_{t-1}^2 + \eta_t - \beta_1\eta_{t-1},$$

END OF PROOF.

Stochastic properties: unconditional variance

Note: ARMA(1,1) representation of y_t^2 is useful for obtaining the unconditional variance of y_t

Theorem (unconditional variance)

The returns $\{y_t\}_{t\in\mathbb{Z}}$ generated by an GARCH(1,1) model with $\alpha_1 + \beta_1 < 1$ have a time-invariant unconditional variance given by $\mathbb{V}ar(y_t) = \omega/(1 - \beta_1 - \alpha_1)$.

Proof:

We know that $\mathbb{V}\mathrm{ar}(y_t) = \mathbb{E}(y_t^2)$.

Hence by the ARMA representation we get (if $\alpha_1 + \beta_1 < 1$)

$$\mathbb{E}(y_t^2) = \omega/(1 - \beta_1 - \alpha_1).$$

END OF PROOF.

Stochastic properties: stationarity

Until now: we established that returns generated by an GARCH(1,1) model:

1. have zero unconditional mean,
2. have fixed unconditional variance (if $\alpha_1 + \beta_1 < 1$),
3. are uncorrelated at any lag.

Conclusion: $\{y_t\}_{t\in\mathbb{Z}}$ is a *weakly stationary* ***white noise*** sequence

Corollary (stationary white noise)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an GARCH(1,1) model with $\alpha_1 + \beta_1 < 1$. Then, $\{y_t\}_{t\in\mathbb{Z}}$ is a weakly stationary white noise sequence.

Stochastic properties: variance and tails

Note: variance and tails increase as $1-\beta_1-\alpha_1$ approaches zero

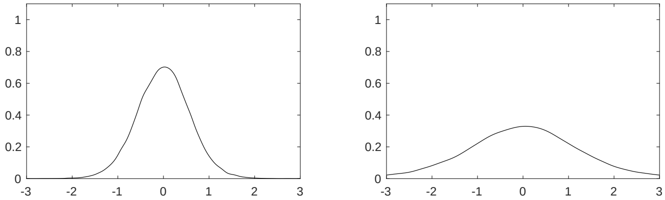


Figure: Unconditional sample density of y_t generated by a GARCH(1,1) model with parameters $(\omega, \alpha_1, \beta_1) = (0.1, 0.2, 0.5)$ [left figure] and $(\omega, \alpha_1, \beta_1) = (0.1, 0.2, 0.75)$ [right figure].

Stochastic properties: ARCH(∞)

Important: GARCH(1,1) model is able to capture high persistence in the conditional variance!

Note: this is highlighted by noting that the GARCH(1,1) can be re-written as an ARCH(∞) model with some constraints on the parameters.

Unfold GARCH(1,1):

$$\begin{aligned}\sigma_t^2 &= \omega + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ &= \omega + \beta_1 \omega + \alpha_1 y_{t-1}^2 + \beta_1 \alpha_1 y_{t-2}^2 + \beta_1^2 \sigma_{t-2}^2 \\ &= \frac{\omega}{(1-\beta_1)} + \alpha_1 \sum_{i=0}^{\infty} \beta_1^i y_{t-1-i}^2,\end{aligned}$$

which is an ARCH(∞).

The GARCH(p,q) model

Question: can we describe additional temporal dynamics?

Answer: Yes! With a GARCH(p,q) model

Definition: A ***GARCH(p,q) model*** is given by

$$y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 \tag{3}$$

where $\omega > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$ are parameters, $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is an $NID(0,1)$ sequence.

Stochastic properties

Again: we can establish several interesting stochastic properties for the GARCH(p,q) model

Lemma (some stochastic properties)

GARCH(p,q) returns have the following properties:

- $\mathbb{E}(y_t|Y^{t-1}) = 0$;
- $\mathbb{V}ar(y_t|Y^{t-1}) = \sigma_t^2$;
- $y_t|Y^{t-1} \sim N(0, \sigma_t^2)$;
- *have zero unconditional mean* $\mathbb{E}(y_t) = 0$.
- *are uncorrelated over time* $\mathbb{C}ov(y_t, y_{t-l}) = 0$ *for* $l \neq 0$.

Proof: the same as for ARCH(1) model!

Stochastic properties: ARMA representation

Lemma (ARMA representation)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by a GARCH(p,q) model. Then $\{y_t^2\}_{t\in\mathbb{Z}}$ admits an ARMA$(\max\{q,p\},p)$ representation

$$y_t^2 = \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{i=1}^{p} \beta_i y_{t-i}^2 + \eta_t - \sum_{i=1}^{p} \beta_i \eta_{t-i}$$

where $\{\eta_t\}_{t\in\mathbb{Z}}$ is a white noise process.

Proof: Define $\eta_t = y_t^2 - \sigma_t^2 \sim$WN.

Plug in $\sigma_t^2 = y_t^2 - \eta_t$ and $\sigma_{t-1}^2 = y_{t-1}^2 - \eta_{t-1}$ in the updating eq

$$y_t^2 = \omega + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{i=1}^{p} \beta_i y_{t-i}^2 + \eta_t - \sum_{i=1}^{p} \beta_i \eta_{t-i},$$

This is an ARMA$(\max\{q,p\},p)$ process! **END OF PROOF.**

Stochastic properties: stationarity

The unconditional variance is $\mathbb{V}ar(y_t) = \omega/(1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i)$

Theorem (unconditional variance)

The returns $\{y_t\}_{t\in\mathbb{Z}}$ generated by a GARCH(p,q) model with $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_j < 1$ have a time-invariant unconditional variance given by $\mathbb{V}ar(y_t) = \omega/(1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i)$.

$GARCH$(p,q) returns are stationary if $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1$!

Corollary (stationary white noise)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by a $GARCH(p,q)$ model satisfying $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1$. Then $\{y_t\}_{t\in\mathbb{Z}}$ is weakly stationary white noise sequence.

Summary on ARCH/GARCH models

- **ARCH and GARCH models** are capable of explaining several features of stock returns such as:
  - *White noise* behavior.
  - Volatility clustering.
  - Autocorrelation in squared returns.
  - Heavy tails in the unconditional distribution.
- The ARCH(q) model allows a more complex dependence structure that is more realistic than the ARCH(1).
- GARCH models are able to describe high persistency in a more parsimonious way (*the GARCH(1,1) is an ARCH(∞) but with only two parameters α_1 and β_1*).

Simulating GARCH with R

Simulating GARCH models with R (i)

Question: How can we generate data from a GARCH(1,1) model?

Answer: Simple! `Simulate_GARCH.R`

First: Define sample size `n` and parameter values ω, α_1 and β_1.

```
n <- 1000
omega <- 0.1
alpha <- 0.2
beta <-  0.75
```

Second: generate `n` errors form $\mathrm{N}(0, 1)$

```
epsilon <- rnorm(n)
```

Simulating GARCH models with R (ii)

Next: define vectors of zeros to contain our simulated data

```
sig2 <- rep(0,n)
x <- rep(0,n)
```

Finally: simulate data using a *for loop*

```
sig2[1] <- omega/(1-alpha-beta)
x[1] <- sqrt(sig2[1]) * epsilon[1]

for(t in 2:n){
  sig2[t] <- omega + alpha * x[t-1]^2 + beta * sig2[t-1]
  x[t] <- sqrt(sig2[t]) * epsilon[t]
}
```

Note: we first set the initial value σ_1^2

A reasonable option is: $\omega/(1 - \alpha_1 - \beta_1)$.

Simulating GARCH models with R (iii)

The full R code is given by

```
n <- 1000
omega <- 0.1
alpha <- 0.2
beta <-  0.75
epsilon <- rnorm(n)
sig2 <- rep(0,n)
x <- rep(0,n)
sig2[1] <- omega/(1-alpha-beta)
x[1] <- sqrt(sig2[1]) * epsilon[1]
for(t in 2:n){
  sig2[t] <- omega + alpha * x[t-1]^2 + beta * sig2[t-1]
  x[t] <- sqrt(sig2[t]) * epsilon[t]
}
```