Financial Econometrics

- Week 5, Lecture 1 -

Stochastic Volatility Models

VU Econometrics and Data Science

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Today's class

1. The Stochastic Volatility model
   - Model specification
   - Stochastic properties
   - Simulate SV with R
2. The multivariate SV model
   - Model specification
   - Simulate multivariate SV with R

The Stochastic Volatility model

Parameter-driven models (i)

Until now: we worked with observation-driven models of conditional variances and conditional covariances (GARCH models).

Univariate case: we considered models of the form

$$y_t = \sigma_t \varepsilon_t,$$

where the time-varying volatility σ_t^2 was specified as a function of past observations $\{y_{t-1}, y_{t-2}, \dots\}$

Simple example: ARCH(1) model $\sigma_t^2 = \omega + \alpha_1 y_{t-1}^2$

Recall: GARCH model are *observation-driven* since volatility σ_t^2 is a given constant when we condition on the past observed data Y^{t-1}.

Parameter-driven models (ii)

Today: we introduce the class of *parameter-driven* for time-varying variance; **Stochastic Volatility** (SV) models.

SV model is *parameter-driven* because the volatility σ_t^2 evolves exogenously according to its own dynamic equation:

- Not determined by the past observed data Y^{t-1}.
- σ_t^2 is random even if we condition on past observations Y^{t-1}.

Problem: estimation of static parameter is more challenging!

- No analytic expression for the likelihood function.
- Tomorrow, we learn how to estimate SV by *indirect inference*.

The Stochastic Volatility model (i)

Observation equation of SV: (*the same as for the GARCH*)

$$y_t = \sigma_t \varepsilon_t, \quad \{\varepsilon_t\}_{t \in \mathbb{Z}} \sim NID(0,1).$$

Updating equation of SV: (*different from the GARCH*)

$$\sigma_t^2 = \exp(f_t), \quad f_t = \omega + \beta f_{t-1} + \eta_t,$$

where $\{\eta_t\}_{t \in \mathbb{Z}}$ is $NID(0, \sigma_\eta^2)$ and independent of $\{\varepsilon_t\}_{t \in \mathbb{Z}}$.

Exponential link function: ensures that σ_t^2 is positive.

Note: The *updating equation* is also called *transition equation* in SV model.

The Stochastic Volatility model (ii)

Important: The SV model can describe **volatility clustering**, time-varying conditional variance and autocorrelation in squared log-returns.

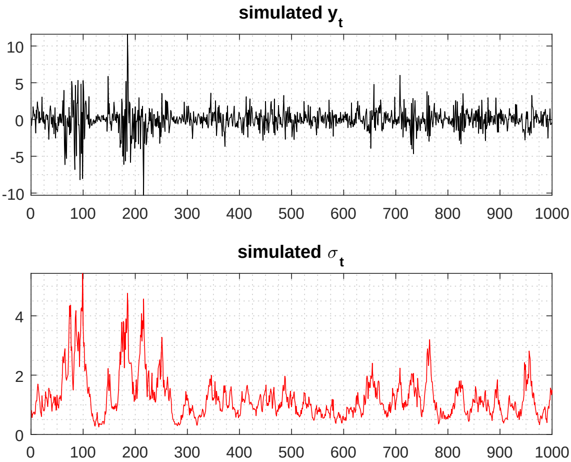
$$y_t = \sigma_t \varepsilon_t,$$

$$\sigma_t^2 = \exp(f_t), \quad f_t = \omega + \beta f_{t-1} + \eta_t.$$

Idea (updating equation): If f_t is large then the volatility of y_t is large as well. Furthermore, if $\beta > 0$, it is likely that also f_{t+1} will be large and thus the volatility of y_{t+1} as well.

Important: σ_t^2 is not a constant given the past Y^{t-1}.

Series generated by SV model



ACF of an SV model

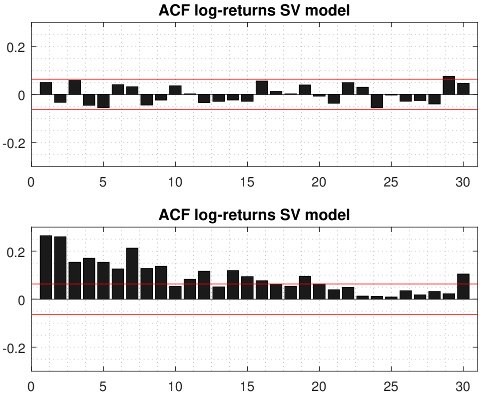


Figure: Sample ACF of y_t (first plot) and Sample ACF of y_t^2 (second plot). The series is generated from an SV model.

SV vs GARCH

Similar: ARCH/GARCH models and SV model are both able to model the volatility clustering observed in financial returns.

Different: Volatility $\sigma_t^2 = \exp(f_t)$ is specified as an unobserved process and not as a function of past observations Y^{t-1}.

Tradition in econometrics:

1. Parameter-driven model is an acceptable DGP.
   - It is seen as an agnostic way of specifying unobserved dynamics.
2. Observation-driven model is a filtering technique.
   - It is seen as making a statement about the exact updating mechanism which should be unknown.

Stochastic properties of f_t

Properties of SV model: requires that we first understand the properties of the unobserved process $\{f_t\}_{t\in\mathbb{Z}}$.

Note: f_t follows a simple AR(1) process ($f_t = \omega + \beta f_{t-1} + \eta_t$).

Simple: recall your intro to time-series courses!

Theorem (stochastic properties of f_t)

When $|\beta| < 1$, the unobserved random sequence $\{f_t\}_{t\in\mathbb{Z}}$ is a weakly stationary AR(1) process with the following properties:

- *The unconditional mean is $\mu_f = \mathbb{E}(f_t) = \omega/(1-\beta)$.*
- *The unconditional variance is $\sigma_f^2 = \mathbb{V}ar(f_t) = \sigma_\eta^2/(1-\beta^2)$.*
- *The unconditional distribution is $f_t \sim N(\mu_f, \sigma_f^2)$.*
- *The autocorrelation function is $\rho_f(l) = \mathbb{C}orr(f_t, f_{t-l}) = \beta^l$.*

Stochastic properties: mean and autocorrelation (i)

Theorem (autocorrelation and conditional mean)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an SV model with $|\beta| < 1$, then y_t is uncorrelated, $\mathbb{C}ov(y_t, y_{t-l}) = 0$ for $l > 0$, and the conditional mean of y_t given the past Y^{t-1} is equal to zero, i.e. $\mathbb{E}(y_t|Y^{t-1}) = 0$.

Proof:

$$\begin{aligned}
\mathbb{E}(y_t|Y^{t-1}) &= \mathbb{E}(\sigma_t\varepsilon_t|Y^{t-1}) \\
&= \mathbb{E}(\sigma_t|Y^{t-1})\mathbb{E}(\varepsilon_t|Y^{t-1}) \qquad (\text{since } \sigma_t \perp \varepsilon_t\ ) \\
&= \mathbb{E}(\sigma_t|Y^{t-1}) \times \mathbb{E}(\varepsilon_t) \qquad (\text{because } \varepsilon_t \perp Y^{t-1}) \\
&\mathbb{E}(\sigma_t|Y^{t-1}) \times 0 = 0
\end{aligned}$$

Stochastic properties: mean and autocorrelation (ii)

Theorem (autocorrelation and conditional mean)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an SV model with $|\beta| < 1$, then y_t is uncorrelated, $\mathbb{C}ov(y_t, y_{t-l}) = 0$ for $l > 0$, and the conditional mean of y_t given the past Y^{t-1} is equal to zero, i.e. $\mathbb{E}(y_t|Y^{t-1}) = 0$.

Proof: (continued)

$$
\begin{aligned}
\mathbb{C}ov(y_t, y_{t-l}) &= \mathbb{E}(y_t y_{t-l}) \\
&= \mathbb{E}(\mathbb{E}(y_t y_{t-l}|Y^{t-1})) \qquad \text{(law of total expectation)} \\
&= \mathbb{E}(y_{t-l}\mathbb{E}(y_t|Y^{t-1})) \qquad (\ y_{t-l} \text{ is constant given } Y^{t-1}) \\
&= \mathbb{E}(y_{t-l} \times 0) \qquad (\text{because } \mathbb{E}(y_t|Y^{t-1}) = 0) \\
&= 0 \qquad \square
\end{aligned}
$$

Stochastic properties: conditional variance

Question: how about the conditional variance?

The conditional variance $\mathbb{V}ar(y_t|Y^{t-1})$ is **time-varying**.

However:

1. $\sigma_t^2 = \exp(f_t)$ is not the conditional variance $\mathbb{V}ar(y_t|Y^{t-1})$.
2. There is no closed form expression for $\mathbb{V}ar(y_t|Y^{t-1})$.

Remark (conditional variance)

*The conditional variance of y_t given Y^{t-1}, i.e. $\mathbb{V}ar(y_t|Y^{t-1})$, is **time-varying** but there is no close form expression available.*

Stochastic properties: unconditional variance (i)

Log-Normal is useful to obtain unconditional variance of SV model:

Log-Normal distribution: is a distribution defined on the basis of the normal distribution.

IF X is normal with mean μ and variance σ^2, i.e. $X \sim N(\mu, \sigma^2)$. **THEN** the variable $Y = \exp(X)$ is a log-normal random variable with parameters μ and σ^2, i.e. $Y \sim \log\text{-}N(\mu, \sigma^2)$.

Note: The mean and the variance of a log-normal random variable Y are

$$\mathbb{E}(Y) = \exp(\mu + \sigma^2/2),$$
$$\mathbb{V}ar(Y) = (\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2).$$

Stochastic properties: unconditional variance (ii)

Theorem (unconditional variance)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an SV model with $|\beta| < 1$, then the unconditional variance of y_t is

$$\mathbb{V}ar(y_t) = \exp\left(\frac{\omega}{1-\beta} + \frac{\sigma_\eta^2}{2(1-\beta^2)}\right).$$

Proof: The unconditional distribution of f_t is normal with mean $\mu_f = \omega/(1-\beta)$ and variance $\sigma_f^2 = \sigma_\eta^2/(1-\beta^2)$

$$f_t \sim N(\mu_f, \sigma_f^2)$$

Hence, the unconditional distribution of $\sigma_t^2 = \exp(f_t)$ is log-normal

$$\sigma_t^2 \sim \log\text{-}N(\mu_f, \sigma_f^2)$$

Stochastic properties: unconditional variance (iii)

Theorem (unconditional variance)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an SV model with $|\beta| < 1$, then the unconditional variance of y_t is

$$\mathbb{V}ar(y_t) = \exp\left(\frac{\omega}{1-\beta} + \frac{\sigma_\eta^2}{2(1-\beta^2)}\right).$$

Proof: (continued)

$$\begin{aligned}
\mathbb{V}ar(y_t) &= \mathbb{E}(y_t^2) = \mathbb{E}(\sigma_t^2\varepsilon_t^2) \\
&= \mathbb{E}(\sigma_t^2)\mathbb{E}(\varepsilon_t^2) \qquad (\text{since } \sigma_t^2 \perp \varepsilon_t^2) \\
&= \mathbb{E}(\sigma_t^2) \times 1 = \mathbb{E}(\sigma_t^2) = \exp(\mu_f + \sigma_f^2/2) \quad (\text{because } \sigma_t^2 \sim \log\text{-}N)
\end{aligned}$$

As a result, $\mathbb{V}ar(y_t) = \exp\left(\frac{\omega}{1-\beta} + \frac{\sigma_\eta^2}{2(1-\beta^2)}\right)$ □

Stochastic properties: stationarity

We have seen that an SV sequence $\{y_t\}$ with $|\beta| < 1$ has:

- Uncoditional mean equal to zero;
- Autocorrelation function equal to zero at any lag;
- Unconditional variance constant over time;

The SV is in fact a weakly stationary White Noise process.

Theorem (stationarity)

*Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an SV model with $|\beta| < 1$, then $\{y_t\}_{t\in\mathbb{Z}}$ is a weakly stationary **white noise** sequence.*

Stochastic properties: kurtosis (i)

Note: the unconditional distribution of returns y_t generated by the SV model is not normal!

In particular: the unconditional distribution has fatter tails than the normal (kurtosis> 3)

Theorem (kurtosis)

Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by an SV model with $|\beta| < 1$, then the Kurtosis of y_t is given by

$$k_u = \frac{\mathbb{E}(y_t^4)}{\mathbb{E}(y_t^2)^2} = 3\exp\left(\frac{\sigma_\eta^2}{1-\beta^2}\right).$$

Therefore $k_u > 3$ as long as $\sigma_\eta^2 > 0$.

Stochastic properties: kurtosis (ii)

Proof: First note that $\sigma_t^4 = \exp(2f_t)$.

The unconditional distribution of $2f_t$ is normal with mean $2\mu_f$ and variance $4\sigma_f^2$, i.e. $f_t \sim N(2\mu_f, 4\sigma_f^2)$.

Therefore, the unconditional distribution of $\sigma_t^4 = \exp(2f_t)$ is log-normal $\sigma_t^2 \sim \text{log-}N(2\mu_f, 4\sigma_f^2)$.

Knowing this we obtain that

$$\begin{aligned}
\mathbb{E}(y_t^4) &= \mathbb{E}(\sigma_t^4 \varepsilon_t^4) \\
&= \mathbb{E}(\sigma_t^4)\mathbb{E}(\varepsilon_t^4) \qquad (\sigma_t^4 \perp \varepsilon_t^4) \\
&= \mathbb{E}(\sigma_t^4) \times 3 \qquad (\varepsilon_t \sim N(0,1)) \\
&= 3\mathbb{E}(\sigma_t^4) \\
&= 3\exp(2\mu_f + 2\sigma_f^2) \qquad (\sigma_t \sim \log\text{-}N)
\end{aligned}$$

Stochastic properties: kurtosis (iii)

Proof: (Continued)

Recall that $\mathbb{E}(y_t^2) = \exp(\mu_f + \sigma_f^2/2)$ and thus $\mathbb{E}(y_t^2)^2 = \exp(2\mu_f + \sigma_f^2)$

As a result

$$k_u = \frac{\mathbb{E}(y_t^4)}{\mathbb{E}(y_t^2)^2} = \frac{3\exp(2\mu_f + 2\sigma_f^2)}{\exp(2\mu_f + \sigma_f^2)} = 3\exp(\sigma_f^2)$$

Therefore we conclude that

$$k_u = \frac{\mathbb{E}(y_t^4)}{\mathbb{E}(y_t^2)^2} = 3\exp(\sigma_f^2) = 3\exp\left(\frac{\sigma_\eta^2}{1-\beta^2}\right) \qquad \square$$

Extensions: the SV-ARMA(p,q) model

Important: temporal dynamics can be generalized

SV-ARMA(p,q) model:

$$y_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \exp(f_t),$$
$$f_t = \omega + \sum_{i=1}^{p} \beta_i f_{t-i} + \eta_t + \sum_{i=1}^{q} \alpha_i \eta_{t-i},$$

where $\varepsilon_t \sim NID(0,1)$ and $\eta_t \sim NID(0, \sigma_\eta^2)$.

Properties of SV-ARMA(p,q): obtained in the same way as before using results from introductory time-series

Problems estimating an SV model by ML (i)

Recall: for GARCH models, estimation was easy!

1. The log-likelihood function could be written by summing up conditional log-densities;
2. The conditional distributions were all Normal $y_t|Y^{t-1} \sim N(0, \sigma_t^2)$;
3. This happened because σ_t^2 is a constant conditional on Y^{t-1}.

Problem: for SV models, estimation is difficult!

1. The stochastic volatility σ_t^2 is not a constant given the past Y^{t-1};
2. The conditional pdf of $y_t|Y^{t-1}$ is intractable.

Problems estimating an SV model by ML (ii)

Note: log-likelihood function of an SV model is given by

$$L(\theta; y_1, \ldots, y_T) = \log\left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{t=1}^{T} p(y_t|f_t) p(f_1, \ldots, f_T; \theta) df_1 \ldots df_T\right)$$

Problem 1: This integral cannot be solved in closed form!

Problem 2: Numerical methods to solve integrals are not practically applicable because integral above is high dimensional (dimension is equal to sample size T).

Possible estimation methods: Quasi ML (Kalman filter) - Simulated ML - Indirect Inference - GMM - Bayesian MCMC.

Next Lecture: We are going to learn and apply Indirect Inference.

Simulate from an SV model with R (i)

Question: How can we simulate data from an SV model using R?

Answer: Code in file `Simulate_SV.R`

First step: Choose sample size `n` and parameter values ω, β, σ_η^2

```
n <- 2500

omega <- 0
beta <- 0.95
sig2f <- 0.3
```

Simulate from an SV model with R (ii)

Second: Generate random innovations $\{\varepsilon_t\}_{t=1}^{T}$ and $\{\eta_t\}_{t=1}^{T}$

```
epsilon <- rnorm(n)
eta <- sqrt(sig2f)*rnorm(n)
```

Next: Define vectors `x` and `f`

```
f <- rep(0,n)
x <- rep(0,n)
```

Additionally: Generate the initial value for `f` and the first observation y_1

```
f[1] <- omega/(1-beta) + sqrt(sig2f/(1-beta^2))*rnorm(1)
x[1] <- exp(f[1]/2) * epsilon[1]
```

Simulate from an SV model with R (iii)

Finally: Use for loop to obtain the simulated series

```
for(t in 2:n){

f[t] <- omega + beta * f[t-1] + eta[t]
x[t] <- exp(f[t]/2) * epsilon[t]

}
```

Note: We could have simulated the processes f_t separately first since it only depends on the sequence of errors $\{\eta_t\}$!

The multivariate SV model

Multivariate Stochastic Volatility model

As mentioned in week 3: we are very often interested in modeling multiple time series $\boldsymbol{y}_t = (y_{1t}, \ldots, y_{nt})^\top$!

Multivariate GARCH models: were used for dynamic portfolio optimization

Question: is there a multivariate extension of the SV model?

Answer: Yes. Several. But we consider one in particular that we will call the *multivariate stochastic volatility* (MSV) model.

Multivariate SV: observation equation (i)

MSV model: features an observation equation similar to multivariate GARCH models

$$\boldsymbol{y}_t = \boldsymbol{\Sigma}_t^{1/2} \boldsymbol{\varepsilon}_t, \quad \{\boldsymbol{\varepsilon}_t\}_{t \in \mathbb{Z}} \sim NID_n(\boldsymbol{0}, \boldsymbol{R}).$$

However: $\boldsymbol{\Sigma}_t^{1/2}$ and $\boldsymbol{\varepsilon}_t$ differ from Multivariate GARCH

- $\boldsymbol{\Sigma}_t$ is a diagonal matrix, containing no cross-terms

$$\boldsymbol{\Sigma}_t = \text{diag}\{\exp(\boldsymbol{f}_t)\},$$

  where $\boldsymbol{f}_t$ is a vector-valued unobserved process.

- $\boldsymbol{\Sigma}_t$ is *not* the conditional covariance matrix of $\boldsymbol{y}_t$ given Y^{t-1}
  - First, the matrix $\boldsymbol{\Sigma}_t$ contains only diagonal elements.
  - Second, $\boldsymbol{\Sigma}_t$ is not constant conditioning on Y^{t-1}.

Multivariate SV: observation equation (ii)

MSV model: features an observation equation similar to multivariate GARCH models

$$\boldsymbol{y}_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t, \quad \{\boldsymbol{\varepsilon}_t\}_{t\in\mathbb{Z}} \sim NID_n(\boldsymbol{0}, \boldsymbol{R}).$$

However: $\boldsymbol{\Sigma}_t^{1/2}$ and $\boldsymbol{\varepsilon}_t$ differ from Multivariate GARCH

- Elements of $\boldsymbol{\varepsilon}_t$ are allowed to be correlated

$$\boldsymbol{\varepsilon}_t \sim N_n(\boldsymbol{0}, \boldsymbol{R}).$$

- The covariance matrix of $\boldsymbol{\varepsilon}_t$ is no longer the identity matrix.
- $\boldsymbol{R}$ defines the covariance structure of $\boldsymbol{\varepsilon}_t$.
- $\boldsymbol{R}$ is normalized as a correlation matrix.

Multivariate SV: observation equation (iii)

Some intuition: if $\boldsymbol{\Sigma}_t$ is given (not random), then...

1. The diagonal elements of $\boldsymbol{\Sigma}_t$ are the variances of the vector of returns $\boldsymbol{y}_t$.
2. $\boldsymbol{R}$ corresponds to the correlation matrix of the returns.

Reasoning: for a constant $\boldsymbol{\Sigma}_t$, we would have

$$\mathbb{V}\mathrm{ar}(\boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t) = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{R}\boldsymbol{\Sigma}_t^{1/2}$$

where $\boldsymbol{R}$ is then the correlation matrix because any covariance matrix $\boldsymbol{C}$ can be decomposed as

$$\boldsymbol{C} = \boldsymbol{V}\boldsymbol{R}\boldsymbol{V}$$

where $\boldsymbol{V}$ is a matrix that contains the standard deviations on the diagonal, and $\boldsymbol{R}$ is the correlation matrix.

Multivariate SV: updating equation

MSV updating equation: is a direct extension of the univariate SV model

$$\boldsymbol{\Sigma}_t = \mathrm{diag}\{\exp(\boldsymbol{f}_t)\},$$

$$\boldsymbol{f}_{t+1} = \boldsymbol{\omega} + \boldsymbol{\beta} \odot \boldsymbol{f}_t + \boldsymbol{\eta}_t, \qquad \{\boldsymbol{\eta}_t\}_{t\in\mathbb{Z}} \sim NID_n(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\eta}}),$$

where $\boldsymbol{w}$ and $\boldsymbol{\beta}$ are n-dimensional vectors of parameters and $\boldsymbol{\Sigma}_{\boldsymbol{\eta}}$ $n \times n$ a covariance matrix.

Multivariate SV: full model

Taking all elements together: the n-dimensional multivariate SV model is given by

$$
\begin{aligned}
\boldsymbol{y}_t &= \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t, \\
\boldsymbol{\Sigma}_t &= \operatorname{diag}\{\exp(\boldsymbol{f}_t)\}, \\
\boldsymbol{f}_{t+1} &= \boldsymbol{\omega} + \boldsymbol{\beta} \odot \boldsymbol{f}_t + \boldsymbol{\eta}_t, \\
\{\boldsymbol{\varepsilon}_t\}_{t\in\mathbb{Z}} &\sim NID_n(\boldsymbol{0}, \boldsymbol{R}), \\
\{\boldsymbol{\eta}_t\}_{t\in\mathbb{Z}} &\sim NID_n(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{\eta}}),
\end{aligned}
$$

where $\{\boldsymbol{\varepsilon}_t\}_{t\in\mathbb{Z}}$ is independent of $\{\boldsymbol{\eta}_t\}_{t\in\mathbb{Z}}$, $\boldsymbol{\Sigma}_t$ is a diagonal matrix, $\boldsymbol{R}$ is a correlation matrix and $\boldsymbol{\Sigma}_{\boldsymbol{\eta}}$ a covariance matrix.

Bivariate SV: observation equation

MSV observation equation: The Bivariate Case ($n = 2$)

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \exp(f_{1t}) & 0 \\ 0 & \exp(f_{2t}) \end{bmatrix}^{1/2} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

$$\text{where} \qquad \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right).$$

Note: f_{1t} and f_{2t} control the *scale* of the returns!

Bivariate SV: updating equation

MSV updating equation: bivariate case ($n = 2$)

$$\begin{bmatrix} f_{1t+1} \\ f_{2t+1} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \beta_1 f_{1t} \\ \beta_2 f_{2t} \end{bmatrix} + \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}$$

$$\text{where} \quad \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{1\eta}^2 & \sigma_{12\eta} \\ \sigma_{12\eta} & \sigma_{2\eta}^2 \end{bmatrix} \right).$$

Extension: VAR(p) updating equation

Note: the updating equation allows for all the dynamics of vector autoregressive (VAR) models that you have learned in your introductory time-series courses!

Recall: stochastic properties for the VAR(p) model are easy to obtain!

Naturally: the dynamics can be easily extended to the VAR(p) case where

$$\boldsymbol{f}_{t+1} = \boldsymbol{\omega} + \boldsymbol{\beta}_1 \odot \boldsymbol{f}_t + \ldots + \boldsymbol{\beta}_p \odot \boldsymbol{f}_{t-p} + \boldsymbol{\eta}_t.$$

Simulate bivariate SV model with R (i)

Question: How can I simulate data from the MSV model?

Answer: See R file `Simulate_multivariate_SV.R`

First step: Choose the sample size T, labeled `n`, and parameter values ω, β_1, β_2, $\sigma^2_{1\eta}$, $\sigma^2_{2\eta}$, $\sigma_{12\eta}$, ρ_{12}

```
omega1 <- 0
omega2 <- 0
beta1 <- 0.95
beta2 <- 0.95
sig2f1 <- 0.10
sig2f2 <- 0.10
sigf12 <- 0.05
rho <- 0.5
```

Simulate bivariate SV model with R (ii)

Second: Define R and Σ_η matrices

```
R <- cbind(c(1,rho),c(rho,1))
Sf <- cbind(c(sig2f1,sigf12),c(sigf12,sig2f2))
```

Next: Generate the sequences of error terms $\{\boldsymbol{\eta}_t\}_{t=1}^T$ and $\{\boldsymbol{\varepsilon}_t\}_{t=1}^T$ from a $NID_2(\mathbf{0}, \boldsymbol{R})$ and a $NID_2(\mathbf{0}, \boldsymbol{\Sigma_\eta})$ respectively:

```
epsilon <- mvrnorm(n,rep(0,2),R)
eta <- mvrnorm(n,rep(0,2),Sf)

x <- matrix(0,nrow=n,ncol=2)
f <- matrix(0,nrow=n,ncol=2)
```

Simulate bivariate SV model with R (iii)

Next: define initial values for $\boldsymbol{f}_t$ by drawing from the unconditional distribution and generate the first observation

```
umf <- c(omega1/(1-beta1), omega2/(1-beta2))
uSf <- matrix(0,nrow=2,ncol=2)
uSf[1,1] <- sig2f1/(1-beta1^2)
uSf[2,2] <- sig2f2/(1-beta2^2)
uSf[2,1] <- sigf12/(1-beta1*beta2)
uSf[1,2] <- sigf12/(1-beta1*beta2)
f[1,] <- mvrnorm(1,umf,uSf)

x[1,1] <- exp(f[1,1]/2) * epsilon[1,1]
x[1,2] <- exp(f[1,2]/2) * epsilon[1,2]
```

Note: Drawing $\boldsymbol{f}_1$ from the unconditional distribution ensures stationarity of the generated series!

Simulate bivariate SV model with R (iv)

Finally: Generate values for $\boldsymbol{f}_t$ and $\boldsymbol{y}_t$ using the observation equation and transition equations

```
for(t in 2:n){
f[t,1] <- omega1 + beta1*f[t-1,1] + eta[t,1]
f[t,2] <- omega2 + beta2*f[t-1,2] + eta[t,2]

x[t,1] <- exp(f[t,1]/2) * epsilon[t,1]
x[t,2] <- exp(f[t,2]/2) * epsilon[t,2]
}
```

Simulate bivariate SV model with R (v)

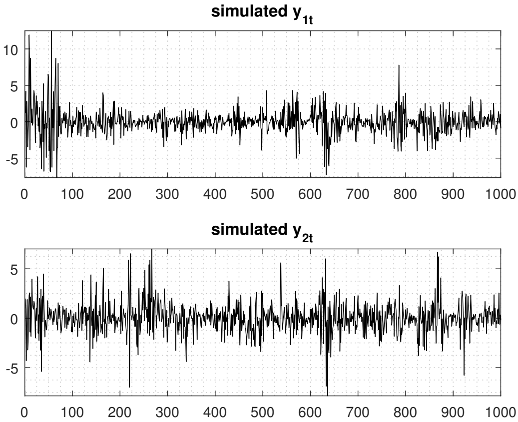


Figure: Simulated series from a bivariate Stochastic Volatility model.

Simulate bivariate SV model with R (vi)

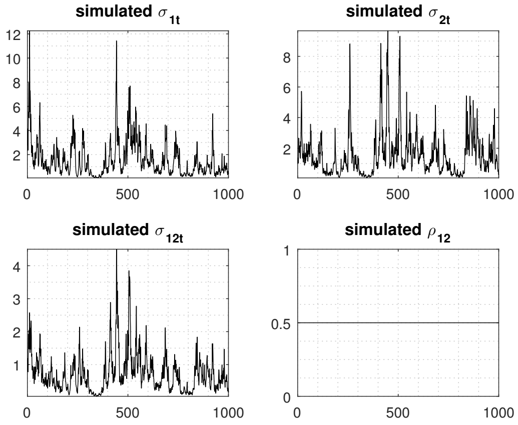


Figure: Simulated series from a bivariate Stochastic Volatility model.