Financial Econometrics

- Week 3, Lecture 1 -

Multivariate GARCH Models

VU Econometrics and Data Science

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Today's class

Multivariate GARCH models

Modeling multiple stocks

In practice: *we often deal with multiple financial assets.*

Notation:

1. Single asset return: $y_{i,t}$, for $i = 1, \dots, n$;
2. Vector of n asset returns: $\boldsymbol{y}_t = (y_{1,t}, \cdots, y_{n,t})^\top$;
3. Observed sample: $\{\boldsymbol{y}_t\}_{t=1}^{T}$.

Question: Can we model returns dynamics using n univariate GARCH models?

Answer: No. This would only be appropriate if the n returns are independent.

Problem: Empirical evidence suggests strong positive correlation across stock returns!... *next slide...*

Correlation across stock returns

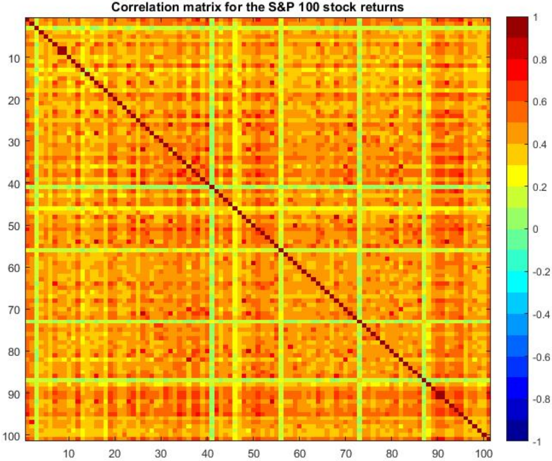

Figure: Correlation of daily stock log-returns in S&P 100 index.

Time-varying variance and covariance (i)

Question: Can we use Vector ARMA (VARMA) models to describe the dependence structure of multiple stock returns?

Answer: No.

Problem: Both conditional variances and conditional covariances change over time!

Example: Conditional covariance and correlation between Microsoft and IBM log-returns changes dramatically over time (see next slide!)

Solution: We need multivariate GARCH models!

Time-varying variance and covariance (ii)

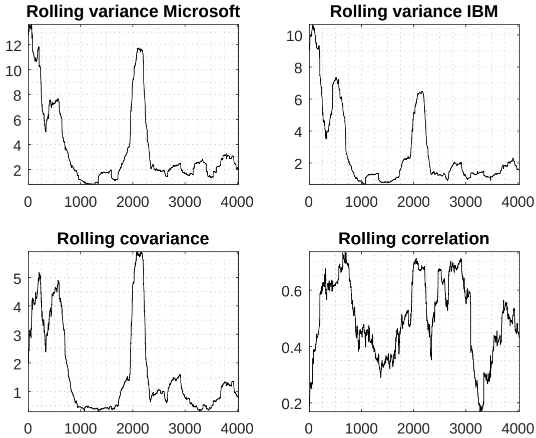

Figure: Rolling window estimate of variances, covariance and correlation between daily returns of MSFT and IBM (window length is 250 obs.).

Multivariate GARCH models (i)

Observation equation:

$$\boldsymbol{y}_t = \boldsymbol{\Sigma}_t^{1/2} \boldsymbol{\varepsilon}_t,$$

1. $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \dots, \varepsilon_{nt})^\top$ is an n-dimensional error vector with multivariate standard normal distribution, i.e. $\boldsymbol{\varepsilon}_t \sim \mathrm{NID}_n(\mathbf{0}_n, \boldsymbol{I}_n)$.
2. $\boldsymbol{\Sigma}_t$ is a *symmetric and positive definite matrix* that depends only on past observations Y^{t-1}.
3. Therefore, $\boldsymbol{\Sigma}_t$ is the conditional covariance matrix of $\boldsymbol{y}_t$

$$\boldsymbol{\Sigma}_t = \mathbb{V}ar(\boldsymbol{y}_t | Y^{t-1}) = \mathbb{E}(\boldsymbol{y}_t \boldsymbol{y}_t^\top | Y^{t-1}).$$

4. The conditional distribution of $\boldsymbol{y}_t$ is $\boldsymbol{y}_t | Y^{t-1} \sim N_n(\mathbf{0}_n, \boldsymbol{\Sigma}_t)$.

Multivariate GARCH models (ii)

Note: The conditional covariance matrix $\mathbf{\Sigma}_t$ has the form

$$\mathbf{\Sigma}_t = \begin{bmatrix} \sigma_{1t}^2 & \sigma_{12t} & \dots & \sigma_{1nt} \\ \sigma_{12t} & \sigma_{2t}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma_{(n-1)nt} \\ \sigma_{1nt} & \dots & \sigma_{(n-1)nt} & \sigma_{nt}^2 \end{bmatrix}.$$

Bivariate case $(n = 2)$:

$$\mathbf{\Sigma}_t = \begin{bmatrix} \sigma_{1t}^2 & \sigma_{12t} \\ \sigma_{12t} & \sigma_{2t}^2 \end{bmatrix}.$$

- **Multivariate GARCH** models differ in the way they specify the **updating equation** of the conditional covariance matrix $\mathbf{\Sigma}_t$.
- There are different types of multivariate GARCH models: **VECH-type models** and **CCC/DCC models**.

VECH models

VECH model: most natural multivariate GARCH!

Idea: we can put the conditional covariance matrix $\boldsymbol{\Sigma}_t$ in vector form and then specify a vector-valued updating equation.

Important: we only need to specify an **updating equation** for the lower triangular elements $\boldsymbol{\Sigma}_t$ because $\boldsymbol{\Sigma}_t$ is a symmetric matrix!

- The specification of **VECH models** is based on the **half vectorization operator** $\text{vech}(\cdot)$.
- The **vech($\cdot$) operator** takes the lower triangular elements of $\boldsymbol{\Sigma}_t$ and stacks them into a vector.

The VECH operator

The **vech(·) operator** stacks the lower triangular elements of a squared matrix into a vector. For instance,

$$\text{if} \quad \boldsymbol{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}, \quad \text{then} \quad \text{vech}(\boldsymbol{A}) = \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \\ \alpha_{22} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix}.$$

In general: vech(·) of $n \times n$ matrix produces $\tilde{n} = n(n+1)/2$ vector that contains all lower triangular elements.

The bivariate VECH(1,1) model (i)

Bivariate case: $\text{vech}(\boldsymbol{\Sigma}_t) = (\sigma^2_{1,t}, \sigma_{12,t}, \sigma^2_{2,t})^\top$.

The updating equation of the **bivariate VECH(1,1)** is

$$\begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{12,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \end{bmatrix} + \begin{bmatrix} \tilde{\beta}_{11} & \tilde{\beta}_{12} & \tilde{\beta}_{12} \\ \tilde{\beta}_{21} & \tilde{\beta}_{22} & \tilde{\beta}_{23} \\ \tilde{\beta}_{31} & \tilde{\beta}_{32} & \tilde{\beta}_{33} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t-1} \\ \sigma_{12,t-1} \\ \sigma^2_{2,t-1} \end{bmatrix} + \begin{bmatrix} \tilde{\alpha}_{11} & \tilde{\alpha}_{12} & \tilde{\alpha}_{13} \\ \tilde{\alpha}_{21} & \tilde{\alpha}_{22} & \tilde{\alpha}_{23} \\ \tilde{\alpha}_{31} & \tilde{\alpha}_{32} & \tilde{\alpha}_{33} \end{bmatrix} \begin{bmatrix} y^2_{1,t-1} \\ y_{1,t-1}y_{2,t-1} \\ y^2_{2,t-1} \end{bmatrix}$$

- $\boldsymbol{\Sigma}_t$ depends only on past observations (and parameters) through $y^2_{1,t-1}$ and $y^2_{2,t-1}$ and $y_{1,t-1}y_{2,t-1}$.
- $y^2_{1,t-1}$ can be seen as an estimate of the variance of $y_{1,t-1}$.
- The product $y_{1,t-1}y_{2,t-1}$ can be seen as estimate of the covariance between $y_{1,t-1}$ and $y_{2,t-1}$.

The VECH(p,q) model

The updating equation of the **n-dimensional VECH(p,q)** model is

$$\text{vech}(\boldsymbol{\Sigma}_t) = \tilde{\boldsymbol{W}} + \sum_{i=1}^{q} \tilde{\boldsymbol{A}}_i \text{vech}(\boldsymbol{y}_{t-i}\boldsymbol{y}_{t-i}^\top) + \sum_{i=1}^{p} \tilde{\boldsymbol{B}}_i \text{vech}(\boldsymbol{\Sigma}_{t-i}),$$

- $\tilde{\boldsymbol{W}}$ is an $\tilde{n}$-dimensional vector of parameters;
- $\tilde{\boldsymbol{B}}_i$ and $\tilde{\boldsymbol{A}}_i$ are $\tilde{n} \times \tilde{n}$ square matrices of parameters with dimension $\tilde{n} = n(n+1)/2$.

Remark (Unconditional variance)

The unconditional covariance matrix $\boldsymbol{\Sigma} = \mathbb{V}ar(\boldsymbol{y}_t)$ of the VECH(p,q) model, when it exists, is given by

$$vech(\boldsymbol{\Sigma}) = \left(\boldsymbol{I}_{\tilde{n}} - \sum_{i=1}^{q} \tilde{\boldsymbol{A}}_i - \sum_{i=1}^{p} \tilde{\boldsymbol{B}}_i\right)^{-1} \tilde{\boldsymbol{W}}.$$

The VECH model: advantages and limitations

Important: The VECH updating equation is very general!

Example (*bivariate VECH(1,1)*): the updating equation of σ_{1t}^2 is:

$$\begin{aligned}\sigma_{1t}^2 &= \tilde{\omega}_1 + \tilde{\beta}_{11}\sigma_{1,t-1}^2 + \tilde{\beta}_{12}\sigma_{12,t-1} + \tilde{\beta}_{12}\sigma_{2,t-1}^2 + \tilde{\alpha}_{11}y_{1,t-1}^2 \\ &+ \tilde{\alpha}_{12}y_{1,t-1}y_{2,t-1} + \tilde{\alpha}_{13}y_{2,t-1}^2.\end{aligned}$$

- The conditional variance σ_{1t}^2 depends on lagged values of $y_{1,t}^2$, $y_{2,t}^2$ and $y_{1,t}y_{2,t}$.
- Therefore, the specification is very flexible since it allows cross effects in the dynamics.
- However, the **VECH(p,q) model has 2 strong limitations**: *the curse of dimensionality* and $\mathbf{\Sigma}_t$ *may not be positive definite.*

The VECH model: limitations (i)

Problem 1: *The curse of dimensionality*

1. Number of parameters to estimate increases very fast with the dimension n of the vector $\boldsymbol{y}_t$;
2. Number of parameters is of order $O(n^4)$;
   - for $n = 2$ stocks and $p = q = 1$, we have 21 parameters!
   - for $n = 4$ stocks and $p = q = 1$, we have 210 parameters!
   - for $n = 10$ stocks and $p = q = 1$, we have 6105 parameters!!!
3. Infeasible in practical applications!

The VECH model: limitations (ii)

Problem 2: *Positivity constraints*

- It is not clear which restrictions we should impose on $\tilde{\boldsymbol{W}}$, $\tilde{\boldsymbol{B}}_i$ and $\tilde{\boldsymbol{A}}_i$ to ensure that $\boldsymbol{\Sigma}_t$ is a *positive definite matrix* for every t;
- If $\boldsymbol{\Sigma}_t$ is not a positive definite matrix, then it cannot be a conditional covariance!

Solution: Impose restrictions on the parameters and obtain other (less flexible) multivariate GARCH models!

1. DVECH model
2. BEKK model

The DVECH model (i)

DVECH model: stands for *Diagonal VECH model!*

DVECH model: imposes that $\tilde{\boldsymbol{B}}_i$ and $\tilde{\boldsymbol{A}}_i$ are diagonal

Bivariate DVECH(1,1) :

$$\begin{bmatrix} \sigma_{1,t}^2 \\ \sigma_{21,t} \\ \sigma_{2,t}^2 \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \end{bmatrix} + \begin{bmatrix} \tilde{\beta}_{11} & 0 & 0 \\ 0 & \tilde{\beta}_{22} & 0 \\ 0 & 0 & \tilde{\beta}_{33} \end{bmatrix} \begin{bmatrix} \sigma_{1,t-1}^2 \\ \sigma_{21,t-1} \\ \sigma_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} \tilde{\alpha}_{11} & 0 & 0 \\ 0 & \tilde{\alpha}_{22} & 0 \\ 0 & 0 & \tilde{\alpha}_{33} \end{bmatrix} \begin{bmatrix} y_{1,t-1}^2 \\ y_{1,t-1} y_{2,t-1} \\ y_{2,t-1}^2 \end{bmatrix}$$

Positive: number of parameters dropped from 21 to 9!

Negative: DVECH model does not allow for cross-causality in variance ($y_{1,t-1}^2$ does not affect $\sigma_{2,t}^2$)

The DVECH model (ii)

Bivariate DVECH: can easily be written as system of equations

$$\begin{aligned}
\sigma_{1,t}^2 &= \omega_{11} + \beta_{11}\sigma_{1,t-1}^2 + \alpha_{11}y_{1,t-1}^2,\\
\sigma_{2,t}^2 &= \omega_{22} + \beta_{22}\sigma_{2,t-1}^2 + \alpha_{22}y_{2,t-1}^2,\\
\sigma_{21,t} &= \omega_{21} + \beta_{21}\sigma_{21,t-1} + \alpha_{21}y_{1,t-1}y_{2,t-1}.
\end{aligned}$$

Important: DVECH model can be rewritten in matrix form using *Hadamard* matrix product $\odot$

$$\begin{aligned}
\begin{bmatrix} \sigma_{1,t}^2 & \sigma_{21,t} \\ \sigma_{21,t} & \sigma_{2,t}^2 \end{bmatrix} &= \begin{bmatrix} \omega_{11} & \omega_{21} \\ \omega_{21} & \omega_{22} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{21} & \beta_{22} \end{bmatrix} \odot \begin{bmatrix} \sigma_{1,t-1}^2 & \sigma_{21,t-1} \\ \sigma_{21,t-1} & \sigma_{2,t-1}^2 \end{bmatrix}\\
&+ \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \odot \begin{bmatrix} y_{1,t-1}^2 & y_{1,t-1}y_{2,t-1} \\ y_{1,t-1}y_{2,t-1} & y_{2,t-1}^2 \end{bmatrix}
\end{aligned}$$

Hadamard product

Note: In the Hadamard product $\odot$ each element ij of a matrix is the product of the elements ij of the two original matrices.

Example: given two 3×3 matrices $\boldsymbol{A} = (\alpha_{ij})$ and $\boldsymbol{B} = (\beta_{ij})$, we have that

$$\boldsymbol{A} \odot \boldsymbol{B} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \odot \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \begin{bmatrix} \alpha_{11}\beta_{11} & \alpha_{12}\beta_{12} & \alpha_{13}\beta_{13} \\ \alpha_{21}\beta_{21} & \alpha_{22}\beta_{22} & \alpha_{23}\beta_{23} \\ \alpha_{31}\beta_{31} & \alpha_{32}\beta_{32} & \alpha_{33}\beta_{33} \end{bmatrix}$$

Note: In DVECH formulation all matrices are symmetric because the conditional covariance must be symmetric!

Multivariate DVECH model

General multivariate DVECH(1,1) model:

$$\boldsymbol{\Sigma}_t = \boldsymbol{W} + \boldsymbol{A}_1 \odot (\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}^{\top}) + \boldsymbol{B}_1 \odot \boldsymbol{\Sigma}_{t-1} \ ,$$

where $\boldsymbol{W}$, $\boldsymbol{B}_1$ and $\boldsymbol{A}_1$ are symmetric $n \times n$ matrices of parameters.

General multivariate DVECH(p,q) model:

$$\boldsymbol{\Sigma}_t = \boldsymbol{W} + \sum_{i=1}^{q} \boldsymbol{A}_i \odot (\boldsymbol{y}_{t-i}\boldsymbol{y}_{t-i}^{\top}) + \sum_{i=1}^{p} \boldsymbol{B}_i \odot \boldsymbol{\Sigma}_{t-i},$$

where $\boldsymbol{W}$, $\boldsymbol{B}_i$ and $\boldsymbol{A}_i$ are symmetric $n \times n$ matrices of parameters.

The sDVECH model

Question: Can we further attenuate the curse of dimensionality?

Answer: Yes, we replace the matrices $\boldsymbol{B}_i$ and $\boldsymbol{A}_i$ of the DVECH with scalar parameters β_i and α_i. The resulting model is called the **scalar DVECH (sDVECH)** model.

The sDVECH(1,1) model is:

$$\boldsymbol{\Sigma}_t = \boldsymbol{W} + \alpha_1\, \boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}^\top + \beta_1\, \boldsymbol{\Sigma}_{t-1} \ ,$$

where $\boldsymbol{W}$ is a symmetric matrix, and α_1 and β_1 are scalar parameters.

The sDVECH(1,1) is the DVECH(1,1) with $\boldsymbol{B}_1$ and $\boldsymbol{A}_1$ given by

$$\boldsymbol{B}_1 = \begin{bmatrix} \beta_1 & \dots & \beta_1 \\ \vdots & \ddots & \vdots \\ \beta_1 & \dots & \beta_1 \end{bmatrix}, \qquad \boldsymbol{A}_1 = \begin{bmatrix} \alpha_1 & \dots & \alpha_1 \\ \vdots & \ddots & \vdots \\ \alpha_1 & \dots & \alpha_1 \end{bmatrix}.$$

Simulated conditional covariance DVECH

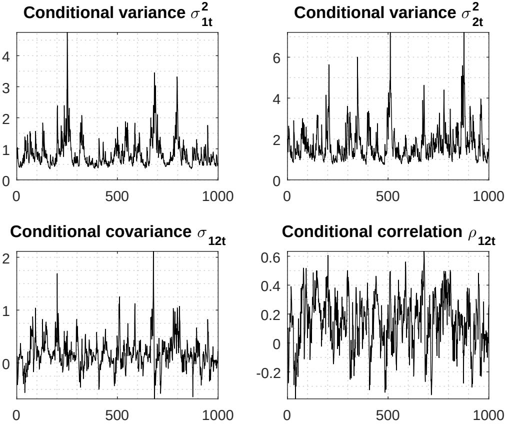

Figure: Conditional variances, covariance and correlation generated from a DVECH model.

DVECH model: limitations

Recall: DVECH is a special case of VECH model where matrices of parameters are imposed to be diagonal!

Important: DVECH attenuates the *curse of dimensionality*

Problem: DVECH does not ensure that the conditional covariance $\boldsymbol{\Sigma}_t$ be positive definite!

Conclusion: DVECH only solves one of the problems of the VECH!
We need another model!

Exercise: DVECH model

Suppose you have portfolio of \$ 100,000 that is composed for 25% of Google's stocks and for 75% of IBM's stocks.

Stock prices went up in the last month 3% for Google and 1.3% for IBM.

The monthly **log-returns** of Google $\{y_{1t}\}$ and IBM stocks $\{y_{2t}\}$ are well described by the following DVECH(0,1) model

$$\begin{bmatrix} \sigma^2_{1,t} & \sigma_{21,t} \\ \sigma_{21,t} & \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} 0.002 & 0.001 \\ 0.001 & 0.003 \end{bmatrix} + \begin{bmatrix} 0.8 & 0.5 \\ 0.5 & 0.9 \end{bmatrix} \odot \begin{bmatrix} y^2_{1,t-1} & y_{1,t-1}y_{2,t-1} \\ y_{1,t-1}y_{2,t-1} & y^2_{2,t-1} \end{bmatrix}$$

Question: What is the probability that you'll loose more than \$10,000 in the next moth?

The bivariate BEKK(1,1) model

BEKK model: stands for Baba-Engle-Kraft-Kroner model.

BEKK model: solves the issue of having a positive definite conditional covariance matrix $\mathbf{\Sigma}_t$.

Bivariate BEKK(1,1):

$$\begin{bmatrix} \sigma^2_{1,t} & \sigma_{21,t} \\ \sigma_{21,t} & \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \omega_{11} & 0 \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} \omega_{11} & \omega_{21} \\ 0 & \omega_{22} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t-1} & \sigma_{21,t-1} \\ \sigma_{21,t-1} & \sigma^2_{2,t-1} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{bmatrix}$$
$$+ \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} y^2_{1,t-1} & y_{1,t-1}y_{2,t-1} \\ y_{1,t-1}y_{2,t-1} & y^2_{2,t-1} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix}.$$

Homework Exercise: write explicitly the bivariate BEKK above as a system of three equations.

The multivariate BEKK(p,q)

Multivariate BEKK(1,1) model:

$$\boldsymbol{\Sigma}_t = \boldsymbol{W}\boldsymbol{W}^\top + \boldsymbol{A}_1(\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}^\top)\boldsymbol{A}_1^\top + \boldsymbol{B}_1\boldsymbol{\Sigma}_{t-1}\boldsymbol{B}_1^\top,$$

Multivariate BEKK(p,q) model:

$$\boldsymbol{\Sigma}_t = \boldsymbol{W}\boldsymbol{W}^\top + \sum_{i=1}^{q}\boldsymbol{A}_i(\boldsymbol{y}_{t-i}\boldsymbol{y}_{t-i}^\top)\boldsymbol{A}_i^\top + \sum_{i=1}^{p}\boldsymbol{B}_i\boldsymbol{\Sigma}_{t-i}\boldsymbol{B}_i^\top,$$

where $\boldsymbol{W}$ is a lower triangular $n \times n$ matrix and $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ are $n \times n$ matrices.

Remark

The BEKK(p,q) model has a positive definite conditional covariance matrix $\boldsymbol{\Sigma}_t$ for any $t \in \mathbb{N}$ if $\boldsymbol{\Sigma}_0, \ldots, \boldsymbol{\Sigma}_{-p-1}$ are positive definite and $\boldsymbol{W}$ or any $\boldsymbol{B}_i$ is a full rank matrix.

BEKK model: limitations

Advantage: BEKK model can ensure a positive definite covariance matrix!

Advantage: BEKK can address curse of dimensionality by imposing restrictions on parameters (like the DVECH model!).

Limitation: disadvantages of BEKK is that the parameters are difficult to interpret.

Solution: mmm... not really... topic of research! :)

For now: let's look at other models!

The CCC model

CCC model: another approach to specify Multivariate GARCH models and deal with the curse of dimensionality.

Important: The CCC is not a special case of the very general VECH model.

CCC: stands for *Constant Conditional Correlation*

Main property: conditional correlation matrix is constant

- Time-variation in the conditional covariance matrix $\boldsymbol{\Sigma}_t$ is only provided by dynamic variances;

The bivariate CCC model (i)

Bivariate CCC model:

$$\boldsymbol{y}_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\Sigma}_t = \boldsymbol{D}_t\boldsymbol{R}\boldsymbol{D}_t,$$

$$\boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma_{1t}^2 & \sigma_{12t} \\ \sigma_{12t} & \sigma_{2t}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix} \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix} \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix}$$

$$\begin{aligned} \sigma_{1,t}^2 &= \omega_1 + \beta_1\sigma_{1,t-1}^2 + \alpha_1 y_{1,t-1}^2 \\ \sigma_{2,t}^2 &= \omega_2 + \beta_2\sigma_{2,t-1}^2 + \alpha_2 y_{2,t-1}^2 \end{aligned}$$

- $\boldsymbol{R}$ is a 2×2 correlation matrix;
- $\boldsymbol{D}_t$ is a 2×2 diagonal matrix containing the conditional standard deviation.

The bivariate CCC model (ii)

Note: The model is called CCC because the conditional correlation matrix $\boldsymbol{R}$ is constant

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix}.$$

Homework exercise: show that

$$\begin{bmatrix} \sigma_{1t}^2 & \sigma_{12t} \\ \sigma_{12t} & \sigma_{2t}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix} \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix} \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix}$$

- $\mathbb{C}orr(y_{1t}, y_{2t}|Y^{t-1}) = \rho_{12}$
- $\mathbb{C}ov(y_{1t}, y_{2t}|Y^{t-1}) = \sigma_{12t} = \sigma_{1t}\sigma_{2t}\rho_{12}$

The bivariate CCC model (iii)

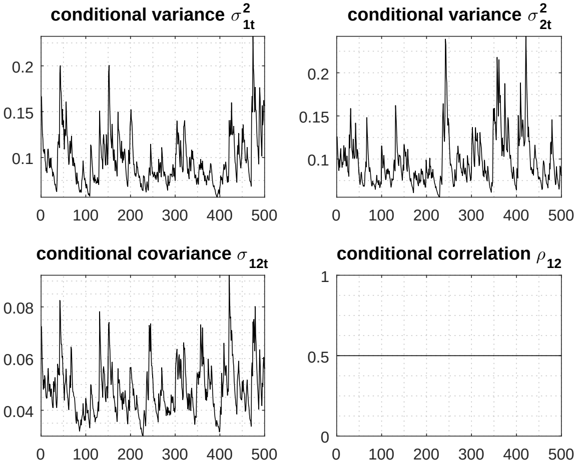

Figure: Conditional variances, covariance and correlation generated from a CCC model.

Multivariate CCC model

Multivariate CCC model:

$$\boldsymbol{\Sigma}_t = \boldsymbol{D}_t \boldsymbol{R} \boldsymbol{D}_t,$$

$$\sigma_{i,t}^2 = \omega_i + \beta_i \sigma_{i,t-1}^2 + \alpha_i y_{i,t-1}^2.$$

- $\boldsymbol{D}_t = \mathrm{diag}(\sigma_{1,t}, \ldots, \sigma_{n,t})$ is an $n \times n$ diagonal matrix containing the conditional standard deviation.
- $\boldsymbol{R}$ is an $n \times n$ correlation matrix.

Advantage: CCC model handles both the curse of dimensionality and the positive definiteness of $\boldsymbol{\Sigma}_t$;

Disadvantage: assumption that conditional correlation matrix is constant can be very restrictive!

The DCC model (i)

DCC model: *Dynamic Conditional Correlation* model

DCC model: allows for time-varying conditional correlation matrix $\boldsymbol{R}_t$

$$\boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma_{1t}^2 & \sigma_{12t} \\ \sigma_{12t} & \sigma_{2t}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix} \begin{bmatrix} 1 & \rho_{12t} \\ \rho_{12t} & 1 \end{bmatrix} \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix}$$

$$\sigma_{1,t}^2 = \omega_1 + \beta_1 \sigma_{1,t-1}^2 + \alpha_1 y_{1,t-1}^2$$
$$\sigma_{2,t}^2 = \omega_2 + \beta_2 \sigma_{2,t-1}^2 + \alpha_2 y_{2,t-1}^2$$

The DCC model (ii)

Conditional correlation:

$$\rho_{12t} = q_{12t} / \sqrt{q_{11t}} \sqrt{q_{22t}},$$

where

$$
\begin{aligned}
q_{11t} &= \omega_q + \beta_q q_{11t-1} + \alpha_q v_{1,t-1}^2, \\
q_{22t} &= \omega_q + \beta_q q_{22t-1} + \alpha_q v_{2,t-1}^2, \\
q_{12t} &= \omega_q + \beta_q q_{12t-1} + \alpha_q v_{1,t-1} v_{2,t-1}.
\end{aligned}
$$

- $v_{1t} = y_{1t}/\sigma_{1t}$ and $v_{2t} = y_{2t}/\sigma_{2t}$ are standardized observations.
- This formulation is needed to ensure that ρ_{12t} is between -1 and 1.
- Each equation for q_{ijt} has same static parameters ω_q, β_q and α_q.

Other extensions

Important: Multivariate GARCH can be extended in many other ways:

1. Observation equation:
    - Time-varying conditional mean $\boldsymbol{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t$.
    - Non-Gaussian fat-tailed innovations $\epsilon_t \sim t(\nu)$.
2. Updating equation:
    - Nonlinear, robust, leverage-effect, thresholds, etc.
    - Use of link functions such as multivariate exponential function.

Simulation: bivariate DVECH(1,1) with R (i)

R file: `generate_DVECH.R`

Set: Sample size and parameter values

```
n <- 1000
w11 <- 0.1
w22 <- 0.2
w12 <- 0.02
b11 <- 0.7
b22 <- 0.7
b12 <- 0.7
a11 <- 0.2
a22 <- 0.2
a12 <- 0.15
```

Define matrices `x` and `VECHt` for $\{\boldsymbol{y}_t\}_{t=1}^{T}$ and $\{\mathrm{vech}(\boldsymbol{\Sigma}_t)\}_{t=1}^{T}$.

```
x <- matrix(0,nrow = n, ncol = 2)
VECHt <- matrix(0,nrow=n,ncol=3)
```

Simulation: bivariate DVECH(1,1) with R (ii)

Initialize: $\text{vech}(\Sigma_1)$ at unconditional covariance matrix.

```
VECHt[1,1] <- w11/(1-b11-a11)
VECHt[1,3] <- w22/(1-b22-a22)
VECHt[1,2] <- w12/(1-b12-a12)
```

Next: Generate first observation $\boldsymbol{y}_1$

```
SIGMAt <- cbind(c(VECHt[1,1],VECHt[1,2]),c(VECHt[1,2],VECHt[1,3]))
x[1,] <- mvrnorm(1, rep(0,2), SIGMAt)
```

Note: The R function `mvrnorm()` generates from Multivariate normal. This function is part of package `MASS`.

Simulation: bivariate DVECH(1,1) with R (iii)

Finally: Run a *for loop* and simulate the series iterating the equations of $\boldsymbol{y}_t$ and $\text{vech}(\boldsymbol{\Sigma}_t)$

```
for(t in 2:n){
  VECHt[t,1] <- w11 + b11*VECHt[t-1,1] + a11*x[t-1,1]^2
  VECHt[t,3] <- w22 + b22*VECHt[t-1,3] + a22*x[t-1,2]^2
  VECHt[t,2] <- w12 + b12*VECHt[t-1,2] + a12*x[t-1,1]*x[t-1,2]

  SIGMAt <- cbind(c(VECHt[t,1],VECHt[t,2]),c(VECHt[t,2],VECHt[t,3]))
  x[t,] <- mvrnorm(1, rep(0,2), SIGMAt)
}
```

Simulation: bivariate DVECH(1,1) with R (iv)

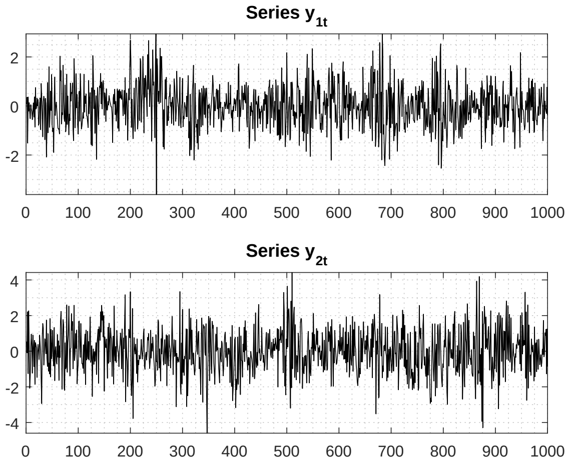

Figure: Series generated from a bivariate DVECH model.