Exercises and Solutions: week 5

Financial Econometrics

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CHAPTER 9 and 10: Stochastic Volatility models

1. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the following Stochastic Volatility model

$$y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \exp(f_t),$$
$$f_t = 0.8 f_{t-1} + \eta_t.$$

Fill in the empty cells of the following table.

t	1	2	3	4	5
ϵ_t	0.5	-0.7	-1.2	0.9	-0.1
η_t	1.3	0.3	-0.9	-0.1	0.7
σ_t^2	1.1				
y_t					

Solution:
First we need to find y_t at time $t = 1$. This can be done through the observation equation.
$t = 1)$ *Observation equation:* $y_1 = \sigma_1\epsilon_1 = \sqrt{1.1} \times 0.5 \approx 0.52$
$t = 2)$ *Transition equation:* $f_2 = 0.8 f_1 + \eta_2 = 0.8 \times \log(1.1) + 0.3 \approx 0.38$, thus $\sigma_2^2 = \exp(0.38) = 1.46$.
Observation equation: $y_2 = \sigma_2\epsilon_2 = \sqrt{1.46} \times (-0.7) \approx -0.85$
$t = 3)$ *Transition equation:* $f_3 = 0.8 f_2 + \eta_3 = 0.8 \times 0.38 - 0.9 \approx -0.60$, thus $\sigma_3^2 = \exp(-0.60) = 0.55$.
Observation equation: $y_3 = \sigma_3\epsilon_3 = \sqrt{0.55} \times (-1.2) \approx -0.90$
$t = 4)$ *Transition equation:* $f_4 = 0.8 f_3 + \eta_4 = 0.8 \times (-0.6) - 0.1 \approx -0.58$, thus $\sigma_4^2 = \exp(-0.58) = 0.56$.
Observation equation: $y_4 = \sigma_4\epsilon_4 = \sqrt{0.56} \times 0.9 \approx 0.67$
$t = 5)$ *Transition equation:* $f_5 = 0.8 f_4 + \eta_5 = 0.8 \times (-0.58) + 0.7 \approx 0.24$, thus $\sigma_5^2 = \exp(0.24) = 1.27$.
Observation equation: $y_5 = \sigma_5\epsilon_5 = \sqrt{1.27} \times (-0.1) \approx -0.11$

2. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the following SV model

$$y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \exp(f_t),$$
$$f_t = \omega + \beta f_{t-1} + \eta_t,$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is $NID(0, 1)$ and $\{\eta_t\}_{t\in\mathbb{Z}}$ is $NID(0, \sigma_\eta^2)$.

(a) Show that the unconditional mean $\mathbb{E}(y_t)$ is equal to zero.

(b) Derive the unconditional variance $\mathbb{V}ar(y_t)$.

(c) Derive the kurtosis of y_t, i.e. $k_u = \mathbb{E}(y_t^4)/\mathbb{E}(y_t^2)^2$.

Solution:

(a) We obtain that

$$\mathbb{E}(y_t) = \mathbb{E}(\sigma_t\epsilon_t) = \mathbb{E}(\sigma_t)\mathbb{E}(\epsilon_t) = \mathbb{E}(\sigma_t) \times 0 = 0,$$

where the second equality follows since σ_t and ϵ_t are independent.

(b) The unconditional variance is

$$\mathbb{V}ar(y_t) = \mathbb{E}(y_t^2) = \mathbb{E}(\sigma_t^2\epsilon_t^2) = \mathbb{E}(\sigma_t^2)\mathbb{E}(\epsilon_t^2) = \mathbb{E}(\sigma_t^2) \times 1 = \mathbb{E}(\sigma_t^2),$$

where the third equality follows since σ_t and ϵ_t are independent and the fourth since the variance of ϵ_t is 1. Next, we compute $\mathbb{E}(\sigma_t^2)$. First, we notice that f_t is a Gaussian AR(1) process with unconditional normal distribution $N(\mu_f, \sigma_f^2)$ with $\mu_f = \frac{\omega}{1-\beta}$ and $\sigma_f^2 = \frac{\sigma_\eta^2}{1-\beta^2}$. Therefore, $\sigma_t^2 = \exp(f_t)$ has a log-normal distribution, log-$N(\mu_f, \sigma_f^2)$, with unconditional expectation given by $\mathbb{E}(\sigma_t^2) = \exp(\mu_f + \sigma_f^2/2)$. Therefore, we conclude that

$$\mathbb{V}ar(y_t) = \exp(\mu_f + \sigma_f^2/2) = \exp\left(\frac{\omega}{1-\beta} + \frac{\sigma_\eta^2}{2(1-\beta^2)}\right).$$

(c) We obtain the fourth moment as follows

$$\mathbb{E}(y_t^4) = \mathbb{E}(\sigma_t^4 \epsilon_t^4) = \mathbb{E}(\sigma_t^4)\mathbb{E}(\epsilon_t^4) = 3\mathbb{E}(\sigma_t^4),$$

where the third equality follows since $\epsilon_t \sim N(0,1)$ and therefore $\mathbb{E}(\epsilon_t^4) = 3$. Next, we notice that $\sigma_t^4 = \exp(2f_t)$ and $2f_t \sim N(2\mu_f, 4\sigma_f^2)$. Therefore, $\sigma_t^4 \sim \text{log-}N(2\mu_f, 4\sigma_f^2)$ and hence $\mathbb{E}(\sigma_t^4) = \exp(2\mu_f + 2\sigma_f^2)$. As a result, we conclude that

$$\begin{aligned} k_u &= \frac{\mathbb{E}(y_t^4)}{\mathbb{E}(y_t^2)^2} = \frac{3\exp(2\mu_f + 2\sigma_f^2)}{\exp(\mu_f + \sigma_f^2/2)^2} = \frac{3\exp(2\mu_f + 2\sigma_f^2)}{\exp(2\mu_f + \sigma_f^2)} \\ &= 3\exp(\sigma_f^2) = 3\exp\left(\frac{\sigma_\eta^2}{1-\beta^2}\right). \end{aligned}$$

3. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the SV model of Question 2 with $\beta = 0$.
   (a) What is the autocorrelation function of the squared observations y_t^2.
   (b) Can you say something about the unconditional distribution of y_t? Is it Normal?

Solution:

(a) First, we notice that y_t is independent of y_{t-l} for any $l > 0$ since η_t and ϵ_t are iid sequences. Therefore, for any $l > 0$, we obtain that

$$\mathbb{C}ov(y_t^2, y_{t-l}^2) = \mathbb{E}(y_t^2 y_{t-l}^2) - \mathbb{E}(y_t^2)\mathbb{E}(y_{t-l}^2) = \mathbb{E}(y_t^2)\mathbb{E}(y_{t-l}^2) - \mathbb{E}(y_t^2)\mathbb{E}(y_{t-l}^2) = 0,$$

where the second equality follows since y_t and y_{t-l} are independent. This implies that the autocorrrelation function is zero for any $l > 0$.

(b) The unconditional distribution is not normal. In particular, the kurtosis is $k_u = 3\exp(\sigma_\eta^2) > 3$ and therefore the distribution cannot be normal since the kurtosis of the normal is 3.

4. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the SV model of Question 2 with $\sigma_\eta^2 = 0$.
   (a) What is the autocorrelation function of the squared observations y_t^2.
   (b) Can you say something about the unconditional distribution of y_t? Is it Normal?

Solution:

(a) If the variance of the error is zero $\sigma_\eta^2 = 0$, f_t is constant and given by $f_t = \omega/(1-\beta)$. As a result y_t is an iid sequence and therefore the autocorrelation function of y_t^2 is zero at any lag.

(b) The unconditional distribution is normal since ϵ_t is normal and σ_t is constant.

5. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the following SV model with an MA(1) specification for f_t

$$\begin{aligned} y_t = \sigma_t \epsilon_t, \ \ \sigma_t^2 = \exp(f_t), \\ f_t = \eta_t + 0.3\eta_{t-1}, \end{aligned}$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is $NID(0,1)$ and $\{\eta_t\}_{t\in\mathbb{Z}}$ is $NID(0,0.1)$.
   (a) Obtain the unconditional variance $\mathbb{V}ar(y_t)$.
   (b) Obtain the kurtosis of y_t, i.e. $k_u = \mathbb{E}(y_t^4)/\mathbb{E}(y_t^2)^2$.
   (c) Obtain the autocorrelation function of the squared observations y_t^2, i.e. $\mathbb{C}orr(y_t^2, y_{t-l}^2)$, $l = 1, 2, \ldots$.

Solution:

(a) The unconditional distribution of f_t is normal with mean $\mathbb{E}(f_t) = 0$ and variance

$$\mathbb{V}ar(f_t) = \mathbb{V}ar(\eta_t) + 0.3^2\mathbb{V}ar(\eta_{t-1}) = 0.1 + 0.09 \times 0.1 = 0.109.$$

Therefore, we have that $\sigma_t^2 \sim \text{log-}N(0, 0.109)$. The unconditional variance is

$$\mathbb{V}ar(y_t) = \mathbb{E}(\sigma_t^2) = \exp(0 + 0.109/2) = 1.056.$$

(b) The fourth moment is

$$\mathbb{E}(y_t^4) = 3\mathbb{E}(\sigma_t^4) = \exp(2\mu_f + 2\sigma_f^2) = 3\exp(2 \times 0.109) = 3.731.$$

Therefore, the kurtosis is

$$k_u = \frac{\mathbb{E}(y_t^4)}{\mathbb{E}(y_t^2)^2} = \frac{3.731}{1.056^2} = 3.35.$$

(c) First, we notice that y_t^2 and y_{t-l}^2 are independent for $l > 1$ and hence uncorrelated. Therefore, we can just calculate $\mathbb{C}orr(y_t^2, y_{t-l}^2)$ for $l = 1$. We obtain that

$$\mathbb{C}ov(y_t^2, y_{t-1}^2) = \mathbb{E}(y_t^2 y_{t-1}^2) - \mathbb{E}(y_t^2)\mathbb{E}(y_{t-1}^2),$$

and

$$\mathbb{E}(y_t^2 y_{t-1}^2) = \mathbb{E}(\sigma_t^2 \epsilon_t^2 \sigma_{t-1}^2 \epsilon_{t-1}^2) = \mathbb{E}(\sigma_t^2 \sigma_{t-1}^2) = \mathbb{E}(\exp(f_t + f_{t-1})).$$

We notice that $f_t + f_{t-1}$ has an unconditional normal distribution with mean zero $\mathbb{E}(f_t + f_{t-1}) = 0$ and variance given by

$$\begin{aligned}\mathbb{V}ar(f_t + f_{t-1}) &= \mathbb{V}ar(\eta_t + 0.3\eta_{t-1} + \eta_{t-1} + 0.3\eta_{t-2}) \\ &= \mathbb{V}ar(\eta_t + 1.3\eta_{t-1} + 0.3\eta_{t-2}) = 0.1 + 1.3^2 \times 0.1 + 0.3^2 \times 0.1 = 0.278.\end{aligned}$$

As a result, we get

$$\mathbb{E}(y_t^2 y_{t-1}^2) = \mathbb{E}(\exp(f_t + f_{t-1})) = \exp(0 + 0.278/2) = 1.15.$$

Finally, we conclude that

$$\mathbb{C}ov(y_t^2, y_{t-1}^2) = \mathbb{E}(y_t^2 y_{t-1}^2) - \mathbb{E}(y_t^2)\mathbb{E}(y_{t-1}^2) = 1.15 - 1.056^2 = 0.0349.$$

6. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the following SV model with an ARMA(1,1) specification for f_t

$$y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \exp(f_t),$$
$$f_t = 0.6 f_{t-1} + \eta_t + 0.4\eta_{t-1}.$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is $NID(0,1)$ and $\{\eta_t\}_{t\in\mathbb{Z}}$ is $NID(0, 0.1)$. Find the unconditional variance of y_t, i.e. $\mathbb{V}ar(y_t)$.

Solution:
First, we need to obtain the unconditional distribution of f_t. In particular, unfolding the recursion of f_t we get that

$$f_t = \sum_{i=0}^{\infty} 0.6^i \eta_{t-i-1} + \eta_t.$$

Thus the unconditional mean μ_f is

$$\mu_f = \mathbb{E}(f_t) = \sum_{i=0}^{\infty} 0.6^i \mathbb{E}(\eta_{t-i-1}) + \mathbb{E}(\eta_t) = 0,$$

whereas the unconditional variance σ_f^2 is

$$\sigma_f^2 = \mathbb{V}ar(f_t) = \sum_{i=0}^{\infty} 0.6^{2i} \mathbb{V}ar(\eta_{t-i-1}) + \mathbb{V}ar(\eta_t) = \frac{0.1}{1 - 0.6^2} + 0.1 = 0.26.$$

Therefore we conclude that $f_t \sim N(0, 0.26)$ and $\exp(f_t) \sim \text{log-}N(0, 0.26)$.
We can now obtain the unconditional variance of y_t as

$$\mathbb{V}ar(y_t) = \mathbb{E}(y_t^2) = \mathbb{E}(\sigma_t^2 \varepsilon_t^2) = \mathbb{E}(\sigma_t^2)\mathbb{E}(\varepsilon_t^2) = \mathbb{E}(\sigma_t^2) = \mathbb{E}[\exp(f_t)] = \exp(\mu_f + \sigma_f^2/2) = \exp(0.26/2) = 1.14.$$

7. Let $\{\boldsymbol{y}_t\}_{t\in\mathbb{Z}}$ be generated by a bivariate SV model[1].

(a) Show that the unconditional mean $\mathbb{E}(\boldsymbol{y}_t)$ is equal to the zero vector $(0,0)^\top$.

(b) Derive the unconditional covariance matrix $\mathbb{V}ar(\boldsymbol{y}_t)$.

[1]Consider the specification you have in the Lecture Notes.

Solution:

The bivariate SV model is given by

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \exp(f_{1t}/2) & 0 \\ 0 & \exp(f_{2t}/2) \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right).$$

$$\begin{bmatrix} f_{1t+1} \\ f_{2t+1} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} \beta_1 f_{1t} \\ \beta_2 f_{2t} \end{bmatrix} + \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{1\eta}^2 & \sigma_{12\eta} \\ \sigma_{12\eta} & \sigma_{2\eta}^2 \end{bmatrix} \right).$$

(a) The unconditional mean $\mathbb{E}(\boldsymbol{y}_t)$ is given by

$$\mathbb{E}(\boldsymbol{y}_t) = \begin{bmatrix} \mathbb{E}(y_{1t}) \\ \mathbb{E}(y_{2t}) \end{bmatrix}.$$

Then we obtain that

$$\mathbb{E}(y_{1t}) = \mathbb{E}[\exp(f_{1t}/2)\varepsilon_{1t}] = \mathbb{E}[\exp(f_{1t}/2)]\mathbb{E}[\varepsilon_{1t}] = \mathbb{E}[\exp(f_{1t}/2)] \times 0 = 0.$$

$$\mathbb{E}(y_{2t}) = \mathbb{E}[\exp(f_{2t}/2)\varepsilon_{2t}] = \mathbb{E}[\exp(f_{2t}/2)]\mathbb{E}[\varepsilon_{2t}] = \mathbb{E}[\exp(f_{2t}/2)] \times 0 = 0.$$

Therefore we conclude that $\mathbb{E}(\boldsymbol{y}_t) = (0,0)^\top$.

(b) The unconditional variance $\mathbb{V}ar(\boldsymbol{y}_t)$ is given by

$$\mathbb{E}(\boldsymbol{y}_t \boldsymbol{y}_t^\top) = \begin{bmatrix} \mathbb{E}(y_{1t}^2) & \mathbb{E}(y_{1t}y_{2t}) \\ \mathbb{E}(y_{1t}y_{2t}) & \mathbb{E}(y_{2t}^2) \end{bmatrix}.$$

First we study the unconditional distribution of $(f_{1t}, f_{2t})^\top$. In particular, unfolding the recursions of f_{1t} and f_{2t}, we obtain that

$$f_{1t} = \frac{\omega_1}{1-\beta_1} + \sum_{i=0}^{\infty} \beta_1^i \eta_{1t-i},$$

$$f_{2t} = \frac{\omega_2}{1-\beta_2} + \sum_{i=0}^{\infty} \beta_2^i \eta_{2t-i}.$$

Therefore, we conclude that

$$\mu_{1f} = \mathbb{E}(f_{1t}) = \frac{\omega_1}{1-\beta_1},$$

$$\mu_{2f} = \mathbb{E}(f_{2t}) = \frac{\omega_2}{1-\beta_2},$$

$$\sigma_{1f}^2 = \mathbb{V}ar(f_{1t}) = \sum_{i=0}^{\infty} \beta_1^{2i} \mathbb{V}ar(\eta_{1t-i}) = \frac{\sigma_{1\eta}^2}{1-\beta_1^2},$$

$$\sigma_{2f}^2 = \mathbb{V}ar(f_{2t}) = \sum_{i=0}^{\infty} \beta_2^{2i} \mathbb{V}ar(\eta_{2t-i}) = \frac{\sigma_{2\eta}^2}{1-\beta_2^2},$$

$$\sigma_{12f} = \mathbb{C}ov(f_{1t}, f_{2t}) = \sum_{i=0}^{\infty} (\beta_1\beta_2)^i \mathbb{E}(\eta_{1t-i}\eta_{2t-i}) = \frac{\sigma_{12\eta}}{1-\beta_1\beta_2},$$

and obtain that the unconditional distribution of $(f_{1t}, f_{2t})^\top$ is given by

$$\begin{bmatrix} f_{1t} \\ f_{2t} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_{1f} \\ \mu_{2f} \end{bmatrix}, \begin{bmatrix} \sigma_{1f}^2 & \sigma_{12f} \\ \sigma_{12f} & \sigma_{2f}^2 \end{bmatrix} \right).$$

We are now ready to obtain the unconditional variance $\mathbb{V}ar(\boldsymbol{y}_t)$. In particular,

$$\mathbb{E}(y_{1t}^2) = \mathbb{E}[\exp(f_{1t})\varepsilon_{1t}^2] = \mathbb{E}[\exp(f_{1t})]\mathbb{E}[\varepsilon_{1t}^2] = \mathbb{E}[\exp(f_{1t})] = \exp(\mu_{1f} + \sigma_{1f}^2/2),$$

where the last equality follows from the fact that $f_{1t} \sim N(\mu_{1f}, \sigma_{1f}^2)$ and therefore $\exp(f_{1t}) \sim \log\text{-}N(\mu_{1f}, \sigma_{1f}^2)$. In the same way we obtain that

$$\mathbb{E}(y_{2t}^2) = \mathbb{E}[\exp(f_{2t})\varepsilon_{2t}^2] = \mathbb{E}[\exp(f_{2t})]\mathbb{E}[\varepsilon_{2t}^2] = \mathbb{E}[\exp(f_{2t})] = \exp(\mu_{2f} + \sigma_{2f}^2/2).$$

Finally, we obtain that

$$\begin{aligned}\mathbb{E}(y_{1t}y_{2t}) &= \mathbb{E}[\exp(f_{1t}/2)\varepsilon_{1t}\exp(f_{2t}/2)\varepsilon_{2t}] = \mathbb{E}[\exp(f_{1t}/2)\exp(f_{2t}/2)]\mathbb{E}[\varepsilon_{1t}\varepsilon_{2t}] = \mathbb{E}[\exp\left((f_{1t}+f_{2t})/2\right)] \times \rho \\ &= \rho \exp\left(\frac{\mu_{1f}+\mu_{2f}}{2} + \frac{1}{8}(\sigma_{1f}^2 + \sigma_{2f}^2 + 2\sigma_{12f})\right),\end{aligned}$$

where the last equality follows from the fact that $(f_{1t}+f_{2t})/2$ has a normal distribution with mean

$$\mathbb{E}\left(\frac{f_{1t}+f_{2t}}{2}\right) = \frac{\mu_{1f}+\mu_{2f}}{2}$$

and variance

$$\mathbb{V}ar\left(\frac{f_{1t}+f_{2t}}{2}\right) = \frac{1}{4}\left(\sigma_{1f}^2 + \sigma_{2f}^2 + 2\sigma_{12f}\right).$$

Therefore, $\exp\left((f_{1t}+f_{2t})/2\right)$ has a log-normal distribution and the result follows immediately.

CHAPTER 11 and 12: Indirect Inference Estimation

1. Consider the following MA(1) model:

$$y_t = \varepsilon_t + \phi\varepsilon_{t-1},$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is $NID(0,1)$ and $\phi \in (-1,1)$.
We want to estimate the "true" parameter ϕ_0 by indirect inference. For each of the following auxiliary statistics, show whether or not the indirect inference estimator is consistent.

(a) $\hat{B}_T = T^{-1}\sum_{t=1}^{T} y_t$
(b) $\hat{B}_T = T^{-1}\sum_{t=1}^{T} y_t^2$
(c) $\hat{B}_T = T^{-1}\sum_{t=2}^{T} y_t y_{t-1}$

Solution:
First, we recall that to show the consistency of the indirect inference estimator, $\hat{\theta}_{HT} \xrightarrow{p} \theta_0$, we need to: **(i)** Obtain the binding functions $B(\theta_0)$ and $B(\theta)$. **(ii)** Show that $\theta = \theta_0$ is the unique minimizer of $d(B(\theta), B(\theta_0))$.

(a) By the Law of Large Numbers we obtain that

$$\hat{B}_T = T^{-1}\sum_{t=1}^{T} y_t \xrightarrow{p} B(\theta_0) = \mathbb{E}(y_t) = 0.$$

Similarly for the auxiliary statistics from the simulated data

$$\tilde{B}_H(\theta) = H^{-1}\sum_{h=1}^{H} \tilde{y}_h(\theta) \xrightarrow{p} B(\theta) = \mathbb{E}[\tilde{y}_h(\theta)] = 0.$$

As a result, we conclude that

$$d(B(\theta), B(\theta_0)) = d(0,0) = 0,$$

for any θ. Therefore $\theta = \theta_0$ is not the unique minimum. The indirect inference estimator is inconsistent.

(b) By the Law of Large Numbers we obtain that

$$\hat{B}_T = T^{-1}\sum_{t=1}^{T} y_t^2 \xrightarrow{p} B(\theta_0) = \mathbb{E}(y_t^2) = 1 + \phi_0^2.$$

Similarly for the auxiliary statistics from the simulated data

$$\tilde{B}_H(\theta) = H^{-1}\sum_{h=1}^{H} \tilde{y}_h(\theta)^2 \xrightarrow{p} B(\theta) = \mathbb{E}[\tilde{y}_h(\theta)^2] = 1 + \phi^2.$$

Now we note that $d(B(\theta), B(\theta_0)) = 0$ if $\phi = \phi_0$ but also if $\phi = -\phi_0$. Therefore $\theta = \theta_0$ is not the unique minimum. The indirect inference estimator is inconsistent.

(c) By the Law of Large Numbers we obtain that

$$\hat{B}_T = T^{-1}\sum_{t=2}^{T} y_t y_{t-1} \xrightarrow{p} B(\theta_0) = \mathbb{E}(y_t y_{t-1}) = \phi_0.$$

Similarly for the auxiliary statistics from the simulated data

$$\tilde{B}_H(\theta) = H^{-1}\sum_{h=1}^{H} \tilde{y}_h(\theta)\tilde{y}_{h-1}(\theta) \xrightarrow{p} B(\theta) = \mathbb{E}[\tilde{y}_h(\theta)\tilde{y}_{h-1}(\theta)] = \phi.$$

Now we note that $d(B(\theta), B(\theta_0)) = 0$ if and only if $\phi = \phi_0$. Therefore $\theta = \theta_0$ is the unique minimum. The indirect inference estimator is consistent.

2. Consider the following AR(1) model:

$$y_t = \rho y_{t-1} + \varepsilon_t,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is $NID(0,1)$ and $\rho \in (-1,1)$.
We want to estimate the "true" parameter ρ_0 by indirect inference. For each of the following auxiliary statistics, show whether or not the indirect inference estimator is consistent.

(a) $\hat{B}_T = T^{-1} \sum_{t=1}^{T} y_t$

(b) $\hat{B}_T = T^{-1} \sum_{t=1}^{T} y_t^2$

(c) $\hat{B}_T = T^{-1} \sum_{t=2}^{T} y_t y_{t-1}$

Solution:
First, we recall that to show the consistency of the indirect inference estimator, $\hat{\theta}_{HT} \xrightarrow{p} \theta_0$, we need to: **(i)** Obtain the binding functions $B(\theta_0)$ and $B(\theta)$. **(ii)** Show that $\theta = \theta_0$ is the unique minimizer of $d(B(\theta), B(\theta_0))$.

(a) By the Law of Large Numbers we obtain that

$$\hat{B}_T = T^{-1} \sum_{t=1}^{T} y_t \xrightarrow{p} B(\theta_0) = \mathbb{E}(y_t) = 0.$$

Similarly for the auxiliary statistics from the simulated data

$$\tilde{B}_H(\theta) = H^{-1} \sum_{h=1}^{H} \tilde{y}_h(\theta) \xrightarrow{p} B(\theta) = \mathbb{E}[\tilde{y}_h(\theta)] = 0.$$

As a result, we conclude that

$$d(B(\theta), B(\theta_0)) = d(0, 0) = 0,$$

for any θ. Therefore $\theta = \theta_0$ is not the unique minimum. The indirect inference estimator is inconsistent.

(b) By the Law of Large Numbers we obtain that

$$\hat{B}_T = T^{-1} \sum_{t=1}^{T} y_t^2 \xrightarrow{p} B(\theta_0) = \mathbb{E}(y_t^2) = (1 - \rho_0^2)^{-1}.$$

Similarly for the auxiliary statistics from the simulated data

$$\tilde{B}_H(\theta) = H^{-1} \sum_{h=1}^{H} \tilde{y}_h(\theta)^2 \xrightarrow{p} B(\theta) = \mathbb{E}[\tilde{y}_h(\theta)^2] = (1 - \rho^2)^{-1}.$$

Now we note that $d(B(\theta), B(\theta_0)) = 0$ if $\rho = \rho_0$ but also if $\rho = -\rho_0$. Therefore $\theta = \theta_0$ is not the unique minimum. The indirect inference estimator is inconsistent.

(c) By the Law of Large Numbers we obtain that

$$\hat{B}_T = T^{-1} \sum_{t=2}^{T} y_t y_{t-1} \xrightarrow{p} B(\theta_0) = \mathbb{E}(y_t y_{t-1}) = \rho_0 (1 - \rho_0^2)^{-1}.$$

Similarly for the auxiliary statistics from the simulated data

$$\tilde{B}_H(\theta) = H^{-1} \sum_{h=1}^{H} \tilde{y}_h(\theta) \tilde{y}_{h-1}(\theta) \xrightarrow{p} B(\theta) = \mathbb{E}[\tilde{y}_h(\theta) \tilde{y}_{h-1}(\theta)] = \rho (1 - \rho^2)^{-1}.$$

Now we note that $d(B(\theta), B(\theta_0)) = 0$ if and only if $\rho = \rho_0$. Therefore $\theta = \theta_0$ is the unique minimum. The indirect inference estimator is consistent.

3. Consider the following MA(1) model:

$$y_t = \varepsilon_t + \phi \varepsilon_{t-1},$$

where $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ is $NID(0, \sigma^2)$ with $\phi \in (-1, 1)$ and $\sigma^2 > 0$.
We want to estimate the "true" parameter vector $\theta_0 = (\phi_0, \sigma_0^2)^\top$ by indirect inference. Show that the indirect inference estimator based on the following auxiliary statistic is consistent.

$$\hat{B}_T = \frac{1}{T} \begin{bmatrix} \sum_{t=1}^{T} y_t^2 \\ \sum_{t=2}^{T} y_t y_{t-1} \end{bmatrix}.$$

Solution:
First we need to obtain the binding functions $B(\theta_0)$ and $B(\theta)$. In particular, we obtain that by the Law of Large Numbers

$$\hat{B}_T = \frac{1}{T}\begin{bmatrix} \sum_{t=1}^T y_t^2 \\ \sum_{t=2}^T y_t y_{t-1} \end{bmatrix} \xrightarrow{p} \begin{bmatrix} \mathbb{E}(y_t^2) \\ \mathbb{E}(y_t y_{t-1}) \end{bmatrix} = \begin{bmatrix} \sigma_0^2(1+\phi_0^2) \\ \sigma_0^2\phi_0 \end{bmatrix} = B(\theta_0).$$

Similarly for the auxiliary statistic from simulated data we obtain

$$\tilde{B}_H(\theta) = \frac{1}{H}\begin{bmatrix} \sum_{h=1}^H \tilde{y}_h(\theta)^2 \\ \sum_{h=2}^H y_h y_{h-1} \end{bmatrix} \xrightarrow{p} \begin{bmatrix} \mathbb{E}[\tilde{y}_h(\theta)^2] \\ \mathbb{E}[\tilde{y}_h(\theta)\tilde{y}_{h-1}(\theta)] \end{bmatrix} = \begin{bmatrix} \sigma^2(1+\phi^2) \\ \sigma^2\phi \end{bmatrix} = B(\theta).$$

As discussed before, the indirect inference estimator is consistent if $\theta = \theta_0$ is the unique minimizer of $d(B(\theta), B(\theta_0))$. In particular, this is equivalent to show that $\theta = \theta_0$ is the unique value of θ such that $B(\theta) = B(\theta_0)$. Thus $\theta = \theta_0$ should be the only solution of the following system of equations

$$\begin{cases} \sigma^2(1+\phi^2) = \sigma_0^2(1+\phi_0^2) \\ \sigma^2\phi = \sigma_0^2\phi_0 \end{cases}.$$

From the above equations we obtain that

$$\sigma^2 = \sigma_0^2(1+\phi_0^2)/(1+\phi^2) \tag{1}$$

and that

$$\phi_0\phi^2 - (1+\phi_0^2)\phi + \phi_0 = 0. \tag{2}$$

We can now distinguish two cases: $\phi_0 = 0$ and $\phi_0 \neq 0$.
$\phi_0 = 0$) From the equation in (2) we immediately get that $\phi = 0$, i.e. $\phi = \phi_0$, and this also implies that $\sigma^2 = \sigma_0^2$ from the equation in (1).
$\phi_0 \neq 0$) We can rewrite the equation in (2) as $\phi^2 - (\phi_0^{-1} + \phi_0)\phi + 1 = 0$ and we then obtain that $(\phi - \phi_0)(\phi - \phi_0^{-1}) = 0$. Therefore the solutions are $\phi = \phi_0$ and $\phi = \phi_0^{-1}$. However, we recall that $\phi \in (-1, 1)$ (see the text of the exercise) and also $\phi_0 \in (-1, 1)$. Therefore, we can exclude the solution $\phi = \phi_0^{-1}$ as $\phi_0^{-1} \notin (-1, 1)$. We then conclude that $\phi = \phi_0$ is the only solution and from the equation in (1) we also get that $\sigma^2 = \sigma_0^2$.
This implies that $\theta = \theta_0$ is the unique minimizer of $d(B(\theta), B(\theta_0))$ and therefore the indirect inference estimator is consistent.

4. Comment on the following statement: "I can use the set of auxiliary statistics (a) to consistently estimate the parameters of the structural model (b)"

   (a) Auxiliary statistics: sample mean, sample variance, sample kurtosis, first-order autocorrelation for y_t, first-order autocorrelation for y_t^2.

   (b) Structural model:

$$\begin{aligned} y_t &= \mu + g_t + \exp(f_t/2)\epsilon_t, \\ g_t &= \phi g_{t-1} + \eta_t + \theta\eta_t, \\ f_t &= \beta f_{t-1} + v_t, \end{aligned}$$

   where $\epsilon_t \sim N(0, 1)$, $\eta_t \sim N(0, \sigma_\eta^2)$ and $v_t \sim N(0, \sigma_v^2)$.

   Solution:
   The claim is incorrect. We cannot identify the parameters of the model using that set of auxiliary statistics. This because the number of auxiliary statistics must be equal to or larger than the number of parameters. In this case we have 6 parameters to be estimated and only 5 auxiliary statistics.

5. Argue that the parameters of the following model cannot be identified from the proposed set of auxiliary statistics:
   Model:

$$\begin{aligned} y_t &= v_t + z_t \\ v_t &\sim N(\mu, \sigma^2) \\ z_t &\sim t_v \end{aligned}$$

Auxiliary statistics: sample mean, sample variance, sample skewness.

Solution:
In order to identify the degrees of freedom v of the Student-t distribution we would need a statistics that measures the kurtosis. We also note that the skewness is a useless statistics since the skewness of y_t is zero for any parameter value θ. Therefore, we conclude that the parameters of the model cannot be identified using those statistics.

6. Argue that the parameters of the structural model in the previous question could be identified from the following set of auxiliary statistics:
   Auxiliary statistics: sample mean, sample variance, sample kurtosis.

   Solution:
   As mentioned in the previous answer the degrees of freedom v of the Student-t distribution can be identified using the kurtosis. Furthermore, μ and σ^2 can be identified using the sample a mean and variance. Therefore we conclude that the parameters can be identified using this set of auxiliary statistics.

7. Which of the following auxiliary models do you think are useful to estimate the SV model:
   (a) AR(1) for y_t
   (b) AR(p) for y_t
   (c) AR(1) for y_t^2
   (d) ARMA(2,2) for y_t^2
   (e) GARCH model

   Solution:
   When we use a model as auxiliary model it means that we use the MLE parameter estimate of that model as auxiliary statistics. Indeed, to estimate an SV model we need an auxiliary model that capture the autocovariace in squared y_t^2. This is indeed not the case for the auxiliary models (a) and (b). Instead, the auxiliary models (c), (d) and (e) can be used to identify the parameters of the SV as they all capture information on the autocovariance of y_t^2.

8. Which of the following auxiliary models do you think are useful to estimate the multivariate SV model:
   (a) VAR(1) for y_t
   (b) VAR(p) for y_t
   (c) VAR(1) for y_t^2
   (d) Bivariate VECH(1,1)
   (e) Bivariate BEKK(1,1)
   (f) Bivariate CCC

   Solution:
   In a similar way as for the univariate case of the previous question, we need auxiliary models that capture dependence on $y_t y_t^\top$. This is indeed not the case for the auxiliary models (a) and (b). Instead, the auxiliary models (c), (d), (e) and (f) can be used to identify the parameters of the multivariate SV.