Exercises and Solutions: week 6

Financial Econometrics

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CHAPTER 13: Dynamic regression models

1. Consider the following *parameter-driven* dynamic regression model

$$y_t = \beta_t x_t + \varepsilon_t$$

$$\beta_t = 0.1 + 0.9\beta_{t-1} + \eta_t,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a $NID(0,1)$ sequence and $\{\eta_t\}_{t\in\mathbb{Z}}$ is a $NID(0,0.5)$. Furthermore, assume that also $\{x_t\}_{t\in\mathbb{Z}}$ is a $NID(0,1)$ sequence.

(a) Find the unconditional mean of y_t, i.e. $\mathbb{E}(y_t)$.

(b) Find the unconditional variance of y_t, i.e. $\mathbb{V}ar(y_t)$.

(c) Find the unconditional covariance between y_t and x_t, i.e. $\mathbb{C}ov(y_t, x_t)$.

Solution:

(a) The unconditional mean of y_t is given by

$$\mathbb{E}(y_t) = \mathbb{E}(\beta_t x_t + \varepsilon_t) = \mathbb{E}(\beta_t)\mathbb{E}(x_t) + \mathbb{E}(\varepsilon_t) = \mathbb{E}(\beta_t) \times 0 + 0 = 0.$$

(b) The unconditional variance of y_t is given by

$$\begin{aligned}\mathbb{V}ar(y_t) &= \mathbb{E}(y_t^2) = \mathbb{E}[(\beta_t x_t + \varepsilon_t)^2] = \mathbb{E}[\beta_t^2 x_t^2 + \varepsilon_t^2 + 2\beta_t x_t \varepsilon_t] = \mathbb{E}[\beta_t^2]\mathbb{E}[x_t^2] + \mathbb{E}[\varepsilon_t^2] + 2\mathbb{E}[\beta_t]\mathbb{E}[x_t]\mathbb{E}[\varepsilon_t] \\ &= \mathbb{E}[\beta_t^2] \times 1 + 1 + 0 = 1 + \mathbb{E}[\beta_t^2] = 1 + \mathbb{V}ar[\beta_t] + \mathbb{E}[\beta_t]^2 = 1 + \frac{0.5}{1-0.9^2} + \left(\frac{0.1}{1-0.9}\right)^2 = 4.63,\end{aligned}$$

where $\mathbb{E}[\beta_t]$ and $\mathbb{V}ar[\beta_t]$ can be easily obtained noticing that β_t is an AR(1) process.

(c) The covariance between y_t and x_t is given by

$$\mathbb{C}ov(y_t, x_t) = \mathbb{E}(y_t x_t) = \mathbb{E}[(\beta_t x_t + \varepsilon_t)x_t] = \mathbb{E}[\beta_t x_t^2] + \mathbb{E}[\varepsilon_t x_t] = \mathbb{E}[\beta_t]\mathbb{E}[x_t^2] + \mathbb{E}[\varepsilon_t]\mathbb{E}[x_t] = \mathbb{E}[\beta_t] = \frac{0.1}{1-0.9} = 1.$$

2. Consider the following *parameter-driven* dynamic regression model

$$y_t = \beta_t x_t + \varepsilon_t$$

$$\beta_t = 0.1 + \eta_t + 0.5\eta_{t-1},$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a $NID(0,1)$ sequence and $\{\eta_t\}_{t\in\mathbb{Z}}$ is a $NID(0,0.5)$. Furthermore, assume that also $\{x_t\}_{t\in\mathbb{Z}}$ is a $NID(0,1)$ sequence.

(a) Find the unconditional mean of y_t, i.e. $\mathbb{E}(y_t)$.

(b) Find the unconditional variance of y_t, i.e. $\mathbb{V}ar(y_t)$.

(c) Find the unconditional covariance between y_t and x_t, i.e. $\mathbb{C}ov(y_t, x_t)$.

Solution:

(a) The unconditional mean of y_t is given by

$$\mathbb{E}(y_t) = \mathbb{E}(\beta_t x_t + \varepsilon_t) = \mathbb{E}(\beta_t)\mathbb{E}(x_t) + \mathbb{E}(\varepsilon_t) = \mathbb{E}(\beta_t) \times 0 + 0 = 0.$$

(b) The unconditional variance of y_t is given by

$$\begin{aligned}\mathbb{V}ar(y_t) &= \mathbb{E}(y_t^2) = \mathbb{E}[(\beta_t x_t + \varepsilon_t)^2] = \mathbb{E}[\beta_t^2 x_t^2 + \varepsilon_t^2 + 2\beta_t x_t \varepsilon_t] = \mathbb{E}[\beta_t^2]\mathbb{E}[x_t^2] + \mathbb{E}[\varepsilon_t^2] + 2\mathbb{E}[\beta_t]\mathbb{E}[x_t]\mathbb{E}[\varepsilon_t] \\ &= \mathbb{E}[\beta_t^2] \times 1 + 1 + 0 = 1 + \mathbb{E}[\beta_t^2] = 1 + \mathbb{V}ar[\beta_t] + \mathbb{E}[\beta_t]^2 = 1 + (0.5 + 0.5^2 \times 0.5) + 0.1^2 = 1.63,\end{aligned}$$

where $\mathbb{E}[\beta_t]$ and $\mathbb{V}ar[\beta_t]$ can be easily obtained noticing that β_t is an MA(1) process.

(c) The covariance between y_t and x_t is given by

$$\mathbb{C}ov(y_t, x_t) = \mathbb{E}(y_t x_t) = \mathbb{E}[(\beta_t x_t + \varepsilon_t)x_t] = \mathbb{E}[\beta_t x_t^2] + \mathbb{E}[\varepsilon_t x_t] = \mathbb{E}[\beta_t]\mathbb{E}[x_t^2] + \mathbb{E}[\varepsilon_t]\mathbb{E}[x_t] = \mathbb{E}[\beta_t] = 0.1.$$

3. Consider the *observation-driven* dynamic regression model

$$y_t = \beta_t x_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2),$$

$$\beta_t = \omega + \phi\beta_{t-1} + \alpha(y_{t-1} - \beta_{t-1}x_{t-1})x_{t-1}.$$

Derive the conditional mean $\mathbb{E}(y_t|Y^{t-1}, X^t)$, conditional variance $\mathbb{V}ar(y_t|Y^{t-1}, X^t)$ and the conditional distribution of $y_t|(Y^{t-1}, X^t)$.

Solution:
The conditional mean is

$$\mathbb{E}(y_t|Y^{t-1}, X^t) = \mathbb{E}(\beta_t x_t + \varepsilon_t|Y^{t-1}, X^t) = \beta_t x_t + \mathbb{E}(\varepsilon_t|Y^{t-1}, X^t) = \beta_t x_t,$$

where the second equality follows since β_t and x_t are constants conditional on Y^{t-1} and X^t, and the third equality follows since ε_t is iid with mean zero.

The conditional variance is

$$\mathbb{V}ar(y_t|Y^{t-1}, X^t) = \mathbb{V}ar(\beta_t x_t + \varepsilon_t|Y^{t-1}, X^t) = 0 + \mathbb{V}ar(\varepsilon_t|Y^{t-1}, X^t) = \sigma^2,$$

where the second equality follows since β_t and x_t are constants conditional on Y^{t-1} and X^t, and the third equality follows since ε_t is iid with variance σ^2.

Finally, we obtain that the conditional distribution is normal with mean $\beta_t x_t$ and variance σ^2, that is, $y_t|(Y^{t-1}, X^t) \sim N(\beta_t x_t, \sigma^2)$. This follows since the error distribution is normal.