Exercises and Solutions: week 1

Financial Econometrics

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CHAPTER 1: Introduction

1. Let the price p_t of a stock be given by the following model

$$p_t = 0.1 + 0.99 p_{t-1} + \epsilon_t \quad \text{for every } t \in \mathbb{Z},$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is an $NID(0, \sigma^2)$ sequence with variance $\sigma^2 = 2$.

(a) Show that the unconditional mean and unconditional variance are invariant in time. Derive their values.

(b) Show that the conditional mean changes over time. Calculate $\mathbb{E}(p_t|p_{t-1})$ when $p_{t-1} = 8.7$.

(c) Show that the conditional variance is invariant in time. Calculate $\mathbb{V}ar(p_t|p_{t-1})$ when $p_{t-1} = 8.7$.

Solution:

(a) To obtain the unconditional mean and variance we first unfold the recursion

$$p_t = 0.1 + 0.99 p_{t-1} + \epsilon_t,$$

and obtain

$$\begin{aligned} p_t &= 0.1 \sum_{i=0}^{\infty} 0.99^i + \sum_{i=0}^{\infty} 0.99^i \epsilon_{t-i} = \frac{0.1}{1-0.99} + \sum_{i=0}^{\infty} 0.99^i \epsilon_{t-i} \\ &= 10 + \sum_{i=0}^{\infty} 0.99^i \epsilon_{t-i}. \end{aligned}$$

Therefore, we calculate the unconditional mean as

$$\mathbb{E}(p_t) = \mathbb{E}\left(10 + \sum_{i=0}^{\infty} 0.99^i \epsilon_{t-i}\right) = 10 + \sum_{i=0}^{\infty} 0.99^i \mathbb{E}\left(\epsilon_{t-i}\right) = 10,$$

where the last equality follows from the fact that $\epsilon_t \sim N(0, 2)$ and therefore $\mathbb{E}(\epsilon_t) = 0$ for any $t \in \mathbb{Z}$. Finally, we obtain the unconditional variance as

$$\mathbb{V}ar(p_t) = \mathbb{V}ar\left(10 + \sum_{i=0}^{\infty} 0.99^i \epsilon_{t-i}\right) = \sum_{i=0}^{\infty} 0.99^{2i} \mathbb{V}ar\left(\epsilon_{t-i}\right) = 2\sum_{i=0}^{\infty} 0.99^{2i} = \frac{2}{1-0.9801} \approx 100.5,$$

where the second equality follows from the fact that the elements of the sequence $\{\epsilon_t\}$ are independent. This shows that the unconditional mean and variance are constant.

(b) The conditional mean is given by

$$\mathbb{E}(p_t|p_{t-1}) = \mathbb{E}(0.1 + 0.99p_{t-1} + \epsilon_t|p_{t-1}) = 0.1 + 0.99\mathbb{E}(p_{t-1}|p_{t-1}) + \mathbb{E}(\epsilon_t|p_{t-1}) = 0.1 + 0.99p_{t-1}.$$

This shows that the conditional mean changes over time. Finally, for $p_{t-1} = 8.7$ we obtain

$$\mathbb{E}(p_t|p_{t-1} = 8.7) = 0.1 + 0.99 \times 8.7 = 8.713.$$

(c) The conditional variance is given by

$$\mathbb{V}ar(p_t|p_{t-1}) = \mathbb{E}\left((p_t - \mathbb{E}(p_t|p_{t-1}))^2|p_{t-1}\right) = \mathbb{E}\left(\epsilon_t^2|p_{t-1}\right) = \mathbb{E}\left(\epsilon_t^2\right) = \sigma^2 = 2.$$

Therefore we conclude that the conditional variance is constant. The conditional variance for $y_{t-1} = 8.7$ is given by $\mathbb{V}ar(p_t|p_{t-1}) = 2$.

2. Let the price p_t of a stock be given by the following model

$$p_t = 0.5 + \epsilon_t + 0.4\epsilon_{t-1} \quad \text{for every } t \in \mathbb{Z},$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is an $NID(0, \sigma^2)$ sequence with variance $\sigma^2 = 1$.

(a) Show that the unconditional mean and unconditional variance are invariant in time. Derive their values.

(b) Show that the conditional mean changes over time (assume ϵ_1 is observed). Derive the conditional mean of p_t given $\{p_{t-1}, \dots, p_2\}$ and ϵ_1.

Solution:

(a) Fist, we obtain the unconditional mean $\mathbb{E}(p_t)$ as follows

$$\mathbb{E}(p_t) = \mathbb{E}(0.5 + \epsilon_t + 0.4\epsilon_{t-1}) = 0.5 + \mathbb{E}(\epsilon_t) + 0.4\mathbb{E}(\epsilon_{t-1}) = 0.5.$$

Next we obtain the unconditional mean $\mathbb{V}ar(p_t)$ as follows

$$\mathbb{V}ar(p_t) = \mathbb{V}ar(0.5 + \epsilon_t + 0.4\epsilon_{t-1}) = \mathbb{V}ar(\epsilon_t) + 0.4^2\mathbb{V}ar(\epsilon_{t-1}) = 1 + 0.16 = 1.16,$$

where the second equality follows since ϵ_t and ϵ_{t-1} are independent (and hence $\mathbb{C}ov(\epsilon_t, \epsilon_{t-1}) = 0$), and the third inequality holds since $\mathbb{V}ar(\epsilon_t) = 1$.

(b) First we note that if we observe $\{p_{t-1}, \ldots, p_1\}$ and ϵ_1, then also $\{\epsilon_{t-1}, \ldots, \epsilon_1\}$ is observed. This because we can write ϵ_t as

$$\epsilon_t = p_t - 0.5 - 0.4\epsilon_{t-1}.$$

Therefore, knowing p_2 and ϵ_1 we can obtain ϵ_2. Once we have ϵ_2, knowing p_3, we can get ϵ_3, and so on up to ϵ_{t-1}. Therefore, the conditional expectation becomes

$$\begin{aligned}\mathbb{E}(p_t|p_{t-1}, \ldots, p_1, \epsilon_1) &= \mathbb{E}(p_t|\epsilon_{t-1}, \ldots, \epsilon_1) = \mathbb{E}(0.5 + \epsilon_t + 0.4\epsilon_{t-1}|\epsilon_{t-1}) \\ &= 0.5 + \mathbb{E}(\epsilon_t|\epsilon_{t-1}) + 0.4\mathbb{E}(\epsilon_{t-1}|\epsilon_{t-1}) \\ &= 0.5 + 0.4\epsilon_{t-1}.\end{aligned}$$

3. Let the price p_t of a stock be given by the following random-walk model

$$p_t = p_{t-1} + \epsilon_t, \qquad \text{for } t \in \mathbb{N}$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is an an $NID(0, \sigma^2)$ sequence with variance $\sigma^2 = 1$ and p_1 is a known constant.

(a) Show that the conditional mean changes over time, but the conditional variance does not.

(b) Show that the unconditional mean is constant and equal to p_1 for all t.

(c) Show that the unconditional variance is time-varying and equal to $(t-1)$.

Solution:

(a) The conditional mean is

$$\mathbb{E}(p_t|p_{t-1}) = \mathbb{E}(p_{t-1} + \epsilon_t|p_{t-1}) = \mathbb{E}(p_{t-1}|p_{t-1}) + \mathbb{E}(\epsilon_t|p_{t-1}) = p_{t-1}.$$

The conditional variance is

$$\mathbb{V}ar(p_t|p_{t-1}) = \mathbb{V}ar(p_{t-1} + \epsilon_t|p_{t-1}) = \mathbb{V}ar(p_{t-1}|p_{t-1}) + \mathbb{V}ar(\epsilon_t|p_{t-1}) = \mathbb{V}ar(\epsilon_t) = 1.$$

(b) We can unfold the recursion of p_t and obtain

$$p_t = p_{t-1} + \epsilon_t = p_{t-2} + \epsilon_{t-1} + \epsilon_t = p_1 + \epsilon_2 + \cdots + \epsilon_t = p_1 + \sum_{i=2}^{t} \epsilon_i.$$

Therefore, we have that

$$\mathbb{E}(p_t) = \mathbb{E}\left(p_1 + \sum_{i=2}^{t} \epsilon_i\right) = p_1 + \sum_{i=2}^{t} \mathbb{E}(\epsilon_i) = p_1.$$

(c) The variance is given by

$$\mathbb{V}ar(p_t) = \mathbb{V}ar\left(p_1 + \sum_{i=2}^{t} \epsilon_i\right) = \sum_{i=2}^{t} \mathbb{V}ar(\epsilon_i) = \sum_{i=2}^{t} 1 = t - 1,$$

where the variance of the sum of the ϵ_t is equal to the sum of the variances since $\{\varepsilon_t\}$ is an iid sequence and therefore uncorrelated.

4. Let the dynamics of a stock price p_t and a market index i_t be given by the following ADL-AR system

$$p_t = 0.5p_{t-1} + 0.2i_t + \epsilon_t$$
$$i_t = 0.6i_{t-1} + v_t,$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ and $\{v_t\}_{t\in\mathbb{Z}}$ are independent $NID(0,1)$ sequences of random variables. Show that the conditional covariance between p_t and i_t given p_{t-1} and i_{t-1} is constant in time.

Solution:
First we find the conditional means of p_t and i_t. The conditional mean of p_t is given by

$$\begin{aligned}\mathbb{E}(p_t|p_{t-1}, i_{t-1}) &= \mathbb{E}(0.5p_{t-1} + 0.2[0.6i_{t-1} + v_t] + \epsilon_t|p_{t-1}, i_{t-1})\\ &= 0.5p_{t-1} + 0.12i_{t-1}.\end{aligned}$$

This means that

$$p_t - \mathbb{E}(p_t|p_{t-1}, i_{t-1}) = \epsilon_t + 0.2v_t.$$

Similarly, for i_t we obtain

$$\begin{aligned}\mathbb{E}(i_t|p_{t-1}, i_{t-1}) &= \mathbb{E}(0.6i_{t-1} + v_t|p_{t-1}, i_{t-1})\\ &= 0.6i_{t-1}.\end{aligned}$$

This means that

$$i_t - \mathbb{E}(i_t|p_{t-1}, i_{t-1}) = v_t.$$

Finally, we are ready to obtain the conditional covariance as

$$\begin{aligned}\mathbb{C}ov(p_t, i_t|p_{t-1}, i_{t-1}) &= \mathbb{E}\left([p_t - \mathbb{E}(p_t|p_{t-1}, i_{t-1})][i_t - \mathbb{E}(i_t|p_{t-1}, i_{t-1})]|p_{t-1}, i_{t-1}\right)\\ &= \mathbb{E}\left([\epsilon_t + 0.2v_t]v_t|p_{t-1}, i_{t-1}\right)\\ &= \mathbb{E}\left(\epsilon_t v_t + 0.2v_t^2\right)\\ &= \mathbb{E}\left(\epsilon_t v_t\right) + 0.2\mathbb{E}\left(v_t^2\right) = 0.2.\end{aligned}$$

This shows that the conditional covariance is constant in time.

5. Let the dynamics of a stock price p_t and a market index i_t be given by the following ADL-AR system

$$p_t = 0.3p_{t-1} + 0.3p_{t-2} + 0.2i_t + 0.1i_{t-1} + \epsilon_t$$
$$i_t = 0.6i_{t-1} + 0.1i_{t-2} + v_t,$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ and $\{v_t\}_{t\in\mathbb{Z}}$ are independent $NID(0,1)$ sequences of random variables. Show that the conditional covariance between p_t and i_t given p_{t-1}, p_{t-2}, i_{t-1} and i_{t-2} is constant in time.

Solution:
First we find the conditional means of p_t and i_t. The conditional mean of p_t is given by

$$\begin{aligned}\mathbb{E}(p_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2}) &= \mathbb{E}(0.3p_{t-1} + 0.3p_{t-2} + 0.12i_{t-1} + 0.02i_{t-2} + 0.2v_t + 0.1i_{t-1} + \epsilon_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2})\\ &= 0.3p_{t-1} + 0.3p_{t-2} + 0.12i_{t-1} + 0.02i_{t-2} + 0.1i_{t-1}.\end{aligned}$$

This means that

$$p_t - \mathbb{E}(p_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2}) = \epsilon_t + 0.2v_t.$$

Similarly, for i_t we obtain

$$\begin{aligned}\mathbb{E}(i_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2}) &= \mathbb{E}(0.6i_{t-1} + 0.1i_{t-2} + v_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2})\\ &= 0.6i_{t-1} + 0.1i_{t-2}.\end{aligned}$$

This means that

$$i_t - \mathbb{E}(i_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2}) = v_t.$$

Finally, we are ready to obtain the conditional covariance as

$$\begin{aligned}\mathbb{C}ov(p_t, i_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2}) &= \mathbb{E}\left([p_t - \mathbb{E}(p_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2})][i_t - \mathbb{E}(i_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2})]|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2}\right)\\ &= \mathbb{E}\left([\epsilon_t + 0.2v_t]v_t|p_{t-1}, p_{t-2}, i_{t-1}, i_{t-2}\right)\\ &= \mathbb{E}\left(\epsilon_t v_t + 0.2v_t^2\right)\\ &= \mathbb{E}\left(\epsilon_t v_t\right) + 0.2\mathbb{E}\left(v_t^2\right) = 0.2.\end{aligned}$$

This shows that the conditional covariance is constant in time.

CHAPTER 2: ARCH models

1. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the following ARCH(1) model

$$y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.1 + 0.3 y_{t-1}^2.$$

The table below reports the error shocks ϵ_t for $t = 1, \dots 5$ and σ_1^2 for the above ARCH model. Fill in the empty cells of the table.

t	1	2	3	4	5
ϵ_t	0.5	-0.7	-1.2	0.9	-0.1
σ_t^2	1.1				
y_t					

Solution: We can iterate observation equation and updating equation to generate the series of σ_t^2 and y_t. First, we need to find y_t at time $t = 1$. This can be done through the observation equation.

$t = 1$) *Observation equation:* $y_1 = \sigma_1 \epsilon_1 = \sqrt{1.1} \times 0.5 \approx 0.52$

$t = 2$) *Updating equation:* $\sigma_2^2 = 0.1 + 0.3 y_1^2 = 0.1 + 0.3 \times 0.52^2 \approx 0.18$

Observation equation: $y_2 = \sigma_2 \epsilon_2 = \dots$

$t = 3$) *Updating equation:* $\dots$ *and so on.*

2. Obtain the AR representation of the following ARCH models.

(a) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.5 + 0.7 y_{t-1}^2$;

(b) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = \omega + 0.9 y_{t-2}^2$;

(c) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.1 + 0.7 y_{t-1}^2 + 0.3 y_{t-4}^2$;

State also the order q of each ARCH model as well as the order of the corresponding AR representation.

Solution:

(a) We first define the White Noise error $\eta_t = y_t^2 - \sigma_t^2$. Then we note that $\sigma_t^2 = y_t^2 - \eta_t$. Finally, we plug-in this expression in the updating equation and obtain

$$y_t^2 - \eta_t = 0.5 + 0.7 y_{t-1}^2.$$

Rearranging the above equation yields

$$y_t^2 = 0.5 + 0.7 y_{t-1}^2 + \eta_t.$$

This is an AR(1) model and the ARCH model is an ARCH(1).

(b) As before, we first define the White Noise error $\eta_t = y_t^2 - \sigma_t^2$. Then we note that $\sigma_t^2 = y_t^2 - \eta_t$. Finally, we plug-in this expression in the updating equation and obtain

$$y_t^2 - \eta_t = \omega + 0.9 y_{t-2}^2.$$

Rearranging the above equation yields

$$y_t^2 = \omega + 0.9 y_{t-2}^2 + \eta_t.$$

This is an AR(2) model and the ARCH model is a ARCH(2).

(c) As before, we first define the White Noise error $\eta_t = y_t^2 - \sigma_t^2$. Then we note that $\sigma_t^2 = y_t^2 - \eta_t$. Finally, we plug-in this expression in the updating equation and obtain

$$y_t^2 - \eta_t = 0.1 + 0.7 y_{t-1}^2 + 0.3 y_{t-4}^2.$$

Rearranging the above equation yields

$$y_t^2 = 0.1 + 0.7 y_{t-1}^2 + 0.3 y_{t-4}^2 + \eta_t.$$

This is an AR(4) model and the ARCH model is a ARCH(4).

3. Obtain the unconditional mean, variance, autocovariance and autocorrelation of y_t for each of the models in the previous exercise. Which models are weakly stationary?

Solution:

The mean, autocovariance and autocorrelation are the same for all the ARCH models in the exercise. This because these quantities do not depend on the specification of the updating equation. In particular, $\mathbb{E}(y_t) = 0$, $\mathbb{C}ov(y_t, y_{t-l}) = 0$ and $\mathbb{C}orr(y_t, y_{t-l}) = 0$ for any $l > 0$. The derivation of these results can be found in the lecture notes. In the following, we find the unconditional variance for each of the three cases.

(a) First we note that this ARCH model is weakly stationary. This can be seen as $\alpha_1 = 0.7 < 1$. The unconditional variance of y_t can be obtained using the AR representation because $\mathbb{V}ar(y_t) = \mathbb{E}(y_t^2)$. Therefore, using the AR representation obtained in the previous exercise we conclude that $\mathbb{E}(y_t^2) = 0.5/(1-0.7) = 1.67$.

(b) First we note that this ARCH model is weakly stationary. This can be seen as $\alpha_2 = 0.9 < 1$. The unconditional variance of y_t can be obtained using the ARMA representation because $\mathbb{V}ar(y_t) = \mathbb{E}(y_t^2)$. Therefore, using the ARMA representation obtained in the previous exercise we conclude that $\mathbb{E}(y_t^2) = \omega/(1-0.9) = 10\omega$.

(c) This ARCH model is not weakly stationary. This can be immediately seen as $\alpha_1+\alpha_4 = 0.7+0.3 = 1 \geq 1$. Therefore the unconditional variance is not finite.

4. Give conditions for the following ARCH models to be weakly stationary

(a) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = \omega + |1-\alpha| y_{t-1}^2;$

(b) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = \omega + 0.7 y_{t-1}^2 + \alpha y_{t-2}^2.$

(c) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = \alpha + \alpha y_{t-2}^2 + \alpha y_{t-3}^2;$

Solution:

(a) The model is an ARCH(1) model with $\alpha_1 = |1-\alpha|$. Therefore it is weakly stationary if α_1, which in this case is $|1-\alpha| < 1$. The restriction $|1-\alpha| < 1$ is equivalent to $1-\alpha < 1$ and $1-\alpha > -1$. Hence we obtain the condition $0 < \alpha < 2$.

(b) The model in question is an ARCH(2) model with $\alpha_1 = 0.7$ and $\alpha_2 = \alpha$. The weak stationarity condition is $\alpha_1 + \alpha_2 < 1$. In our case $0.7 + \alpha < 1$ and thus $\alpha < 0.3$.

(c) The model is an ARCH(3) model with $\alpha_1 = \alpha$, $\alpha_2 = 0$ and $\alpha_3 = \alpha$. The weak stationarity condition is $\alpha_1 + \alpha_2 + \alpha_3 < 1$. In our case $2\alpha < 1$ and thus $\alpha < 0.5$.

5. Consider the following ARCH model

$$y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.1 + 0.7 y_{t-1}^2 + 0.3 y_{t-3}^2.$$

Calculate the conditional probability that $y_t < -0.7$ given that $y_{t-1} = 1.1$, $y_{t-2} = -1.2$ and $y_{t-3} = -1.0$.

Solution:
We obtain the conditional probability as follows

$$\begin{aligned} P(y_t < -0.7 | Y^{t-1}) &= P\left(\frac{y_t}{\sigma_t} < \frac{-0.7}{\sigma_t} | Y^{t-1}\right) \\ &= \Phi\left(\frac{-0.7}{\sigma_t}\right), \end{aligned}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal and the second equality follows since $y_t | Y^{t-1} \sim N(0, \sigma_t^2)$ and therefore $\frac{y_t}{\sigma_t} | Y^{t-1} \sim N(0,1)$. Next, we can obtain the value of σ_t as follows

$$\sigma_t^2 = 0.1 + 0.7 \times 1.1^2 + 0.3 \times (-1)^2 = 1.247.$$

Therefore, the desired probability is

$$P(y_t < -0.7 | Y^{t-1}) = \Phi\left(\frac{-0.7}{\sqrt{1.247}}\right) = 0.265.$$

6. Assume $\{y_t\}_{t\in\mathbb{Z}}$ is generated by an ARCH(1) model with positive $\alpha_1 > 0$. Consider these 4 situations

   (a) $y_{t-1} = 1.1$;

   (b) $y_{t-1} = -1.1$;

   (c) $y_{t-1} = 0.5$;

   (d) $y_{t-1} = -0.5$.

   For which of these four cases do we have the highest conditional probability of having $y_t < -0.7$, i.e. $P(y_t < -0.7|Y^{t-1})$? and the lowest? Explain why.

   Solution:
   We know that y_t is an ARCH(1) and therefore $y_t|Y^{t-1} \sim N(0, \sigma_t^2)$ with $\sigma_t^2 = \omega + \alpha_1 y_{t-1}^2$. From the expression of σ_t^2, we obtain that $\sigma_{a,t}^2 = \sigma_{b,t}^2$ and $\sigma_{c,t}^2 = \sigma_{d,t}^2$. Furthermore, since $\alpha_1 > 0$, we have that $\sigma_{a,t}^2 = \sigma_{b,t}^2 > \sigma_{c,t}^2 = \sigma_{d,t}^2$. Therefore, since $P(y_t < -0.7|Y^{t-1}) = \Phi\left(\frac{-0.7}{\sigma_t}\right)$ and $\Phi(\cdot)$ is a positive and increasing function, we conclude that $P(y_t < -0.7|Y^{t-1})$ for (a) and (b) is bigger than for (c) and (d).

CHAPTER 3: GARCH models

1. Let $\{y_t\}_{t\in\mathbb{Z}}$ be generated by the following GARCH(1,1) model

$$y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = 0.1 + 0.3 y_{t-1}^2 + 0.6\sigma_{t-1}^2.$$

The table below reports the error shocks ϵ_t, for $t = 1, \dots 5$, and the initial values σ_1^2 and y_1 for the above GARCH(1,1) model. Fill in the empty cells of the table.

t	1	2	3	4	5
ϵ_t	0.5	-0.7	-1.2	0.9	-0.1
σ_t^2	1.1				
y_t					

Solution:
First we need to find y_t at time $t = 1$. This can be done through the observation equation.
$t = 1$) *Observation equation:* $y_1 = \sigma_1\epsilon_1 = \sqrt{1.1} \times 0.5 \approx 0.52$
$t = 2$) *Updating equation:* $\sigma_2^2 = 0.1 + 0.3y_1^2 + 0.6\sigma_1^2 = 0.1 + 0.3 \times 0.52^2 + 0.6 \times 1.1 \approx 0.84$
Observation equation: $y_2 = \sigma_2\epsilon_2 = \sqrt{0.84} \times (-0.7) \approx -0.64$
$t = 3$) *Updating equation:* $\sigma_3^2 = 0.1 + 0.3y_2^2 + 0.6\sigma_2^2 = 0.1 + 0.3 \times (-0.64)^2 + 0.6 \times 0.85 \approx 0.73$
Observation equation: $y_3 = \sigma_3\epsilon_3 = \sqrt{0.73} \times (-1.2) \approx -1.03$
$t = 4$) *Updating equation:* $\sigma_4^2 = 0.1 + 0.3y_3^2 + 0.6\sigma_3^2 = 0.1 + 0.3 \times (-1.03)^2 + 0.6 \times 0.73 \approx 0.86$
Observation equation: $y_4 = \sigma_4\epsilon_4 = \sqrt{0.86} \times 0.9 \approx 0.83$
$t = 5$) *Updating equation:* $\sigma_5^2 = 0.1 + 0.3y_4^2 + 0.6\sigma_4^2 = 0.1 + 0.3 \times 0.83^2 + 0.6 \times 0.86 \approx 0.82$
Observation equation: $y_5 = \sigma_5\epsilon_5 = \sqrt{0.82} \times (-0.1) \approx -0.09$

2. Derive the ARCH(∞) representation of a GARCH(1,1) model.

Solution:
The ARCH(∞) representation of a GARCH(1,1) model is obtained unfolding the GARCH updating equation as follows

$$\begin{aligned}
\sigma_t^2 &= \omega + \beta_1\sigma_{t-1}^2 + \alpha_1 y_{t-1}^2 \\
&= \omega + \beta_1(\omega + \beta_1\sigma_{t-2}^2 + \alpha_1 y_{t-2}^2) + \alpha_1 y_{t-1}^2 \\
&= \omega + \beta_1\omega + \beta_1^2\sigma_{t-2}^2 + \alpha_1\beta_1 y_{t-2}^2 + \alpha_1 y_{t-1}^2 \\
&= \omega + \beta_1\omega + \beta_1^2(\omega + \beta_1\sigma_{t-3}^2 + \alpha_1 y_{t-3}^2) + \alpha_1\beta_1 y_{t-2}^2 + \alpha_1 y_{t-1}^2 \\
&= \omega + \beta_1\omega + \beta_1^2\omega + \beta_1^3\sigma_{t-3}^2 + \alpha_1\beta_1^2 y_{t-3}^2 + \alpha_1\beta_1 y_{t-2}^2 + \alpha_1 y_{t-1}^2.
\end{aligned}$$

Therefore after k iterations we get

$$\begin{aligned}
\sigma_t^2 &= \omega + \beta_1\omega + \dots + \beta_1^{k-1}\omega + \beta_1^k\sigma_{t-k}^2 + \alpha_1\beta_1^{k-1} y_{t-k}^2 + \dots + \alpha_1\beta_1 y_{t-2}^2 + \alpha_1 y_{t-1}^2 \\
&= \omega\sum_{i=0}^{k-1}\beta_1^i + \beta_1^k\sigma_{t-k}^2 + \alpha_1\sum_{i=0}^{k-1}\beta_1^i y_{t-i-1}^2.
\end{aligned}$$

Now we can take the limit $k \to \infty$ and obtain

$$\begin{aligned}
\sigma_t^2 &= \omega\sum_{i=0}^{\infty}\beta_1^i + \alpha_1\sum_{i=0}^{\infty}\beta_1^i y_{t-i-1}^2 \\
&= \frac{\omega}{1-\beta_1} + \alpha_1\sum_{i=0}^{\infty}\beta_1^i y_{t-i-1}^2.
\end{aligned}$$

This concludes the proof and show that a GARCH(1,1) can be rewritten as an ARCH(∞).

3. Obtain the ARMA representation of the following GARCH models.

(a) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.1 + 0.3 y_{t-1}^2 + 0.3 \sigma_{t-1}^2$;

(b) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.1 + 0.3 y_{t-2}^2 + 0.3 \sigma_{t-2}^2$;

(c) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.1 + 0.5 y_{t-1}^2 + 0.3 y_{t-3}^2 + 0.2 \sigma_{t-1}^2$;

State also the order p and q of each GARCH model as well as the order of the corresponding ARMA representation.

Solution:

(a) We first define the White Noise error $\eta_t = y_t^2 - \sigma_t^2$. Then we note that $\sigma_t^2 = y_t^2 - \eta_t$ and $\sigma_{t-1}^2 = y_{t-1}^2 - \eta_{t-1}$. Finally, we plug-in these expressions in the updating equation and obtain

$$y_t^2 - \eta_t = 0.1 + 0.3 y_{t-1}^2 + 0.3(y_{t-1}^2 - \eta_{t-1}).$$

Rearranging the above equation yields

$$y_t^2 = 0.1 + 0.6 y_{t-1}^2 + \eta_t - 0.3 \eta_{t-1}.$$

This is an ARMA(1,1) model and the GARCH model is a GARCH(1,1).

(b) As before, we first define the White Noise error $\eta_t = y_t^2 - \sigma_t^2$. Then we note that $\sigma_t^2 = y_t^2 - \eta_t$ and $\sigma_{t-2}^2 = y_{t-2}^2 - \eta_{t-2}$. Finally, we plug-in these expressions in the updating equation and obtain

$$y_t^2 - \eta_t = 0.1 + 0.3 y_{t-2}^2 + 0.3(y_{t-2}^2 - \eta_{t-2}).$$

Rearranging the above equation yields

$$y_t^2 = 0.1 + 0.6 y_{t-2}^2 + \eta_t - 0.3 \eta_{t-2}.$$

This is an ARMA(2,2) model and the GARCH model is a GARCH(2,2).

(c) As before, we first define the White Noise error $\eta_t = y_t^2 - \sigma_t^2$. Then we note that $\sigma_t^2 = y_t^2 - \eta_t$ and $\sigma_{t-1}^2 = y_{t-1}^2 - \eta_{t-1}$. Finally, we plug-in these expressions in the updating equation and obtain

$$y_t^2 - \eta_t = 0.1 + 0.5 y_{t-1}^2 + 0.3 y_{t-3}^2 + 0.2(y_{t-1}^2 - \eta_{t-1}).$$

Rearranging the above equation yields

$$y_t^2 = 0.1 + 0.7 y_{t-1}^2 + 0.3 y_{t-3}^2 + \eta_t - 0.2 \eta_{t-1}.$$

This is an ARMA(3,1) model and the GARCH model is a GARCH(1,3).

4. Obtain the unconditional mean, variance, autocovariance and autocorrelation of y_t for each of the models in the previous exercise. Which models are weakly stationary?

Solution:

The mean, autocovariance and autocorrelation are the same for all the GARCH models in the exercise. This because these quantities do not depend on the specification of the updating equation. In particular, $\mathbb{E}(y_t) = 0$, $\mathbb{C}ov(y_t, y_{t-l}) = 0$ and $\mathbb{C}orr(y_t, y_{t-l}) = 0$ for any $l > 0$. The derivation of these results can be found in the lecture notes. In the following, we find the unconditional variance for each of the three cases.

(a) First we note that this GARCH model is weakly stationary. This can be seen as $\alpha_1 + \beta_1 = 0.3 + 0.3 = 0.6 < 1$. The unconditional variance of y_t can be obtained using the ARMA representation because $\mathbb{V}ar(y_t) = \mathbb{E}(y_t^2)$. Therefore, using the ARMA representation obtained in the previous exercise we conclude that $\mathbb{E}(y_t^2) = 0.1/(1 - 0.6) = 0.25$.

(b) First we note that this GARCH model is weakly stationary. This can be seen as $\alpha_2 + \beta_2 = 0.3 + 0.3 = 0.6 < 1$. The unconditional variance of y_t can be obtained using the ARMA representation because $\mathbb{V}ar(y_t) = \mathbb{E}(y_t^2)$. Therefore, using the ARMA representation obtained in the previous exercise we conclude that $\mathbb{E}(y_t^2) = 0.1/(1 - 0.6) = 0.25$.

(c) This GARCH model is not weakly stationary. This can be immediately seen as $\alpha_1 + \alpha_3 + \beta_1 = 0.5 + 0.3 + 0.2 = 1 \geq 1$. Therefore the unconditional variance is not finite.

5. Give conditions for the following GARCH models to be weakly stationary?

(a) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = \omega + 0.7 y_{t-1}^2 + \beta \sigma_{t-1}^2$.

(b) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = \omega + |1-\beta| \sigma_{t-1}^2 + 0.1 y_{t-1}^2$;

(c) $y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = \alpha + \beta y_{t-2}^2 + \beta \sigma_{t-3}^2$;

Solution:

(a) The model in question is a GARCH(1,1) with $\alpha_1 = 0.7$ and $\beta_1 = \beta$. Therefore, we have weak stationarity if $0.7 + \beta < 1$ and thus $\beta < 0.3$.

(b) The model in question is a GARCH(1,1) with $\alpha_1 = 0.1$ and $\beta_1 = |1-\beta|$. Therefore, we have weak stationarity if $0.1 + |1-\beta| < 1$ and thus $0.1 < \beta < 1.9$.

(c) The model in question is a GARCH(3,2) with $\alpha_1 = 0$, $\alpha_2 = \beta$, $\beta_1 = 0$, $\beta_2 = 0$ and $\beta_3 = \beta$. Therefore, we have weak stationarity if $\beta + \beta < 1$ and thus $\beta < 0.5$.

6. Consider the following GARCH model

$$y_t = \sigma_t \epsilon_t, \;\; \sigma_t^2 = 0.4 + 0.2 y_{t-2}^2 + 0.5 \sigma_{t-1}^2.$$

Calculate the conditional probability that $y_t < -0.7$ given that $y_{t-1} = 1.1$, $y_{t-2} = -1.2$, $\sigma_{t-1}^2 = 1.2$ and $\sigma_{t-2}^2 = 0.9$. Can you also exactly calculate the unconditional probability that $y_t < -0.7$? Explain why.

Solution:

We know that the conditional distribution of a GARCH model is

$$y_t | Y^{t-1} \sim N(0, \sigma_t^2).$$

Therefore we obtain

$$P\{y_t < -0.7 | Y^{t-1}\} = P\{y_t/\sigma_t < -0.7/\sigma_t | Y^{t-1}\} = \Phi(-0.7/\sigma_t),$$

where $\Phi(\cdot)$ is the cdf of a standard normal distribution. Finally, we obtain the value of σ_t^2 as

$$\sigma_t^2 = 0.4 + 0.2 \times (-1.2)^2 + 0.5 \times 1.2 \approx 1.29.$$

Therefore we conclude that

$$P\{y_t < -0.7 | Y^{t-1}\} = \Phi(-0.7/\sqrt{1.29}) = \Phi(-0.62).$$

We cannot exactly calculate the unconditional probability that y_t is smaller than -0.7 because the unconditional distribution of GARCH model has an unknown functional form.