Acquisition and Analysis of Neural Data

Bernstein Center for Computational Neuroscience Berlin
Lecturer: Prof. Dr. Richard Kempter
Tutors: Gaspar Cano, Stefano Masserini

Problem Set 1 Deadline: Tuesday 25th April, 2023 at 23:55

Please work in groups of ideally 2 students and submit a single joint solution per group. Include the names of all group members in the files you submit. Please specify how much time you needed to finish each part.
Analytical Part: scan and upload your solution as a single PDF. Make sure all text is clearly readable.
Numerical Part: submit both a Python code file (or notebook) and a PDF or HTML file including figures and code. Use clear, descriptive variable names and comments.

Analytical Part

Convolution and Dirac delta

The convolution of two functions $u(t)$ and $v(t)$ is defined as

$$(u * v)(t) = \int_{-\infty}^{\infty} u(t-s)v(s)\, \mathrm{d}s\,. \tag{1}$$

Convolution is used widely in mathematics, physics, and engineering, e.g., in signal processing and linear systems' analysis. In neuroscience, it could be used to estimate firing rates from discrete spike times. With the following exercises, you will get an intuitive understanding of the convolution operation.

1. Graphical convolution (3 points)
The response $A = x * h$ of a linear, time invariant, system is given by the convolution of the input signal $x(t)$ and the system's impulse response $h(t)$. Here, we perform a "graphical" convolution between x and h, by computing the area under the product of the two functions. Consider the following functions:

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ 3 & \text{for } 0 \leqslant t \leqslant 3 \\ 0 & \text{for } t > 3 \end{cases} \quad \text{and} \quad h(t) = \begin{cases} 0 & \text{for } t < 0 \\ 4 & \text{for } 0 \leqslant t \leqslant 1 \\ 0 & \text{for } t > 1\,. \end{cases}$$

(a) Sketch $x(t-s)$ and $h(s)$ as a function of s for a fixed, generic value of t. Identify the intervals of t for which the system has a different linear response. For each interval, sketch again $x(t-s)$ and $h(s)$ as a function of s. This will help you to compute the system's response.

(b) Compute the system's response $A(t) = \int_{-\infty}^{\infty} x(t-s)h(s)\, \mathrm{d}s$ and sketch the result.

2. Properties of the convolution (3 points)
Prove the following properties of the convolution:

(i) Commutative: $(u * v)(t) = (v * u)(t)$
(ii) Distributive: $(u * (\alpha v + \beta w))(t) = \alpha\,(u * v)(t) + \beta\,(u * w)(t)$ for all $\alpha, \beta \in \mathbb{R}$
(iii) Associative: $(u * (v * w))(t) = ((u * v) * w)(t)$

where $*$ stands for convolution. Hint: use the definition of the convolution and change of variables.

3. The Dirac delta (4 points)
The Dirac delta is commonly used to analytically describe the spike times of a neuron. This exercise gets you acquainted with two ways of defining the Dirac delta and how one of them is not mathematically rigorous.

(a) Consider a rectangular function $R_A(t)$ centered at 0 with width A and height $1/A$ such that

$$R_A(t) = \begin{cases} 1/A & \text{for} \quad -\frac{A}{2} \leq t \leq \frac{A}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

  (i) Sketch and explain what happens to R_A for arbitrarily small A.
  (ii) Calculate $\int_{-\infty}^{\infty} R_A(t)\,\mathrm{d}t$ for any $A > 0$.

(b) **Extra:** The Dirac delta often appears defined as the limit of functions $\delta(t) = \lim_{A\to 0} R_A(t)$, while satisfying the condition $\int_{-\infty}^{\infty} \delta(s)\,\mathrm{d}s = 1$. **(1 extra point)**

  (i) Why can we not calculate the Riemann integral $\int_{-\infty}^{\infty} R_A(t)\,\mathrm{d}t$ for $A = 0$?
  (ii) Explain why the property $\int_{-\infty}^{\infty} D(t)\,\mathrm{d}t = 1$ is not fulfilled by any real-valued function D with $D(t) = 0$ for all $t \neq 0$.
  (iii) Use (i) and (ii) to argue why the definition in (b) cannot be mathematically rigorous.

(c) Let's now look at a mathematically correct definition: The Dirac delta $\delta(t)$ has the defining property

$$\int_{-\infty}^{\infty} f(t-s)\delta(s)\,\mathrm{d}s = f(t). \tag{3}$$

$\delta(t)$ is not a function in the common sense but a *distribution* (or "generalized function"). Thus, the left hand side of equation (3) is not a Riemann integral but can instead be defined as:

$$\int_{-\infty}^{\infty} f(t-s)\delta(s)\,\mathrm{d}s := \lim_{A\to 0} \int_{-\infty}^{\infty} f(t-s)R_A(s)\,\mathrm{d}s. \tag{4}$$

  (i) Show that $\int_{-\infty}^{\infty} f(t-s)R_A(s)\,\mathrm{d}s = \frac{F(t+A/2)-F(t-A/2)}{A}$, where F is a primitive function of f.
  (ii) Show that the defining property (3) of the Dirac delta holds for (4).
  Hint: Use (i) and the definition of derivative $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lim_{h\to 0} \frac{x(t+h)-x(t-h)}{2h}$.
  (iii) Use (3) to show that

$$\int_{-\infty}^{\infty} \delta(s)\,\mathrm{d}s = 1. \tag{5}$$

Keep in mind that the Dirac delta is not a function but a *distribution*. Even though it is often intuitively also used on its own, it is an object that rigorously only makes sense when applied to a so-called test function f (see (3)).

Numerical Part

Estimating firing rates from spike data

1. Firing rate estimation and convolution (10 points)
The goal of this exercise is to learn how to estimate firing rates from spike train data. To start, you need to download ExampleSpikeTimes1.dat from the Moodle course page.

This .dat-file can be loaded in Python with the numpy function loadtxt(). The file contains a variable named SpikeTimes, which is a vector of spike times t_i (in ms) with $i = 1, 2, \ldots, n$ in response to a stimulus with a duration of $T = 4000\,\mathrm{ms}$. The temporal resolution of the spike times is 0.1 ms. Hint: it is convenient to convert the data in seconds. Here, as in all other exercises, use numpy arrays and avoid loops as much as possible.

Your task is to estimate the firing rate with different methods. Hint: Your plots should be organized similar to Fig. 1.4 on page 12 of the Dayan and Abbott book.

(a) (1 point) Plot the raw spike train. Have a look at the pyplot function eventplot. You may need to set the axes limits manually for correct visualization.

(b) (3 points) Construct spike-count histograms with non-overlapping bins of widths $\Delta t = 20$, 50 and 150 ms, and normalize them to obtain a firing rate in spikes/s. Hint: Have a look at the pyplot function hist and the weights parameter.

(c) (5 points) Use the window functions defined below to estimate an approximate firing rate

$$r_{\text{approx}}(t) = \sum_{\text{i}=1}^{n} w(t - t_{\text{i}}) = \int_0^T \mathrm{d}\tau \, w(t-\tau)\rho(\tau)$$

where $\rho(t) := \sum_{i=1}^{n} \delta(t - t_{\text{i}})$ is the neural response function and T is the trial length. Hint: Have a look at the analytical exercises on the convolution and the Dirac delta on this sheet. You can use the numpy function convolve. Make sure you understand the differences between the convolution modes full and same, particularly regarding time offsets and boundary effects. You could plot the spikes on top of the rates to check that you obtained the correct time offset. Firing rates shall be plotted in units of spikes/s.

(i.) Estimate the rate with a rectangular window

$$w(\tau) = \begin{cases} 1/\Delta t & \text{if } -\Delta t/2 \leq \tau < \Delta t/2 \\ 0 & \text{otherwise} \end{cases}$$

with widths of $\Delta t = 20$, 50 and 150 ms. The rectangular window is *centered* at each spike, i.e., the filtering is non causal.

(ii.) Estimate the rate with a Gaussian window

$$w(\tau) = \frac{1}{\sqrt{2\pi}\sigma_{\text{w}}} \exp\left(-\frac{\tau^2}{2\sigma_{\text{w}}^2}\right)$$

with $\sigma_{\text{w}} = 10$, 20 and 50 ms. Here the Gaussian is centered at each spike (non causal filtering).

(iii.) Estimate the rate with an alpha function

$$w(\tau) = \left[\alpha^2 \, \tau \, \exp(-\alpha\tau)\right]_+$$

where $1/\alpha$ is a time scale and $[x]_+$ is the half-wave rectification function. Use $1/\alpha = 10$, 20 and 50 ms. Here the rate starts rising only *after* a spike is emitted, i.e., the filter is causal. Why would one want to use a causal rather than a non-causal filter?

(d) (1 point) Calculate the spike count-rate $r = \frac{n}{T} = \frac{1}{T}\int_0^T \mathrm{d}t \, \rho(t)$.

2. Extra: estimation of firing rate and trial averaging (3 extra points)

(a) (1 extra point) Estimate firing rates for the spike data in ExampleSpikeTimes2.dat to see if your code is flexible enough to handle it. Spike timings are in milliseconds, the trial length is $T = 10$ s, and the resolution is 0.1 ms.

(b) (2 extra points) Download ExampleSpikeTimes3.dat and calculate first the trial-averaged neural response and then the trial-averaged firing rate (only one convolution). You can use a filter of your choice. Load the data using the numpy function loadtxt() with the keyword parameter delimiter=';'. The file contains a matrix of spike times in milliseconds. The dimensions are e͡xtsf[m x n, where m=100 is the maximum spike-time index and n=1000 is the number of trials. Note: each trial has a different number of spikes, the remaining entries are filled with nan. The trial length is $T = 1$ s, the resolution is 0.1 ms.

Gaspar Cano | Phone: 2093 98406 | Email: gaspar.cano at hu-berlin.de | Office: Haus 4, Room 106
Stefano Masserini | Phone: 2093 98406 | Email: stefanomasse at gmail.com | Office: Haus 4, Room 106