Fitting the distribution of heights data

Instructions

In this assessment you will write code to perform a steepest descent to fit a Gaussian model to the distribution of heights data that was first introduced in *Mathematics for Machine Learning: Linear Algebra*.

The algorithm is the same as you encountered in *Gradient descent in a sandpit* but this time instead of descending a pre-defined function, we shall descend the χ^2 (chi squared) function which is both a function of the parameters that we are to optimise, but also the data that the model is to fit to.

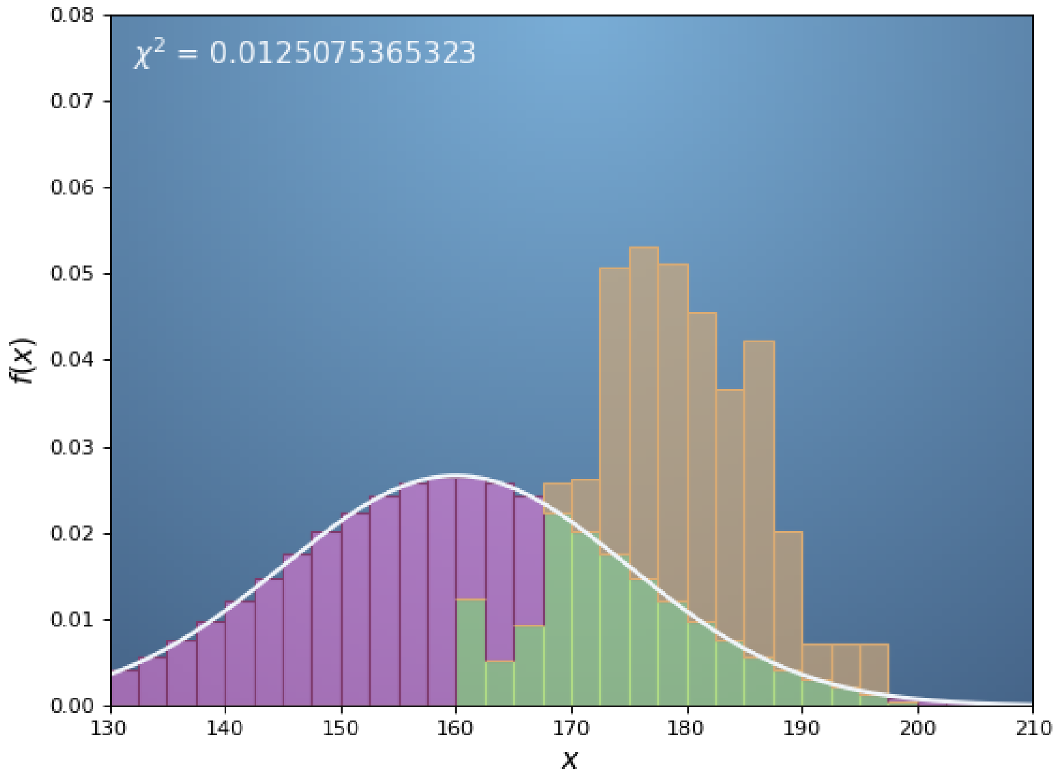
How to submit

Complete all the tasks you are asked for in the worksheet. When you have finished and are happy with your code, press the **Submit Assingment** button at the top of this notebook.

Get started

Run the cell below to load dependancies and generate the first figure in this worksheet.

In [1]:
```
# Run this cell first to load the dependancies for this assessment,
# and generate the first figure.
from readonly.HeightsModule import *
```



Background

If we have data for the heights of people in a population, it can be plotted as a histogram, i.e., a bar chart where each bar has a width representing a range of heights, and an area which is the probability of finding a person with a height in that range. We can look to model that data with a function, such as a Gaussian, which we can specify with two parameters, rather than holding all the data in the histogram.

The Gaussian function is given as,

$$f(\mathbf{x};\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\mathbf{x}-\mu)^2}{2\sigma^2}\right)$$

The figure above shows the data in orange, the model in magenta, and where they overlap in green. This particular model has not been fit well - there is not a strong overlap.

Recall from the videos the definition of χ^2 as the squared difference of the data and the model, i.e $\chi^2 = |\mathbf{y} - f(\mathbf{x};\mu,\sigma)|^2$. This is represented in the figure as the sum of the squares of the pink and orange bars.

Don't forget that $\mathbf{x}$ an $\mathbf{y}$ are represented as vectors here, as these are lists of all of the data points, the |*abs-squared*|^2 encodes squaring and summing of the residuals on each bar.

To improve the fit, we will want to alter the parameters μ and σ, and ask how that changes the χ^2. That is, we will need to calculate the Jacobian,

$$\mathbf{J} = \left[\frac{\partial(\chi^2)}{\partial\mu}, \frac{\partial(\chi^2)}{\partial\sigma}\right] \ .$$

Let's look at the first term, $\frac{\partial(\chi^2)}{\partial\mu}$, using the multi-variate chain rule, this can be written as,

$$\frac{\partial(\chi^2)}{\partial\mu} = -2(\mathbf{y} - f(\mathbf{x};\mu,\sigma)) \cdot \frac{\partial f}{\partial\mu}(\mathbf{x};\mu,\sigma)$$

With a similar expression for $\frac{\partial(\chi^2)}{\partial\sigma}$; try and work out this expression for yourself.

The Jacobians rely on the derivatives $\frac{\partial f}{\partial\mu}$ and $\frac{\partial f}{\partial\sigma}$. Write functions below for these.

In [2]:
```
# PACKAGE
import matplotlib.pyplot as plt
import numpy as np
```

In [4]:
```
# GRADED FUNCTION

# This is the Gaussian function.
def f (x,mu,sig) :
    return np.exp(-(x-mu)**2/(2*sig**2)) / np.sqrt(2*np.pi) / sig

# Next up, the derivative with respect to μ.
# If you wish, you may want to express this as f(x, mu, sig) multiplied by cha
in rule terms.
# === COMPLETE THIS FUNCTION ===
def dfdmu (x,mu,sig) :
    return f(x, mu, sig) * ((x-mu) / sig**2)

# Finally in this cell, the derivative with respect to σ.
# === COMPLETE THIS FUNCTION ===
def dfdsig (x,mu,sig) :
    return -f(x, mu, sig)/sig + f(x, mu, sig) * (x-mu)**2/sig**3
```

Next recall that steepest descent shall move around in parameter space proportional to the negative of the Jacobian, i.e., $\begin{bmatrix} \delta\mu \\ \delta\sigma \end{bmatrix} \propto -\mathbf{J}$, with the constant of proportionality being the *aggression* of the algorithm.

Modify the function below to include the $\frac{\partial(\chi^2)}{\partial\sigma}$ term of the Jacobian, the $\frac{\partial(\chi^2)}{\partial\mu}$ term has been included for you.

In [5]:
```
# GRADED FUNCTION

# Complete the expression for the Jacobian, the first term is done for you.
# Implement the second.
# === COMPLETE THIS FUNCTION ===
def steepest_step (x, y, mu, sig, aggression) :
    J = np.array([
        -2*(y - f(x,mu,sig)) @ dfdmu(x,mu,sig),
        -2*(y - f(x,mu,sig)) @ dfdsig(x,mu,sig) # Replace the ??? with the sec
ond element of the Jacobian.
    ])
    step = -J * aggression
    return step
```

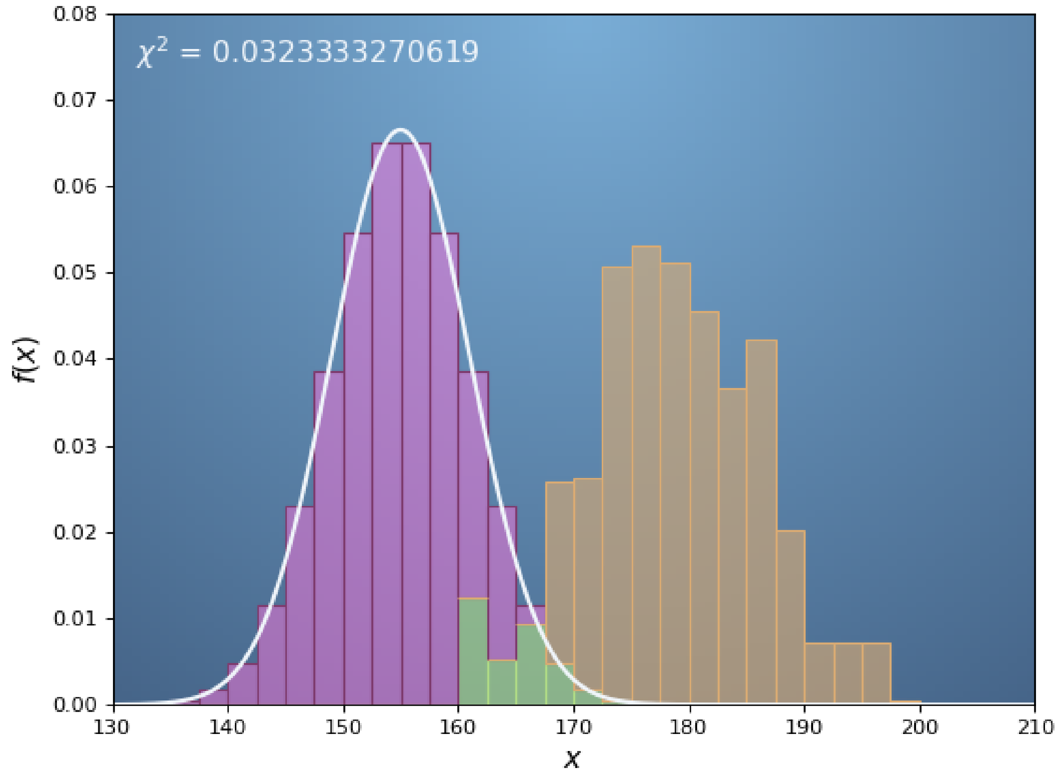
Test your code before submission

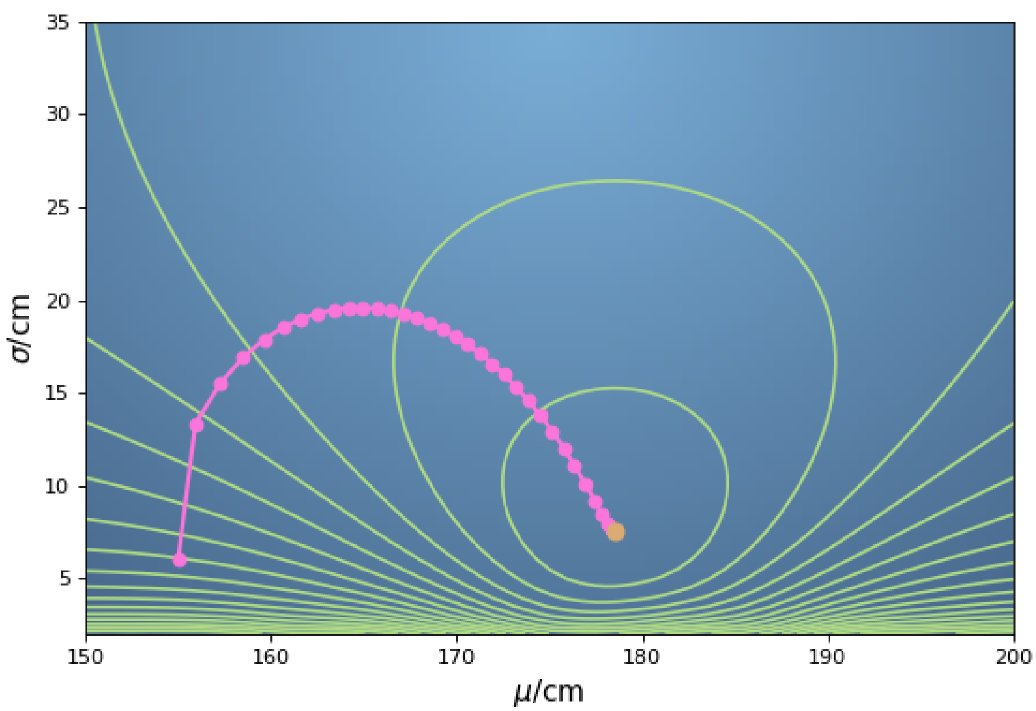
To test the code you've written above, run all previous cells (select each cell, then press the play button [▶|] or press shift-enter). You can then use the code below to test out your function. You don't need to submit these cells; you can edit and run them as much as you like.

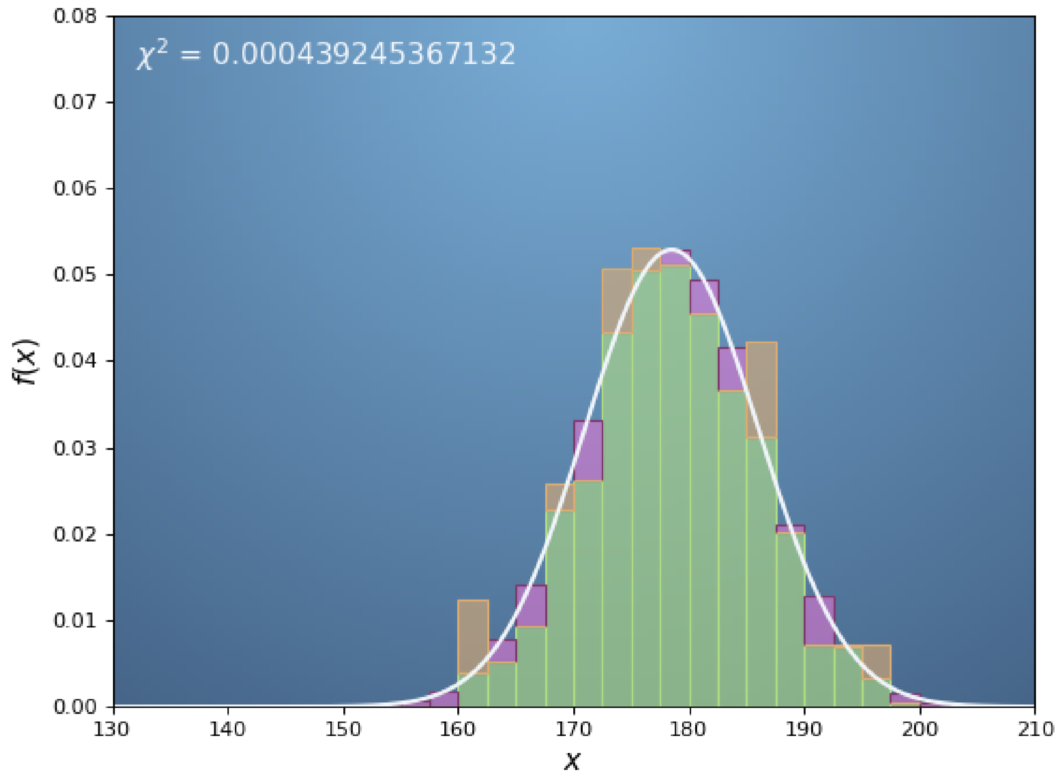
In [6]:
```
# First get the heights data, ranges and frequencies
x,y = heights_data()

# Next we'll assign trial values for these.
mu = 155 ; sig = 6
# We'll keep a track of these so we can plot their evolution.
p = np.array([[mu, sig]])

# Plot the histogram for our parameter guess
histogram(f, [mu, sig])
# Do a few rounds of steepest descent.
for i in range(50) :
    dmu, dsig = steepest_step(x, y, mu, sig, 2000)
    mu += dmu
    sig += dsig
    p = np.append(p, [[mu,sig]], axis=0)
# Plot the path through parameter space.
contour(f, p)
# Plot the final histogram.
histogram(f, [mu, sig])
```


χ^2 = 0.0323333270619
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
130
140
150
160
170
180
190
200
210
$f(x)$
x


35
30
25
20
15
10
5
150
160
170
180
190
200
σ/cm
μ/cm



Note that the path taken through parameter space is not necesarily the most direct path, as with steepest descent we always move perpendicular to the contours.