Project 2 (Z5399014)

B2

Process Model (State Equation)

The process model for the attitude estimator is given by:

$$\frac{dX}{dt} = F(X, U)$$

where:

- X is the state vector representing the attitude (roll, pitch, yaw) in radians.
- u represents the gyroscope measurements (angular rates in rad/s).
- F(X,u) is the nonlinear function describing the dynamics of the attitude.

The Jacobian matrix $\mathrm{J} = \frac{\partial \mathrm{X}}{\partial \mathrm{F}}$ is assumed to be provided.

Covariance Matrix Update

The covariance matrix for the prediction step of the EKF is calculated as:

$$P(\, k+1 \mid k\,) = J \cdot P(\, k \mid k\,) \cdot J^T + Q$$

where:

- P(k|k) is the current covariance matrix.
- Q is the process noise covariance matrix.

Calculation of J_u:

The Jacobian J_u maps gyroscope measurement noise to the state uncertainty. For the augmented state $X = [\phi, \theta, \psi]^T$, it is derived from the **attitude dynamics** (Euler angle rates):

$$\dot{\phi} = \omega_x + b_x + \left(\omega_y \cdot sin\phi + \omega_z \cdot cos\phi\right) \cdot tan\theta$$

$$\dot{\theta} = \omega_y \cdot cos\phi - \omega_z \cdot sin\phi$$

$$\dot{\psi} = \frac{\omega_y \cdot sin\phi + \omega_z \cdot cos\phi}{cos\theta}$$

The Jacobian J_u is the partial derivative of these rates w.r.t. the gyroscope inputs $\omega_x, \omega_y, \omega_z$

$$J_u = \frac{\partial[\dot{\phi}, \dot{\theta}, \dot{\psi}]}{\partial[\omega_x, \omega_y, \omega_z]}.$$

Hence on calculating partial derivative for the same we get:

$$J_u = \begin{bmatrix} 1 & sin\phi \cdot tan\theta & cos\phi \cdot tan\theta \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi / cos\theta & cos\theta / cos\phi \end{bmatrix}$$

Calculation of Q

The process noise covariance matrix Q accounts for the noise in the gyroscope measurements. Each gyroscope measurement is assumed to be polluted by white Gaussian noise (WGN) with a standard deviation of $4\,\frac{deg}{s}$nverting this to radians:

$$\sigma\omega = 4 \times \frac{\pi}{180} \text{ rad/s}$$

The Q matrix is derived from the Jacobian of the input u (gyroscope measurements) and the noise characteristics:

$$Q = Ju \cdot (\sigma\omega^2 \cdot I_3) \cdot Ju^T \cdot \Delta t$$

where:

- Ju is the Jacobian of the state equation with respect to the gyroscope inputs.
- Δt is the time step (5 ms = 0.005 s).

B3

A) Initialization of the EKF

The initial state and covariance matrix are set as follows:

- **Expected initial attitude:**

$$\hat{X}(0 \mid 0) = [0; 0; 0]\, rad$$

This assumes the platform starts at zero roll, pitch, and yaw.

Explanation:

- This represents an initial guess of zero roll, pitch, and yaw.
- If additional sensors (e.g., accelerometers, magnetometers) provide a better initial estimate, it can be adjusted.

B) Initial covariance matrix:

Given:

- Each attitude component may be wrong by 5 degrees.
- We model this uncertainty as independent Gaussian noise for each Euler angle.

$$P(0 \mid 0) = \begin{bmatrix} \sigma_\phi^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\psi^2 \end{bmatrix}$$

where each standard deviation σϕ, σθ, σψ corresponds to an uncertainty of 5° in each attitude component:

$$\sigma\phi = \sigma\theta = \sigma\psi = 5 \times \frac{\pi}{180}\, rad \approx 0.0873 rad$$

Thus:

$$P(0 \mid 0) = (5 \times \pi/180)^2 \cdot I_3 = \begin{bmatrix} 0.0076 & 0 & 0 \\ 0 & 0.0076 & 0 \\ 0 & 0 & 0.0076 \end{bmatrix}$$

This means we are 95% confident that the true attitude lies within $\pm 10^\circ (2\sigma)$ of the initial estimate.

B4

Output Equation for Floor Detection

When the floor is detected, the normal vector of the floor is measured in the platform's coordinate frame (CF). The output equation is:

$$Y = h(X) = R^T(X) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where:

- R(X) is the rotation matrix from the global coordinate frame (GCF) to the platform's CF, parameterized by the attitude $X = [\phi, \theta, \psi]^T$
- $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the normal vector of the floor in the GCF.

Measurement Noise and R Matrix

The measurement noise is assumed to be WGN with a standard deviation of 0.05 for each component of the normal vector. The measurement noise covariance matrix is:

$$R = \begin{bmatrix} 0.05^2 & 0 & 0 \\ 0 & 0.05^2 & 0 \\ 0 & 0 & 0.05^2 \end{bmatrix}$$

Analytical Expression for H Matrix

The 3d rotation matrix can be described as follows:

$$R_x(\psi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi_x & sin\psi_x \\ 0 & -sin\psi_x & cos\psi_x \end{bmatrix}$$

$$R_y(\psi_y) = \begin{bmatrix} cos\psi_y & 0 & -sin\psi_x \\ 0 & 1 & 0 \\ sin\psi_x & 0 & cos\psi_x \end{bmatrix}$$

$$R_z(\psi_z) = \begin{bmatrix} cos\psi_x & sin\psi_x & 0 \\ -sin\psi_x & cos\psi_x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The H matrix is the Jacobian of the output equation with respect to the state X:

$$H = \frac{\partial h(X)}{\partial X}$$

$$h(x) = \left(R_x(\psi_x)R_y(\psi_y)R_z(\psi_z)\right)^{\mathrm{T}} \cdot \boldsymbol{n_b}$$

$$h(x) = \left(R_x(\psi_x)R_y(\psi_y)R_z(\psi_z)\right)^{\mathrm{T}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$h(x) = \begin{bmatrix} sin(\phi) \cdot sin(x) + cos(\phi) \cdot cos(x) \cdot sin(y) \\ cos(x) \cdot sin(\phi) \cdot sin(y) - cos(\phi) \cdot sin(x) \\ cos(x) \cdot cos(y) \end{bmatrix}$$

$$H = \begin{bmatrix} \partial_{h_1}/\partial x & \partial_{h_1}/\partial y & \partial_{h_1}/\partial \phi \\ \partial_{h_2}/\partial x & \partial_{h_2}/\partial y & \partial_{h_2}/\partial \phi \\ \partial_{h_3}/\partial x & \partial_{h_3}/\partial y & \partial_{h_3}/\partial \phi \end{bmatrix}$$

H=

$$\begin{bmatrix} \sin(\phi)\cdot\cos(x) - \cos(\phi)\cdot\sin(x)\cdot\sin(y) & \cos(\phi)\cdot\cos(x)\cdot\cos(y) & \cos(\phi)\cdot\sin(x) - \sin(\phi)\cdot\cos(x)\cdot\sin(y) \\ -\sin(x)\cdot\sin(\phi)\cdot\sin(y) - \cos(\phi)\cdot\cos(x) & \cos(x)\cdot\sin(\phi)\cdot\cos(y) & \cos(x)\cdot\cos(\phi)\cdot\sin(y) + \sin(\phi)\cdot\sin(x) \\ -\sin(x)\cdot\cos(y) & -\cos(x)\cdot\sin(y) & 0 \end{bmatrix}$$

B5

Output Equation for Wall Detection

When a wall parallel to the YZ plane in the GCF is detected, its normal vector in the GCF is $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}^T$. The output equation is:

$$Y = h(X) = R^T(X)\cdot\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

where:

- R(X) is the rotation matrix from the global coordinate frame (GCF) to the platform's CF, parameterized by the attitude $X = [\phi, \theta, \psi]^T$
- $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ is the normal vector of the floor in the GCF.

Measurement Noise and R Matrix

The measurement noise is the same as in B4, with R given by:

$$R = \begin{bmatrix} 0.05^2 & 0 & 0 \\ 0 & 0.05^2 & 0 \\ 0 & 0 & 0.05^2 \end{bmatrix}$$

Analytical Expression for H Matrix

The 3d rotation matrix is the same as described in B4:

$$R_x(\psi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi_x & sin\psi_x \\ 0 & -sin\psi_x & cos\psi_x \end{bmatrix}$$

$$R_y(\psi_y) = \begin{bmatrix} cos\psi_y & 0 & -sin\psi_x \\ 0 & 1 & 0 \\ sin\psi_x & 0 & cos\psi_x \end{bmatrix}$$

$$R_z(\psi_z) = \begin{bmatrix} cos\psi_x & sin\psi_x & 0 \\ -sin\psi_x & cos\psi_x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The H matrix is the Jacobian of the output equation with respect to the state X:

$$H = \frac{\partial h(X)}{\partial X}$$

$$h(x) = \left(R_x(\psi_x)R_y(\psi_y)R_z(\psi_z)\right)^T \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$h(x) = \begin{bmatrix} -cos(\phi) \cdot cos(y) \\ -cos(y) \cdot sin(\phi) \\ sin(y) \end{bmatrix}$$

$$H = \begin{bmatrix} \partial_{h_1}/\partial x & \partial_{h_1}/\partial y & \partial_{h_1}/\partial\phi \\ \partial_{h_2}/\partial x & \partial_{h_2}/\partial y & \partial_{h_2}/\partial\phi \\ \partial_{h_3}/\partial x & \partial_{h_3}/\partial y & \partial_{h_3}/\partial\phi \end{bmatrix}$$

$$\mathrm{H}=\begin{bmatrix} 0 & cos(\phi) \cdot sin(y) & \sin(\phi) \cdot \cos(y) \\ 0 & \sin(\mathrm{y}) \cdot cos(\phi) & -\cos(y) \cdot \cos(\phi) \\ 0 & \cos(y) & 0 \end{bmatrix}$$

EKF Update Step

1. **Compute Kalman Gain:**

$$K = PH^T\,(HPH^T + R)^{-1}$$

2. **Update State Estimate:**

$$\hat{X}^+ = \hat{X}^- + \left(K\left(z - h(\hat{X}^-)\right)\right)$$

3. **Update Covariance:**

$$P^+ = (I - KH)P^-$$

B6

a) Definition of Augmented State Vector

The augmented state vector includes both the attitude (roll, pitch, yaw) and the gyroscope biases. The state vector is defined as:

$$X = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\theta} \\ \boldsymbol{\psi} \\ \boldsymbol{b_x} \\ \boldsymbol{b_y} \\ \boldsymbol{b_z} \end{bmatrix}$$

- **ϕ,θ,ψ**: Roll, pitch, and yaw angles (attitude).
- **bx,by,bz**: Biases for the gyroscope measurements in the x, y, and z axes, respectively.

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b) Discrete Time Process Model

The process model describes how the state evolves over time. Using Euler approximation with a time step of 5 milliseconds (0.005 seconds), the discrete-time state equation is:

$$X_{k+1} = X_k + \Delta t \ \cdot \ F(X_k, u_k)$$

Where:

- $F(X_k, u_k)$ is the continuous-time state equation.
- $u_k = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$ are the gyroscope measurements (angular rates).

The state equations are:

1. **Attitude Dynamics**

   $\dot{\phi} = \omega_x + b_x + \left(\omega_y \cdot sin\phi + \omega_z \cdot cos\phi\right) \cdot tan\theta$

   $\dot{\theta} = \omega_y \cdot cos\phi - \omega_z \cdot sin\phi$

   $\dot{\psi} = \dfrac{\omega_y \cdot sin\phi + \omega_z \cdot cos\phi}{cos\theta}$

2. **Bias Dynamics:**
   Since the biases are assumed constant:

$$\dot{b}_x = 0, \dot{b}_y = 0, \dot{b}_z = 0$$

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c) Q Matrix for the EKF Prediction Step

The process noise covariance matrix Q accounts for the uncertainty in the gyroscope measurements and biases. It is derived as follows:

1. **Gyroscope Noise**:
   The gyroscope measurements are polluted by white Gaussian noise (WGN) with a standard deviation of 2.5 degrees/second (≈0.0436 radians/second).
   The noise covariance for the gyroscope measurements is:

$$Q_u = diag((0.0436)^2, (0.0436)^2, (0.0436)^2).$$

2. **Jacobian Matrix ($\mathbf{J_u}$):**
   a) Understand the Jacobian's Role
   The Jacobian J_u maps gyroscope measurement noise to the state uncertainty. For the augmented state $X = \left[\phi, \theta, \psi, b_x, b_y, b_z\right]^T$, it is derived from the **attitude dynamics** (Euler angle rates):

$$\dot{\phi} = \omega_x + b_x + \left(\omega_y \cdot sin\phi + \omega_z \cdot cos\phi\right) \cdot tan\theta$$

$$\dot{\theta} = \omega_y \cdot cos\phi - \omega_z \cdot sin\phi$$

$$\dot{\psi} = \frac{\omega_y \cdot sin\phi + \omega_z \cdot cos\phi}{cos\theta}$$

   The Jacobian J_u is the partial derivative of these rates w.r.t. the gyroscope inputs $\omega_x, \omega_y, \omega_z$

$$J_u = \frac{\partial\left[\dot{\phi}, \dot{\theta}, \dot{\psi}\right]}{\partial\left[\omega_x, \omega_y, \omega_z\right]}.$$

   b) Analytical Verification of $\boldsymbol{J_\omega}$

   Step-by-Step Verification:

   First Row $\left(\frac{\partial\dot{\phi}}{\partial\omega}\right)$:

   - $\frac{\partial\dot{\phi}}{\partial\omega_x}$=1 (direct dependency).
   - $\frac{\partial\dot{\phi}}{\partial\omega_y} = sin\phi \cdot tan\theta \left(from\ \omega_y sin\phi \cdot tan\theta\right).$
   - $\frac{\partial\dot{\phi}}{(\partial\omega_z)} = cos\phi tan\theta\ (from\ \omega_z cos\phi \cdot tan\theta).$
   -

Second Row $\left(\frac{\partial\dot{\theta}}{\partial\omega}\right)$:

- $\frac{\partial\dot{\theta}}{\partial\omega_x} = 0\ (no\ dependency)$.
- $\frac{\partial\dot{\theta}}{\partial\omega_y} = cos\phi\ \left(from\ \omega_y \cdot cos\phi\right)$.
- $\frac{\partial\dot{\theta}}{\omega_z} = -sin\phi\ (from\ -\omega_z \cdot sin\phi)$.

Third Row $\left(\frac{\partial\dot{\psi}}{\partial\omega}\right)$:

- $\frac{\partial\dot{\psi}}{\partial\omega_x} = 0\ (no\ dependency)$.
- $\frac{\partial\dot{\psi}}{\partial\omega_y} = \frac{sin\phi}{cos\theta}\ \left(from\ \omega_y \cdot \frac{sin\phi}{cos\theta}\right)$.
- $\frac{\partial\dot{\psi}}{\omega_z} = \frac{cos\phi}{cos\theta}\ \left(from\ \omega_z \cdot \frac{cos\phi}{cos\theta}\right)$.

The proposed Jacobian in your document is:

$$J_u = \begin{bmatrix} 1 & sin\phi \cdot tan\theta & cos\phi \cdot tan\theta \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi/cos\theta & cos\theta/cos\phi \end{bmatrix}$$

3. **Bias Noise**:
   The biases are assumed constant, but small noise is added to allow for slow drift. The noise covariance for the biases is:

   $$Q_b = diag((0.001)^2, (0.001)^2, (0.001)^2).$$

4. **Combined Q Matrix**:
   The total process noise covariance matrix is:

   $$Q = \begin{bmatrix} J_u \cdot Q_u \cdot J_u^T & 0_{3\times3} \\ 0_{3\times3} & Q_b \end{bmatrix}$$

where J_u is the Jacobian of the attitude dynamics with respect to the gyroscope measurements.

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d) Initialization: $\widehat{X}(0 \mid 0)$ and $P(0 \mid 0)$

- **Initial State Estimate**:

$$\hat{X}(0 \mid 0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

  - The initial attitude is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (radians).
  - The initial biases are assumed to be $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

- **Initial Covariance Matrix**:

$$P(0 \mid 0) = diag\left(\sigma_{\phi}^2, \sigma_{\theta}^2, \sigma_{\psi}^2, \sigma_{b_x}^2, \sigma_{b_y}^2, \sigma_{b_z}^2\right),$$

where:

  - $\sigma\phi = \sigma\theta = \sigma\psi = 5° \approx 0.0873$ radians (uncertainty in initial attitude).
  - $\sigma_{b_x} = \sigma_{b_y} = \sigma_{b_z} = \frac{2°}{second} \approx 0.0349 \frac{radians}{second}$ (uncertainty in biases).

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e) Output Equation, H Matrix, and R Matrix for B5-like Measurement

1. **Output Equation**:
   The measured normal vector in the platform's coordinate frame (CF) is:

$$z = h(X) + v,$$

   where h(X) is the expected normal vector in the platform CF, and vv is measurement noise.

2. **Expected Normal Vector**:

For a floor detected on the XY plane(normal vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in GCF), the measurement model is:

The expected normal vector in the platform CF is obtained by rotating the global normal vector using the current attitude estimate:

$$h(X) = R^T(\phi, \theta, \psi) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$h(x) = \begin{bmatrix} sin(\phi) \cdot sin(x) + cos(\phi) \cdot cos(x) \cdot sin(y) \\ cos(x) \cdot sin(\phi) \cdot sin(y) - cos(\phi) \cdot sin(x) \\ cos(x) \cdot cos(y) \end{bmatrix}$$

where R is the rotation matrix from the platform CF to GCF.

For a wall parallel to the YZ plane(normal vector $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ in GCF), the measurement model is:

The expected normal vector in the platform CF is obtained by rotating the global normal vector using the current attitude estimate:

$$h(X) = R^T(\phi, \theta, \psi) \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$h(x) = \begin{bmatrix} -cos(\phi) \cdot cos(y) \\ -cos(y) \cdot sin(\phi) \\ sin(y) \end{bmatrix}$$

where R is the rotation matrix from the platform CF to GCF.

3. **H Matrix**:
The Jacobian H is the derivative of h(X) with respect to the state vector X.

$$H = \begin{bmatrix} \frac{\partial h_x}{\partial \phi} & \frac{\partial h_x}{\partial \theta} & \frac{\partial h_x}{\partial \psi} & 0 & 0 & 0 \\ \frac{\partial h_y}{\partial \phi} & \frac{\partial h_y}{\partial \theta} & \frac{\partial h_y}{\partial \psi} & 0 & 0 & 0 \\ \frac{\partial h_z}{\partial \phi} & \frac{\partial h_z}{\partial \theta} & \frac{\partial h_z}{\partial \psi} & 0 & 0 & 0 \end{bmatrix}$$

The Jacobian H for part B4 along with bias is given as:

H=

$$\begin{bmatrix} \sin(\phi)\cdot\cos(x)-\cos(\phi)\cdot\sin(x)\cdot\sin(y) & \cos(\phi)\cdot\cos(x)\cdot\cos(y) & \cos(\phi)\cdot\sin(x)-\sin(\phi)\cdot\cos(x)\cdot\sin(y) & 0 & 0 & 0 \\ -\sin(x)\cdot\sin(\phi)\cdot\sin(y)-\cos(\phi)\cdot\cos(x) & \cos(x)\cdot\sin(\phi)\cdot\cos(y) & \cos(x)\cdot\cos(\phi)\cdot\sin(y)+\sin(\phi)\cdot\sin(x) & 0 & 0 & 0 \\ -\sin(x)\cdot\cos(y) & -\cos(x)\cdot\sin(y) & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Jacobian H for part B5 along with bias is given as:

$$H=\begin{bmatrix} 0 & cos(\phi)\cdot sin(y) & \sin(\phi)\cdot\cos(y) & 0 & 0 & 0 \\ 0 & \sin(y)\cdot cos(\phi) & -\cos(y)\cdot\cos(\phi) & 0 & 0 & 0 \\ 0 & \cos(y) & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. **R Matrix**:
   The measurement noise covariance matrix is:

$$R = diag((0.05)^2, (0.05)^2, (0.05)^2),$$

   reflecting the uncertainty in the normal vector components (std = 0.05).