

# 1 Predicate Logic

50 points

## 1.1 First Order Logic

14 points

Translate the following sentences into first order logic:

- a. There is an ice cream flavor loved by everyone.

$$\exists flavor \forall x : Loves(x, flavor)$$

- b. It is not true that everyone loves some ice cream flavor.

$$\exists x \forall flavor : \neg Loves(x, icecream)$$

- c. Anyone who does not shave himself must be shaved by the barber.

Let *barber* be a constant.

$$\forall x : \neg(Shaves(x, x)) \rightarrow Shaves(barber, x)$$

- d. Whomever the barber shaves, must not shave himself.

Let *barber* be a constant.

$$\forall x : Shaves(barber, x) \rightarrow \neg(Shaves(x, x))$$

- e. Rudolph is a reindeer and Rudolph has a red nose.

Let *rudolph* be a constant.

$$Reindeer(rudolph) \wedge RedNose(rudolph)$$

- f. Anyone with a red nose is weird or is a clown.

$$\forall x : RedNose(x) \rightarrow (Weird(x) \vee Clown(x))$$

- g. No reindeer is a clown.

$$\forall x : Reindeer(x) \rightarrow \neg Clown(x)$$

Note: You are not required to specify which object a variable refers to, or what a function returns, if it is obvious from the context. For instance, it's obvious from the context that *x* is a person and *Likes(x, y)* is true if *x* likes some *y*. However, if you think that the context is insufficient, you might start providing definitions.

## 1.2 Clausal Forms

**6 points**

Reduce the following sentences to clausal form:

a.  $\forall x \forall y (R(x, y) \rightarrow (R(x, y) \wedge Q(y)))$

$$\begin{aligned}
 & \forall x \forall y (R(x, y) \rightarrow (R(x, y) \wedge Q(y))) \\
 & \equiv \forall x \forall y (\neg R(x, y) \vee (R(x, y) \wedge Q(y))) \\
 & \equiv \forall x \forall y ((\neg R(x, y) \vee R(x, y)) \wedge (\neg R(x, y) \vee Q(y))) \\
 & \equiv \forall x \forall y (1 \wedge (\neg R(x, y) \vee Q(y))) \\
 & \equiv \forall x \forall y ((\neg R(x, y) \vee Q(y))) \\
 & \equiv \boxed{\neg R(x, y) \vee Q(y)}
 \end{aligned}$$

b.  $\forall x \exists y \forall z (P(x, y, z) \rightarrow \exists u R(x, u, z))$

$$\begin{aligned}
 & \forall x \exists y \forall z (P(x, y, z) \rightarrow \exists u R(x, u, z)) \\
 & \equiv \forall x \exists y \forall z (\neg P(x, y, z) \vee \exists u R(x, u, z)) \\
 & \equiv \forall x \forall z (\neg P(x, F(x), z) \vee R(x, G(x, z), z)) \\
 & \equiv \boxed{\neg P(x, F(x), z) \vee R(x, G(x, z), z)}
 \end{aligned}$$

c.  $\forall x (\neg \exists y P(x, y) \wedge \neg(Q(x) \wedge \neg R(x)))$

$$\begin{aligned}
 & \forall x (\neg \exists y P(x, y) \wedge \neg(Q(x) \wedge \neg R(x))) \\
 & \equiv \forall x (\neg \exists y P(x, y) \wedge (\neg Q(x) \vee R(x))) \\
 & \equiv \forall x (\forall y \neg P(x, y) \wedge (\neg Q(x) \vee R(x))) \\
 & \equiv \boxed{\neg P(x, y) \wedge (\neg Q(x) \vee R(x))}
 \end{aligned}$$

## 1.3 Knowledge Extraction via Resolution

**30 points**

Let *flee*, *stay*, *now*, *lion* be constants. You are given the following Knowledge Base:

**S1**  $\forall t \forall x (Observe(t, x) \wedge Danger(x) \rightarrow Suggestion(t, flee))$

**S2**  $\forall t (\neg \exists x (Observe(t, x) \wedge Danger(x)) \rightarrow Suggestion(t, stay))$

**S3**  $Danger(lion)$

**S4**  $Observe(now, lion)$

Use Resolution to show that the KB entails  $\exists x Suggestion(now, x)$  and extract an answer.

**S1:**

$$\begin{aligned} \forall t \forall x (\text{Observe}(t, x) \wedge \text{Danger}(x) \rightarrow \text{Suggestion}(t, \text{flee})) \\ \equiv \neg \text{Observe}(t, x) \vee \neg \text{Danger}(x) \vee \text{Suggestion}(t, \text{flee}) \end{aligned}$$

**S2:**

$$\begin{aligned} \forall t (\neg \exists x (\text{Observe}(t, x) \wedge \text{Danger}(x)) \rightarrow \text{Suggestion}(t, \text{stay})) \\ \equiv (\text{Observe}(t, f(t)) \wedge \text{Danger}(f(t))) \vee \text{Suggestion}(t, \text{stay}) \end{aligned}$$

**S3:**

$$\text{Danger}(\text{lion})$$

**S3:**

$$\text{Observe}(\text{now}, \text{lion})$$

**Negated Query:**

$$\begin{aligned} \neg \exists x \text{Suggestion}(\text{now}, x) \\ \equiv \forall x \neg \text{Suggestion}(\text{now}, x) \\ \equiv \neg \text{Suggestion}(\text{now}, x) \end{aligned}$$

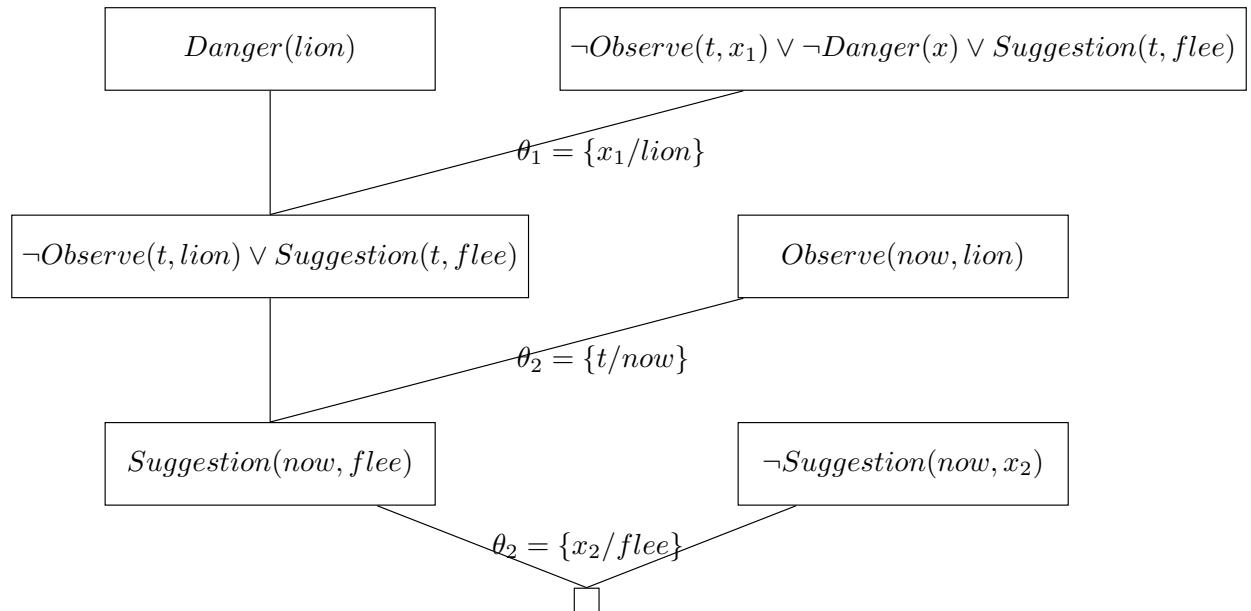


Figure 1: FOL-Resolution

The empty set at the end proofs, that the  $KB \wedge \neg \text{Query}$  is unsatisfiable, but that implies that the Query is satisfiable. The extracted answer is  $x = \text{flee}$ . It was not necessary to use the entire knowledge base, since a subset already proofed the unsatisfiability.

## 2 Predicate Logic

**50 points**

### 2.1 Probability

**30 points**

Examiners running the game show “*Ring the golden bell*” have been taking bribes from some participants. A given participant is either allowed to stay or kicked off in each episode. If the participant has been bribing the examiners she will be allowed to stay with probability  $4/5$ . If the participant has not been bribing the examiners, she will be allowed to stay with probability  $1/3$ .

Suppose that  $1/4$  of the participants have been bribing the examiners. The same participants bribe the examiners in both rounds, i.e., if a participant bribes them in the first round, she bribes them in the second round too (and vice versa).

- a. If you pick a random participant who was allowed to stay during the first episode, what is the probability that she was bribing the examiners?

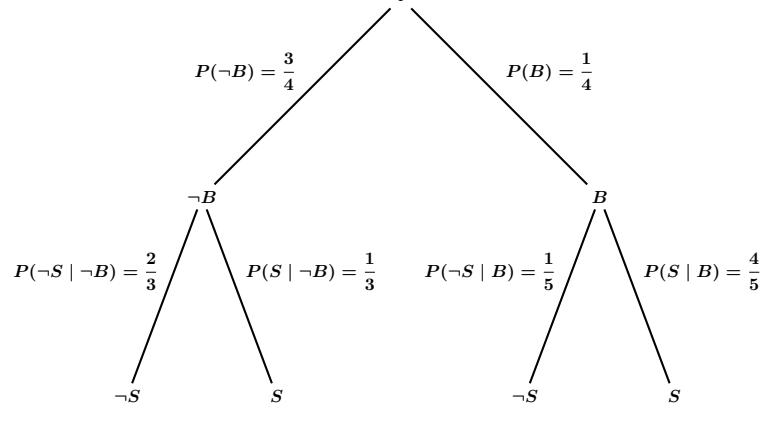
$B = \text{is bribing}$ ,  $\neg B = \text{is not bribing}$ ,  $S = \text{is staying}$ ,  $\neg S = \text{is not staying}$ .

We are looking for and want to compute  $P(B|S)$ .

Given:

- $P(B) = \frac{1}{4}$
- $P(S | B) = \frac{4}{5}$
- $P(S | \neg B) = \frac{1}{3}$

With the given information we can create the tree in Figure 2.



$$\frac{3}{4} \cdot \frac{2}{3} = \boxed{\frac{1}{2}} \quad \frac{3}{4} \cdot \frac{1}{3} = \boxed{\frac{1}{4}} \quad \frac{1}{4} \cdot \frac{1}{5} = \boxed{\frac{1}{20}} \quad \frac{1}{4} \cdot \frac{4}{5} = \boxed{\frac{1}{5}}$$

Figure 2: Probability tree

Before computing  $P(B|S)$  we need to compute  $P(S)$  first:

$$P(S) = P(\neg B) \cdot P(S | \neg B) + P(B) \cdot P(S | B) = \frac{1}{4} \cdot \frac{4}{5} + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{5} + \frac{1}{4} = \frac{4+5}{5 \cdot 4} = \frac{9}{20}$$

we computing  $P(B | S)$  using Bayes' theorem:

$$P(B | S) = \frac{P(S|B) \cdot P(B)}{P(S)} = \frac{\frac{4}{5} \cdot \frac{1}{4}}{\frac{9}{20}} = \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{20}{9} = \frac{4}{9}$$

- b. If you pick a random participant, what is the probability that she is allowed to stay during both of the first two episodes?

$$\begin{aligned} P(\text{stay 2 times}) &= P(\neg B) \cdot P(S | \neg B)^2 + P(B) \cdot P(S | B)^2 \\ &= \frac{3}{4} \cdot \left(\frac{1}{3}\right)^2 + \frac{1}{4} \cdot \left(\frac{4}{5}\right)^2 \\ &= \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{4}{5} \cdot \frac{4}{5} \\ &= \frac{1}{4 \cdot 3} + \frac{4}{5^2} \\ &= \frac{1}{12} + \frac{4}{25} \\ &= \frac{25 + 4 \cdot 12}{12 \cdot 25} \\ &= \frac{73}{300} \end{aligned}$$

- c. If you pick random participant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

$$\begin{aligned} P(\text{kicked off second time}) &= P(\neg B) \cdot P(S | \neg B) \cdot P(\neg S | \neg B) \\ &\quad + P(B) \cdot P(S | B) \cdot P(\neg S | B) \\ &= \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{4}{5} \cdot \frac{1}{5} \\ &= \frac{1}{2 \cdot 3} + \frac{1}{5^2} \\ &= \frac{1}{6} + \frac{1}{25} \\ &= \frac{25 + 6}{6 \cdot 25} \\ &= \frac{31}{150} \end{aligned}$$

## 2.2 Naïve Bayes' Classifier

**20 points**

Given this data table about buying a car. Predict the class of the following new example using Naïve Bayes' Classification:

age  $\leq 30$ , income = high, Married = yes, credit-rating = fair

RID	age	income	married	credit_rating	class: buy_car
1	$\leq 30$	high	no	fair	no
2	$\leq 30$	high	no	excellent	no
3	31, 32, ..., 40	high	no	fair	yes
4	$> 40$	medium	no	fair	yes
5	$> 40$	low	yes	fair	yes
6	$> 40$	low	yes	excellent	no
7	31, 32, ..., 40	low	yes	excellent	yes
8	$\leq 30$	medium	no	fair	no
9	$\leq 30$	low	yes	fair	yes
10	$> 40$	medium	yes	fair	yes
11	$\leq 30$	medium	yes	excellent	yes
12	31, 32, ..., 40	medium	yes	excellent	yes
13	31, 32, ..., 40	high	yes	fair	yes
14	$> 40$	medium	no	excellent	no

We want to predict the class of the given example:

$$e = (\underbrace{\text{age } \leq 30}_{e_0}, \underbrace{\text{income } = \text{high}}_{e_1}, \underbrace{\text{married } = \text{yes}}_{e_2}, \underbrace{\text{credit } = \text{fair}}_{e_3})$$

by classifying it into one of the two categories:

$$\text{buy\_car} \in \{\text{yes}, \text{no}\}.$$

- Prior over categories ( $P(\text{category})$ ):

$$P(\text{buy\_car} = \text{yes}) = \frac{9}{14}, \quad P(\text{buy\_car} = \text{no}) = \frac{5}{14}.$$

- Conditional probabilities ( $P(e_i \mid \text{category})$ ):

- $\text{buy\_car} = \text{yes}$ :

$$P(\underbrace{\text{age } \leq 30}_{e_0} \mid \text{buy\_car} = \text{yes}) = \frac{2}{9}, \quad P(\underbrace{\text{income } = \text{high}}_{e_1} \mid \text{buy\_car} = \text{yes}) = \frac{2}{9},$$

$$P(\underbrace{\text{married } = \text{yes}}_{e_2} \mid \text{buy\_car} = \text{yes}) = \frac{7}{9}, \quad P(\underbrace{\text{credit } = \text{fair}}_{e_3} \mid \text{buy\_car} = \text{yes}) = \frac{6}{9}.$$

- $buy\_car = no$ :

$$P(\underbrace{age \leq 30}_{e_0} | buy\_car = no) = \frac{3}{5}, \quad P(\underbrace{income = high}_{e_1} | buy\_car = no) = \frac{2}{5},$$

$$P(\underbrace{married = yes}_{e_2} | buy\_car = no) = \frac{1}{5}, \quad P(\underbrace{credit = fair}_{e_3} | buy\_car = no) = \frac{2}{5}.$$

Using the Naïve Bayes formula

$$P(Category | e) = \alpha P(Category) \prod_j P(e_j | Category)$$

we can compute  $P(buy\_car = yes | e)$ :

$$P(e | buy\_car = yes) = \prod_j P(e_j | buy\_car = yes) = \frac{2}{9} \cdot \frac{2}{9} \cdot \frac{7}{9} \cdot \frac{6}{9} = \frac{56}{2187} \approx 0.0256,$$

$$P(buy\_car = yes | e) = P(buy\_car = yes) \cdot P(e | buy\_car = yes) = \frac{9}{14} \cdot \frac{56}{2187} \approx 0.0165.$$

and  $P(buy\_car = no | e)$ :

$$P(e | buy\_car = no) = \prod_j P(e_j | buy\_car = no) = \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} \cdot \frac{2}{5} = \frac{12}{625} \approx 0.0192,$$

$$P(buy\_car = no | e) = P(buy\_car = no) \cdot P(e | buy\_car = no) = \frac{5}{14} \cdot \frac{12}{625} \approx 0.00686.$$

Now we can predict the class of the given example:

$$P(buy\_car = yes | e) = 0.0165 > 0.00686 = P(buy\_car = no | e).$$

Predicted Category:  $buy\_car = yes$

The Naïve Bayes classifier predicts that the customer will buy a car.