Cheatsheet — Core formulas & concepts

Cheatsheet — Core formulas & concepts (Casini + JT hybrid project)

Author: Ammar Hamza Saad Abdulraheem **Purpose:** Quick reference for the hybrid Casini (relative entropy) + JT island project.

Quantum information basics

Von Neumann entropy

$$ S(\rho) = -\mathrm{Tr}\, (\rho \ln \rho) $$

Relative entropy (quantum Kullback–Leibler)

$$ S(\rho \| \sigma) = \mathrm{Tr}\,(\rho \ln \rho) - \mathrm{Tr}\,(\rho \ln \sigma) \ge 0 $$ (positivity: equality iff $\rho = \sigma$.)

Key identity (Casini-style)

If $\sigma = e^{-K}/\mathrm{Tr}(e^{-K})$ (modular state), define $$ \Delta S = S(\rho) - S(\sigma), \qquad \Delta K = \mathrm{Tr}(K\rho) - \mathrm{Tr}(K\sigma). $$ Then $$ S(\rho\|\sigma) = \Delta K - \Delta S \ge 0 \quad\Rightarrow\quad \Delta S \le \Delta K. $$

Modular Hamiltonian (intuition & common forms)

- **Definition (formal):** $\sigma = e^{-K}/\mathrm{Tr}(e^{-K})$; K is the modular Hamiltonian for σ.
- **Rindler wedge (heuristic, QFT vacuum restricted to $x>0$):**

$$ K = 2\pi \int_{x>0} x\, T_{00}(x)\, dx $$ (This gives the intuitive weight-by-distance factor appearing in Casini's arguments.)

- **Interval in 1+1D CFT:** modular Hamiltonian is known for an interval in a CFT; uses conformal maps — see references.

Bekenstein-style bound (schematic)

For a weakly gravitating localized system of energy E and size R, a common schematic form is $$ S \le 2\pi \frac{E R}{\hbar} $$ Casini's derivation recasts such bounds in terms of relative entropy and modular Hamiltonians in QFT.

JT gravity (very short sketches)

JT action (schematic, conventions vary):

$$I_{\mathrm{JT}} = -\frac{1}{2}\int d^2x\sqrt{-g}\,\phi(R+2) + \text{boundary terms}$$ where ϕ is the dilaton field.

Island formula (schematic):

For radiation region Rad, $$S(\mathrm{Rad}) = \min_{\text{islands}} \mathrm{Ext}_I\left[\frac{\mathrm{Area}(\partial I)}{4G_N} + S_{\text{matter}}(\mathrm{Rad}\cup I) \right]$$ (Used to compute fine-grained entropy of radiation including semiclassical gravitational saddle contributions.)

Useful symbolic/numeric checks (toy ideas)

- Compute relative entropy between two Gaussians as a toy model:

$S(p\|q)=\int p(x)\ln\frac{p(x)}{q(x)}dx$ — analytically tractable and good for series expansion checks.

- For modular Hamiltonian checks, compare ΔK integrals for small excitations to the perturbative expansion of ΔS (relative entropy series).

Short list of references (start here)

1. H. Casini, "Relative entropy and the Bekenstein bound" (2008). 2. R. Bousso, "The holographic principle" (Rev. Mod. Phys., 2002). 3. Almheiri / Engelhardt / Marolf / Maxfield / Penington papers (2019–2020) — replica wormholes & islands. 4. Lecture notes: David Tong (QFT), Srednicki (entanglement intro), Maldacena/Stanford (SYK/JT notes).

Quick reminders

- Keep units consistent (I often use $\hbar = c = 1$ in derivations; restore factors when comparing to physical bounds).
- If analytic integrals diverge, isolate vacuum divergence and compute finite differences (Casini's method subtracts vacuum contributions to get finite ΔS).
- Always provide a short README and a 1-page non-technical summary for supervisors.

If you want, I can also create a printable PDF of this cheatsheet and put it in the repo.

Prepared for: Ammar Hamza Saad Abdulraheem