Machine Learning

4771

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Topic 20

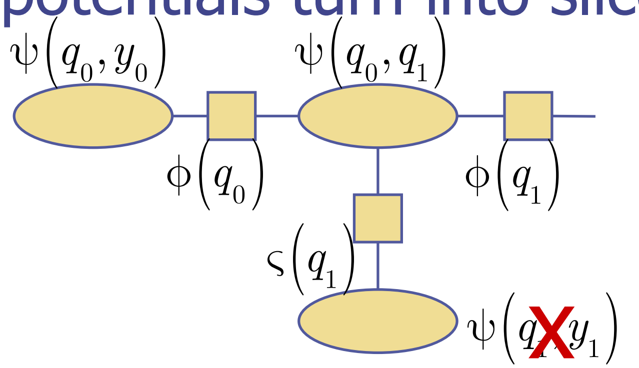
- HMMs with Evidence
- HMM Collect
- HMM Evaluate
- HMM Distribute
- HMM Decode
- HMM Parameter Learning via JTA & EM

HMMs: JTA with Evidence

- If y sequence is observed (in problems 1,2,3) get evidence:

$$p(q,\bar{y}) = p(q_0)\prod_{t=1}^{T} p(q_t \mid q_{t-1})\prod_{t=0}^{T} p(\bar{y}_t \mid q_t)$$

- The potentials turn into slices:



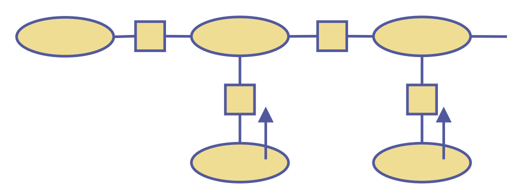
$\longrightarrow \psi(q_1,\bar{y}_1)$

$$\varsigma^*(q_t) \cancel{=} \sum_{y_t} \psi(q_t,y_t)$$

$$\varsigma^*(q_t) = \psi(q_t,\bar{y}_t) = p(\bar{y}_t \mid q_t)$$

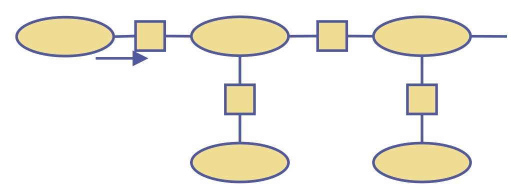
- Next, pick a root, for example *rightmost* one: $\psi(q_{T-1},q_T)$

- Collect all zeta separators bottom up:



$$\varsigma^*(q_t) = \psi(q_t,\bar{y}_t) = p(\bar{y}_t \mid q_t)$$

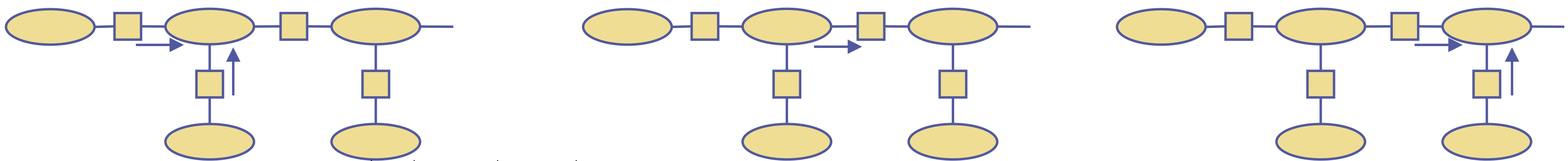
- Collect leftmost phi separator to the right:



$$\phi^*(q_0) = \sum_{y_0} \psi(q_0,\bar{y}_0)\delta(y_0-\bar{y}_0) = p(\bar{y}_0,q_0)$$

HMMs: Collect with Evidence

- Now, we will collect (*) along the backbone left to right
- Update each clique with its left and bottom separators:



$$\psi^*\left(q_t, q_{t+1}\right) = \frac{\phi^*\left(q_t\right)}{1} \frac{\varsigma^*\left(q_{t+1}\right)}{1} \psi\left(q_t, q_{t+1}\right) = \phi^*\left(q_t\right) p\left(\bar{y}_{t+1} \mid q_{t+1}\right) \alpha_{q_t, q_{t+1}}$$

$$\phi^*\left(q_{t+1}\right) = \sum_{q_t} \psi^*\left(q_t, q_{t+1}\right) = \sum_{q_t} \phi^*\left(q_t\right) p\left(\bar{y}_{t+1} \mid q_{t+1}\right) \alpha_{q_t, q_{t+1}}$$

- Keep going along chain until right most node
- Note: above formula for phi is recursive, could use as is.
- Property: recall we had $\phi^*\left(q_0\right) = p\left(\bar{y}_0, q_0\right)$

$$\phi^*\left(q_1\right) = \sum_{q_0} p\left(\bar{y}_0, q_0\right) p\left(\bar{y}_1 \mid q_1\right) p\left(q_1 \mid q_0\right) = p\left(\bar{y}_0, \bar{y}_1, q_1\right)$$

$$\phi^*\left(q_2\right) = \sum_{q_1} p\left(\bar{y}_0, \bar{y}_1, q_1\right) p\left(\bar{y}_2 \mid q_2\right) p\left(q_2 \mid q_1\right) = p\left(\bar{y}_0, \bar{y}_1, \bar{y}_2, q_2\right)$$

$$\phi^*\left(q_{t+1}\right) = \sum_{q_t} p\left(\bar{y}_0, \ldots, \bar{y}_t, q_t\right) p\left(\bar{y}_{t+1} \mid q_{t+1}\right) p\left(q_{t+1} \mid q_t\right) = p\left(\bar{y}_0, \ldots, \bar{y}_{t+1}, q_{t+1}\right)$$

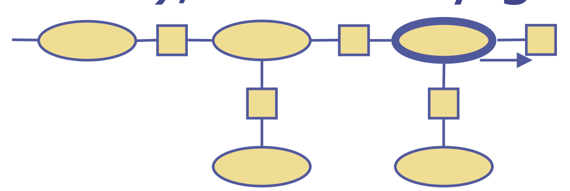
HMMs: Evaluate with Evidence

- Say we are solving the first HMM problem:
  1) Evaluate: given $y_0,\ldots,y_T$ & θ compute $p(y_0,\ldots,y_T|\theta)$
- If we want to compute the likelihood, we are already done!
- We really just need to do collect (not even distribute).
- From previous slide we had:

$$\phi^*\left(q_{t+1}\right) = \sum_{q_t} p\left(\bar{y}_0,\ldots,\bar{y}_t,q_t\right) p\left(\bar{y}_{t+1} \mid q_{t+1}\right) p\left(q_{t+1} \mid q_t\right) = p\left(\bar{y}_0,\ldots,\bar{y}_{t+1},q_{t+1}\right)$$

- As we collect to the root (rightmost node), we finally get:

$$\phi^*\left(q_T\right) = p\left(\bar{y}_0,\ldots,\bar{y}_T,q_T\right)$$



- Can compute the likelihood just by marginalizing this phi

$$p\left(\bar{y}_0,\ldots,\bar{y}_T\right) = \sum_{q_T} p\left(\bar{y}_0,\ldots,\bar{y}_T,q_T\right) = \sum_{q_T} \phi^*\left(q_T\right)$$

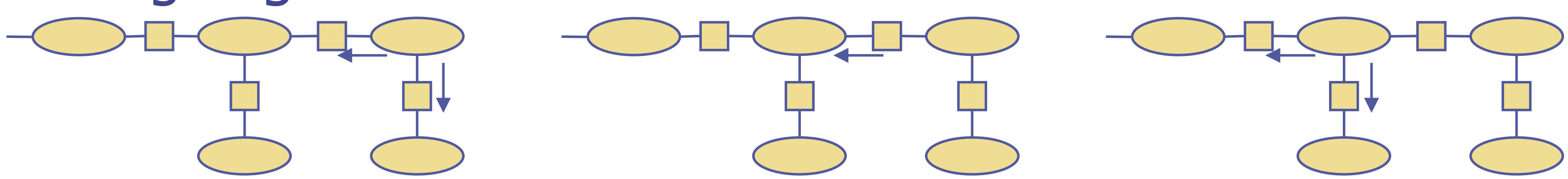
- So, adding up the entries in last ϕ^* gives us the likelihood

HMMs: Distribute with Evidence

- Back to collecting… say just finished collecting to the root with our last update formula:

$$\psi^*\left(q_{T-1},q_T\right)=\frac{\phi^*\left(q_{T-1}\right)}{1}\frac{\varsigma^*\left(q_T\right)}{1}\psi\left(q_{T-1},q_T\right)=\phi^*\left(q_{T-1}\right)p\left(\bar{y}_T\mid q_T\right)\alpha_{q_{T-1},q_T}$$

- Now, we distribute (**) along the backbone right to left
- Have first ** for root (stays the same): $\psi^{**}\left(q_{T-1},q_T\right)=\psi^*\left(q_{T-1},q_T\right)$
- Start going to the left from there:



$$\phi^{**}\left(q_t\right)=\sum_{q_{t+1}}\psi^{**}\left(q_t,q_{t+1}\right)$$

$$\varsigma^{**}\left(q_{t+1}\right)=\sum_{q_t}\psi^{**}\left(q_t,q_{t+1}\right)$$

$$\psi^{**}\left(q_t,q_{t+1}\right)=\frac{\phi^{**}\left(q_{t+1}\right)}{\phi^*\left(q_{t+1}\right)}\psi^*\left(q_t,q_{t+1}\right)$$

HMMs: Marginals & MaxDecoding

- Now that JTA is finished, we have the following:

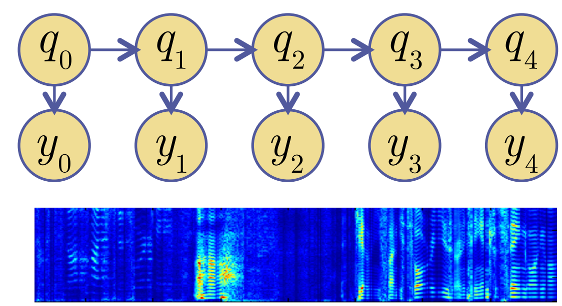
$$\phi^{**}(q_t) \propto p(q_t \mid \bar{y}_1,\ldots,\bar{y}_T) \qquad \varsigma^{**}(q_{t+1}) \propto p(q_{t+1} \mid \bar{y}_1,\ldots,\bar{y}_T)$$

$$\psi^{**}(q_t, q_{t+1}) \propto p(q_t, q_{t+1} \mid \bar{y}_1,\ldots,\bar{y}_T)$$

- We have done part of the HMM Problem:

  2) Decode: given $y_0,\ldots,y_T$ & θ find $p(q_0),\ldots,p(q_T)$ and $q_0,\ldots,q_T$
- The separators define a distribution over the hidden states
- This tells us the probability the audio y_t was phoneme q_t
- We can also decode to find the most likely path $q_0 \ldots q_T$
- Here, we use the ArgMax JTA algorithm
- Run JTA but replace sums with max
- Then, find biggest entry in separators:

110002220000111111



$$\hat{q}_t = \arg\max_{q_t} \phi^{**}(q_t) \quad \forall t = 0 \ldots T$$

HMMs: EM Learning

- Finally 3) Max Likelihood: given $y_0,\ldots,y_T$ learn parameters θ
- Recall max likelihood: $\hat{\theta} = \arg\max_\theta \log p\left(\bar{y} \mid \theta\right)$
- If observe q, it's easy to maximize the *complete* likelihood:

$$
\begin{aligned}
l(\theta) &= \log\left(p\left(q,y\right)\right) \\
&= \log\left(p\left(q_0\right)\prod_{t=1}^{T} p\left(q_t \mid q_{t-1}\right)\prod_{t=0}^{T} p\left(\bar{y}_t \mid q_t\right)\right) \\
&= \log p\left(q_0\right) + \sum_{t=1}^{T} \log p\left(q_t \mid q_{t-1}\right) + \sum_{t=0}^{T} \log p\left(\bar{y}_t \mid q_t\right) \\
&= \log \prod_{i=1}^{M}\left[\pi_i\right]^{q_0^i} + \sum_{t=1}^{T} \log \prod_{i=1}^{M}\prod_{j=1}^{M}\left[\alpha_{ij}\right]^{q_{t-1}^i q_t^j} + \sum_{t=0}^{T} \log \prod_{i=1}^{M}\prod_{j=1}^{N}\left[\eta_{ij}\right]^{q_t^i y_t^j} \\
&= \sum_{i=1}^{M} q_0^i \log \pi_i + \sum_{t=1}^{T}\sum_{i,j=1}^{M} q_{t-1}^i q_t^j \log \alpha_{ij} + \sum_{t=0}^{T}\sum_{i=1}^{M}\sum_{j=1}^{N} q_t^i y_t^j \log \eta_{ij}
\end{aligned}
$$

Introduce Lagrange & take derivatives $\longrightarrow$ $\sum_{i=1}^{M} \pi_i = 1 \qquad \sum_{j=1}^{M} \alpha_{ij} = 1 \qquad \sum_{j=1}^{N} \eta_{ij} = 1$

$$
\hat{\pi}_i = q_0^i \qquad \hat{\alpha}_{ij} = \frac{\sum_{t=0}^{T-1} q_t^i q_{t+1}^j}{\sum_{k=1}^{M}\sum_{t=0}^{T-1} q_t^i q_{t+1}^k} \qquad \hat{\eta}_{ij} = \frac{\sum_{t=0}^{T} q_t^i y_t^j}{\sum_{k=1}^{N}\sum_{t=0}^{T} q_t^i y_t^k}
$$

HMMs: EM Learning

- But, we don't observe the q's, incomplete…

$$p\left(\bar{y} \mid \theta\right) = \sum_{q} p\left(q, \bar{y} \mid \theta\right) = \sum_{q_0} \cdots \sum_{q_T} p\left(q_0\right) \prod_{t=1}^{T} p\left(q_t \mid q_{t-1}\right) \prod_{t=0}^{T} p\left(\bar{y}_t \mid q_t\right)$$

- EM: Max expected complete likelihood given current p(q)

$$E\left\{l\left(\theta\right)\right\} = E_{p\left(q_0,\ldots,q_T \mid y\right)}\left\{\log p\left(q,y\right)\right\} = \sum_{q_0} \cdots \sum_{q_T} p\left(q \mid y\right) \log p\left(q,y\right)$$

$$= E\left\{\sum_{i=1}^{M} q_0^i \log \pi_i + \sum_{t=1}^{T} \sum_{i,j=1}^{M} q_{t-1}^i q_t^j \log \alpha_{ij} + \sum_{t=0}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} q_t^i y_t^j \log \eta_{ij}\right\}$$

$$= \sum_{i=1}^{M} E\left\{q_0^i\right\} \log \pi_i + \sum_{t=1}^{T} \sum_{i,j=1}^{M} E\left\{q_{t-1}^i q_t^j\right\} \log \alpha_{ij} + \sum_{t=0}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} E\left\{q_t^i\right\} y_t^j \log \eta_{ij}$$

- M-step is maximizing as before:

$$\hat{\pi}_i = E\left\{q_0^i\right\} \qquad \hat{\alpha}_{ij} = \frac{\sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^j\right\}}{\sum_{k=1}^{M} \sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^k\right\}} \qquad \hat{\eta}_{ij} = \frac{\sum_{t=0}^{T} E\left\{q_t^i\right\} y_t^j}{\sum_{k=1}^{N} \sum_{t=0}^{T} E\left\{q_t^i\right\} y_t^k}$$

- What are E{}'s?

HMMs: EM Learning

- But, we don't observe the q's, incomplete…

$$p\left(\bar{y} \mid \theta\right) = \sum_q p\left(q, \bar{y} \mid \theta\right) = \sum_{q_0} \cdots \sum_{q_T} p\left(q_0\right) \prod_{t=1}^{T} p\left(q_t \mid q_{t-1}\right) \prod_{t=0}^{T} p\left(\bar{y}_t \mid q_t\right)$$

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$$E\left\{l\left(\theta\right)\right\} = E_{p\left(q_0,\ldots,q_T \mid y\right)}\left\{\log p\left(q, y\right)\right\} = \sum_{q_0} \ldots \sum_{q_T} p\left(q \mid y\right) \log p\left(q, y\right)$$

$$= E\left\{\sum_{i=1}^{M} q_0^i \log \pi_i + \sum_{t=1}^{T} \sum_{i,j=1}^{M} q_{t-1}^i q_t^j \log \alpha_{ij} + \sum_{t=0}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} q_t^i y_t^j \log \eta_{ij}\right\}$$

$$= \sum_{i=1}^{M} E\left\{q_0^i\right\} \log \pi_i + \sum_{t=1}^{T} \sum_{i,j=1}^{M} E\left\{q_{t-1}^i q_t^j\right\} \log \alpha_{ij} + \sum_{t=0}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} E\left\{q_t^i\right\} y_t^j \log \eta_{ij}$$

- M-step is maximizing as before:

$$\hat{\pi}_i = E\left\{q_0^i\right\} \qquad \hat{\alpha}_{ij} = \frac{\sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^j\right\}}{\sum_{k=1}^{M} \sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^k\right\}} \qquad \hat{\eta}_{ij} = \frac{\sum_{t=0}^{T} E\left\{q_t^i\right\} y_t^j}{\sum_{k=1}^{N} \sum_{t=0}^{T} E\left\{q_t^i\right\} y_t^k}$$

- What are E{}'s? $E_{p(x)}\left\{x^i\right\} = \sum_x p\left(x\right) x^i = \sum_x p\left(x\right) \delta\left(x = x^i\right) = p\left(x^i\right)$

HMMs: EM Learning

- But, we don't observe the q's, incomplete…

$$p\left(\bar{y} \mid \theta\right) = \sum_{q} p\left(q, \bar{y} \mid \theta\right) = \sum_{q_0} \cdots \sum_{q_T} p\left(q_0\right) \prod_{t=1}^{T} p\left(q_t \mid q_{t-1}\right) \prod_{t=0}^{T} p\left(\bar{y}_t \mid q_t\right)$$

- EM: Max expected complete likelihood given current p(q)

$$E\left\{l\left(\theta\right)\right\} = E_{p\left(q_0,\ldots,q_T \mid y\right)}\left\{\log p\left(q, y\right)\right\} = \sum_{q_0} \cdots \sum_{q_T} p\left(q \mid y\right) \log p\left(q, y\right)$$

$$= E\left\{\sum_{i=1}^{M} q_0^i \log \pi_i + \sum_{t=1}^{T} \sum_{i,j=1}^{M} q_{t-1}^i q_t^j \log \alpha_{ij} + \sum_{t=0}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} q_t^i y_t^j \log \eta_{ij}\right\}$$

$$= \sum_{i=1}^{M} E\left\{q_0^i\right\} \log \pi_i + \sum_{t=1}^{T} \sum_{i,j=1}^{M} E\left\{q_{t-1}^i q_t^j\right\} \log \alpha_{ij} + \sum_{t=0}^{T} \sum_{i=1}^{M} \sum_{j=1}^{N} E\left\{q_t^i\right\} y_t^j \log \eta_{ij}$$

- M-step is maximizing as before:

$$\hat{\pi}_i = E\left\{q_0^i\right\} \qquad \hat{\alpha}_{ij} = \frac{\sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^j\right\}}{\sum_{k=1}^{M} \sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^k\right\}} \qquad \hat{\eta}_{ij} = \frac{\sum_{t=0}^{T} E\left\{q_t^i\right\} y_t^j}{\sum_{k=1}^{N} \sum_{t=0}^{T} E\left\{q_t^i\right\} y_t^k}$$

- What are E{}'s? $E_{p(x)}\left\{x^i\right\} = \sum_x p\left(x\right) x^i = \sum_x p\left(x\right) \delta\left(x = x^i\right) = p\left(x^i\right)$
- Our JTA ψ & ϕ marginals! (JTA is the E-Step for given θ)

$$E\left\{q_t^i q_{t+1}^j\right\} = p\left(q_t = i, q_{t+1} = j \mid \bar{y}\right) \qquad E\left\{q_t^i\right\} = p\left(q_t = i \mid \bar{y}\right)$$

HMMs: Gaussian Emissions

- Instead of table for emissions, have Gaussian:

$$p\left(\bar{y} \mid \theta\right) = \sum_q p\left(q, \bar{y} \mid \theta\right) = \sum_{q_0} \cdots \sum_{q_T} p\left(q_0\right) \prod_{t=1}^{T} p\left(q_t \mid q_{t-1}\right) \prod_{t=0}^{T} p\left(\bar{y}_t \mid q_t\right)$$

where $p\left(\bar{y}_t \mid q_t\right) = N\left(\bar{y}_t \mid \mu_{q_t}, I\right)$

- Clique initialization: $\psi\left(q_t, \bar{y}_t\right) = \psi\left(q_t\right) = N\left(\bar{y}_t \mid \mu_{q_t}, I\right)$

- M-step is maximizing as before:

$$\hat{\pi}_i = E\left\{q_0^i\right\} \qquad \hat{\alpha}_{ij} = \frac{\sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^j\right\}}{\sum_{k=1}^{M} \sum_{t=0}^{T-1} E\left\{q_t^i q_{t+1}^k\right\}} \qquad \vec{\mu}_i = \frac{\sum_{t=0}^{T} E\left\{q_t^i\right\} \bar{y}_t}{\sum_{t=0}^{T} E\left\{q_t^i\right\}}$$

- Can thus handle continuous time series as in speech recognition