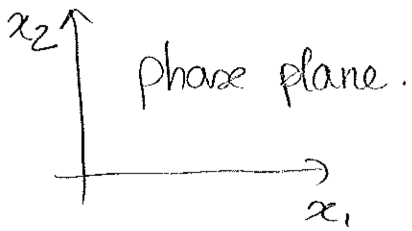
Lecture 2 Phase Plane Analysis

Use : Graphically observe how 2nd order system behave for different initial conditions

States of the system
$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) = \text{nonlinear function} \\ \dot{x}_2 = f_2(x_1, x_2) = \text{nonlinear function} \end{cases}$$



Example Spring mass system

$$\ddot{x} + x = 0$$

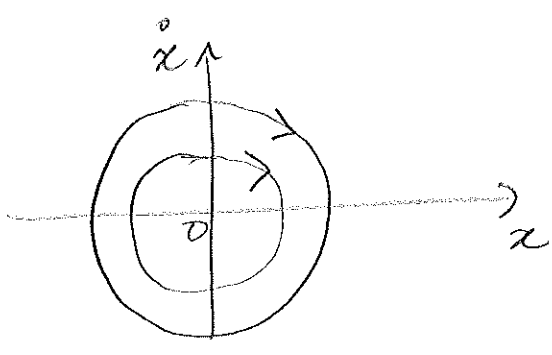
$x_1 = x$

$x_2 = \dot{x}$

$\dot{x}_2 + x_1 = 0.$

$\dot{x}_1 = x_1$

$\dot{x}_2 = -x_1$



Singular Points

Equilibrium pt. in the phase plane.

$$\dot{x} = 0 \Rightarrow f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$

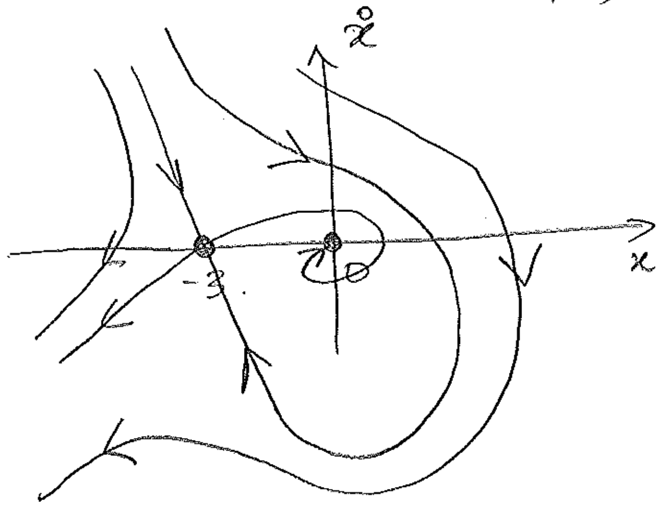
Why called singular pt? Slope in phase plane $\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2}{f_1}$

At equm pt, $\frac{dx_2}{dx_1} \Rightarrow$ indeterminate ($0/0$ form)

Example

$$\ddot{x} + 0.6\dot{x} + 3x + x^2 = 0$$

Two singular pts: $(0,0)$
$(-3,0)$



Symmetry in Phase-Plane Portraits

$$\ddot{x} + f(x,\dot{x}) = 0 \quad 2^{nd} \text{ order dynamics}$$

$x_1 = x$
$x_2 = \dot{x}$
$\dot{x}_2 = f(x_1, x_2)$

$$\frac{dx_2}{dx_1} = \frac{-f(x_1,x_2)}{\dot{x}}$$

Symmetry about x_1 axis

$$f(x_1,x_2) = f(x_1,-x_2)$$

Symmetry about x_2 axis

$$f(x_1,x_2) = -f(-x_1,x_2)$$

Symmetry about origin

$$f(x_1,x_2) = -f(-x_1,-x_2)$$

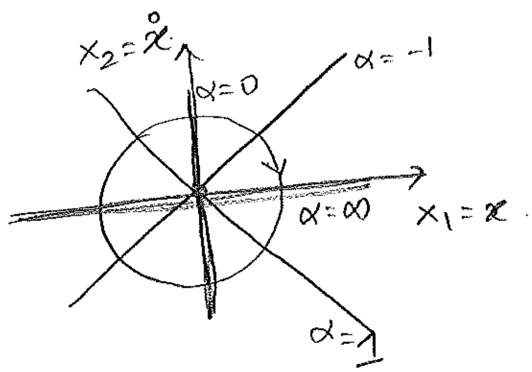
Method of isoclines to construct phase portrait

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)} = \alpha \implies \text{Locus of pts w/ slope } \alpha$$

$$f_2(x_1,x_2) = \alpha f_1(x_1,x_2)$$

For mass-spring system

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$



Phase plane analysis for linear system

second order
$$\begin{cases} \dot{x}_1 = a x_1 + b x_2 \\ \dot{x}_2 = c x_1 + d x_2 \end{cases}$$

can be rewritten as $\longrightarrow$ $\ddot{x}_1 \pm (a+d)\dot{x}_1 + (cb-ad)x_1$

$\underbrace{\ddot{x} + a\dot{x} + bx = 0}_{\text{generalize}}$

Solution $\quad x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}, \quad \lambda_1 \neq \lambda_2$

$$x(t) = (k_1 + k_2 t)e^{\lambda_1 t}, \quad \lambda_1 = \lambda_2$$

where λ_1 & λ_2 are solutions of characteristic eq^n

$$s^2 + as + b = 0$$

Roots are

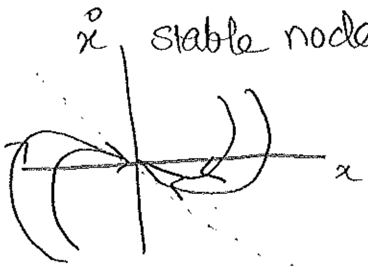
$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

There is only one equlibrium pt. ($x = 0$)

However, system behavior is different depending on

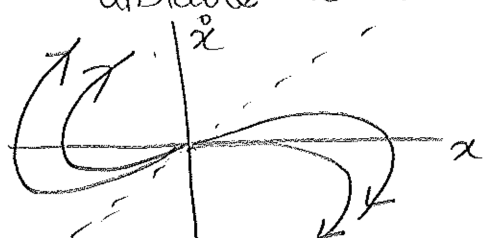
λ_1, λ_2.

stable node



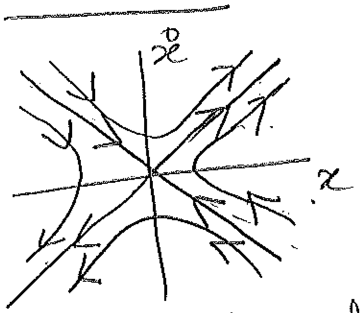
λ_1, λ_2 : negative real

Unstable node



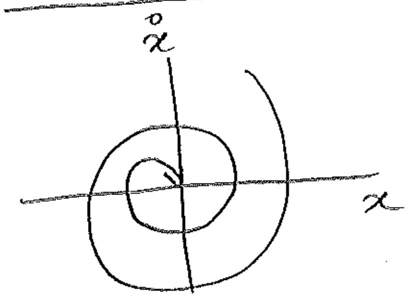
λ_1, λ_2 : positive real.

Saddle pt.



λ_1 and λ_2 real & opposite sign.

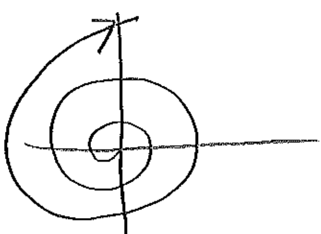
Stable focus



λ_1, λ_2 complex

real parts < 0

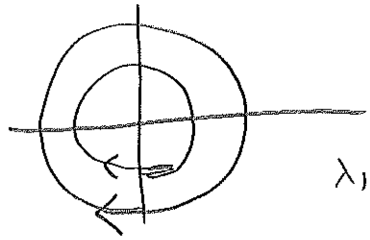
Unstable focus



λ_1, λ_2 complex conjugate

real parts > 0

Center point



λ_1, λ_2 complex conjugate

real parts $= 0$

Phase Plane Analysis of Nonlinear Systems

- Local behavior can be approximated by linear system
- can exhibit multiple equm pts + limit cycles

Limit cycle → $\underbrace{\text{Isolated}}$ $\underbrace{\text{closed}}$ curve.

Isolated: nearby curves converging or diverging from it

closed: periodic → motion starting on the curve stays on it forever.

Stable limit cycle → All trajectories nearby converge to it as $t \to \infty$

Unstable " " → " " " diverge from it as $t \to \infty$

semistable " " → " " " either converge or diverge as $t \to \infty$

Three theorems on limit cycles

Poincare : If a limit cycle exists in the 2nd order autonomous system, then $N = S+1$

autonomous: (no notion of time) $\dot{x} = f(x)$

N = # of nodes, center, foci enclosed by limit cycle

S = # of enclosed saddle points

Utility for us: Limit cycle must enclose at least one equm pt.

Poincare - Bendixson : If a trajectory of a 2^{nd} order autonomous system remains in a finite region Ω, then one of the following is true:

(a) the trajectory goes to an equm pt.

(b) " " tends to an asymptotically stable limit cycle

(c) trajectory is itself a limit cycle.

Bendixson : For the nonlinear system
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

no limit cycle can exist in a region Ω in which $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ does not vanish and does not change sign.