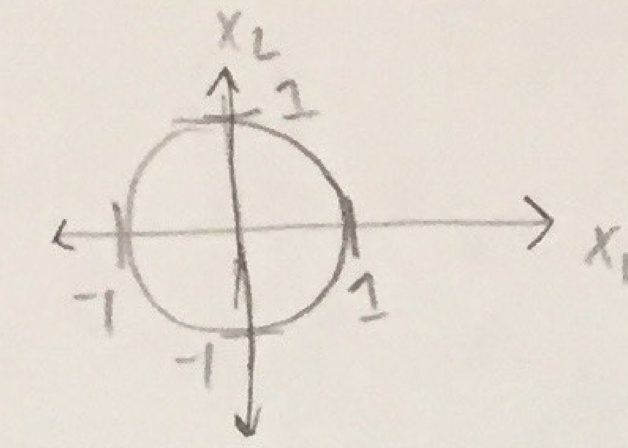
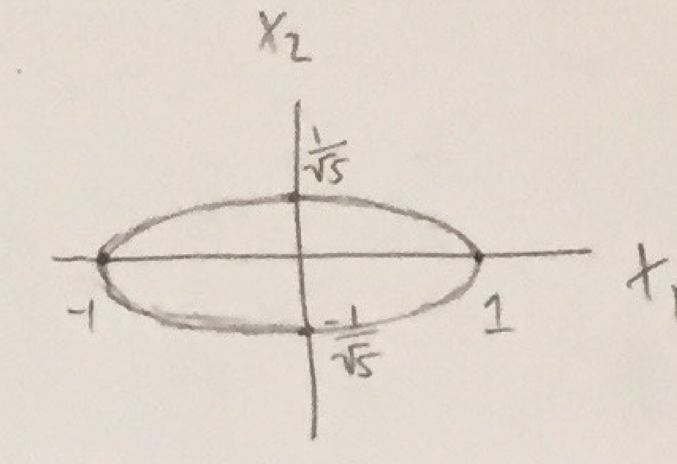
ME 533 HW2

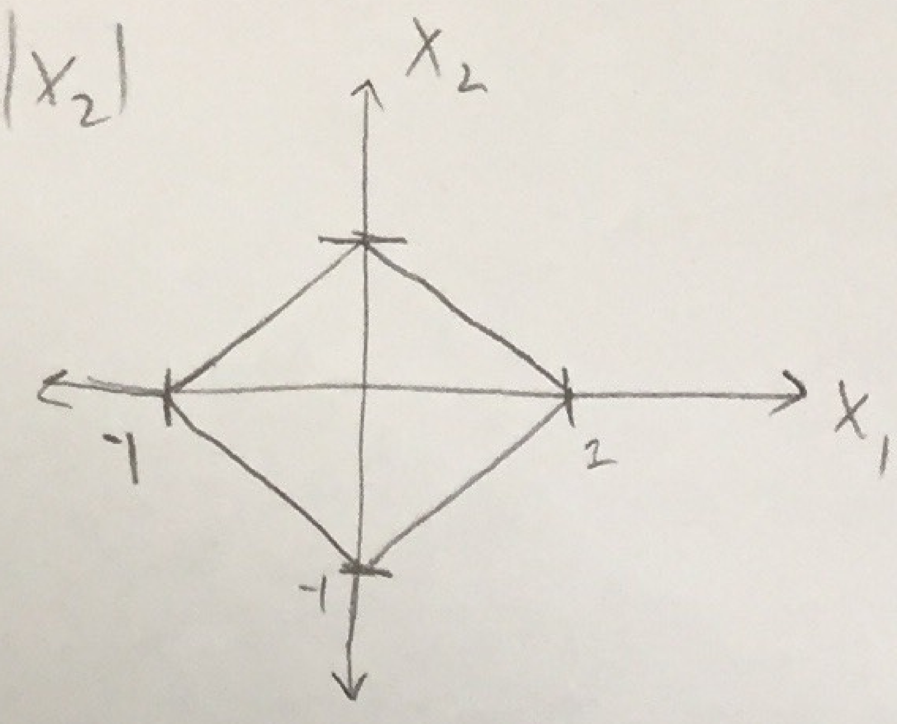
① i) $\|x\|^2 = x_1^2 + x_2^2$



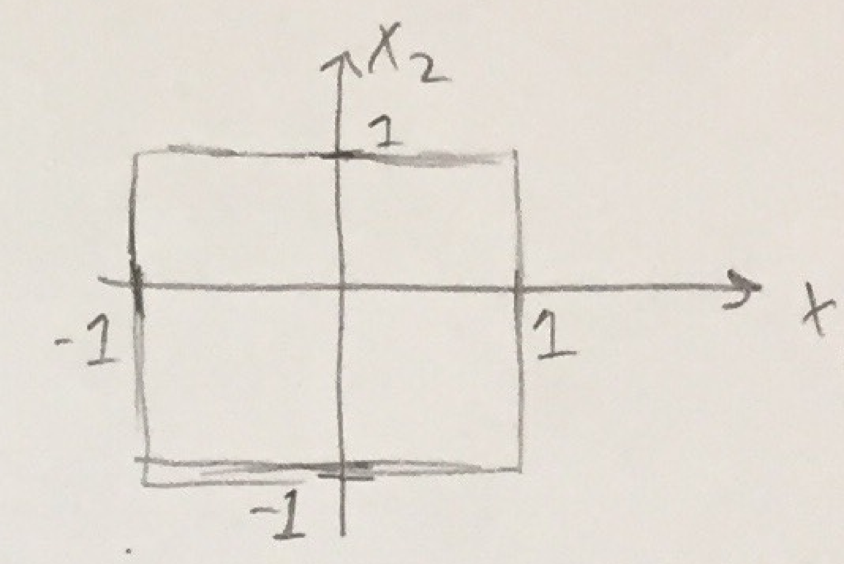
ii) $\|x\|^2 = x_1^2 + 5x_2^2$



iii) $\|x\| = |x_1| + |x_2|$



iv) $\|x\| = \mathrm{Sup}(|x_1|, |x_2|)$



② i) $\dot{x} = -x^3 + \sin^4 x$

equillibrium points : $[0]$

$V = x^2 \geq 0$ for all x

$\dot{V} = 2x\dot{x}$

$\dot{V} = 2x(-x^3 + \sin^4 x)$

$\sin^4 x \leq |\sin^4 x| \leq |\sin^3 x| \leq |x^3|$

$= -2x^4 + 2x\sin^4 x \leq -2x^4 + x|x^3| = -2x^4 + x^4 = \underbrace{-x^4}_{\text{neg everywhere.}}$

according to invariant set theorem
asymptotically stable because M contains only one equillibrium pt
at the origin

$V(x) \rightarrow$ as $\|x\| \rightarrow \infty$ so globally stable

$\dot{V}(x) \leq 0$ for all x

ii) $\dot{x} = (5-x)^5$

eq pt $= [5]$

$y = x - 5$

$\dot{y} = \dot{x} = (5-x)^5 = (-y)^5 = -y^5$

eq pt for y: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ ✓

$V = y^2 > 0$ for all y

$\dot{V} = 2y\dot{y} = 2y(-y^5) = -2y^6 \leq 0$ for all y

$V(y) \rightarrow \infty$ as $\|y\| \rightarrow \infty$

$\dot{V}(y) \leq 0$ for all y

$\boxed{\text{globally asymptotically stable according to global invariant set theorem}}$

iii) $\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8 x \cos^2 3x$

$y^2 \sin^8 y \cos^2 3y \leq y^2 \cdot y^5 \sin^6 y \cos^2 3y \leq y^7$

$\int_0^x y^7 - y^7 \, dy = 0$

$V = KE + PE$

$V = \underbrace{\tfrac{1}{2}\dot{x}^2}_{\text{pos everywhere}} + \underbrace{\int_0^x y^7 - y^2 \sin^8 y \cos^2 3y \, dy}_{\text{greater than 0 for all } x}$

$V \geq 0$ for all x

$V(x) \rightarrow \infty$ as $\|x\| \rightarrow 0$

$\dot{V} = \tfrac{1}{2} \cdot 2\dot{x}\ddot{x} + \dot{x}[x^7 - x^2 \sin^8 x \cos^2 3x]$

$\dot{V} = \dot{x}[-\dot{x}^5 - \cancel{x^7} + \cancel{x^2 \sin^8 x \cos^2 3x}] + \dot{x}[\cancel{x^7} - \cancel{x^2 \sin^8 x \cos^2 3x}]$

$\dot{V} = -\dot{x}^6 \leq 0$ for all x

if $\dot{x} = 0$

$\ddot{x} + x^7 = x^2 \sin^6 x \cos^2 3x$

$\ddot{x} = 0$ if $-x^7 = x^2 \sin^6 x \cos^2 3x$

which is only true when $x = 0$

$\boxed{\text{globally asymptotically stable}}$

eq pt: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

iv) $\ddot{x} + (x-1)^4 \cdot \dot{x}^7 + x^5 = x^3 \sin^3 x$

$V = KE + PE$

$V = \underbrace{\frac{1}{2}\dot{x}^2}_{\text{pos for all } \dot{x}} + \underbrace{\int_0^x y^5 - y^3 \sin^3 y \, dy}_{\text{pos for all } x} \geq 0$ for whole state space

$y^3 \sin^3 y \leq y^3 \cdot y^2 \sin y \leq y^5$

$\int_0^x y^5 - y^3 \sin^3 y \, dy \geq 0$ for all x

$\dot{V} = \dot{x}\ddot{x} + \dot{x}[x^5 - x^3 \sin^3 x]$

$\dot{V} = \dot{x}\left[-(x-1)^4 \cdot \dot{x}^7 + \cancel{-x^5} + \cancel{x^3 \sin^3 x}\right] + \dot{x}\left[\cancel{x^5} - \cancel{x^3 \sin^3 x}\right]$

$= -\dot{x}^8 (x-1)^4 \leq 0$ for whole state space

if $\dot{x} = 0$: $\ddot{x} + x^5 = x^3 \sin^3 x$

$\ddot{x}$ is only zero when x is zero

$$\boxed{\text{globally asymptotically stable} \quad \text{eq pt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

v) $\ddot{x} + (x-1)^2 \cdot \dot{x}^7 + x = \sin\left(\frac{\pi x}{2}\right)$

$\int_0^x y - \sin\left(\frac{\pi y}{2}\right) dy$

$\frac{1}{2}y^2 + \cos\left(\frac{\pi y}{2}\right) \cdot \frac{2}{\pi}\Big|_0^x = \underbrace{\frac{1}{2}x^2 + \cos\left(\frac{\pi x}{2}\right)\frac{2}{\pi} - \frac{2}{\pi}}_{\text{not positive everywhere}}$

$V = KE + PE$

$V = \frac{1}{2}\dot{x}^2 + \int_0^x y - \sin\left(\frac{\pi y}{2}\right) dy$

Krasovskii

$\dot{x} = 0$

$\ddot{x} + x = \sin\left(\frac{\pi x}{2}\right)$

$\ddot{x} = 0$ when $x = -1, 0, 1$

$f = \begin{bmatrix} \dot{x} \\ -(x-1)^2 \dot{x}^7 - x + \sin\left(\frac{\pi x}{2}\right) \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -2(x-1)\dot{x}^7 - 1 + \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) & -(x-1)^2 7\dot{x}^6 \end{bmatrix}$

$F = \begin{bmatrix} 0 & -2(x-1)\dot{x}^7 + \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) \\ -2(x-1)\dot{x}^7 + \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) & -(x-1)^2 7\dot{x}^6 \end{bmatrix}$

not negative definite over state space so unstable?

3) $\dot{v} + 2a|v|v + bv = \cancel{c}$ $a>0$ $b>0$

$V = \frac{1}{2}v^2$

$\dot{V} = v\dot{v} = v(-2a|v|v - bv + c) = -2a|v|v^2 - bv^2 + cv$

$\dot{V} \leq 0$ when $v = 0$ or $c < 2a|v|v + bv$ if $v \neq 0$

4) i) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 \\ e^{3t} & -2 \end{bmatrix} = \begin{bmatrix} -20 & e^{3t} \\ e^{3t} & -4 \end{bmatrix}$$

$$\lambda = -12 \pm \sqrt{e^{6t} + 64}$$

$\boxed{\text{unstable because eigenvalues are not negative for all } t}$

ii) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2\sin(t) \\ 0 & -(t+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\lambda = -t-2 \pm \left[4\sin^2 t + t^2\right]^{1/2}$$

$\boxed{\lambda \text{ is negative for all } t \text{ so asymptotically stable}}$

$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 & 2\sin t \\ 0 & -(t+1) \end{bmatrix}$

$\boxed{A_2 \not\to 0 \text{ as } t \to 0 \text{ not globally}}$

iii) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\lambda = 18 \pm \sqrt{1 + e^{4t}}$$

$\boxed{\text{unstable not negative everywhere}}$