Lecture #5

Robust Control of Nonlinear Systems

Models are usually imprecise

- actual plant uncertainty (unknown parameters)
- purposeful simplified representation

Robust controller deals w/ such imprecision

- through a nominal controller + additional terms to deal w/ uncertainty

For example, (Lots of systems come in this form)

~~xxxx~~ Consider the scalar n^{th} order d.e.

$$x^{(n)} = f(x, \dot{x}, \ldots, x^{(n-1)}, t) + b(x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}, t)\, u$$

If f and b are known, and we had a trajectory tracking problem, we can define

$$\tilde{x}(t) = x(t) - x_{des}(t)$$

and send $\tilde{x}(t) \to 0$

We do this by setting $v = f + bu$ and then solve for u.

But of course, we do not know f & b

e.g., Robot in the presence of underwater currents

- robots w/ timevarying or unknown loads.

So you define

$$\| f(x,t) - \hat{f}(x,t) \| \leq F(x,t)$$

unknown

Some estimate of drift

worst-case

and similarly for $b(x,t)$

Typical problem is to track trajectory and keep error small.

The key idea is to replace n^{th} order system w/ first order sys.

- and take advantage of simplicity of first order problem

($\rightarrow$ if output negative, apply +ve input
if " + , " -ve ")

Otherwise, this new variable when controlled represents original system

So create intermediate variable s:

such that (1) s contains u

and

(2) $s \to 0 \Rightarrow \tilde{x}(t) \to 0$

<u>Example</u>

$n=2$ $\qquad \ddot{x} = f + bu$

Pick $s = \dot{x}$

or

$s = \dot{x}_d - \dot{x}$

But this does not satisfy requirement (2)

If $s \to 0$ only $\dot{\tilde{x}} \to 0$, not $\tilde{x}(t)$

Suppose we choose to satisfy the 2nd condition

$$s = \tilde{x}$$

Then as $s \to 0$, $\tilde{x} \to 0$. So that is good

But that does not satisfy the 1st condition

So let's mix the two.

Choose $s = \dot{\tilde{x}} + \underbrace{\lambda}_{\text{scalar} > 0} \tilde{x}$

this would satisfy condition #1

Regarding condition #2, this equation is like $s \to \boxed{\frac{1}{\frac{d}{dt} + \lambda}} \to \tilde{x}$

filter

therefore as $s \to 0$, $\tilde{x} \to 0$.

For the n^{th} order case,

choose

$$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}$$

$n = 3 \Rightarrow s = \ddot{\tilde{x}} + 2\lambda\dot{\tilde{x}} + \lambda^2 \tilde{x}$

The reasoning is the same.

(1) $\dot{s}$ contains $\left(\frac{d}{dt}\right)^n \tilde{x}$ and therefore contains u.

(2) Now you have n^{th} order filter

$$s \to \boxed{\frac{1}{\left(\frac{d}{dt} + \lambda\right)^{n-1}}} \to \tilde{x}$$

Stable linear filter

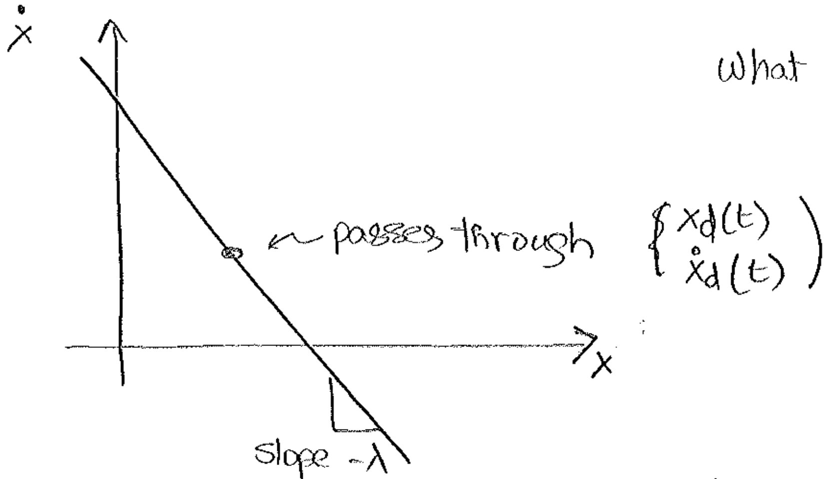
So you replace n^{th} order problem w/ first order problem.

If you send $s \to 0$, $\tilde{x} \to 0$

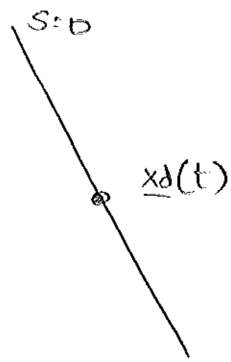
Let's look @ this geometrically : (second order system)

$s = \dot{\tilde{x}} + \lambda \tilde{x}$

What does $s = 0$ represent ?



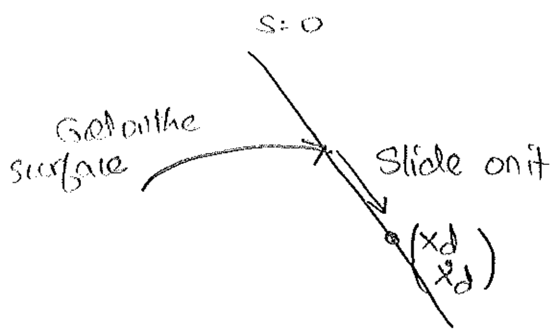
Let's go to n^{th} order system. The line $(s=0)$ becomes a hyper plane



You want the system to tend to $\underline{x_d}$ and stay there.

What does this mean ?

You want to make it an invariant set.



(may be spiral in n-dimensions)

So $s=0$ is both a place & a dynamics

Once you get onto hyperplane, motion is completely defined.

What if I can only guarantee

$$|s| \leq \phi_{\text{constant}} \ ?$$

How does the residual error in s translate to x?

Let's examine $n = 2$ case

$$s = \dot{\tilde{x}} + \lambda \tilde{x}$$

$s = (p+\lambda)\tilde{x}$

$\dfrac{\tilde{x}}{s} = \dfrac{1}{p+\lambda}$

$$s \longrightarrow \boxed{\dfrac{1}{\frac{d}{dt}+\lambda}} \longrightarrow \tilde{x}$$

We know that $|s| \leq \phi$ after transient. (effect of initial conditions go away)

So the form for

$$\tilde{x}(t) = \underbrace{\begin{pmatrix}\text{effect of initial}\\ \text{cond'n}\end{pmatrix}}_{\longrightarrow 0 \text{ since } \lambda > 0} + \int_0^t e^{-\lambda(t-r)} s(r)\,dr$$

Now $$\left| \int_0^t e^{-\lambda(t-r)} s(r)\,dr \right| \leq \int_0^t \left| e^{-\lambda(t-r)} s(r)\,dr \right|$$

$$\leq \phi \int_0^t \left| e^{-\lambda(t-r)} \right| dr$$

$$\leq \frac{\phi}{\lambda}$$

$\Rightarrow$ $\tilde{x}$ bounded by $\dfrac{\phi}{\lambda}$

$$s \longrightarrow \boxed{\dfrac{1}{\frac{d}{dt}+\lambda}} \longrightarrow \cdots \longrightarrow \boxed{\dfrac{1}{\frac{d}{dt}+\lambda}} \longrightarrow \tilde{x}$$

What about n^{th} order sys? Similar calculation w/ $(n-1)$ consecutive filters.

If $\underset{\text{(error)}}{|s| \leq \phi}$, then $\underset{\text{(error)}}{|\tilde{x}|} \leq \dfrac{\phi}{\lambda^{n-1}}$

So let's solve the first order problem (we have delayed it all this while!)

consider second order sys.

7.1.3 $\ddot{x} = f(x,t) + b(x,t)u$

$$\ddot{x} + a(t)\dot{x}^2\cos 3x = u \quad \text{w/} \quad \underbrace{1 \le a(t) \le 2}_{\text{similar to a drag term}}$$

Choose $\hat{f} = \underbrace{-1.5}_{\text{middle of range}}\,\dot{x}^2\cos 3x$

So $F = 0.5\,\dot{x}^2|\cos 3x|$

$$s = \dot{\tilde{x}} + \lambda\tilde{x}$$

$$\dot{s} = \ddot{\tilde{x}} + \lambda\dot{\tilde{x}} = f + bu - \ddot{x}_d + \lambda\dot{\tilde{x}}$$

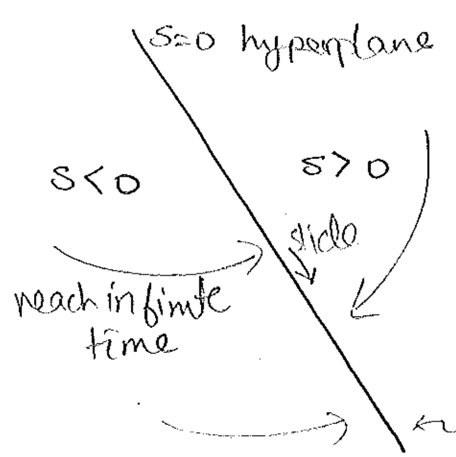
Choose u such that $\frac{d}{dt}s^2 \le 0$ (we want $|s|$ to always decrease)

or more precisely

$$\boxed{\frac{1}{2}\frac{d}{dt}s^2 \le -\underbrace{\eta}_{\text{constant} > 0}|s|}$$

called sliding condition [slide after reaching surface $s=0$]

What does this mean?
What if we start w/ $s > 0$



$\Rightarrow s\dot{s} \le -\eta s$.

If $s > 0$.

$\dot{s} \le -\eta$

$\Rightarrow s=0$ is reached in finite time $\left(\le \frac{s(t=0)}{\eta}\right)$

← trapped on this surface (invariant set)
then first order system takes over.

Choose $b = 1$ for simplicity

Define $\hat{u}$ such that $\dot{s} = 0$ if you know f + b exactly.
$\Downarrow$ (i.e. $f = \hat{f}$)

i.e. $\hat{u} = \underbrace{-\hat{f}}_{\text{best guess}} + \ddot{x}_d - \lambda \dot{\tilde{x}}$

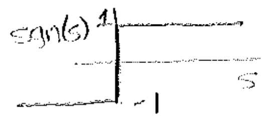
with $u = \hat{u}$

$$\dot{s} = f - \hat{f}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dt} s^2 = s\dot{s} = s(f - \hat{f})$$

Also $f - \hat{f}$ may be of arbitrary sign

But this does not necessarily mean that $|f - \hat{f}| \leq \eta$



Add one more term.

$$u = \hat{u} - k\,\text{sign}(s)$$ → push in the right direction

Then $$s\dot{s} = s(f - \hat{f}) - k|s| \underset{\text{needs to be}}{\leq} -\eta|s|$$

Choose $k = F + \eta$ to satisfy this condition
F → uncertainty in f

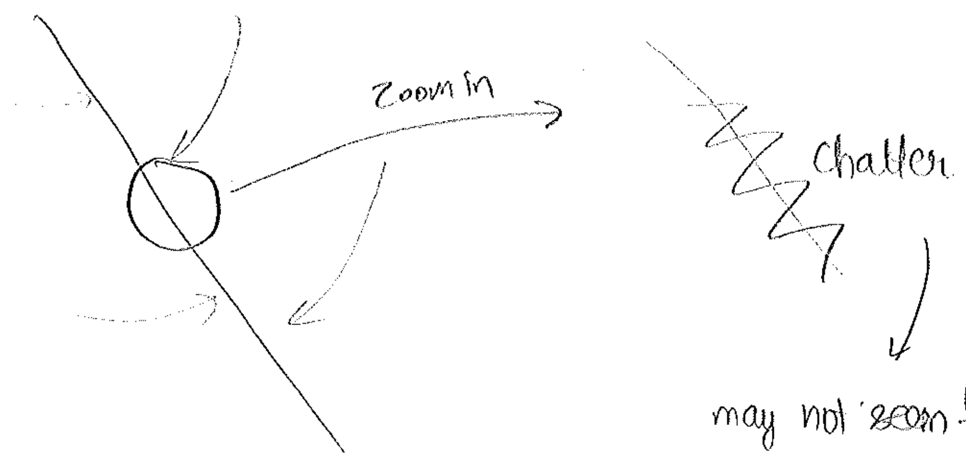
So for the example.

$$u = 1.5\,\dot{x}^2\cos 3x + \ddot{x}_d - \lambda\dot{\tilde{x}} - (0.5\,\dot{x}^2|\cos 3x| + \eta)\,\text{sign}(\dot{\tilde{x}} + \lambda\tilde{x})$$

Why does this work?

You have chosen control law so that sys. is pushed on to the surface $s = 0$ always (the effect of the term $k|s|$)

Let's look closely @ performance though.



may not seem like a big deal in the s space but may be a big deal in the actuator space.

- large oscillations

May be you can ignore small tracking error and change the control law to ~~ensure~~ not act for such small errors.

(ignore $|s| \leq \phi$)