Lecture #4

Stability of Linear Time-Variant Systems.

The methods for linear time invariant systems do not apply.
– (such as eigenvalue determination)

– Thus, it is useful to try to apply Lyapunov's direct method.

Consider $\dot{x} = A(t)\, x$

For LTI systems, if eigenvalues all have negative real parts, then LTI system is stable.

However, this is not the case for LTV systems even if they always have negative (even constant) eigenvalues.

For example,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues are (-1) and (-1) at all times.

However, the system sol'n is

$$x_2 = x_2(0)\, e^{-t}$$

$$\dot{x}_1 + x_1 = \underbrace{x_2(0)\, e^{t}}_{\text{blows up as } t \to \infty}$$

However, the LTV system is asymptotically stable if eigenvalues of the symmetric matrix $A + A^T$ (all of which are real) remain strictly in the left-half plane.

$$\lambda_i(A + A^T) \le -\lambda$$

Then the system $x \to 0$ exponentially w/ rate λ.

<u>How to show this?</u>

Pick scalar function $V = x^T x$

$$\dot{V} = x^T \dot{x} + \dot{x}^T x$$

$$= x^T (A + A^T) x \le -2\lambda x^T x$$

$$\le -2\lambda V$$

Thus, $\dot{V} + 2\lambda V = 0$

Thus $0 \le V \le V(0) e^{-2\lambda t}$

$\Rightarrow$ $V \to 0$ as $t \to \infty$

$\Rightarrow$ $x \to 0$ as $t \to \infty$

Advanced Stability Analysis using Barbalat's Lemma

The invariant set theorems are not applicable to non-autonomous systems.

Barbalat's lemma helps address these issues.

First: some general properties of functions

(1) $\dot{f} \to 0 \not\Rightarrow f$ converges $\quad$ [e.g. $f(t) = \sqrt{t}\sin(\log t)$]

(2) f converges $\not\Rightarrow \dot{f} \to 0$ $\quad$ [e.g. $f(t) = e^{-t}\sin^2(e^{2t})$]

(3) If f lower bounded + $\dot{f} \le 0$, then f converges to a limit

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Barbalat's Lemma

If a differentiable function $f(t)$ has a finite limit as $t \to \infty$ and $\dot{f}$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$

(derivative is bounded)

Lyapunov-Like Lemma

If a scalar function $V(x,t)$ satisfies

→ $V(x,t)$ is lower bounded

→ $\dot{V}(x,t)$ is negative semi-definite

→ $\dot{V}(x,t)$ is uniformly continuous in time

then $\dot{V}(x,t) \to 0$ as $t \to \infty$

[Also V approaches a finite limiting value $V_\infty \leq V(x(0),0)$

See P4.5

Example
$$\dot{e} = -e + \theta w(t)$$
$$\dot{\theta} = -e\, w(t)$$

e = tracking error
θ = parameter "
$w(t)$ = bounded continuous function

Let's investigate asymptotic properties of the system.

$2e\dot{e} + 2\theta\dot{\theta}$

$2e(-e+\theta w) + 2\theta(-e w)$

Choose $V = e^2 + \theta^2$
$$\dot{V} = -2e^2 \leq 0$$

$\Rightarrow$ $V(t) \leq V(0)$ $\Rightarrow$ e & θ are bounded.

But the invariant set theorem cannot be used since the system is non-autonomous.

Let's use Barbalat's Lemma:

Typical strategy is to use $\dot{V} = -e^2$

$= -(\text{error term})^2$

$$\left.\begin{array}{l}\text{Now since } V \text{ is typically lower bounded,} \\ \quad + \dot{V} \leq 0 \\ \quad + \ddot{V} \text{ is bounded}\end{array}\right\} \Rightarrow \dot{V} \to 0 \text{ as } t \to \infty$$

The derivative of $\dot{V}$ is

$$\ddot{V} = -4e(-e+\theta w)$$

$\Rightarrow$ $\ddot{V}$ is bounded $\Rightarrow$ $\dot{V}$ is uniformly continuous

$\Rightarrow$ $\dot{V} \to 0$ as $t \to \infty$

Applying Barbalat's theorem

$\Rightarrow$ $e \to 0$ as $t \to \infty$

[Note that we cannot say more about θ.
It is only bounded]

Two main differences from Lyapunov analysis:

(1) V can simply be lower bounded (not necessarily p.d)

(2) $\dot{V}$ must also be uniformly continuous in addition to being n.s.d. ($\ddot{V}$ bounded)