① Problem 7.7

$\ddot{x} + \alpha_1(t)\,|x|\,\dot{x}^2 + \alpha_2(t)\,x^3\cos 2x = 5\dot{u} + u$

$\forall t \geq 0 \quad |\alpha_1(t)| \leq 1 \quad -1 \leq \alpha_2(t) \leq 5$

Hint: $v = 5\dot{u} + u$

$x = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad \tilde{x} = \begin{bmatrix} x - x_d \\ \dot{x} - \dot{x}_d \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \dot{\tilde{x}} \end{bmatrix} \quad \ddot{\tilde{x}} = \ddot{x} - \ddot{x}_d$

$\ddot{x} = \ddot{\tilde{x}} + \ddot{x}_d$

Aside

$\ddot{x} = f + u$

$\dot{s} = f + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$

$\dot{s} = 0$ if $u = -f + \ddot{x}_d - \lambda\dot{\tilde{x}}$

$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x} \qquad n = 2$

$s = \frac{d}{dt}\tilde{x} + \lambda\tilde{x} = \dot{\tilde{x}} + \lambda\tilde{x}$

$\dot{s} = \ddot{\tilde{x}} + \lambda\dot{\tilde{x}} = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}} =$

$\ddot{x} = \underbrace{-\alpha_1(t)\,|x|\,\dot{x}^2 - \alpha_2(t)\,x^3\cos 2x + 5\dot{u} + u}_{f}$

$\hat{f} = -0|x|\dot{x}^2 + 2x^3\cos 2x$

$F = -|x|\dot{x}^2 - 3x^3\cos 2x$

$\dot{s} = -\alpha_1(t)|x|\dot{x}^2 - \alpha_2 x^3\cos 2x + 5\dot{u} + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$

$\dot{s} = 0 \rightarrow u + 5\dot{u} = \alpha_1(t)|x|\dot{x}^2 + \alpha_2 x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}}$

best guess for $\hat{u}$ by using $\hat{f}$

$\hat{u} + 5\hat{\dot{u}} = \hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\text{sgn}(s)$

plug in $\hat{f}$

$\hat{u} + 5\hat{\dot{u}} = 2x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\text{sgn}(s)$

$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s|$

$\dot{s}s \leq -\eta|s|$

$[-\alpha_1(t)|x|\dot{x}^2 - \alpha_2 x^3\cos 2x + 5\dot{u} + \hat{u} - \ddot{x}_d + \lambda\dot{\tilde{x}}]s$

plug in $\hat{u}$

$[-\alpha_1(t)|x|\dot{x}^2 - \alpha_2 x^3\cos 2x + (-2x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\text{sign}(s)) - \ddot{x}_d + \lambda\dot{\tilde{x}}]s$

$[-\alpha_1(t)|x|\dot{x}^2 - (\alpha_2 + 2)x^3\cos 2x + \cancel{\ddot{x}_d} - \cancel{\ddot{x}_d} - \cancel{\lambda\dot{\tilde{x}}} + \cancel{\lambda\dot{\tilde{x}}} - k\,\text{sign}(s)]s$

$[\underbrace{-\alpha_1(t)|x|\dot{x}^2 - (\alpha_2 + 2)x^3\cos 2x}_{\leq F \Rightarrow \leq |x|\dot{x}^2 - 3x^3\cos 2x} - k\,\text{sign}(s)]s \leq -\eta|s|$

$\eta = 0.1$

$[F - k\,\text{sign}(s)]s \leq -\eta|s|$

choose $k = F + \eta$

$k = -|x|\dot{x}^2 - 3x^3\cos 2x + 0.1$

$s \leq |s|$

$Fs - k|s| \leq -\eta|s|$

$\boxed{\dot{u} = \frac{1}{5}[-2x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\text{sgn}(s) - \int\dot{u}\,du]}$

$k|s| \geq (\eta + F)|s|$

effect of chattering?


Position
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Actual
Velocity
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Actual
Position Tracking Error
Position Tracking Error
Control Input
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9
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400
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-200
-400
-600

7.10) sliding controller for

$|\alpha_1| \le 1 \qquad |\alpha_2| \le 2 \qquad 1 \le b \le 4$

$\dddot{x} + \alpha_1 \ddot{x}^2 + \alpha_2 \dot{x}^5 \sin 4x = bu$

$\dddot{x} = -\alpha_1 \ddot{x}^2 - \alpha_2 \dot{x}^5 \sin 4x + bu$
$= f + bu$

$f = -\alpha_1 \ddot{x}^2 - \alpha_2 \dot{x}^5 \sin 4x$

$\hat{f} = 0\ddot{x}^2 - 0\dot{x}^5 \sin 4x$

$n=3$

$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x} = \ddot{\tilde{x}} + 2\lambda\dot{\tilde{x}} + \lambda^2 \tilde{x}$

$|\hat{f} - f| \le F = \ddot{x}^2 + 2\dot{x}^5 \sin 4x$

$\tilde{x} = [x - x_{des}]$
$\dot{\tilde{x}} = [\dot{x} - \dot{x}_{des}]$
$\ddot{\tilde{x}} = [\ddot{x} - \ddot{x}_{des}]$

Lyapunov

$\dot{s}s = -\eta |s|$

$[f + b[\hat{u} - k\,\mathrm{sat}(s/\phi)]\hat{b}^{-1} - \dddot{x}_{des} + 2\lambda\ddot{\tilde{x}} + \lambda^2\dot{\tilde{x}}]s \le -\eta|s|$

$f + b\hat{b}^{-1}(-\hat{f} + \dddot{x}_{des} - 2\lambda\ddot{\tilde{x}} - \lambda^2\dot{\tilde{x}} - k\,\mathrm{sat}(\frac{s}{\phi})) - \dddot{x}_{des} + 2\lambda\ddot{\tilde{x}} + \lambda^2\dot{\tilde{x}}]s \le -\eta|s|$

$s = \ddot{x} - \ddot{x}_{des} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x}$

$[(f - b\hat{b}^{-1}\hat{f}) + (-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})(1 - b\hat{b}^{-1}) - b\hat{b}^{-1}k\,\mathrm{sat}(\frac{s}{\phi})]s \le -\eta|s|$

$\dot{s} = \dddot{x} - \dddot{x}_{des} + 2\lambda\ddot{\tilde{x}} + \lambda^2\dot{\tilde{x}}$

$s(b\hat{b}^{-1}k\,\mathrm{sat}(\frac{s}{\phi})) \ge s[(f - b\hat{b}^{-1}\hat{f}) + (-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})(1 - b\hat{b}^{-1})] + \eta|s|$

input best guess

$\dot{s} = f + bu - \dddot{x}_{des} + 2\lambda\ddot{\tilde{x}} + \lambda^2\dot{\tilde{x}}$

$s k\,\mathrm{sat}(\frac{s}{\phi}) \ge s[b^{-1}\hat{b}f - \hat{f} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})]s + \eta b^{-1}\hat{b}|s|$

$\dot{s} = 0 \rightarrow 0 = \hat{f} + \hat{b}u - \dddot{x}_{des} + 2\lambda\ddot{\tilde{x}} + \lambda^2\dot{\tilde{x}}$

$s k\,\mathrm{sat}(\frac{s}{\phi}) \ge s[b^{-1}\hat{b}\hat{f} + b^{-1}\hat{b}F - \hat{f} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})]s + \eta b^{-1}\hat{b}|s|$

$u_I = [\underbrace{-\hat{f} + \dddot{x}_{des} - 2\lambda\ddot{\tilde{x}} - \lambda^2\dot{\tilde{x}}}_{\hat{u}}]\hat{b}^{-1}$

inputting $f = \hat{f} + (f - \hat{f}) \le \hat{f} + |f - \hat{f}| \le \hat{f} + F$

control as if we knew system perfectly

$u_A = [\hat{u} - k\,\mathrm{sat}(\frac{s}{\phi})]\hat{b}^{-1}$

know $b^{-1}\hat{b} > 0$ & $\eta > 0$
$s\,\mathrm{sgn}(s) = |s|$

$k \ge \frac{s}{|s|}[b^{-1}\hat{b}\hat{f} + b^{-1}\hat{b}F - \hat{f} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})] + \eta b^{-1}\hat{b}$

$\hat{b}b^{-1}$

$\hat{b} = 2 \quad b^{-1} = 1 \quad 2$
$b^{-1} = 1/4 \quad 1/2$

$k \ge |b^{-1}\hat{b}\hat{f} + b^{-1}\hat{b}F - \hat{f} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})| + \eta b^{-1}\hat{b}$

$k \ge b^{-1}\hat{b}F + b^{-1}\hat{b}\eta + |b^{-1}\hat{b} - 1||\hat{f} - \dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x}|$

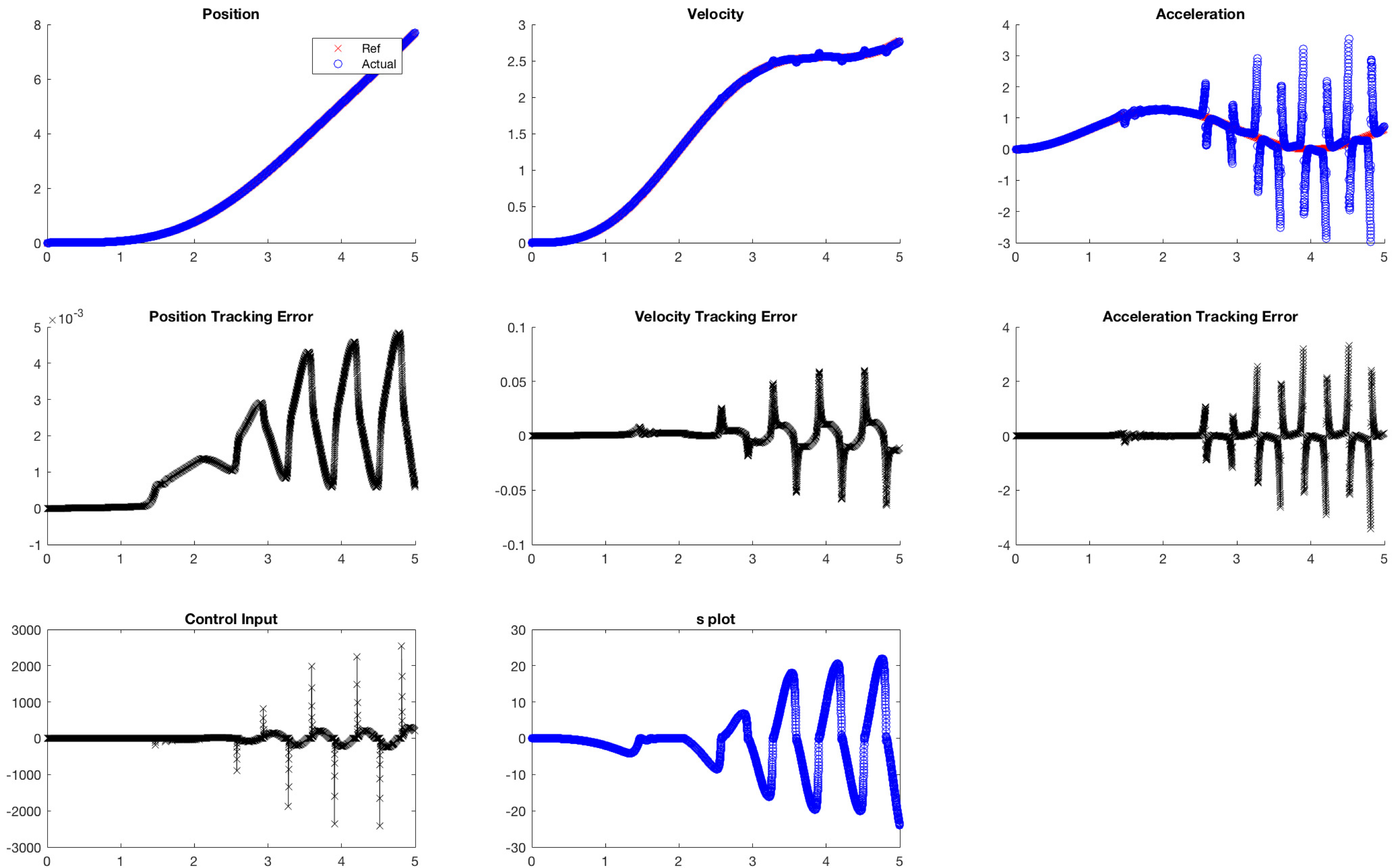
$B = 2$

$b^{-1}\hat{b} = \beta, \quad |-\hat{u}| = |\hat{u}|$

$k \ge \beta F + \beta\eta + |\beta - 1||\hat{u}|$

$k \ge \beta(F + \eta) + |\beta - 1||\hat{u}|$

$$\boxed{\begin{aligned} &\hat{f} = 0 \qquad F = \ddot{x}^2 + 2\dot{x}^5 \sin 4x \\ &s = \ddot{x} - \ddot{x}_{des} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x} \\ &u = \hat{u} - k\,\mathrm{sat}(s/\phi) \\ &\hat{u} = -\hat{f} + \dddot{x}_{des} - 2\lambda\ddot{\tilde{x}} - \lambda^2\dot{\tilde{x}} \\ &k = B(F + \eta) + |B - 1||\hat{u}| \end{aligned}}$$



Position
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Acceleration
Position Tracking Error
×10^-3
Velocity Tracking Error
Acceleration Tracking Error
Control Input
s plot

$$\ddot{y} + a_{p_1}\dot{y} + a_{p_2}y = \dot{u} + b_p u$$

$$\dot{u} + b_p u = \frac{1}{b_p}\underbrace{(b_p\dot{u} + u)}_{v}$$

$$\ddot{y} + a_{p_1}\dot{y} + a_{p_2}y = b_v v$$

$$\frac{1}{b_v}\ddot{y} + \frac{a_{p_1}}{b_v}\dot{y} + \frac{a_{p_2}}{b_v}y = v$$

$$a_v = \begin{bmatrix} a_{v_1} \\ a_{v_2} \\ a_{v_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{b_v} \\ a_{p_1}/b_v \\ a_{p_2}/b_v \end{bmatrix} \qquad a_m = \begin{bmatrix} a_{m_1} \\ a_{m_2} \\ a_{m_3} \end{bmatrix}$$

$$b_v = \frac{1}{a_{m_1}}$$

$$a_{m_2} = \frac{a_{p_1}}{b_v}$$

$$a_{m_2} b_v = a_{p_1}$$

$$a_{m_2} \cdot \frac{1}{a_{m_1}} = a_{p_1}$$

$$a_{v_1}\ddot{y} + a_{v_2}\dot{y} + a_{v_3}y = v$$

$$V = \frac{1}{2}a_{v_1}s^2 \qquad \dot{V} = a_{v_1}s\dot{s} = a_{v_1}s(\ddot{y} - \ddot{y}_r)$$

$$= s(-a_{v_2}\dot{y} - a_{v_3}y + v - a_{v_1}\ddot{y}_r)$$

$$= s\left(\underbrace{\begin{bmatrix} \ddot{y}_r & \dot{y} & y \end{bmatrix}}_{y} \underbrace{\begin{bmatrix} -a_{v_1} \\ -a_{v_2} \\ -a_{v_3} \end{bmatrix}}_{a_v} + v\right)$$

$$s = \dot{\tilde{x}} + \lambda\tilde{x}$$

$$s = \dot{x} - \dot{x}_r \qquad \dot{x}_r = \dot{x}_d - \lambda\tilde{x}$$

$$\dot{s} = \ddot{x} - \ddot{x}_r \qquad \ddot{x}_r = \ddot{x}_d - \lambda\dot{\tilde{x}}$$

choose $v = y a_m - ks$

$$\dot{V} = s(v - y a_v) = s(y a_m - ks - y a_v)$$

$$= s(y\tilde{a} - ks) = sy\tilde{a} - ks^2$$

try $V = \frac{1}{2}a_{v_1}s^2 + \frac{1}{2}\tilde{a}^T p^{-1}\tilde{a}$

$$\dot{V} = a_{v_1}s\dot{s} + \dot{\hat{a}}^T p^{-1}\tilde{a}$$

$$= -ks^2 + sy\tilde{a} + \dot{\hat{a}}^T p^{-1}\tilde{a}$$

$$= -ks^2 + \underbrace{(sy + \dot{\hat{a}}^T p^{-1})}\tilde{a}$$

choose $\dot{\hat{a}}$ so this goes to $\emptyset$

$$sy + \dot{\hat{a}}^T p^{-1} = 0$$

$$\dot{\hat{a}} = (-syp)^T = -py^T s$$

$$v = y a_m - ks$$

$$\dot{\hat{a}} = -py^T s$$


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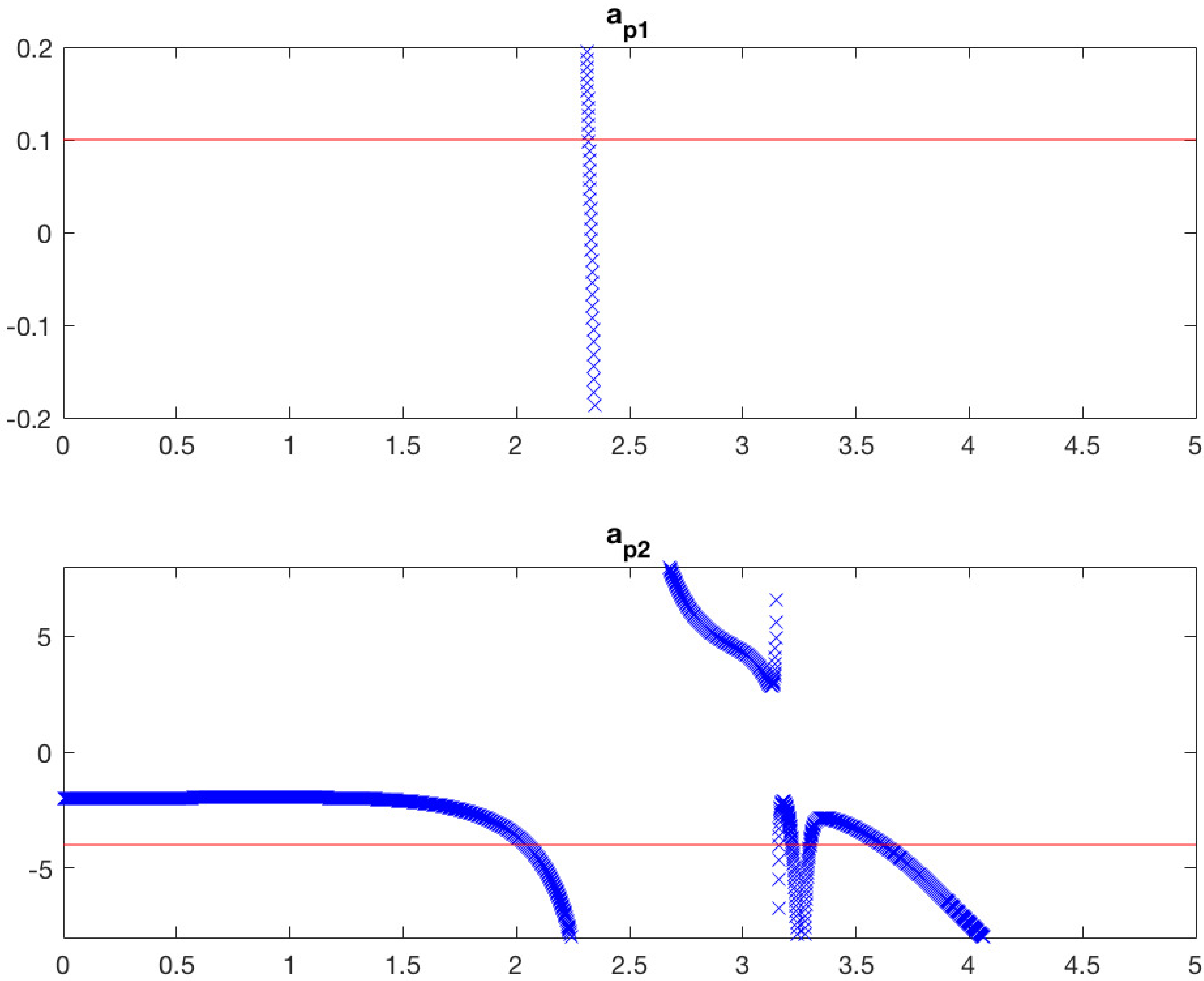

$a_{p1}$
0.2
0.1
0
-0.1
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
$a_{p2}$
5
0
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Parameter estimation can be easier for an unstable system because a smaller set of parameters make the system stable or track a trajectory. For a stable system, many values may achieve the same performance so the adaptive system will only change the values enough to ensure stability. These values are unlikely to be the same as the plant for a stable system.