Lecture #8

Robust Adaptive Control

Last time: 2nd order system

$$\underbrace{J}_{a_1}\ddot{x} + \underbrace{b}_{a_2}\dot{x}|\dot{x}| + \underbrace{mgl}_{a_3}\sin x = u$$

$$\tilde{x}(t) = x(t) - x_d(t)$$

$$s = \dot{\tilde{x}} + \lambda\tilde{x}$$

$$= \dot{x} - \underbrace{\dot{x}_r}_{\dot{x}_d - \lambda\tilde{x}}$$

More generally $s = x^{(n)} - x_r^{(n)}$

$$V = \frac{1}{2}Js^2$$

$$\dot{V} = Js\dot{s} = s(u - Y\underline{a})$$

$$Y = [\ddot{x}_r \quad \dot{x}|\dot{x}| \quad \sin x]$$

$$\boxed{u = Y\hat{a} - ks}$$ control

$$\dot{V} = -ks^2 + sY\underline{\tilde{a}}$$ → we don't know the sign of this

$$\underline{\tilde{a}} = \underline{\hat{a}} - \underline{a}$$

So set $V = \frac{1}{2}Js^2 + \frac{1}{2}\tilde{a}^T P^{-1}\tilde{a}$

Then $\dot{V} = -ks^2 + \underbrace{(sY\tilde{a} + \dots)}_{\text{set this to } 0}$

$$= -ks^2 \leq 0$$

$$\boxed{\begin{array}{l}\text{Adaptation} \\ \dot{\hat{a}} = -PY^T s\end{array}}$$

Barbalat lemma.

$V(x,t)$ lower bounded

$\dot{V} \leq 0$

$\ddot{V}$ bounded

then $\dot{V} \to 0$

(in addition V upper bounded also.)

$\Downarrow$

often used to prove $\ddot{V}$ is bounded given only first two conditions

$\dot{V} \to 0 \Rightarrow s \to 0 \Rightarrow \tilde{x} \to 0$, $\dot{\tilde{x}} \to 0$

So let's look @ this for our system.

$$\ddot{V} = \frac{d}{dt}\dot{V} = \frac{d}{dt}(-ks^2) = -2ks\dot{s}$$

$$\ddot{V} = -2ks(\ddot{x} - \ddot{x}_r)$$

V bounded $\Rightarrow$ s is bounded $\Rightarrow$ $\tilde{x}$ and $\dot{\tilde{x}}$ is bounded

and $\hat{a}$ is " $\Rightarrow$ x and $\dot{x}$ is bounded

$\downarrow$

(Sum of two positive numbers)

we also talked about "sufficient richness" for parameter convergence

$\Rightarrow Y_d^T Y_d \geq 0$ on "average"

Let's go further.

$$\underbrace{\underbrace{J}_{a_1}\ddot{x} + \underbrace{b}_{a_2}\dot{x}|\dot{x}| + \underbrace{mgl}_{a_3}\sin x}_{\text{weak model.}} = \underbrace{f(x,\dot{x},t)}_{\text{additional noise}} + u$$

but you have some bound on it.

$$|\hat{f}(x,t) - f(x,t)| \leq F$$

known

How do we do adaptive control on this?

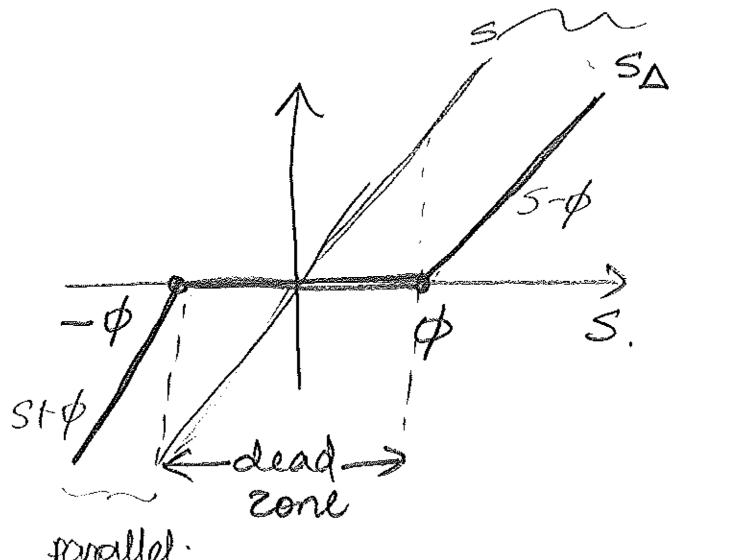
we have both unknown constants and uncertainty
Bounded

- We want the sys. to behave as if $\underline{a}$ is known, i.e. as if we had to be only robust to $f(x,t)$

Create new variable s_Δ

$$\underbrace{|s| \leq \phi}_{\text{Boundary layer.}} \Leftrightarrow s_\Delta = 0$$

$$s_\Delta = s - \phi \, \text{sat}\left(\frac{s}{\phi}\right)$$



Design a controller w/ a boundary layer

Assume $F =$ constant for simplicity

Then $\phi =$ "

What is a reasonable choice of V.

$$V = \frac{1}{2} J s_\Delta^2$$

(One can verify)

$$\frac{d}{dt} s_\Delta^2 = 2 s_\Delta \dot{s}$$

$$\dot{V} = J s_\Delta \dot{s}$$

<u>Control Law</u>

$$u = \underbrace{Y \underline{\hat{a}}}_{\text{adaptive part}} - k\,\mathrm{sat}\left(\frac{s}{\phi}\right) - \hat{f}(x,t)$$

$$\Rightarrow \dot{V} = \underbrace{s_\Delta \left(-k\,\mathrm{sat}\left(\frac{s}{\phi}\right) + Y\underline{\tilde{a}} + f(x,t) - \hat{f}(x,t) \right)}_{\text{How can we rewrite this?}}$$

Note: $s_\Delta\, \mathrm{sat}(s/\phi) = s_\Delta\, \mathrm{sgn}(s) = |s_\Delta|$

$$= -k|s_\Delta| + \underbrace{\overbrace{s_\Delta Y \underline{\tilde{a}}}^{\text{adaptive}} + \overbrace{s_\Delta (f - \hat{f})}^{\text{robust}}}_{\text{we don't like these two terms}}$$

So make

$$V = \tfrac{1}{2} J s_\Delta^2 + \tfrac{1}{2} \underline{\tilde{a}}^T P^{-1} \underline{\tilde{a}}$$

$$\dot{V} = \tfrac{1}{2} J s_\Delta^2 + \underline{\dot{\hat{a}}}^T P^{-1} \underline{\tilde{a}}$$

$$= -k|s_\Delta| + s_\Delta Y \underline{\tilde{a}} + s_\Delta(b - \hat{b}) + \underline{\dot{\hat{a}}}^T P^T \underline{\tilde{a}}$$

add to zero

So the adaptation becomes

$$\underline{\dot{\hat{a}}} = -P Y^T s_\Delta$$

And so: $\dot{V} = \underbrace{-k|s_\Delta|}_{\text{negative}} + s_\Delta(b - \hat{b})$

Suppose $k = F + \eta$

Bound on the error in dynamics.

Then $\dot{V} = -(F+\eta)|s_\Delta| + s_\Delta(b - \hat{b})$

$$\leq -\eta |s_\Delta| \leq 0$$

Always the same story: we don't like a term.

" Introduce a new term that overpowers the first term and makes the sum negative

Then using Barbalat $\dot{V} \to 0 \Rightarrow s_\Delta \to 0$

(s_Δ squeezed betw'n $\dot{V}$ and 0)

Therefore, system converges to same boundary layer as though we knew those constants. Powerful, isn't it? :)

In the adaptation law, what does s_Δ do?

- Inside the boundary layer, there is no adaptation.

  Why? Because we don't have enough info. to modify $\underline{\hat{a}}$. The errors may be due to $\hat{f}$ errors.

[Similar in spirit to overfitting in Machine Learning]

- you learn the params only until some point to improve performance.
- Afterward, performance drops.
- Do not tune params on irrelevant info.

Let's go further! Exploit more info that is available

Suppose

$$J\ddot{x} + b\dot{x}|\dot{x}| + mgl\sin x = \overbrace{\underbrace{\phi(\dot{x})}_{\text{only depends on velocity}} + g(x, \dot{x}, t)}^{f = \phi + g} + u$$

(other variations are possible too { only depends on position, or posn + vel)

Then $|\hat{g} - g| \leq F_g$.

[Note $F_g < F$ since we have reduced uncertainty]

Now, we can do math that enables control as though we know J, b, mgl, and $\phi(\mathring{x})$.

Usually the terms Y and $\phi(\mathring{x})$ have physical meaning.

However, even if $\phi(\mathring{x})$ does not have physical ", it can still be represented.

So "any" $\phi(\mathring{x}) = \underset{\text{infinite sum}}{\sum} \alpha_i \underbrace{g_i(\mathring{x})}_{\text{basis functions}}$

unknown.

For example, if

ϕ is periodic w/ period T, $\Longrightarrow$ Fourier series expansion

So your parameter space can be expanded w/ the α_i (infinite set)

So even if your physics knowledge is exhausted, math can help!

What properties should the expansion have?

Desirable properties

(1) At any $\mathring{x}$, only a few terms are needed for a good approximation. (finite terms + residual)

e.g. $\varsigma_i(\mathring{x}) = \xi(\mathring{x} - \mathring{x}_i)$.

and ξ local

So @ any pt. in the state, only a few of the basis functions are non-zero.

– and you need to only compute some of them @ any time (or estimate)

So only a finite set of α_i are updated.

At any pt $\mathring{x}$,

(2) $$\sum_{\substack{\text{infinite}\\ \text{sum}}} = \Big(\sum_{\substack{\text{finite}\\ N}}\Big) + \underbrace{\Big(\text{residual}\Big)}$$

$| \, | \to 0$ fast as $N \uparrow$

[combining $\underbrace{\text{function approx.}}_{\text{signal processing}}$ w/ control]

3) Orthonormal basis functions $\varsigma_i(x)$

$$\int \varsigma_i \varsigma_j \, d\mathring{x} = \delta_{ij} \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases}$$

Why orthonormal?

$$\left\| \sum_{\text{infinite}} - \sum_{\text{finite } N} \right\|^2 = \left\| \sum_{\text{residual}} \alpha_i^2 \right\|^2$$

↑ true value ↑ approximation

↳ adds to Fg

If not orthonormal, you would have interaction between the basis functions, making it more complicated.

⇒ keep those basis funcs w/ largest coeffs so that residual is small

This is useful to do in state space ($\overset{x}{\dot{x}}$) and not in time, because it is difficult to learn across time series since anything can happen. However, the sys. may pass through the same pts.

<u>Choices for basis functions</u> : radial basis, gaussians, wavelets. } tradeoff computational power w/ precision

<u>Summary</u> : Robust Control → Adaptive control unknown params → Combine

↓

Augment using basis functions