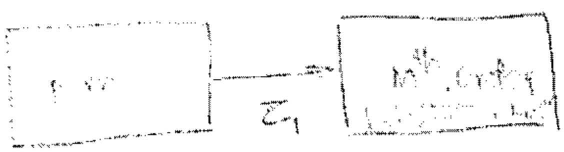
Lecture # 12

Controllability, Observability, Backstepping

Last time: Zero dynamics of linear single-input systems.

Relative degree of {transfer function} = # poles − # zeros = # of times we {need to differentiate the output of interest to get input in the eqn.}



stable by control

left-over dynamics

[illegible]

We derived this in general for linear systems.

Nonlinear

For nonlinear systems, we can define something called zero dynamics

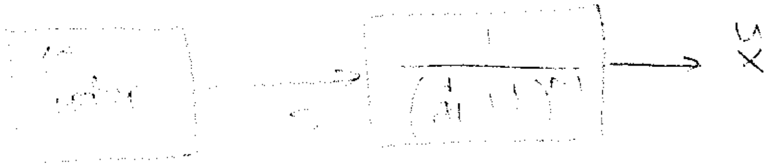
dynamics corresponding to maintaining state @ zero

We can also define that the nonlinear sys. is minimum phase if the zero dynamics is stable.

Warning

In nonlinear sys., if zero dynamics is stable $\not\Rightarrow$ internal dynamics is stable.

Also ensured stability to parameter variations (robust control) (2)



stable internal dynamics.

Section 6.3

$$\dot{x} = f(x) + g(x)u$$

When does state transformation $\underline{z} = \underline{z}(\underline{x})$ exist such that

$$\underline{z} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ \vdots \\ z_1^{(n-1)} \end{bmatrix} \quad \text{and} \quad z_1^{(n)} = v \,.$$

where $u = \alpha(\underline{x}) + \beta(x)v$? [here no zero dynamics]

[Necessary and sufficient]

Some tools we need first

Lie derivative

$L_{\underline{f}}h = \nabla h \cdot \underline{f}$ scalar, directional derivative

$L_f^i h = L_f(L_f^{i-1} h)$

Lie bracket $[\underline{f}, \underline{g}] = \nabla \underline{g}\, \underline{f} - \nabla \underline{f}\, \underline{g}$

Also written as $ad_f\, g = [f, g]$

$\to ad_f^i\, g = [f, ad_f^{i-1} g]$ recursive definition

Involutivity of linearly independent family of vector fields

$\Downarrow$

$[f_i, f_j]$ ∈ in the span of original vector field.

Now for the theorem (6.?) $\quad \dot{x} = f(x) + \underline{g}(x)u.$

(we only state it. Proof in textbook)

A necessary and sufficient conditions:

(1) $\left\{ \underline{g},\ ad_f\, \underline{g},\ ad_f^2\, \underline{g}, \ldots, ad_f^{n-2}\, \underline{g},\ ad_f^{n-1}\, \underline{g} \right\}$ linearly independent.

n vector fields

AND

(2) $\left\{ \underline{g},\ ad_f\, \underline{g}, \ldots, ad_f^{n-2}\, \underline{g} \right\}$ is involutive

Note that these are all vector fields & so everything is dependent on state.

So the result may hold in only some regions. (in nonlinear systems, equilibrium is state dependent)

So what do these conditions mean?

z_1 can be obtained as a solution to

$$\underbrace{\left[\underline{g}\ \ ad_f\, \underline{g}\ \ \cdots\ \ ad_f^{(n-2)}\, \underline{g}\ \ ad_f^{n-1}\, \underline{g} \right]}_{\text{invertible.}} \underbrace{\nabla z_1}_{\text{gradient of scalar}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

normalized.

Interpretation

condition (i) If system was linear $\underline{\dot{x}} = A\underline{x} + \underline{b}u$, (b → constant.)

Then the set of functions is $\{ b, ?, ? \ldots \}$

$$ad_{\underline{f}}\,\underline{b} = [\underline{f}, \underline{b}] = \nabla\underline{b}\,\underline{f} - \nabla\underline{f}\,\underline{b}$$

$$= 0 - A\underline{b}$$

$\Rightarrow$ The set is $\{ \underline{b}, -A\underline{b}, \ldots, (-1)^{n-1}A^{n-1}\underline{b} \}$ linearly independent

$\Updownarrow$

Controllable.

Condition (ii) is trivial in linear case

∇z_1 is a gradient

↳ what does this mean?

Suppose

$$\underline{V}(x) = \text{grad}\,\phi(x) = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

$$\nabla\underline{V} = \begin{bmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial}{\partial x_2}V_1 & \cdots & \frac{\partial V_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial V_n}{\partial x_1} & \cdots & & \frac{\partial V_n}{\partial x_n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2 \phi}{\partial x_1^2} & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\phi & & \vdots \\ \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\phi & \cdots & & \vdots \\ \vdots & & & \\ & & & \frac{\partial^2 \phi}{\partial x_n^2} \end{bmatrix} = \text{symmetric}$$

Should be the same if everything is smooth

Symmetric property $\Rightarrow$

$$\frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1} = 0.$$

If this is not satisfied, $\underline{v}$ cannot be gradient.

[How would you write this condition in 3 dimensions?

$$\frac{\partial V_2}{\partial x_3} - \frac{\partial V_3}{\partial x_2} = 0$$
]

This condition basically means that the curl $\underline{V} = 0$

rotation of $\underline{V} = 0$.

Necessary and sufficient condition for a function to be a vector field.

[i.e. if there exists $\phi(x)$, $\underline{V}(\underline{x}) = \nabla \phi(\underline{x}) \Leftrightarrow$ curl $\underline{V}(\underline{x}) = 0$]

Not straightforward to apply this theorem, but it says that if the coordinates are changed, system is much simpler.

Now, a new topic

Backstepping (p258)

Suppose you have a variable of interest, but you have lost-order dynamics. Well that is still good, because you have at least simplified it.

p260 Example

$$\dot{y} = v.$$
$$\ddot{z} + \dot{z}^3 - z^5 + yz = 0$$
2nd order system

We want to control $y \to 0$

The left-over dynamics is $\ddot{z} + \dot{z}^3 - z^5 + yz = 0$

Suppose I chose $v = -y$, then $\dot{y} + y = 0 \Rightarrow y \to 0$

Will this work?

What is the zero dynamics when $y = 0$?

$$\Rightarrow \ddot{z} + \dot{z}^3 - z^5 = 0$$

This sys. is unstable. Not good.

Let's concentrate then on the hard part, which is the left-over dynamics.

Let's see 'y' as the "control" input to the left-over dynamics. What shd it be?

Suppose $y = +2z^4$

Then $\ddot{z} + \dot{z}^3 + z^5 = 0 \Rightarrow$ globally asymptotically stable.

[The Return of Lyapunov!]

When $z \to 0$, then $\lambda y \to 0$. This is what we want. (p7)

What Lyapunov function should we use?

Let's consider kinetic energy.

$$V_0 = \underbrace{\frac{1}{2}\dot{z}^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{6}z^6}_{\text{potential energy}}.$$

we can show using Lyapunov theorem & la Salle theorem that this sys. is stable.

But this Lyapunov function only talks about 'z'. what about 'y'? Let's augment our Lyapunov function (idea similar to adaptive control where we add terms)

$$V_1 = V_0 + \frac{1}{2}(y - 2z^4)^2$$

$$= -\dot{z}^4 + z\dot{z}(y - 2z^4) + (y - 2z^4)\frac{d}{dt}(y - 2z^4)$$

$$\dot{V}_1 = \underset{\substack{\uparrow \\ \text{damping term.}}}{-\dot{z}^4} + (y - 2z^4)(v - 8z^3\dot{z} - z\dot{z})$$

So this suggests the use of $v = -(y - 2z^4) + 8z^3\dot{z} + z\dot{z}$

then $\dot{V}_1 = -\dot{z}^4 - (y - 2z^4)^2 \leq 0$.

and then use the invariant set theorem.

So what have done?

– We used left-over dynamics to find a control input & also the Lyapunov function to show stability.

Important : This idea can be applied recursively. So it is powerful.

Example 6.15

$$\dot{x} + \boxed{x^2 y^5 z e^{xy} = (x^4+2)u.}$$

this is a joke. Just call it 'v'

$$\dot{y} + y^3 z^2 - x = 0$$

$$\ddot{z} + \dot{z}^3 - z^5 + yz = 0.$$

want to control $x \rightarrow 0$

What about the rest? Choose 'x' as control input for the last 2 eqns.

How about. $x = \underbrace{-y + 2z^4 + 8z^3\dot{z} + z\dot{z}}_{\text{Same as previous problem. Call this } x_0} + y^3 z^2$

Choose new $V_2 = \underset{\substack{\downarrow \\ \text{(from prev. example)}}}{V_1} + \frac{1}{2}(x - x_0)^2$

$$\mathring{V} = -\mathring{z}^4 - (y - 2z^4)^2 + \underbrace{(x - x_0)(v + \dots)}$$

Choose v so that the whole thing becomes $-(x - x_0)^2$

Then $\mathring{V} \leq 0$

Compute u from v

<u>Summary</u>: First we had ~~formal~~ proof for controllability

Then Next we had more informal method of Backstepping.