Lecture # 3 Lyapunov Theory.

Preliminaries

$$\dot{x} = f(x,t)$$ non-autonomous system

$x \in \mathbb{R}^{n \times 1}$

n = Order of system.

note that any input $u(x,t)$ can be bundled into this eqn.

Linear System $\dot{x} = A(t)x$

Autonomous system $\underbrace{\dot{x} = f(x)}$

No notion of time.

Equilibrium pt: $f(x^*) = 0$ Not always easy to solve. Multiple solutions also [what about pendulum? x: 0, 2π, 4π, … and π, 3π, 5π, …]

Note that the nonlinear equations can be transformed such that the equm pt is moved to the origin

[what if you have station? → Regions of equilibrium]

New variable: $y = x - x^*$

$\dot{y} = \dot{x}$

$\Rightarrow \dot{y} = f(y + x^*)$ } one-to-one correspondence

[what about time as "state"? Some math formulations exist, but not in this course!]

Also, the nonlinear system performance can be tracked around a desired trajectory (time-based) also. This error (dynamics) however will be a non-autonomous system though.

$$m\ddot{x} + k_1 x + k_2 x^3 = 0$$

Suppose $x^*(t)$ is the nominal trajectory arising from $x(0) = x_0$

Now, we perturb the initial position to be $x(0) = x_0 + \delta x_0$

Define $e(t) = x(t) - x^*(t)$

$$m\ddot{e} + k_1 e + k_2 (e^3 + 3e^2 x^* + 3e x^{*2}) = 0$$

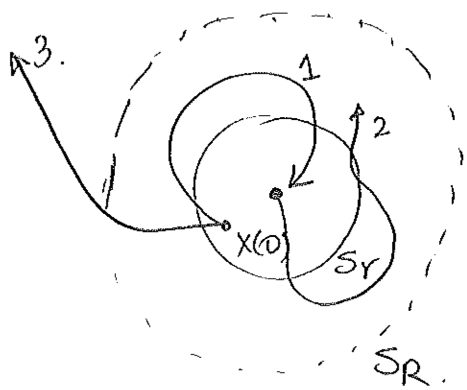
2nd order system around origin (which is inturn around the nominal trajectory x^*)

Concepts of stability

Definition: The equm state $x = 0$ is said to be stable if, for any $R > 0$, there exists $r > 0$, such that if $\|x(0)\| < r$ then $\|x(t)\| < R$ for all $t \geq 0$. Otherwise, the equm pt. is unstable.

[B_R = spherical region defined by $\|x\| < R$
S_R = the sphere itself defined by $\|x\| = R$]

- What does this mean? stability $\Rightarrow$ if system starts sufficiently close to origin, it stays arbitrarily close to it.



curve 1: asymptotically stable.

curve 2: marginally "

curve 3: unstable.

Instability Vs "Blowing up."

$\underbrace{\text{Instability}}$ → trajectories cannot stay arbitrarily close to origin

$\underbrace{\text{Blowing up}}$ → all trajectories close to origin go to ∞

[what happens if $r = R$?
stronger stability condition
But rare due to continuity conditions]

Asymptotic Stability : stable + (if $\|x(0)\| < r$, $r > 0$, then $\underbrace{x(t) \to 0 \text{ as } t \to \infty}$)

↓ state converges to 0 as $t \to \infty$

[Note: If $r = \infty$, then globally asymptotically stable
⇒ Only equm pt !!]

If stable but not asymptotically stable ⇒ marginally stable

Exponential Stability : 2 strictly positive numbers (α, λ) such that

$$\forall t > 0, \quad \|x(t)\| \leq \alpha \|x(0)\| e^{-\lambda t} \quad \text{in some ball } B_r \text{ around origin}$$

α → scaling factor $\alpha \geq 1$

Exponential stability ⇒ Asymptotic stability

There is a time constant to convergence.
write $\alpha = e^{\lambda \tau_0}$ then $\alpha e^{-\lambda t} = e^{-\lambda(\underbrace{t - \tau_0}_{\text{delay}})}$, α ⇒ extra delay.
⇒ Convergence represented in terms of $\tau_0 + \frac{1}{\lambda}$, $\tau_0 + 3/\lambda$, $\tau_0 + 5/\lambda$ ...

Example

$$\dot{x} = -(1+\sin^2 x)\, x$$

Solution: $$x(t) = x(0)\, e^{-\int_0^t [1+\sin^2(x(\tau))]\, d\tau}$$

$$\leq x(0)\, e^{-t}$$

$\Rightarrow$ exponentially stable.

Linearization & Local Stability

- Lyapunov's linearization method: a formalization of the idea that a nonlinear sys. should behave like linear system in small regions

$$\dot{x} = f(x)$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\right)_{x=0}}_{A \in \mathbb{R}^{n\times n}} x + f_{h.o.t}(x)$$

Note $f(0)=0$ since $x=0$ is an equm pt.

$$\dot{x} = Ax \Rightarrow \text{Linear approximation @ } 0$$

Example

$$\dot{x}_1 = x_2^2 + x_1 \cos x_2$$

$$\dot{x}_2 = x_2 + (x_1+1) x_1 + x_1 \sin x_2$$

Linearized approximation about 0

$$\dot{x}_1 = x_1$$

$$\dot{x}_2 = x_2 + x_1 + x_1 x_2 = x_2 + x_1$$

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x.$$

If there is an input

$$\ddot{x} + 4\dot{x}^5 + (x^2+1) u = 0$$

$$\simeq \quad \ddot{x} + u = 0.$$

Now choose $u = \sin x + x^3 + \dot{x} \cos^2 x$

$$\simeq x + \dot{x}$$

$$\Rightarrow \quad \ddot{x} + \dot{x} + x = 0$$

Theorem (Lyapunov Linearization Method)

(1) If linearized system is strictly stable (all eigenvalues of A in left-half complex plane), then the equm pt. for the non-linear system is asymptotically stable.

(2) If linearized system is unstable (at least one eigenvalue is strictly in right-half complex plane), then the equm pl. is unstable for nonlinear sys.

(3) If linearized system is marginally stable (all eigenvalues are in left-half complex plane, but at least one of them is on the $j\omega$ axis), then we cannot conclude anything about nonlinear sys.

[General idea : Proof by continuity]

<u>Example</u> $\mathring{x} = ax + bx^5$

How many equm pts? $ax + bx^5 = 0$

$x(a + bx^4) = 0$

Soln $x = 0$

$a + bx^4 = 0 \Rightarrow x^4 = -\frac{a}{b}$

$x = \pm\left(\frac{-a}{b}\right)^{1/4}$

Another example

$\mathring{v} + v|v| = 0$ stable

$\mathring{v} - v|v| = 0$ unstable.

Linearization is same for both systems $\mathring{v} = 0$

marginally stable

cannot say anything about nonlinear sys

<u>Linearization about origin</u>

$\mathring{x} = ax$

$a < 0 \Rightarrow$ Nonlinear sys. asymptotically stable

$a > 0 \Rightarrow$ unstable

$a = 0 \Rightarrow$ cannot tell from linearization

Summary. Linearization only works in small ranges. This is limiting

Lyapunov's Direct Method

Basic idea – If total energy of a system is continuously dissipated (whether linear/nonlinear), the system must eventually settle down @ an equm pt.

Thus, construct a scalar function & evaluate how it varies.

Example $m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$

Spring-mass system.

* General solution of non-linear eqs. unavailable
* Linearization marginally stable + invalid over large deviations in x.

Let's look @ the system energy.

$$V(x) = \frac{1}{2} m\dot{x}^2 + \int_0^x (k_0 x + k_1 x^3)\,dx$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x^4$$

Note: equm pt $\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = 0 \Rightarrow$ zero energy

$\Rightarrow$ asymptotic stability $\rightarrow$ convergence to zero energy

instability $\Rightarrow$ growth of energy.

$$\dot{V}(x) = m\dot{x}\ddot{x} + (k_0 x + k_1 x^3)\dot{x} = \dot{x}(-b\dot{x}|\dot{x}|)$$

$$= -b|\dot{x}|^3 \leq 0$$

Thus, system settles down @ origin $\underbrace{\text{power dissipated}}$ = thermodynamic laws.

Some important properties of V

(1) strictly positive

(2) $\mathring{V} < 0$ (everywhere other than @ equm pt.)

Locally positive definite $V(x)$

$V(0) = 0$ and in a ball B_{R_0} around $x = 0$

$x \neq 0 \Rightarrow V(x) > 0$

(positive semi-definite $V(x) \geq 0$)

Note: For dynamic system, is KE, positive definite function? NO! Because KE can be = 0 for non-zero position, which is included in X

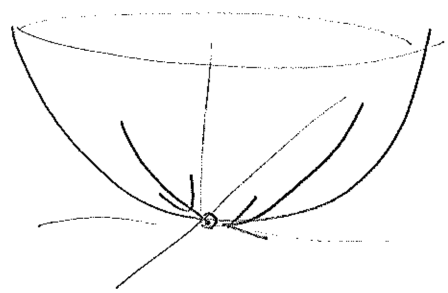
Note $\mathring{V} = \dfrac{dV}{dt} = \dfrac{\partial V}{\partial x}\dfrac{dx}{dt} = \underbrace{\dfrac{\partial V}{\partial x}\mathring{x}}$

Rate of change of energy = power

derivative of V along system traj.

Definition of Lyapunov function [for autonomous system, $\mathring{V}$ = function of x alone]

If in a ball B_{R_0}, $V(x)$ is positive definite + has continuous partial derivatives, and $\underbrace{\mathring{V}(x) \leq 0}_{\text{negative semi-definite}}$, $V(x)$ is a Lyapunov function



Lyapunov Theorem for Local Stability

If, in a ball B_{R_0}, there exists a scalar function $V(x)$ with continuous first partial derivatives such that

→ $V(x)$ is positive definite

→ $\dot{V}(x)$ is negative semi-definite,

then the equm pt. '0' is stable.

[Proof by contradiction
Assume limit "l" $\neq 0$
⇒ There is a no-fly zone where trajectory never enters.
But $\dot{V} < 0$ ⇒ sys. enters no-fly zone.]

If $\dot{V}(x)$ is negative definite in B_{R_0}, then system is asymptotically stable.

[Intuitive proof Interplay of geometry & dynamics
m = minimum of $V(x)$ on the sphere $\|x\| = R$
$m > 0$
Consider a cloud inside ball where $V(x) < m$]

Since $\dot{V} \leq 0$, sys. can never cross the ball.
⇒ sys. stable.

<u>Example</u> Simple pendulum w/ viscous damping

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

$$V(x) = (1 - \cos\theta) + \frac{\dot{\theta}^2}{2} > 0 \text{ except @ } 0.$$

$$\dot{V}(x) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = -\dot{\theta}^2 \leq 0 \quad \text{negative semidefinite}$$

⇒ stable equilibrium

<u>Example</u>

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1x_2^2$$

$$\dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 2)$$

$$V(x) = x_1^2 + x_2^2 \qquad \dot{V}(x) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$$

$\dot{V}(x) < 0$ if $x_1^2 + x_2^2 < 2$ ⇒ asymptotically stable.

Global stability

$V(x)$ continuous first-order derivatives

$V(x)$ = positive definite

$\dot{V}(x)$ = negative "

$V(x) \longrightarrow \infty$ as $\|x\| \rightarrow \infty$. (radially unbounded)

Important: otherwise the system may drift away to ∞

$\Rightarrow$ equm @ $x = 0$ is globally asymptotically stable.

Five types of stability

(1) stability (2) asymptotic stability (3) exponential stability (4) global (2) (5) global (3)

Invariant Set theorem

We need these theorems because if $\dot{V}$ is negative semi-definite, we cannot say anything about asymptotic stability

Definition : Invariant set

A set G is an " " for a dynamic system if every trajectory which starts from a pt. in G remains in G for all future time.

e.g. An equm pt. is an invariant set.

" " " domain of attraction is an invariant set.

Examples for Lyapunov Theorem.

Example 3.9

$\dot{x} + c(x) = 0$

$c(x)$ is continuous.

Also $x \neq 0 \Rightarrow x\,c(x) > 0$.

Show that the equm $x = 0$ is Globally Asymptotically Stable

The challenge is to find. $V(x)$

Choose $V(x) = x^2$

$V(x)$ is p.d. and tends to ∞ as $x \to \infty$

$\dot{V} = 2x\dot{x} = 2x(-c(x)) = -2x\,c(x) < 0$

$\Rightarrow$ system is G.A.S.

For example, $\dot{x} + x = \sin^2(x)$ is G.A.S.

since the sign of $x - \sin^2(x)$ is always the sign of x

$\sin^2 x < |\sin x| < |x|$

Alternate $V(x) = 2x^2$

x^4

$x^2 + x^4$

$\int c(y)\,dy$ (assuming $c(y)$ does not die too quickly)

Local Invariant Set Theorem

Consider

$\dot{x} = f(x)$, f continuous

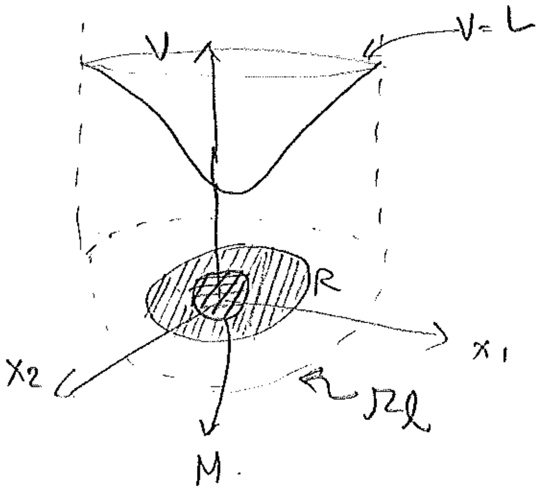
$V(x)$ = scalar function w/ continuous first partial derivatives

Assume that

→ for some $l > 0$, the region Ω_l defined by $V(x) < l$ is bounded

→ $\dot{V}(x) \le 0$ for all x in Ω_l

Let R be the set of all pts w/in Ω_l where $\dot{V}(x) = 0$ and M be the largest invariant set in R. Then every solution $x(t)$ originating in Ω_l tends to M as $t \to \infty$



System may settle down @ $x \neq 0$

<u>Example</u> Mass spring damper system:

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$

$$V(x) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x^4$$

$$\dot{V}(x) = -b|\dot{x}|^3 \quad \text{(negative semidefinite)}$$

So we cannot say anything about asymptotic stability

We will use this new theorem to show that M contains only one pt.

The set R is defined by $\dot{x} = 0$.

Let us show that the largest invariant set M contains only the origin. Assume M contains a pt. w/ non-zero position x_1, then acc'n @ that pt

$$\ddot{x} = -\left(\frac{k_0}{m}\right) x - \left(\frac{k_1}{m}\right) x^3 \neq 0$$

$\Rightarrow$ Trajectory leaves R & M.

The only pt. for which system stays inside is $x = 0$

$\Rightarrow$ Asymptotically stable.

Attractive Limit Cycle

$$\dot{x}_1 = x_2 - x_1^7 \left[x_1^4 + 2x_2^2 - 10 \right]$$

$$\dot{x}_2 = -x_1^3 - 3x_2^5 \left(x_1^4 + 2x_2^2 - 10 \right)$$

First $x_1^4 + 2x_2^2 = 10$ is invariant. Why?

$$\frac{d}{dt}\left(x_1^4 + 2x_2^2 - 10 \right) = -\left(4x_1^{10} + 12x_2^6 \right)\left(x_1^4 + 2x_2^2 - 10 \right)$$

is zero on the set.

The motion on this invariant set is described by either of these eqns: $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1^3$

$\Rightarrow$ The invariant set is a limit cycle, where the sys. moves clockwise.

This limit cycle is actually ~~act~~ attractive as well.

Define $V = (x_1^4 + 2x_2^2 - 10)^2$

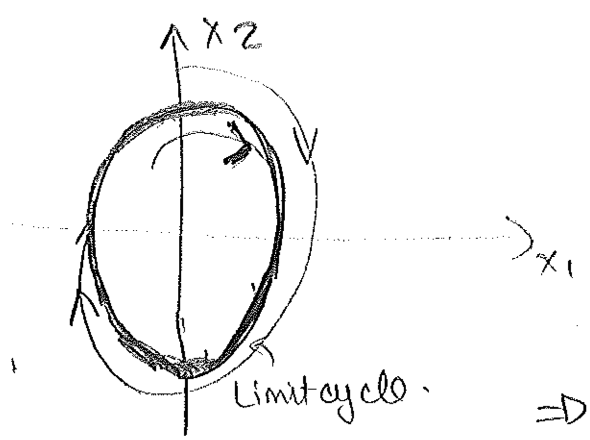
For any positive number l, the region Ω_l which surrounds the limit cycle is bounded.

$$\dot{V} = -8(x_1^{10} + 3x_2^6)(x_1^4 + 2x_2^2 - 10)^2$$

$\dot{V} < 0$, unless $x_1^{10} + 3x_2^6 = 0$ ← only @ origin

or $x_1^4 + 2x_2^2 = 10$ ← Limit cycle.

Since the limit cycle & the origin are invariant sets, the set M consists of their union. Thus, all trajectories starting in Ω_l converges to the limit cycle or the origin



Furthermore, since the origin is a local maximum of V, the set M only contains the limit cycle

$\Rightarrow$ origin is also unstable & ~~every~~ all curves converge to limit cycle.

How to find Lyapunov Function?

Let's start w/ Linear Systems.

Theorem: A necessary & sufficient condition for an LTI system $\dot{x} = Ax$ to be strictly stable is that for any symm. p.d. matrix Q, the unique matrix P which is a solution of $A^TP + PA = -Q$ be symm. p.d.

Notes
(1) $M \in \mathbb{R}^{n\times n}$ is positive def if $x^TMx > 0, x \neq 0$
(2) Symmetric $M = M^T$
(3) skew " $M = -M^T$

$$\left[\begin{array}{ll} \text{Choose} & V = x^TPx \\ & \dot{V} = \dot{x}^TPx + x^TP\dot{x} = -x^TQx \\ & -Q = A^TP + PA \end{array}\right]$$

Example

$$A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

Choose $Q = I$

Then $P = \frac{1}{16}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$ = symm. p.d. $\Rightarrow$ stable.

Now, back to nonlinear sys.

Krasovskii Theorem : For sys $\dot{x} = f(x)$, w/ equm pt at 0, define $A(x) = \frac{\partial f}{\partial x}$

If $F = A + A^T$ is negative definite in a neighborhood Ω, then the equm @ 0 is asymptotically stable. The Lyapunov function for this sys. is $V = f^Tf$

If Ω is the entire state space and $V(x) \to \infty$ as $\|x\| \to \infty$, then the equm pt. is globally asymptotically stable

Generalized Krasovskii Theorem.

$\dot{x} = f(x)$, equm @ 0

$$A(x) = \frac{\partial f}{\partial x}$$

Sufficient condition for origin to be asymptotically stable is that there exist two symm. p.d matrices P and Q such that

$$F(x) = A^T P + PA + Q$$ is negative s.d. in some neighborhood

Ω of the origin. A Lyapunov func'n is $V(x) = f^T P f$.

If $V(x) \to \infty$ as $\|x\| \to \infty$, then globally asymptotically stable.

Variable Gradient Method : Key idea: Assume Lyapunov function has a certain gradient. Integrate gradient to get " ".

$$V(x) = \int_0^x \nabla V \, dx,$$

where $\nabla V = \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \\ \vdots \\ \partial V/\partial x_n \end{bmatrix}$

Of course, the gradient function has to satisfy the "curl" condition

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i}$$

One way to move forward.

(1) Assume $\nabla V_i = \sum_{j=1}^{n} a_{ij} x_j$ for some set of coeffts a_{ij}.

(2) Solve for coefficients that satisfy curl equations

(3) Restrict coeffts such that $\dot{V}$ is negative semi-definite

(4) Compute V from ∇V by integration

(5) Check if V is p.d.

[Integrate along a path that is ‖el to each axis in turn

$$V(x) = \int_0^{x_1} \nabla V_1(x_1, 0, \cdots, 0)\,dx + \int_0^{x_2} \nabla V_2(x_1, x_2, 0 \cdots 0)\,dx_2 + \cdots + \int_0^{x_n} \nabla V_n(x_1, x_2, \cdots, x_n)\,dx_n$$

Example

$$\dot{x}_1 = -2x_1$$
$$\dot{x}_2 = -2x_2 + 2x_1 x_2^2$$

Assume

$$\nabla V_1 = a_{11} x_1 + a_{12} x_2$$

$$\nabla V_2 = a_{21} x_1 + a_{22} x_2$$

$$\frac{\partial V_1}{\partial x_2} = \frac{\partial V_2}{\partial x_1} \Rightarrow a_{12} + x_2 \frac{\partial a_{12}}{\partial x_2} = a_{21} + x_1 \frac{\partial a_{21}}{\partial x_1}$$

One choice

$a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$

$\Rightarrow \nabla V_1 = x_1$, $\nabla V_2 = x_2$

$\Rightarrow \dot{V} = \nabla V \dot{x} = -2x_1^2 - 2x_2^2(1 - x_1 x_2)$

$\dot{V} < 0$ if $1 - x_1 x_2 > 0$

$$V(x) = \int_0^{x_1} x_1 \, dx_1 + \int_0^{x_2} x \, dx_2 = \frac{x_1^2 + x_2^2}{2}$$

> 0 (except @ $x = 0$)

$\Rightarrow$ Asymptotic stability

Robot Example

Dynamics of an n-link robot

$$\underbrace{H(q)\ddot{q}}_{\text{inertia}} + \underbrace{b(q,\dot{q})}_{\text{velocity terms}} + \underbrace{g(q)}_{\text{gravity}} = \tau$$

Choose $\tau = -K_D \dot{q} - K_P q + g(q)$

Choose $V = \frac{1}{2} (\underbrace{\dot{q}^T H \dot{q}}_{\text{kinetic energy}} + \underbrace{q^T K_P q}_{\text{virtual potential energy.}})$

$$\dot{V} = \overbrace{\dot{q}^T (\tau - g)}^{\text{power provided by external forces}} + \dot{q}^T K_P q$$

$\ddot{q}^T H \dot{q} + \dot{q}^T \dot{H}$

$= -\dot{q}^T K_D \dot{q}$

≤ 0 Since sys. cannot "get stuck" @ $q \neq 0$, we use invariant set theorem to prove global asymptotic stability

Lesson $\rightarrow$ use physical quantities as much as possible for Lyapunov functions

Performance Analysis using Lyapunov Methods

$\downarrow$

of linear and nonlinear sys.

Convergence Lemma

If a real function $W(t)$ satisfies

$$\dot{W}(t) + \alpha W(t) \leq 0 \quad , \text{ where } \alpha = \text{real number}$$

Then $$W(t) \leq W(0)\, e^{-\alpha t}$$

Implication : If $W(t)$ is non-negative, $\dot{W}(t) + \alpha W(t) \leq 0$ guarantees exponential convergence of $W(t)$ to zero.

Example :
$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 1) - 4x_1x_2^2$$
$$\dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - 1)$$

choose $V = \|x\|^2 = x_1^2 + x_2^2$

$$\dot{V} = 2V(V-1)$$

$$\Rightarrow \int \frac{dV}{V(1-V)} = -2\int dt$$

$$V(x) = \frac{\alpha e^{-2t}}{1 + \alpha e^{-2t}} \quad \text{where } \alpha = \frac{V(0)}{1 - V(0)}$$

If $\|x(0)\|^2 = V(0) < 1$, i.e. the system starts inside the unit circle, $\alpha > 0$ and $\underbrace{V(t) < \alpha e^{-2t}}_{\text{exponentially converges to zero}}$

If trajectory starts outside circle (if $V(0) > 1$)

then $\alpha < 0$.

Then $V(t)$ and $\|x\|$ tend to infinity

("finite escape" or "explosion")

Regulator design Using Lyapunov method

$$\ddot{x} + \dot{x}^3 + x^2 = u$$

[Similar to the problem $\ddot{x} + c(x) = 0$, $x\,c(x) > 0$]

Need to bring it to equm @ $x = 0$

Choose $u = u_1(\dot{x}) + u_2(x)$

where

damping effects $\longrightarrow$ $\dot{x}\left(\dot{x}^3 + u_1(\dot{x})\right) < 0$ for $\dot{x} \neq 0$

$x\left(x^2 + u_2(x)\right) > 0$ for $x \neq 0$ $\longrightarrow$ spring effects.