Lecture #6

Robust Control (Continued)

Last time:

$$x^{(n)} = f(x,t) + b(x,t)u$$

$$\left|\hat{f}(x,t) - f(x,t)\right| \leq F(x,t)$$

$$\tilde{x} = x(t) - x_d(t)$$

Create new variable: s $\begin{cases} \dot{s} \text{ contains } u \\ \text{and} \\ s \to 0 \Rightarrow \tilde{x} \to 0 \end{cases}$

Choose $s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}$

$$n = 2 \Rightarrow s = \left(\frac{d}{dt} + \lambda\right) \tilde{x} = \dot{\tilde{x}} + \lambda \tilde{x}$$

$$n = 3 \Rightarrow s = \ddot{\tilde{x}} + 2\lambda \dot{\tilde{x}} + \lambda^2 \tilde{x}$$

Also showed $|s| \leq \phi \Rightarrow \tilde{x} \leq \frac{\phi}{\lambda^{n-1}} = \varepsilon$

More generally $\left|\tilde{x}^{(i)}(t)\right| \leq (2\lambda)^i \varepsilon$

$\Rightarrow$ Bounds on s $\Rightarrow$ Bounds on error on state

Once you get on the surface, the surface takes care of the rest.

$\downarrow$ first order problem. n-1 dimensions

Sliding condition

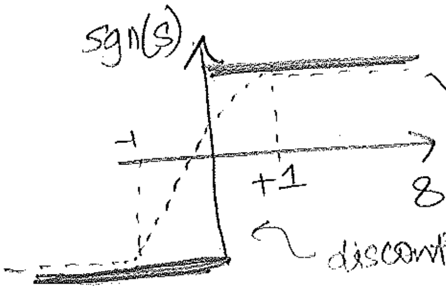
$$\frac{1}{2}\frac{d}{dt}s^2 \le \underbrace{-\eta}_{\text{constant} > 0}|s|$$

$$x^{(n)} = f + bu$$

$$u = \hat{u} - k\,\mathrm{sgn}(s), \quad k = \underbrace{F + \eta}_{\text{overpowers uncertainty} = F}$$

But this creates control chatter. This may be alright if the control method already has chatter (such as pulse width modulation). You might as well chatter the "right way".

But sometimes chattering is not ok.

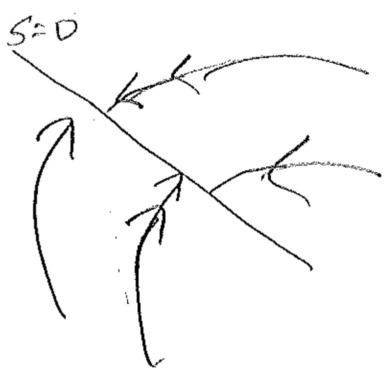


call it saturation function sat(y)

make it continuous

More generally, let it transition betw'n $-\phi, \phi$

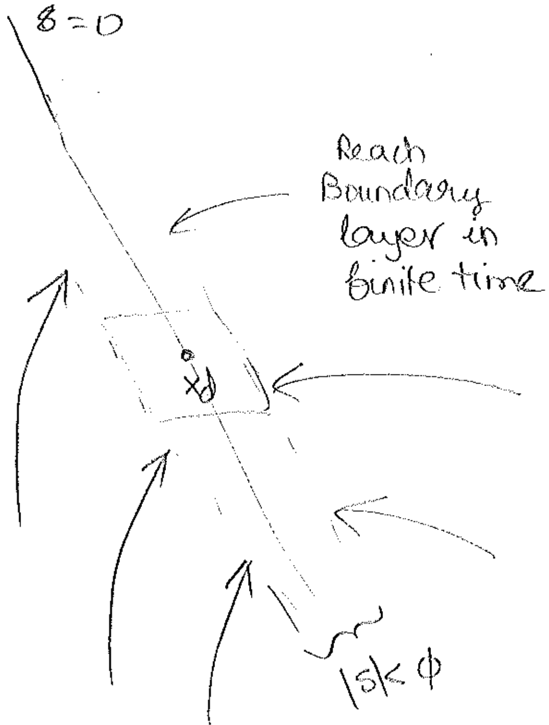
Then the function $\mathrm{sat}\left(\frac{s}{\phi}\right)$



$u = \hat{u} - k\,\mathrm{sgn}(s)$

Instead $u = \hat{u} - k\,\mathrm{sat}(s/\phi)$

What happens then?



So what have we gained? Made controller smoother.

<u>But how do we choose ϕ?</u>

sometimes ϕ can vary as well w/ state & time.

[Note first that k = function of state.]

$= k(x,t)$

$$s \geq \phi \implies \frac{1}{2}\frac{d}{dt}s^2 \leq (\underbrace{\dot{\phi}}_{\text{time derivative of } \phi} - \eta)|s|$$

So if $\dot{\phi}$ is negative, the rate of change of s^2 must be faster than that.

So where does this lead?

Earlier, with $u = \hat{u} - \overset{F+\eta}{\tilde{k}}\,\mathrm{sgn}(s)$

we had $$\dot{s} = f - \hat{f} - k\,\mathrm{sgn}(s)$$

Now, we have
$$u = \hat{u} - (\underbrace{F+\eta-\dot{\phi}}_{K})\,\mathrm{sat}\left(\frac{s}{\phi}\right)$$

$$\Rightarrow \dot{s} = f - \hat{f} - (F+\eta-\dot{\phi})\,\mathrm{sat}\left(\frac{s}{\phi}\right)$$

Once in the boundary layer,

$$\dot{s} = f - \hat{f} - (F+\eta-\dot{\phi})\left(\frac{s}{\phi}\right)$$

Assume that f, and F are smooth.

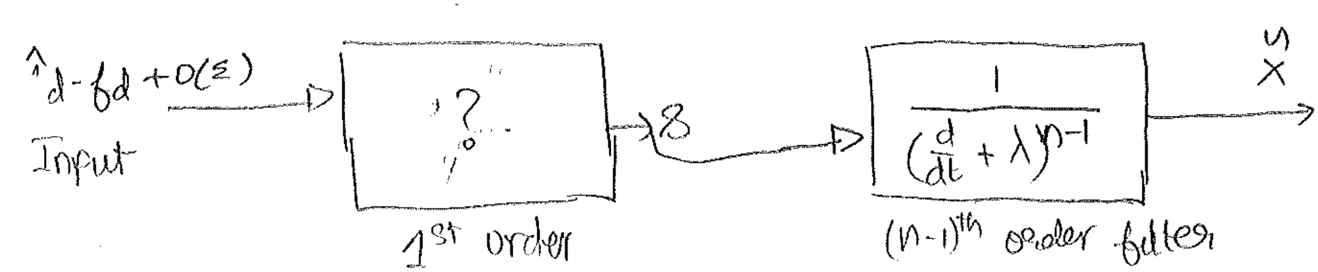
So after transients,

$$\tilde{x} = O(\varepsilon)$$

Also $\hat{f}$ is also close to the desired value.

$$\dot{s} = f_d - \hat{f}_d - (k_d - \dot{\phi})\frac{s}{\phi} + O(\varepsilon)$$

What does this eqn tell us?



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So ofcourse we still assume that we have state feedback, immediate actuation, all this much faster than λ unmodeled dynamics

We can choose λ as needed. (no where near other dynamics in system)

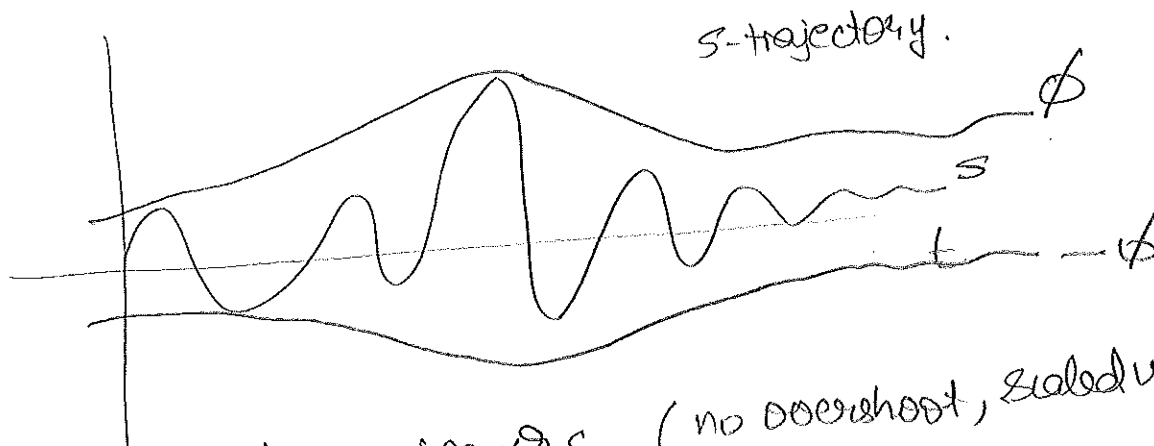
But how to choose ϕ?

Let $\dfrac{k_d - \dot{\phi}}{\phi} = \lambda$

i.e. $\dot{\phi} + \lambda\phi = k_d$

Then the ? in the diagram is $\dfrac{1}{\frac{d}{dt} + \lambda}$ by definition of ϕ

So this is all good. Let's observe how the system now behave.



$\hat{x}$ = filtered version of s. (no overshoot, scaled version)

But s includes elements of $\dot{\hat{x}}$, $\ddot{\hat{x}}$ and so on. So those measurement errors will show up in s.

If $s \ll \phi$, then you are too conservative
If only one large bump in 's', then your modeling is not correct there.

If ϕ jumps around a lot, then modeling is not ok.

Now, let's get back to

$$\dot{\phi} + \lambda \phi = kd$$

If we neglect time constant of the order $\frac{1}{\lambda}$, $\phi \simeq \frac{kd}{\lambda}$

and $\lambda^{n-1}(\max|\tilde{x}|) \simeq \phi$

Then $\lambda^n \max|\tilde{x}| \simeq kd \simeq Fd$

(since η = small value)

$\Rightarrow$ You have a straightforward mapping between external disturbance + effect on error (assuming other constraints are small). This is again similar to first order system performance.

$$|\tilde{x}| \lesssim \frac{Fd}{\lambda^n}$$

To make $\tilde{x}$ small, make Fd small (better parameters) to overcome "parametric uncertainty" = "fast unmodelled dynamics"

Another way, make λ large. (faster unmodeled dynamics)

parametric uncertainty $\rightarrow$ include more terms in model or better

What limits ~~to~~ λ?

- unmodeled fast dynamics

eg. Motor assumes the shaft does not bend.

But there will always be bending.

$$\lambda \leq \frac{2\pi \nu_R}{3}$$ ← freq of first unmodeled mode. (structural)

↳ λ chosen to be slower than this unmodeled freq.

or

$$\lambda \leq \frac{1}{3 T_A}$$ ← motor time constant

Sampling effects $$\lambda \leq \frac{\nu_{sampling}}{5}$$

$$\lambda \leq \frac{\lambda_M}{10}$$ measurement dynamics.

Choose $\lambda = \min(\lambda_s, \lambda_A, \lambda_{sampling}, \lambda_{meas}, \ldots \text{any others})$

For good overall design of system,

$$\lambda_s \simeq \lambda_A \simeq \lambda_{samp} \simeq \lambda_{meas}$$

So that nothing dominates (wasted resources)
(materials, computation, sensing, motor hardware)

Also note that

$|\overset{n}{\lambda}|$ is linear in Fd

but is order n in λ

$\Rightarrow$ changes in λ will rapidly make $|\overset{n}{\lambda}|$ small

<u>Next time</u>: Adaptation $\rightarrow$ learn the constants in the model.