① Problem 7.7

$\ddot{x} + \alpha_1(t)\,|x|\,\dot{x}^2 + \alpha_2(t)\,x^3\cos 2x = 5\dot{u} + u$ $\qquad \forall t \geq 0 \quad |\alpha_1(t)| \leq 1 \quad -1 \leq \alpha_2(t) \leq 5$

$x = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \qquad \tilde{x} = \begin{bmatrix} x - x_d \\ \dot{x} - \dot{x}_d \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \dot{\tilde{x}} \end{bmatrix} \qquad \ddot{\tilde{x}} = \ddot{x} - \ddot{x}_d$

$\ddot{x} = \ddot{\tilde{x}} + \ddot{x}_d$

Hint: $V = 5\dot{u} + u$

$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x} \qquad n = 2$

Aside

$\ddot{x} = f + u$

$\dot{s} = f + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$

$\dot{s} = 0$ if $u = -f + \ddot{x}_d + \lambda\dot{\tilde{x}}$

$s = \frac{d}{dt}\tilde{x} + \lambda\tilde{x} = \dot{\tilde{x}} + \lambda\tilde{x}$

$\dot{s} = \ddot{\tilde{x}} + \lambda\dot{\tilde{x}} = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}} =$

$\ddot{x} = \underbrace{-\alpha_1(t)\,|x|\,\dot{x}^2 - \alpha_2(t)\,x^3\cos 2x}_{f} + 5\dot{u} + u$

$\hat{f} = -0\,|x|\,\dot{x}^2 + 2x^3\cos 2x$

$F = -|x|\,\dot{x}^2 - 3x^3\cos 2x$

$\dot{s} = -\alpha_1(t)\,|x|\,\dot{x}^2 - \alpha_2\,x^3\cos 2x + 5\dot{u} + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$

$\dot{s} = 0 \rightarrow \quad u + 5\dot{u} = \alpha_1(t)\,|x|\,\dot{x}^2 + \alpha_2\,x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}}$

best guess of $\hat{u}$ by $\quad \hat{u} + 5\dot{\hat{u}} = \hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\mathrm{sgn}(s)$

for $\hat{f}$ using $\hat{f}$

plug in $\hat{f}$ $\quad \hat{u} + 5\dot{\hat{u}} = 2x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\mathrm{sgn}(s)$

$\frac{1}{2}\frac{d}{dt}s^2 \leq -\eta|s|$

$\dot{s}s \leq -\eta|s|$

$[-\alpha_1(t)\,|x|\,\dot{x}^2 - \alpha_2\,x^3\cos 2x + 5\dot{\hat{u}} + \hat{u} - \ddot{x}_d + \lambda\dot{\tilde{x}}]\,s$

plug in $\hat{u}$ $\rightarrow [-\alpha_1(t)\,|x|\,\dot{x}^2 - \alpha_2 x^3\cos 2x + (-2x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\mathrm{sign}(s)) - \ddot{x}_d + \lambda\dot{\tilde{x}}]\,s$

$[-\alpha_1(t)\,|x|\,\dot{x}^2 - (\alpha_2 + 2)\,x^3\cos 2x + \cancel{\ddot{x}_d} - \cancel{\ddot{x}_d} - \cancel{\lambda\dot{\tilde{x}}} + \cancel{\lambda\dot{\tilde{x}}} - k\,\mathrm{sgn}(s)]\,s$

$[\underbrace{-\alpha_1(t)\,|x|\,\dot{x}^2 - (\alpha_2 + 2)\,x^3\cos 2x}_{\leq F \Rightarrow \leq |x|\dot{x}^2 - 3x^3\cos 2x} - k\,\mathrm{sign}(s)]\,s \leq -\eta|s|$

$\eta = 0.1$

$[F - k\,\mathrm{sign}(s)]\,s \leq -\eta|s|$ $\qquad$ choose $\quad k = F + \eta$

$k = -|x|\,\dot{x}^2 - 3x^3\cos 2x + 0.1$

$s \leq |s| \qquad Fs - k|s| \leq -\eta|s|$

$k|s| \geq (\eta + F)|s|$

$\boxed{\dot{u} = \frac{1}{5}\left[-2x^3\cos 2x + \ddot{x}_d - \lambda\dot{\tilde{x}} - k\,\mathrm{sgn}(s) - \int \dot{u}\,du\right]}$

effect of chattering?


Position
Ref
Actual
Velocity
Ref
Actual
Position Tracking Error
Position Tracking Error
Control Input

7.10) sliding controller for

$|\alpha_1| \le 1 \qquad |\alpha_2| \le 2 \qquad 1 \le b \le 4$

$\dddot{x} + \alpha_1 \ddot{x}^2 + \alpha_2 \dot{x}^5 \sin 4x = bu$

$\dddot{x} = -\alpha_1 \ddot{x}^2 - \alpha_2 \dot{x}^5 \sin 4x + bu$

$= f + bu$

$n=3$

$s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x} = \ddot{\tilde{x}} + 2\lambda \dot{\tilde{x}} + \lambda^2 \tilde{x}$

$\tilde{x} = [x - x_{des}]$

$\dot{\tilde{x}} = [\dot{x} - \dot{x}_{des}]$

$\ddot{\tilde{x}} = [\ddot{x} - \ddot{x}_{des}]$

$f = -\alpha_1 \ddot{x}^2 - \alpha_2 \dot{x}^5 \sin 4x$

$\hat{f} = 0\ddot{x}^2 - 0\dot{x}^5 \sin 4x$

$|\hat{f} - f| \le F = \ddot{x}^2 + 2\dot{x}^5 \sin 4x$

$s = \ddot{x} - \ddot{x}_{des} + 2\lambda \dot{\tilde{x}} + \lambda^2 \tilde{x}$

$\dot{s} = \dddot{x} - \dddot{x}_{des} + 2\lambda \ddot{\tilde{x}} + \lambda^2 \dot{\tilde{x}}$

input best guess

$\dot{s} = f + bu - \dddot{x}_{des} + 2\lambda \ddot{\tilde{x}} + \lambda^2 \dot{\tilde{x}}$

$\dot{s} = 0 \rightarrow 0 = \hat{f} + \hat{b}u - \dddot{x}_{des} + 2\lambda \ddot{\tilde{x}} + \lambda^2 \dot{\tilde{x}}$

control as if we knew system perfectly

$u_I = [\underbrace{\hat{f} + \dddot{x}_{des} - 2\lambda \ddot{\tilde{x}} - \lambda^2 \dot{\tilde{x}}}_{\hat{u}}] \hat{b}^{-1}$

$u_A = [\hat{u} - k\,sat(\frac{s}{\phi})] \hat{b}^{-1}$

Lyapunov

$\dot{s}s = -\eta|s|$

$[f + b[\hat{u} - k\,sat(s/\phi)]\hat{b}^{-1} - \dddot{x}_{des} + 2\lambda \ddot{\tilde{x}} + \lambda^2 \dot{\tilde{x}}]s \le -\eta|s|$

$f + b\hat{b}^{-1}(-\hat{f} + \dddot{x}_{des} - 2\lambda \ddot{\tilde{x}} - \lambda^2 \dot{\tilde{x}} - k\,sat(\frac{s}{\phi})) - \dddot{x}_{des} + 2\lambda \ddot{\tilde{x}} + \lambda^2 \dot{\tilde{x}}]s \le -\eta|s|$

$[(f - b\hat{b}^{-1}\hat{f}) + (-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})(1 - b\hat{b}^{-1}) - b\hat{b}^{-1}k\,sat(\frac{s}{\phi})]s \le -\eta|s|$

$s(b\hat{b}^{-1}k\,sat(\frac{s}{\phi})) \ge s[(f - b\hat{b}^{-1}\hat{f}) + (-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})(1 - b\hat{b}^{-1})] + \eta|s|$

$s\,k\,sat(\frac{s}{\phi}) \ge s[b^{-1}\hat{b}f - \hat{f} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})]s + \eta b^{-1}\hat{b}|s|$

$s\,k\,sat(\frac{s}{\phi}) \ge s[b^{-1}\hat{b}\hat{f} + b^{-1}\hat{b}F - \hat{f} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})]s + \underline{\eta b^{-1}\hat{b}}|s|$

inputting $f = \hat{f} + (f - \hat{f}) \le \hat{f} + |f - \hat{f}| \le \hat{f} + F$

know $b^{-1}\hat{b} > 0$ & $\eta > 0$

$s\,sgn(s) = |s|$

$k \ge \frac{s}{|s|}[b^{-1}\hat{b}\hat{f} + b^{-1}\hat{b}F - \hat{f} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})] + \eta b^{-1}\hat{b}$

$k \ge |\underline{b^{-1}\hat{b}\hat{f}} + b^{-1}\hat{b}F - \underline{\hat{f}} + (b^{-1}\hat{b} - 1)(-\dddot{x}_{des} + 2\lambda\ddot{x} + \lambda^2\dot{x})| + \eta b^{-1}\hat{b}$

$k \ge b^{-1}\hat{b}F + b^{-1}\hat{b}\eta + |b^{-1}\hat{b} - 1||\hat{f} - \dddot{x}_{des} - 2\lambda\ddot{x} + \lambda^2\dot{x}|$

$b^{-1}\hat{b} = \beta, \quad |-\hat{u}| = |\hat{u}|$

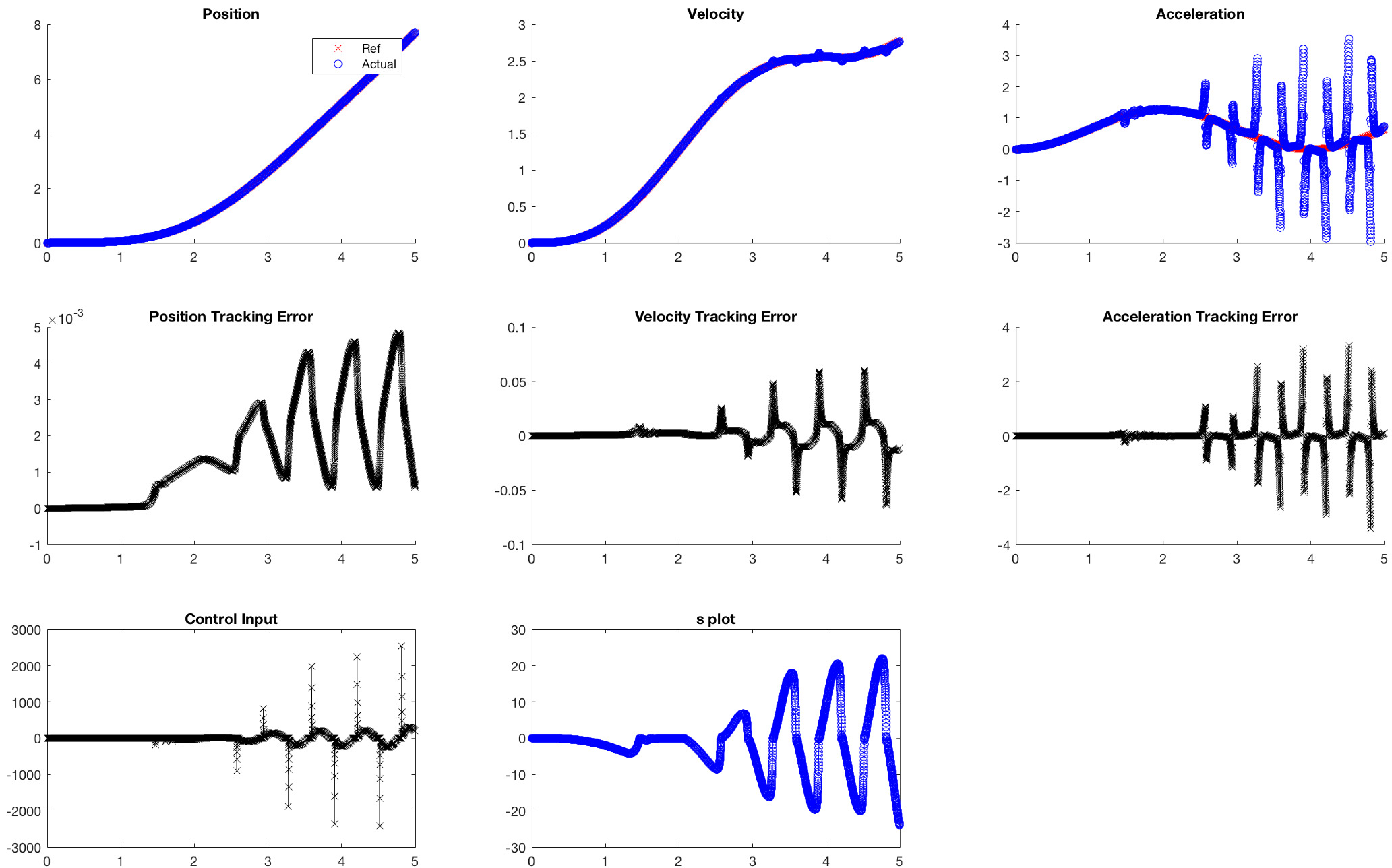
$k \ge \beta F + \beta\eta + |\beta - 1||\hat{u}|$

$k \ge \beta(F + \eta) + |\beta - 1||\hat{u}|$

	$\hat{b}b^{-1}$
$\hat{b} = 2$ $\quad b^{-1} = 1$	2
$b^{-1} = 1/4$	$1/2$

$\beta = 2$

$$\boxed{\begin{array}{l} \hat{f} = 0 \qquad F = \ddot{x}^2 + 2\dot{x}^5 \sin 4x \\ s = \ddot{x} - \ddot{x}_{des} + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x} \\ u = \hat{u} - k\,sat(s/\phi) \\ \hat{u} = -\hat{f} + \dddot{x}_{des} - 2\lambda\ddot{\tilde{x}} - \lambda^2\dot{\tilde{x}} \\ k = \beta(F + \eta) + |\beta - 1||\hat{u}| \end{array}}$$



Position
Ref
Actual
Velocity
Acceleration
Position Tracking Error
×10^-3
Velocity Tracking Error
Acceleration Tracking Error
Control Input
s plot

$\ddot{y} + a_{p_1}\dot{y} + a_{p_2}y = \dot{u} + b_p u$

$\dot{u} + b_p u = \frac{1}{s_p}\underbrace{(b_p\dot{u} + u)}_{v}$

$\ddot{y} + a_{p_1}\dot{y} + a_{p_2}y = b_v v$

$\frac{1}{b_v}\ddot{y} + \frac{a_{p_1}}{b_v}\dot{y} + \frac{a_{p_2}}{b_v}y = v$

$a_{v_1}\ddot{y} + a_{v_2}\dot{y} + a_{v_3}y = v$

$a_v = \begin{bmatrix} a_{v_1} \\ a_{v_2} \\ a_{v_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{b_v} \\ a_{p_1}/b_v \\ a_{p_2}/b_v \end{bmatrix}$

$a_m = \begin{bmatrix} a_{m_1} \\ a_{m_2} \\ a_{m_3} \end{bmatrix}$

$b_v = \frac{1}{a_{m_1}}$

$a_{m_2} = \frac{a_{p_1}}{b_v}$

$a_{m_2} b_v = a_{p_1}$

$a_{m_2} \cdot \frac{1}{a_{m_1}} = a_{p_1}$

$V = \frac{1}{2}a_{v_1}s^2 \qquad \dot{V} = a_{v_1}s\dot{s} = a_{v_1}s(\ddot{y} - \ddot{y}_r)$

$= s(-a_{v_2}\dot{y} - a_{v_3}y + v - a_{v_1}\ddot{y}_r)$

$= s\left(\underbrace{[\ddot{y}_r \quad \dot{y} \quad y]}_{y}\underbrace{\begin{bmatrix} -a_{v_1} \\ -a_{v_2} \\ -a_{v_3} \end{bmatrix}}_{a_v} + v\right)$

$s = \dot{\tilde{x}} + \lambda\tilde{x}$

$s = \dot{x} - \dot{x}_r \qquad \dot{x}_r = \dot{x}_d - \lambda\tilde{x}$

$\dot{s} = \ddot{x} - \ddot{x}_r \qquad \ddot{x}_r = \ddot{x}_d - \lambda\dot{\tilde{x}}$

choose $v = y a_m - ks$

$\dot{V} = s(v - y a_v) = s(y a_m - ks - y a_v)$

$= s(y\tilde{a} - ks) = sy\tilde{a} - ks^2$

try $V = \frac{1}{2}a_{v_1}s^2 + \frac{1}{2}\tilde{a}^T p^{-1}\tilde{a}$

$\dot{V} = a_{v_1}s\dot{s} + \dot{\hat{a}}^T p^{-1}\tilde{a}$

$= -ks^2 + sy\tilde{a} + \dot{\hat{a}}^T p^{-1}\tilde{a}$

$= -ks^2 + (\underbrace{sy + \dot{\hat{a}}^T p^{-1}}_{\text{this}})\tilde{a}$

choose $\dot{\hat{a}}$ so this goes to $\emptyset$

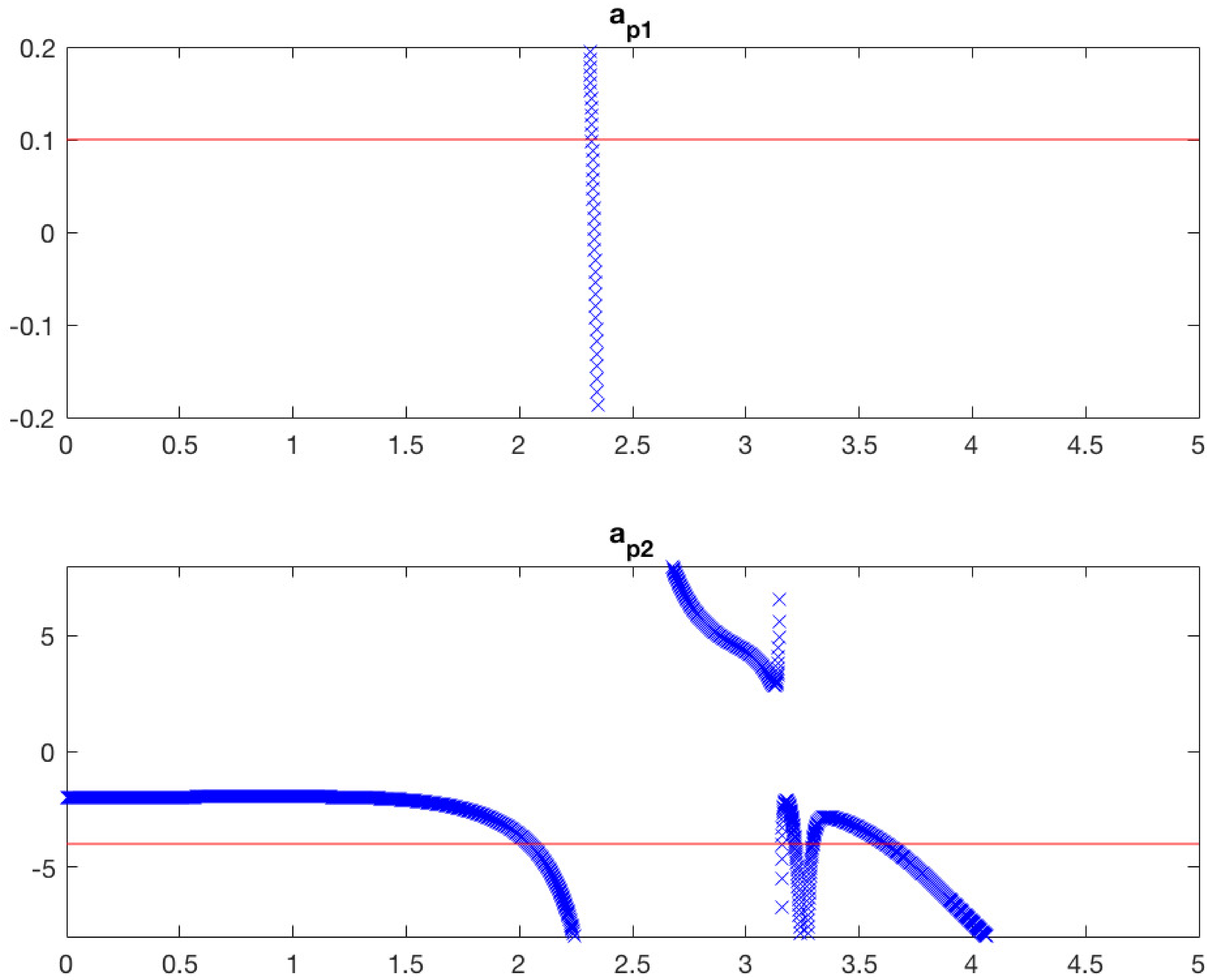
$sy + \dot{\hat{a}}^T p^{-1} = 0$

$\dot{\hat{a}} = (-syp)^T = -py^T s$

$v = y a_m - ks$

$\dot{\hat{a}} = -py^T s$


Position
3
2
1
0
0 1 2 3 4 5
Ref
Actual
Velocity
2
1
0
-1
0 1 2 3 4 5
Acceleration
100
50
0
-50
0 1 2 3 4 5
Position Tracking Error
0.02
0
-0.02
0 1 2 3 4 5
Velocity Tracking Error
1
0
-1
-2
0 1 2 3 4 5
Acceleration Tracking Error
100
50
0
-50
0 1 2 3 4 5
Control Input
100
0
-100
0 1 2 3 4 5
s plot
2
0
-2
-4
0 1 2 3 4 5



a_p1
0.2
0.1
0
-0.1
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
a_p2
5
0
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Parameter estimation can be easier for an unstable system because a smaller set of parameters make the system stable or track a trajectory. For a stable system, many values may achieve the same performance so the adaptive system will only change the values enough to ensure stability. These values are unlikely to be the same as the plant for a stable system.