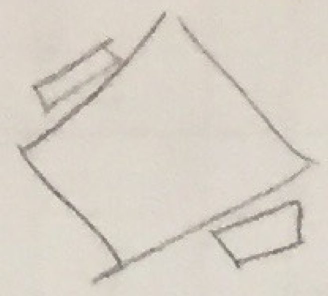
Map approach

HW4

$$g=\begin{bmatrix}a\\b\\c\end{bmatrix}\quad q=\begin{bmatrix}x\\y\\\theta\end{bmatrix}\quad \frac{\partial g}{\partial q}=\begin{bmatrix}\frac{\partial a}{\partial x}&\frac{\partial a}{\partial y}&\frac{\partial a}{\partial \theta}\\ \\ \frac{\partial c}{\partial x}&\frac{\partial c}{\partial y}&\frac{\partial c}{\partial \theta}\end{bmatrix}$$

① Show that the non-holonomic constraints on the differential drive car induce a holonomic constraint between its orientation and its wheel angle.



$$\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}\alpha_f\\\alpha_r\end{bmatrix}\qquad A_1=\begin{bmatrix}0\\0\\1\end{bmatrix}\quad A_2=\begin{bmatrix}c\theta\\s\theta\\0\end{bmatrix}$$

$$\begin{array}{ccc}a&b&c\\ \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix},&\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix},&\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}\end{array}$$

$$[A_1,A_2]=\nabla_{A_1}A_2-\nabla_{A_2}A_1=\begin{bmatrix}0\\0\\0\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

$$=\frac{\partial\begin{bmatrix}0\\0\\1\end{bmatrix}}{\partial(x,y,\theta)}\cdot\begin{bmatrix}c\theta\\s\theta\\0\end{bmatrix}\Bigg|_e=\underset{3\times3}{\begin{bmatrix}0\\0\\0\end{bmatrix}}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$=\frac{\partial\begin{bmatrix}c\theta\\s\theta\\0\end{bmatrix}}{\partial(x,y,\theta)}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\Bigg|_e=\begin{bmatrix}0&0&-\sin\theta\\0&0&\cos\theta\\0&0&0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\Bigg|_e=\begin{bmatrix}-s\theta\\c\theta\\0\end{bmatrix}\Bigg|_e=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

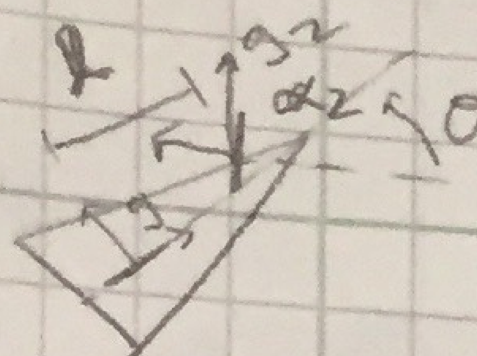
$$[a,[a,b]]:\ \frac{\partial\begin{bmatrix}\alpha_f\\0\\x\\0\\0\end{bmatrix}}{\partial(\alpha_f,\alpha_r,x,y,\theta)}\begin{bmatrix}0\\0\\0\\-y\\0\end{bmatrix}\Bigg|_e=\begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\0\\y\\0\end{bmatrix}\Bigg|_e=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$$\frac{\partial\begin{bmatrix}0\\0\\0\\-y\\0\end{bmatrix}}{\partial(\sim)}\begin{bmatrix}\alpha_f\\0\\x\\0\\0\end{bmatrix}\Bigg|_e=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}\alpha_f\\0\\x\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

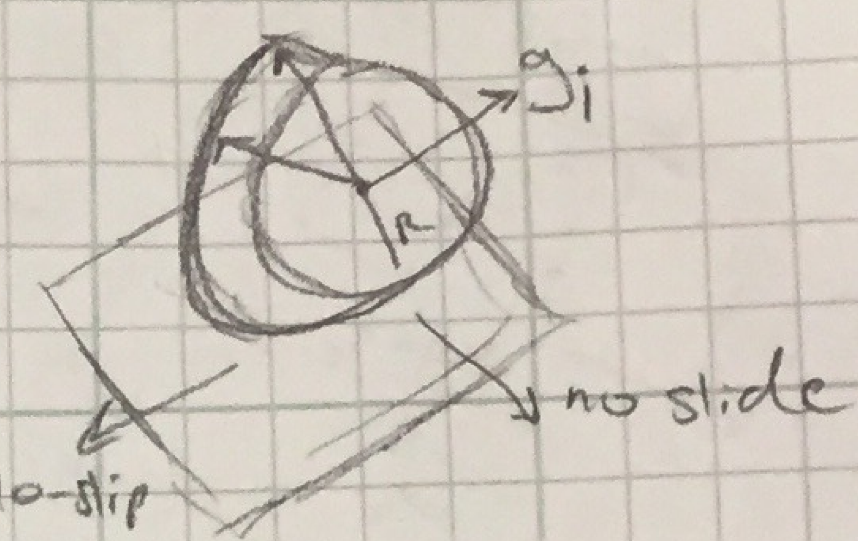
$$[b,[a,b]]:\ \frac{\partial\begin{bmatrix}0\\\alpha_r\\0\\0\\\theta\end{bmatrix}}{\partial(\alpha_f,\alpha_r,x,y,\theta)}\begin{bmatrix}0\\0\\0\\y\\0\end{bmatrix}\Bigg|_e=\begin{bmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\0\\y\\0\end{bmatrix}\Bigg|_e=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$$\frac{\partial\begin{bmatrix}0\\0\\0\\y\\0\end{bmatrix}}{\partial(\alpha_f,\alpha_r,x,y,\theta)}\begin{bmatrix}0\\\alpha_r\\0\\0\\\theta\end{bmatrix}\Bigg|_e=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}0\\\alpha_r\\0\\0\\\theta\end{bmatrix}$$

Ackerman Steering rear wheel drive



$g = g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad g_{2g} = \begin{bmatrix} l \\ 0 \\ \alpha_2 \end{bmatrix}$ $\quad$ from g to g_2



no slip: $\vec{\mathring{g}}_1^{x} - R\dot{\alpha}_1 = 0$ $\qquad$ no slide: $\vec{\mathring{g}}_1^{y} = 0$

$\vec{\mathring{g}}_2^{y} = 0$

$\vec{\mathring{g}}_1 = Ad_{g_{1,g}}^{-1} \vec{\mathring{g}}_g \qquad Ad_{g_{1,g}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\vec{\mathring{g}}_1 = \vec{\mathring{g}}_g$

$\vec{\mathring{g}}_1^{x} = \vec{\mathring{g}}^{x}$

$\vec{\mathring{g}}_1^{y} = \vec{\mathring{g}}^{y}$

$\vec{\mathring{g}}^{\theta} = \vec{\mathring{g}}^{\theta}$

$\vec{\mathring{g}}_2 = Ad_{g_{2,g}}^{-1} \vec{\mathring{g}}_g \qquad Ad_{g_{2g}}^{-1} = \begin{bmatrix} c\alpha_2 & s\alpha_2 & l s\alpha_2 \\ -s\alpha_2 & c\alpha_2 & l c\alpha_2 \\ 0 & 0 & 1 \end{bmatrix}$

$\vec{\mathring{g}}_2 = \begin{bmatrix} c\alpha_2 & s\alpha_2 & l s\alpha_2 \\ -s\alpha_2 & c\alpha_2 & l c\alpha_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{\mathring{g}}^{x} \\ \vec{\mathring{g}}^{y} \\ \vec{\mathring{g}}^{\theta} \end{bmatrix}$

$\vec{\mathring{g}}_2^{x} = c\alpha_2 \vec{\mathring{g}}^{x} + s\alpha_2 \vec{\mathring{g}}^{y} + l s\alpha_2 \vec{\mathring{g}}^{\theta}$

$\vec{\mathring{g}}_2^{y} = -s\alpha_2 \vec{\mathring{g}}^{x} + c\alpha_2 \vec{\mathring{g}}^{y} + l c\alpha_2 \vec{\mathring{g}}^{\theta}$

$\vec{\mathring{g}}_2^{\theta} = \vec{\mathring{g}}^{\theta}$

rewrite constraints

$\vec{\mathring{g}}^{x} - R\dot{\alpha}_1 = 0$ $\qquad$ $\vec{\mathring{g}}^{y} = 0$

$-s\alpha_2 \vec{\mathring{g}}^{x} + c\alpha_2 \vec{\mathring{g}}^{y} + l c\alpha_2 \vec{\mathring{g}}^{\theta} = 0$

~~4 constraints on SE(2) → over constrained ⇒ can only follow certain shape trajectories?~~

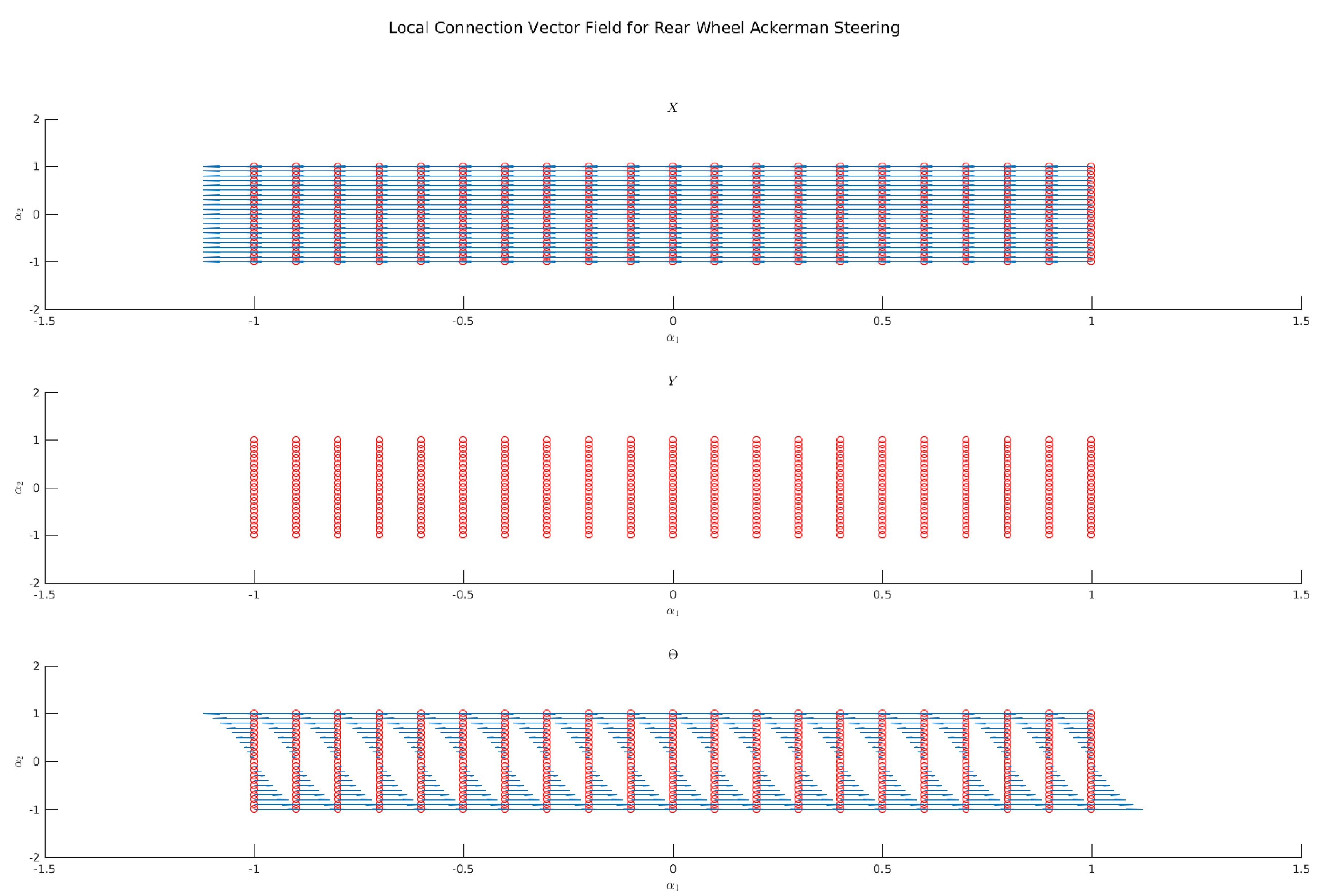
express in pfaffian

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -R & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -s\alpha_2 & c\alpha_2 & l c\alpha_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{\mathring{g}}^{x} \\ \vec{\mathring{g}}^{y} \\ \vec{\mathring{g}}^{\theta} \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$

$w_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s\alpha_2 & c\alpha_2 & l c\alpha_2 \end{bmatrix} \qquad w_b = \begin{bmatrix} -R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

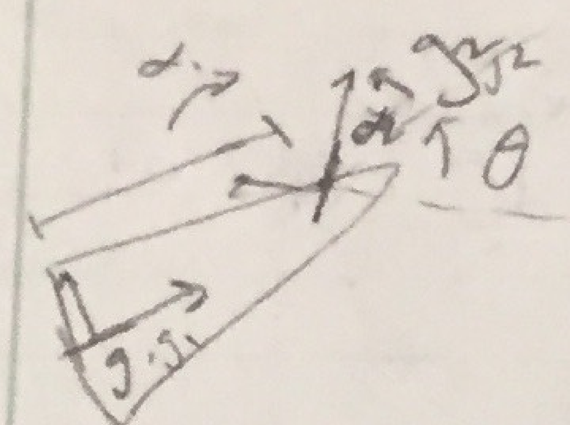
$w_g^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l^{-1}\tan(\alpha_2) & -\frac{1}{l} & \frac{1}{l\cos(\alpha_2)} \end{bmatrix}$

$A = w_g^{-1} \cdot w_b = \begin{bmatrix} -R & 0 \\ 0 & 0 \\ -\frac{R}{l}\tan(\alpha_2) & 0 \end{bmatrix}$


Local Connection Vector Field for Rear Wheel Ackerman Steering
X
Y
Θ
α1
α2
-1.5
-1
-0.5
0
0.5
1
1.5
-2
2

Ackerman Steering front wheel drive

same as rear except different constraint



$$g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad g_2 = \begin{bmatrix} l \\ 0 \\ \alpha_2 \end{bmatrix}$$

no slip : $\overset{\circ\to}{g}{}_2^{x} - R\dot{\alpha}_1 = 0$ no slide : $\overset{\circ\to}{g}{}_1^{y} = 0$

$\overset{\circ\to}{g}{}_2^{y} = 0$

$$Ad^{-1}_{g_2 \cdot g} = \begin{bmatrix} c\alpha_2 & s\alpha_2 & l s\alpha_2 \\ -s\alpha_2 & c\alpha_2 & l c\alpha_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\overset{\circ\to}{g}{}_1^{x} = \overset{\circ\to}{g}{}^{x}$ $\qquad \overset{\circ\to}{g}{}_2^{x} = c\alpha_2 \overset{\circ\to}{g}{}^{x} + s\alpha_2 \overset{\circ\to}{g}{}^{y} + l s\alpha_2 \overset{\circ\to}{g}{}^{\theta}$

$\overset{\circ\to}{g}{}_1^{y} = \overset{\circ\to}{g}{}^{y}$ $\qquad \overset{\circ\to}{g}{}_2^{y} = -s\alpha_2 \overset{\circ\to}{g}{}^{x} + c\alpha_2 \overset{\circ\to}{g}{}^{y} + l c\alpha_2 \overset{\circ\to}{g}{}^{\theta}$

$\overset{\circ\to}{g}{}_1^{\theta} = \overset{\circ\to}{g}{}^{\theta}$ $\qquad \overset{\circ\to}{g}{}_2^{\theta} = \overset{\circ\to}{g}{}^{\theta}$

rewrite constraints

$$c\alpha_2 \overset{\circ\to}{g}{}^{x} + s\alpha_2 \overset{\circ\to}{g}{}^{y} + l s\alpha_2 \overset{\circ\to}{g}{}^{\theta} - R\dot{\alpha}_1 = 0$$

$$\overset{\circ\to}{g}{}^{y} = 0$$

$$-s\alpha_2 \overset{\circ\to}{g}{}^{x} + c\alpha_2 \overset{\circ\to}{g}{}^{y} + l c\alpha_2 \overset{\circ\to}{g}{}^{\theta} = 0$$

express in pfaffian

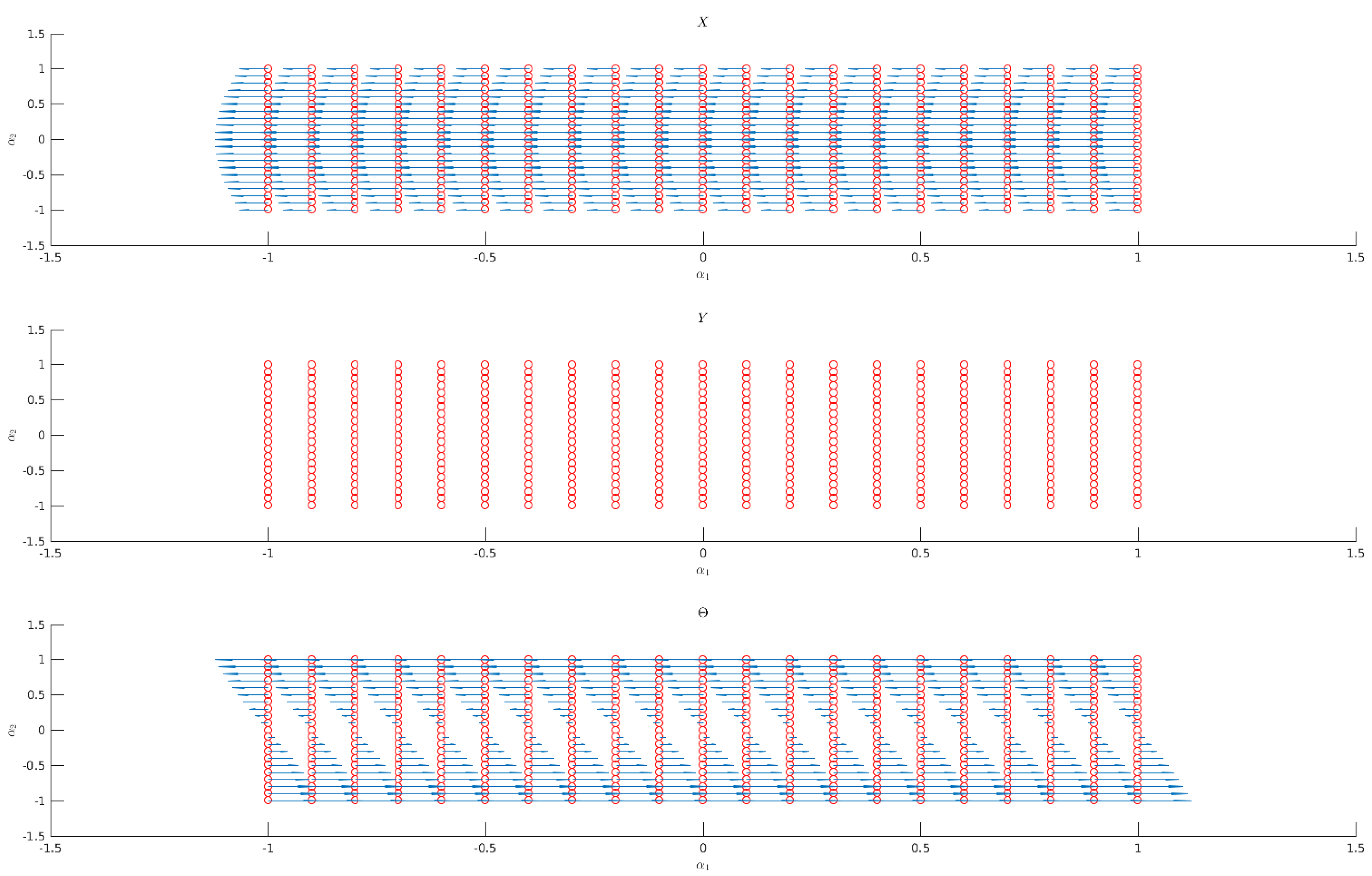
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{ccc|cc} c\alpha_2 & s\alpha_2 & l s\alpha_2 & -R & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -s\alpha_2 & c\alpha_2 & l c\alpha_2 & 0 & 0 \end{array}\right] \begin{bmatrix} \overset{\circ\to}{g}{}^{x} \\ \overset{\circ\to}{g}{}^{y} \\ \overset{\circ\to}{g}{}^{\theta} \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

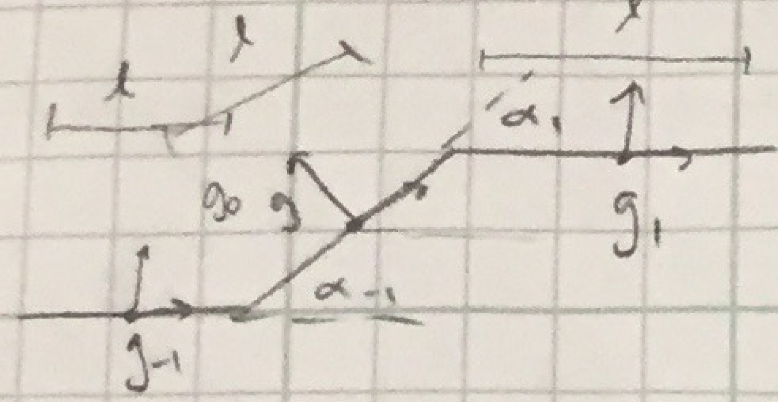
$$\underbrace{}_{w_g} \qquad \underbrace{}_{w_b}$$

$$w_g^{-1} = \begin{bmatrix} c\alpha_2 & 0 & -s(\alpha_2) \\ 0 & 1 & 0 \\ \frac{1}{l} s\alpha_2 & -\frac{1}{l} & \frac{1}{l}\cos(\alpha_2) \end{bmatrix}$$

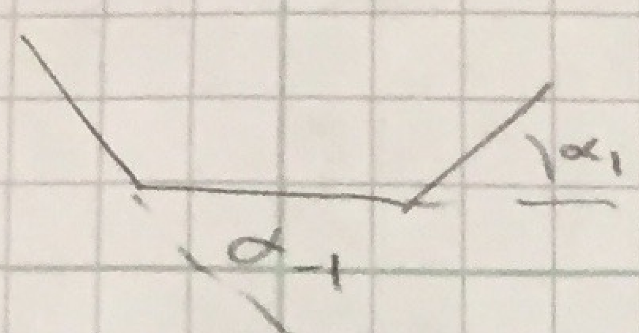
$$A = w_g^{-1} \cdot w_b = \begin{bmatrix} -\cos(\alpha_2) & 0 \\ 0 & 0 \\ -\sin(\alpha_2) & 0 \end{bmatrix}$$

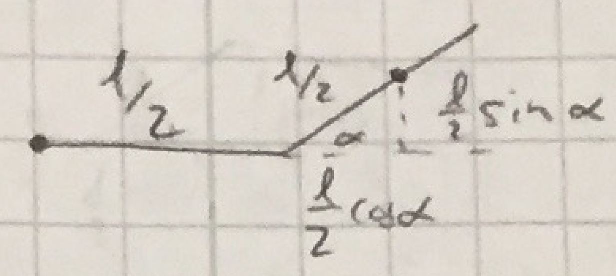
Local Connection Vector Field for Front Wheel Ackerman Steering

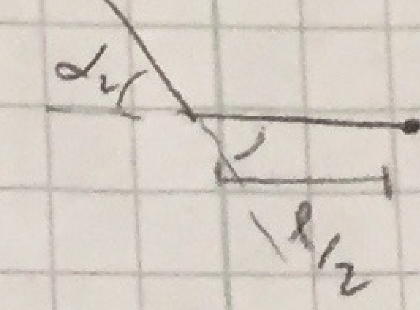




$$g_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad g_1 = \begin{bmatrix} \frac{l}{2}(1+\cos\alpha_1) \\ \frac{l}{2}\sin\alpha_1 \\ \alpha_1 \end{bmatrix} \qquad g_{-1} = \begin{bmatrix} -\frac{l}{2}(1+\cos\alpha_{-1}) \\ \frac{l}{2}\sin\alpha_{-1} \\ \alpha_{-1} \end{bmatrix}$$







pinned ground link: $\overset{\circ\rightarrow}{g}{}_0^{x} = 0$ conservation of angular momentum $0 = \overset{\circ\rightarrow}{g}{}^{\theta} - m\cdot\overset{\circ\rightarrow}{g}{}_1^{x} + m\overset{\circ\rightarrow}{g}{}_{-1}^{x}$

$\overset{\circ\rightarrow}{g}{}_0^{y} = 0$

$$\vec{g}_1 = Ad^{-1}_{g_1,g}\,\vec{g}_g$$

$$Ad^{-1}_{g_1,g} = \begin{bmatrix} c\alpha_1 & s\alpha_1 & \frac{l}{2}(1+\cos\alpha_1)s\alpha_1 - \frac{l}{2}\sin\alpha_1 c\alpha_1 \\ -s\alpha_1 & c\alpha_1 & \frac{l}{2}(1+c\alpha_1)c\alpha_1 + \frac{l}{2}s\alpha_1 s\alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \qquad Ad^{-1}_{g_0,g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ad^{-1}_{g_{-1},g} = \begin{bmatrix} c\alpha_{-1} & s\alpha_{-1} & -\frac{l}{2}(1+\cos\alpha_{-1})s\alpha_{-1} - \frac{l}{2}\sin\alpha_{-1}c\alpha_{-1} \\ -s\alpha_{-1} & c\alpha_{-1} & -\frac{l}{2}(1+\cos\alpha_{-1})c\alpha_{-1} + \frac{l}{2}\sin\alpha_{-1}s\alpha_{-1} \\ 0 & 0 & 1 \end{bmatrix}$$

$\overset{\circ\rightarrow}{g}{}^{x} = 0$

$\overset{\circ\rightarrow}{g}{}^{y} = 0$

$$0 = \overset{\circ\rightarrow}{g}{}^{\theta} - m\left[c\alpha_1\overset{\circ\rightarrow}{g}{}^{x} + s\alpha_1\overset{\circ\rightarrow}{g}{}^{y} + \frac{l}{2}(s\alpha_1)\overset{\circ\rightarrow}{g}{}^{\theta}\right]$$
$$+\, m\left[c\alpha_{-1}\overset{\circ\rightarrow}{g}{}^{x} + s\alpha_{-1}\overset{\circ\rightarrow}{g}{}^{y} + \frac{-l}{2}\big(s\alpha_{-1} + \underbrace{s\alpha_{-1}c\alpha_{-1} + s\alpha_{-1}c\alpha_{-1}}_{\sin 2\alpha_{-1}}\big)\right]$$

not symmetric