Geometric Mechanics HW 3 Due: 10/30/17

Comprehension

1) first constraint restricts the system to points that are a fixed distance from $(0,0,0) \in \mathbb{R}^3$

Same accessible manifold as

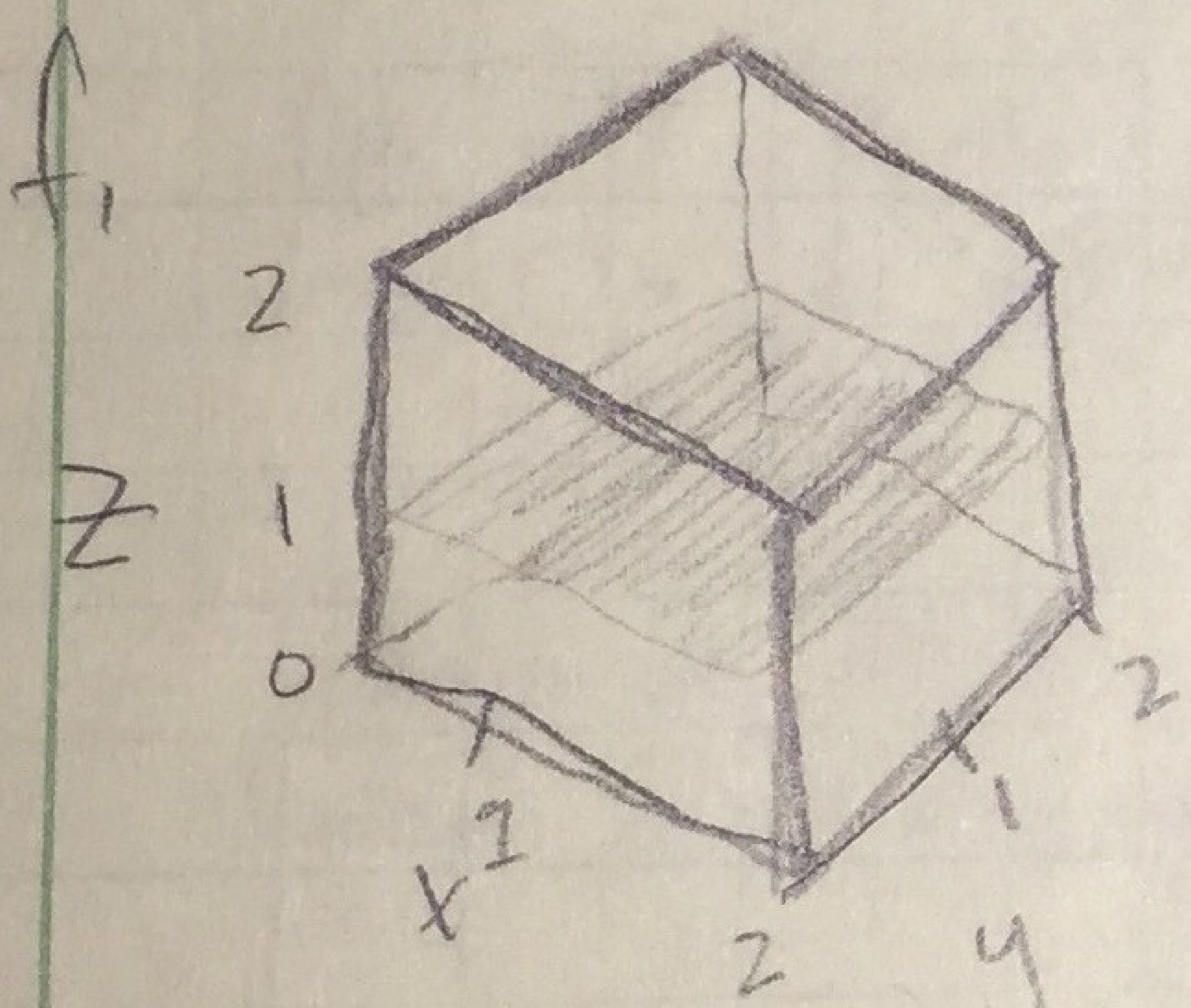
$$f_1(p,t) = z - 1$$

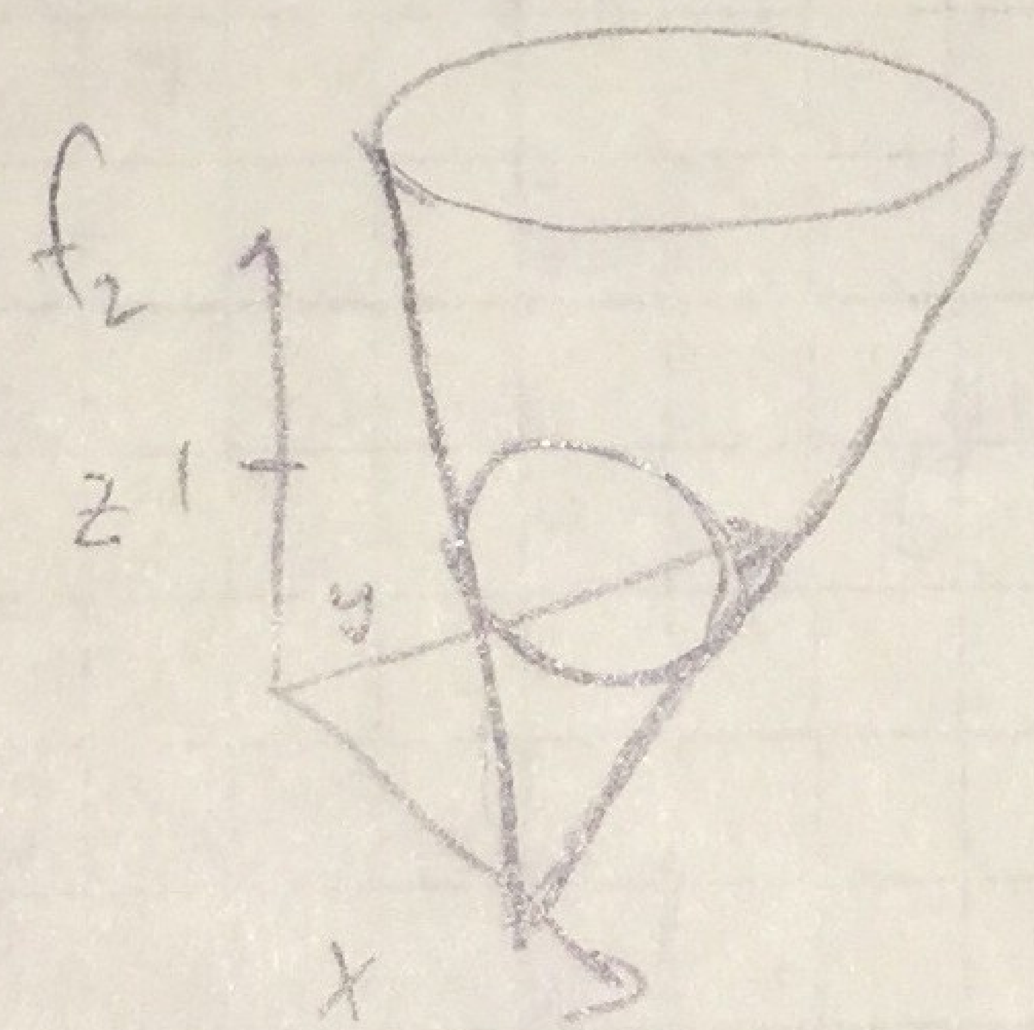
$$f_2(p_1,t) = \sqrt{x^2+y^2} - 1$$

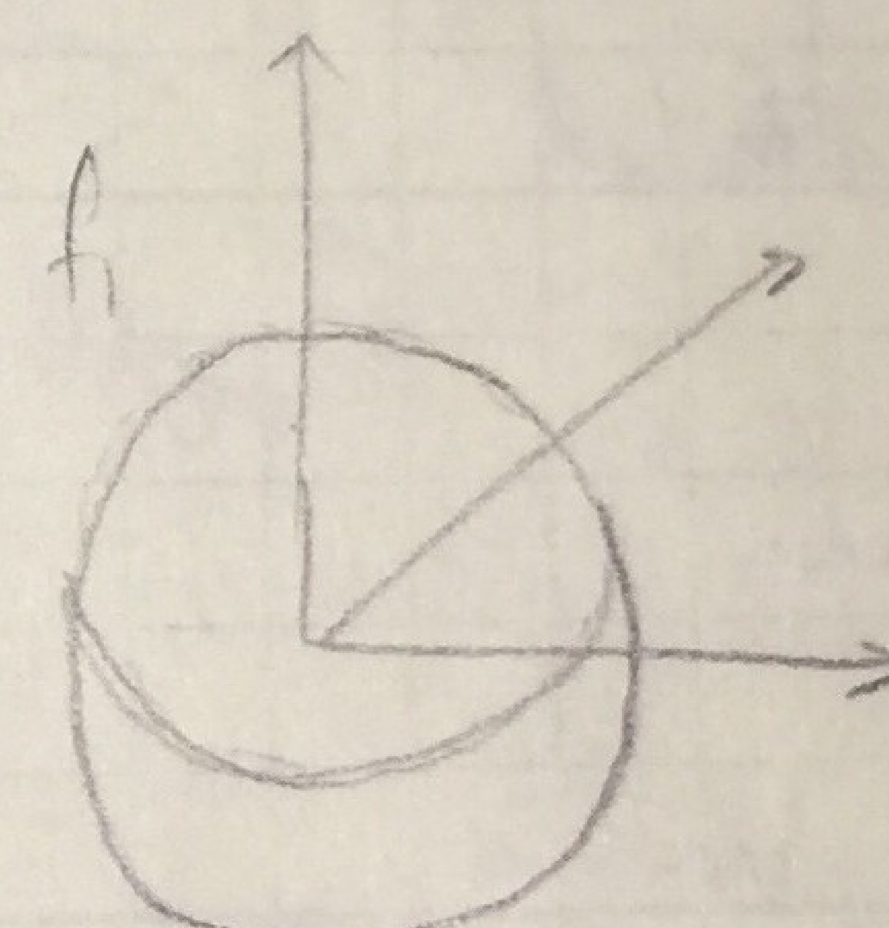
Equation should be zero when constraints satisfied

$$f_1(p,t) = \sqrt{x^2+y^2+z^2} - d$$

$$f_2(p_1,t) = z - 1$$







$$0 = z - 1$$
$$z = 1$$

$$0 = \sqrt{x^2+y^2+(1)^2} - d$$
$$d^2 = x^2+y^2+1$$
$$d^2 - 1 = x^2+y^2$$

want unit circle so

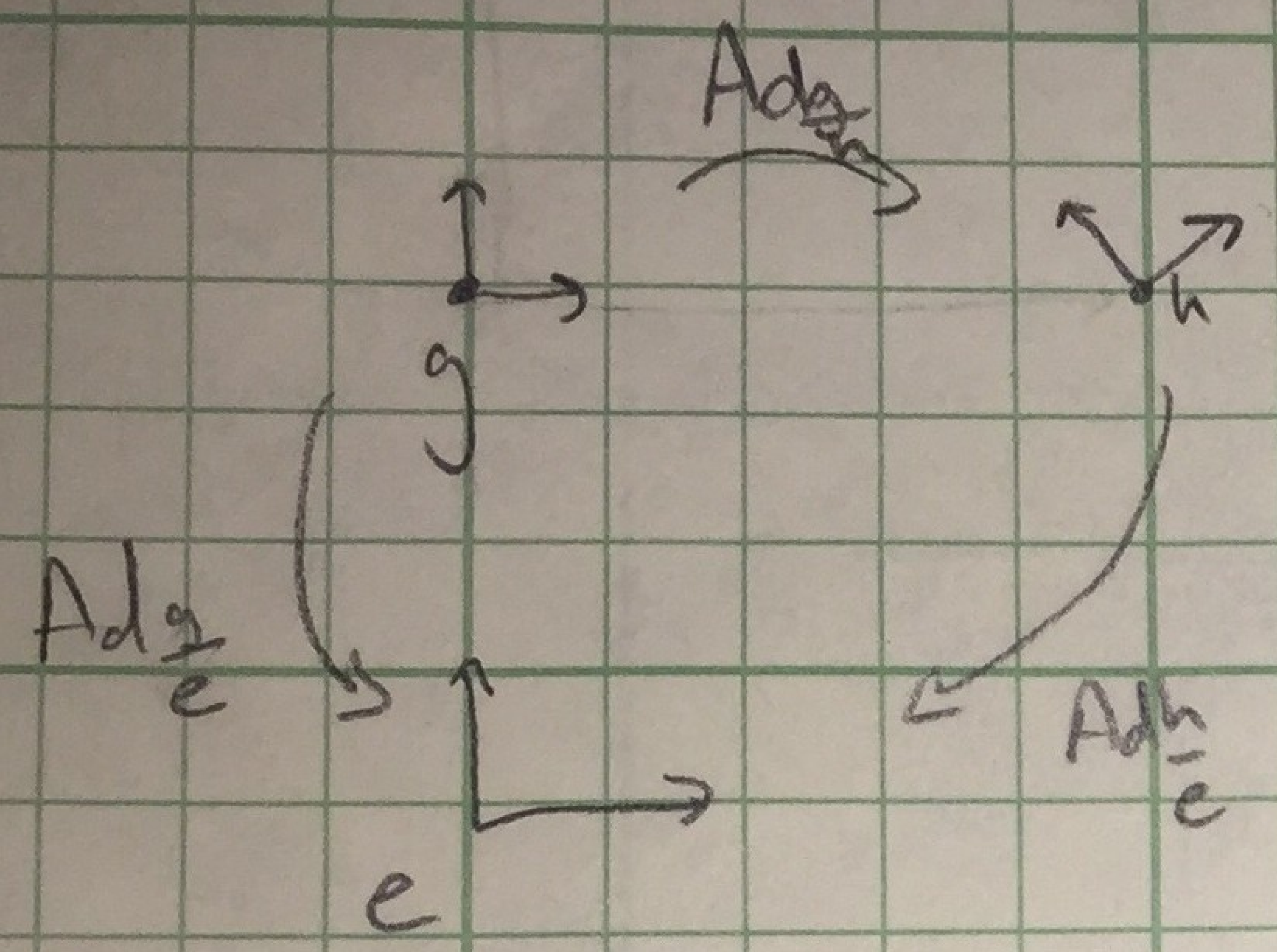
$$d^2 - 1 = 1$$
$$d^2 = 2$$
$$d = \sqrt{2}$$

Sphere with radius d, and center at $c = (0,0,0)$

$$\boxed{\begin{aligned} f_1(p,t) &= \sqrt{x^2+y^2+z^2} - \sqrt{2} \\ f_2(p_1,t) &= z - 1 \end{aligned}}$$

2)

$$\mathrm{Ad}_{\frac{g}{e}}\, \overset{\circ}{\vec{g}} = \mathrm{Ad}_{\frac{h}{e}}\, \overset{\circ}{\vec{h}} \qquad \overset{\circ}{\vec{h}} = \mathrm{Ad}_{\frac{g}{h}}\, \overset{\circ}{\vec{g}}$$



$$\overset{\circ}{\vec{h}} = \mathrm{Ad}^{-1}_{(\frac{h}{e})}\, \mathrm{Ad}_{\frac{g}{e}}\, \overset{\circ}{\vec{g}} \quad \Rightarrow \quad \mathrm{Ad}^{-1}_{(\frac{h}{e})}\, \mathrm{Ad}_{\frac{g}{e}} = \mathrm{Ad}_{\frac{g}{h}}$$

$$\mathrm{Ad}_{(\frac{e}{h})}\, \mathrm{Ad}_{\frac{g}{e}} = \mathrm{Ad}_{\frac{g}{h}}$$

$\overset{\leftarrow\circ}{g} = \overset{\leftarrow\circ}{h}$ because rigidly attached

$$\left.\begin{aligned} \overset{\leftarrow\circ}{g} &= (T_g R_{g^{-1}})(T_e L_g)\, \overset{\circ}{\vec{g}} \\ \overset{\leftarrow\circ}{h} &= (T_h R_{h^{-1}})(T_e L_h)\, \overset{\circ}{\vec{h}} \end{aligned}\right\} \text{equal}$$

$$(T_g R_{g^{-1}})(T_e L_g) = \overset{ad_g}{\begin{bmatrix} c\theta & -s\theta & y \\ s\theta & c\theta & -x \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}$$

$\theta = 0$ for g (assume no rotation)

$$(T_h R_{h^{-1}})(T_e L_h) = \overset{ad_{h^{-1}}}{\begin{bmatrix} c\alpha & \sin\alpha & x\sin\alpha - y\cos\alpha \\ -s\alpha & \cos\alpha & x\cos\alpha + y\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\mathrm{Ad}_{h^{-1}}\mathrm{Ad}_g = \begin{bmatrix} c\alpha & s\alpha & x\sin\alpha - y\cos\alpha \\ -s\alpha & c\alpha & x\,c\alpha + y\,s\alpha \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha & s\alpha & \cancel{y\,c\alpha} - \cancel{x\,s\alpha} + \cancel{x\,s\alpha} - \cancel{y\,c\alpha} \\ -s\alpha & c\alpha & -\cancel{y\,s\alpha} - \cancel{x\,c\alpha} + \cancel{x\,c\alpha} + \cancel{y\,s\alpha} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark}$$

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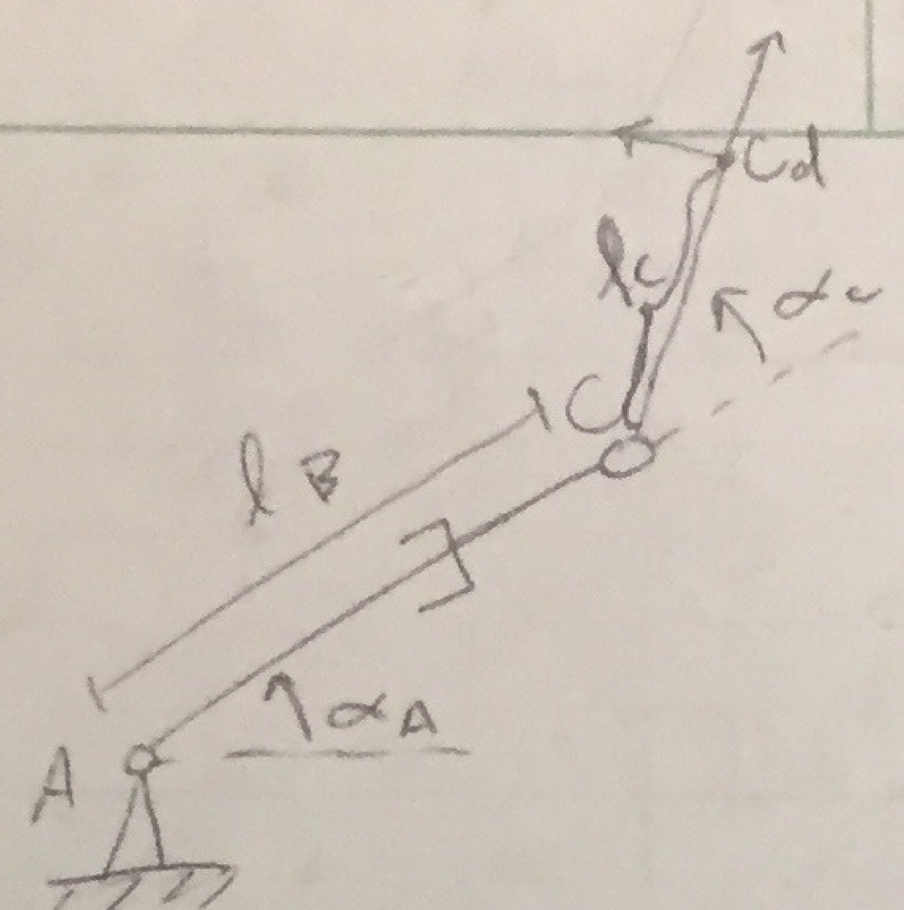

$$\alpha_{AC} = \alpha_A + \alpha_C$$

$$C_d = \begin{pmatrix} l_B \cos\alpha_A + l_C \cos\alpha_{AC} \\ l_B \sin\alpha_A + l_C \sin\alpha_{AC} \\ \alpha_{AC} \end{pmatrix} \qquad S = \begin{pmatrix} \alpha_A \\ l_B \\ \alpha_C \end{pmatrix}$$

$$\dot{C}_d = J_{C_d}(s)\,\dot{S}$$

$$\boxed{\dot{C}_d = \begin{array}{ccc} \left[\begin{matrix} -l_B \sin\alpha_A - l_C \sin\alpha_{AC} \\ l_B \cos\alpha_A + l_C \cos\alpha_A \\ 1 \end{matrix}\right. & \begin{matrix} \cos\alpha_A \\ \sin\alpha_A \\ 0 \end{matrix} & \left.\begin{matrix} -l_C \sin\alpha_{AC} \\ l_C \cos\alpha_{AC} \\ 1 \end{matrix}\right] \\ \frac{\partial c_d}{\partial \alpha_A} & \frac{\partial c_d}{\partial l_B} & \frac{\partial c_d}{\partial \alpha_C} \end{array}}$$

local axis approach

$$\vec{g}_{Cd} = Ad_{h_C}^{-1} Ad_{a_C}^{-1} \left(\vec{g}_{Bd} + \dot{\alpha}_C \vec{a}_C\right)$$

$$\vec{g}_{Bd} = Ad_{h_B}^{-1} Ad_{a_B}^{-1} \left(\vec{g}_{Ad} + \dot{\alpha}_B \vec{a}_B\right)$$

$$\vec{g}_{Ad} = Ad_{h_A}^{-1} Ad_{a_A}^{-1} \left(\vec{g}_0 + \dot{\alpha}_A \vec{a}_A\right)$$

$$h_C = \begin{pmatrix} l_C \\ 0 \\ 0 \end{pmatrix} \quad \vec{a}_C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad ad_{a_C}^{-1} = \begin{bmatrix} C\alpha_C & S\alpha_C & 0 \\ -S\alpha_C & C\alpha_C & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$ad_{h_C}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & l_C \\ 0 & 0 & 1 \end{bmatrix}$$

$$h_B = \begin{pmatrix} l_B \\ 0 \\ 0 \end{pmatrix} \quad \vec{a}_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad ad_{a_B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$ad_{h_B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & l_B \\ 0 & 0 & 0 \end{bmatrix}$$

$$h_A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{a}_A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad ad_{a_A}^{-1} = \begin{bmatrix} C\alpha_A & S\alpha_A & 0 \\ -S\alpha_A & C\alpha_A & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$ad_{h_A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{g}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} \vec{g}_{Cd} &= Ad_{h_C}^{-1} Ad_{a_C}^{-1}\left(Ad_{h_B}^{-1} Ad_{a_B}^{-1}\left(Ad_{h_A}^{-1} Ad_{a_A}^{-1}\left(\dot{\alpha}_A \vec{a}_A\right) + \dot{\alpha}_B \vec{a}_B\right) + \dot{\alpha}_C \vec{a}_C\right) \\ \vec{g}_{Cd} &= J_{Cd,0}^{b}\,\dot{\alpha} \\ \dot{g}_{Cd} &= T_e L_{Cd} J_{Cd,0}^{b}\,\dot{\alpha} \end{aligned}}$$

$$Ad_{h_C}^{-1} Ad_{a_C}^{-1} Ad_{h_B}^{-1} Ad_{a_B}^{-1}\left(\vec{g}_{Ad}\dot{\alpha}_A + \dot{\alpha}_B \vec{a}_B\right)$$

$$\underbrace{\begin{bmatrix} C\alpha_C & S\alpha_C & 0 \\ -S\alpha_C & C\alpha_C & l_C \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & l_B \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ l_C & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_A \\ \dot{\alpha}_B \end{bmatrix}$$

$$\to \begin{bmatrix} C\alpha_C & S\alpha_C & l_B S\alpha_C \\ -S\alpha_C & C\alpha_C & l_B C\alpha_C + l_C \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} C\alpha_C & S\alpha_C & S\alpha_C \\ -S\alpha_C & C\alpha_C & C\alpha_C \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} S\alpha_C l_C + S\alpha_C & C\alpha_C \\ S\alpha_C l_C + C\alpha_C & -S\alpha_C \\ 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} C\alpha_{AC}(S\alpha_C l_C + S\alpha_C) - S\alpha_{AC}(C\alpha_C l_C + C\alpha_C) + 0 \\ \\ \end{bmatrix}$$

$$Ad_{h_C}^{-1} Ad_{a_C}^{-1} \vec{a}_C$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & l_C \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\alpha_C & S\alpha_C & 0 \\ -S\alpha_C & C\alpha_C & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} C\alpha_C & S\alpha_C & 0 \\ -S\alpha_C & C\alpha_C & l_C \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ l_C \\ 1 \end{bmatrix}}_{\vec{g}_{Ad}}$$

$$T_e L_{C_d} \cdot J_{g_{Cd}}^{b} \cdot \dot{\alpha}$$

$$= \begin{bmatrix} C\alpha_{AC} & -S\alpha_{AC} & 0 \\ S\alpha_{AC} & C\alpha_{AC} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_C \\ 1 \end{bmatrix} = \begin{bmatrix} -l_C \sin\alpha_{AC} \\ l_C \cos\alpha_{AC} \\ 1 \end{bmatrix}$$

$\vec{a}_C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\vec{a}_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{a}_A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

spatial axis approach:

$$\overset{\leftarrow}{g}_C = \overset{\leftarrow}{g}_B + Ad_{g_{Bd}} (\dot{\alpha}_C \vec{a}_C)$$
$$\overset{\leftarrow}{g}_B = \overset{\leftarrow}{g}_A + Ad_{g_{Ad}} (\dot{\alpha}_B \vec{a}_B)$$
$$\overset{\leftarrow}{g}_A = \overset{\leftarrow}{g}_e + Ad_{g_{od}} (\dot{\alpha}_A \vec{a}_a)$$
$$J = T_e R_{cd} J^s$$

$$Ad_{g_{od}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Ad_{g_{Ad}} = \begin{bmatrix} c\alpha_A & -s\alpha_A & 0 \\ s\alpha_A & c\alpha_A & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ad_{g_{Bd}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_B \\ 0 & 0 & 1 \end{bmatrix}$$

$$Ad_{g_{Ad}} \cdot \dot{\alpha}_B (\vec{a}_B) = \begin{bmatrix} c\alpha_A & -s\alpha_A & 0 \\ s\alpha_A & c\alpha_A & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\alpha_A \\ s\alpha_A \\ 0 \end{bmatrix}$$

$$Ad_{g_{od}} \dot{\alpha} \vec{a}_A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\alpha}_A = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\alpha}_A$$

$$T_e R_{cd} = \begin{bmatrix} 1 & 0 & -(l_B \sin\alpha_A + l_C \sin\alpha_{AC}) \\ 0 & 1 & l_B \cos\alpha_A + l_C \cos\alpha_{AC} \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_B = T_e R_{cd} J^s_B$$
$$= \begin{bmatrix} c\alpha_A \\ s\alpha_A \\ 0 \end{bmatrix} \checkmark$$

$$J_A = T_e R_{cd} J^s_A$$

exponential approach

body

$$J^b_{g_{nd},i} = \left[\left(\prod_{j=i}^{n} h_j a_j \right) \vec{a}_i \left(\prod_{j=i}^{n} a_j h_j \right) \right]$$

$$g_{cd} = a_A h_A a_B h_B a_C h_C$$
$$g_{cd}^{-1} = h_C^{-1} a_C^{-1} h_B^{-1} a_B^{-1} h_A^{-1} a_A^{-1}$$

$$\dot{a}_A = \dot{\alpha}_A \vec{a}_A a_A \quad \dot{a}_C = \dot{\alpha}_C \vec{a}_C a_C$$
$$\dot{a}_B = \dot{\alpha}_B \vec{a}_B a_B$$

$$\dot{g} = \vec{g} g$$

$$\vec{g}_{cd} = g_{cd}^{-1} \dot{g}_{cd} = h_C^{-1} a_C^{-1} h_B^{-1} a_B^{-1} h_A^{-1} a_A^{-1} (\dot{a}_A h_A a_B h_B a_C h_C + a_A h_A \dot{a}_B h_B a_C h_C + a_A h_A a_B h_B \dot{a}_C h_C)$$
$$Ad^{-1}_{h_C} Ad^{-1}_{a_C} Ad^{-1}_{h_B} Ad^{-1}_{a_B} Ad^{-1}_{h_A} Ad^{-1}_{a_A} \vec{a}_A \dot{\alpha}_A + Ad^{-1}_{h_C} Ad^{-1}_{a_C} Ad^{-1}_{h_B} Ad^{-1}_{a_B} \vec{a}_B \dot{\alpha}_B + Ad^{-1}_{h_C} Ad^{-1}_{a_C} \vec{a}_C \dot{\alpha}_C$$

spatial