New Robust Statistics for Change Detection in Time Series of Multivariate SAR Images

Supplementary material

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Abstract

This document corresponds to a supplementary material for the paper *New Robust Statistics for Change Detection in Time Series of Multivariate SAR Images* submitted to Transactions on Signal Processing. Detailed derivation for the detectors presented in the main paper are given here and detailled proof for the convergence property of a fixed-point algorithm estimating $\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}$.

I. GLRT FOR OMNIBUS PROBLEM 1

In this problem, we test a change in both texture and covariance parameters. Thus, the GLRT for this problem has the following form:

$$\hat{\Lambda} = \frac{\underset{\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T}{\max} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T\right)}{\underset{\boldsymbol{\theta}_0}{\max} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_0\right)} \tag{1}$$

where $\boldsymbol{\theta}_0 = \left\{\tau_1,\dots,t_N,\mathbf{\Sigma}_0\right\}$ and $\forall t \in [\![1,T]\!]$, $\boldsymbol{\theta}_t = \left\{\tau_1^{(t)},\dots,\tau_N^{(t)},\mathbf{\Sigma}_t\right\}$.

Using the assumption that all observations are independent, we can rewrite:

$$\hat{\Lambda} = \frac{\underset{\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T}{\max} \quad \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_t\right)}{\underset{\boldsymbol{\theta}_0}{\max} \quad \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_0\right)}.$$

This expression can be computed by optimising the numerator and denominator separately. Then, the idea is to estimate each unknown parameter separately and plugging back the

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estimates. Indeed, as we show in the main paper, the negative log of the likelihood functions considered here are jointly g-convex with regards to the covariance and texture parameters. In this case, each stationary-point of the negative log-likelihood correspond to a global minima which in turn correspond to the global maxima of the likelihoods. Thus we can compute:

$$\hat{\Lambda} = \frac{\mathcal{L}_1\left(\hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_T\right)}{\mathcal{L}_0\left(\hat{\boldsymbol{\theta}}_0\right)}, \tag{2}$$

where

$$\begin{aligned}
\mathcal{L}_1(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_T) &= \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)}; \boldsymbol{\theta}_t\right),\\
\mathcal{L}_0(\boldsymbol{\theta}_0) &= \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)}; \boldsymbol{\theta}_0\right),\\
\hat{\boldsymbol{\theta}}_0 &= \underset{\boldsymbol{\theta}_0}{\operatorname{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0),\\
\forall t \in [\![1, T]\!],\ \hat{\boldsymbol{\theta}}_t &= \underset{\boldsymbol{\theta}_t}{\operatorname{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_T).
\end{aligned}$$

We optimise $\mathcal{L}_0$ and $\mathcal{L}_1$ separately:

- Consider

$$\log \mathcal{L}_0 = -\pi^{TNp} - T\,N\,\log|\boldsymbol{\Sigma}_0| - T\,p\sum_{k=1}^{N}\log(\tau_k) - \sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}} \frac{q\left(\boldsymbol{\Sigma}_0, \mathbf{x}_k^{(t)}\right)}{\tau_k}\,.$$

  Let $k \in [\![1, N]\!]$, we solve:

$$\frac{\partial \log \mathcal{L}_0}{\partial \tau_k} = -Tp\sum_{k=1}^{N}\frac{1}{\tau_k} + \sum_{t=1}^{T}\frac{q\left(\boldsymbol{\Sigma}_0, \mathbf{x}_k^{(t)}\right)}{\tau_k^2} = 0\,,$$

  which leads to:

$$\hat{\tau}_k = \frac{1}{Tp}\sum_{t=1}^{T} q\left(\boldsymbol{\Sigma}_0, \mathbf{x}_k^{(t)}\right)\,. \tag{3}$$

  Now we consider the optimisation with regards to $\boldsymbol{\Sigma}_0$. Recall complex differentiation results [1]:

$$\begin{aligned}
\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\Sigma}} &= \boldsymbol{\Sigma}^{-1},\\
\frac{\partial q\left(\boldsymbol{\Sigma}, \mathbf{x}_k^{(t)}\right)}{\partial \boldsymbol{\Sigma}} &= -\mathbf{S}_k^{(t)}\,\boldsymbol{\Sigma}^{-2}\,.
\end{aligned} \tag{4}$$

  We solve:

$$\frac{\partial \log \mathcal{L}_0}{\partial \boldsymbol{\Sigma}_0} = -T\,N\,{\boldsymbol{\Sigma}_0}^{-1} + \sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}} \frac{\mathbf{S}_k^{(t)}}{\tau_k}\,\boldsymbol{\Sigma}_0^{-2} = \mathbf{0}_{p^2}\,,$$

which yields:

$$\hat{\boldsymbol{\Sigma}}_0 = \frac{1}{T\,N} \sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}} \frac{\mathbf{S}_k^{(t)}}{\tau_k} \,. \tag{5}$$

Then by plugging back the estimates of textures at eq. (3) in eq. (5), we obtain:

$$\hat{\boldsymbol{\Sigma}}_0 = \frac{p}{N} \sum_{k=1}^{N} \frac{\sum_{t=1}^{T} \mathbf{S}_k^{(t)}}{\sum_{t=1}^{T} q\left(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)}\right)} \,, \tag{6}$$

that we denote $\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}$.

- For $\mathcal{L}_1$, we consider the same procedure and optimize alternatively for each $\tau_k^{(t)}$ and $\boldsymbol{\Sigma}_t$. We have:

$$\log \mathcal{L}_1 = -\pi^{T\,N\,p} - N \sum_{t=1}^{T} \log |\boldsymbol{\Sigma}_t| - p \sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}} \log\left(\tau_k^{(t)}\right) - \sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}} \frac{q\left(\boldsymbol{\Sigma}_t, \mathbf{x}_k^{(t)}\right)}{\tau_k^{(t)}} \,.$$

Let $k \in [\![1, N]\!], t \in [\![1, T]\!]$, solving

$$\frac{\partial \log \mathcal{L}_1}{\partial \tau_k^{(t)}} = 0 \,,$$

yields:

$$\hat{\tau}_k^{(t)} = \frac{1}{p}\, q\left(\boldsymbol{\Sigma}_t, \mathbf{x}_k^{(t)}\right) \,. \tag{7}$$

Let $t \in [\![1, T]\!]$, we have to solve:

$$\frac{\partial \log \mathcal{L}_1}{\partial \boldsymbol{\Sigma}_t} = N\,{\boldsymbol{\Sigma}_t}^{-1} + \sum_{k=1}^{k=N} \frac{\mathbf{S}_k^{(t)}}{\tau_k}\, \boldsymbol{\Sigma}_t^{-2} = \mathbf{0}_{p^2}$$

which yields:

$$\hat{\boldsymbol{\Sigma}}_t = \frac{1}{N} \sum_{k=1}^{k=N} \frac{\mathbf{S}_k^{(t)}}{\tau_k^{(t)}} \,. \tag{8}$$

Then by plugging estimates of eq. (7) in (8), we obtain:

$$\hat{\boldsymbol{\Sigma}}_t = \frac{p}{N} \sum_{k=1}^{N} \frac{\mathbf{S}_k^{(t)}}{q\left(\hat{\boldsymbol{\Sigma}}_t, \mathbf{x}_k^{(t)}\right)} \,, \tag{9}$$

that we denote $\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}$.

Finally, we have to compute:

$$\hat{\Lambda} = \frac{\mathcal{L}_1\left(\hat{\boldsymbol{\theta}}_1, \dots, \hat{\boldsymbol{\theta}}_T\right)}{\mathcal{L}_0\left(\hat{\boldsymbol{\theta}}_0\right)}$$

$$= \frac{\prod\limits_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{1}{\pi^p \left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right| \left(\hat{\tau}_k^{(t)}\right)^p} \exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k^{(t)}}\right\}}{\prod\limits_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{1}{\pi^p \left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}\right| \left(\hat{\tau}_k\right)^p} \exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k}\right\}}$$

$$= \frac{\left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}\right|^{TN}}{\prod\limits_{t=1}^{T}\left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right|^N} \prod\limits_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{(\hat{\tau}_k)^p}{\left(\hat{\tau}_k^{(t)}\right)^p} \frac{\exp\left\{-\sum\limits_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k^{(t)}}\right\}}{\exp\left\{-\sum\limits_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k}\right\}}$$

Now, if we replace the texture estimates by their expression at eq. (3) and eq. (7), we have:

$$\hat{\Lambda} = \frac{\left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}\right|^{TN}}{\prod\limits_{t=1}^{T}\left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right|^N} \prod_{k=1}^{N} \frac{\left(\sum\limits_{t=1}^{T} q\left(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)}\right)\right)^{Tp}}{T^{Tp}\prod\limits_{t=1}^{T}\left(q\left(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)}\right)\right)^p} \frac{\exp\left\{-p\sum\limits_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)})}{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)})}\right\}}{\exp\left\{-Tp\sum\limits_{k=1}^{k=N} \frac{\sum\limits_{t=1}^{t=T} q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}{\sum\limits_{t=1}^{t=T} q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}\right\}}$$

$$= \frac{\left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}\right|^{TN}}{\prod\limits_{t=1}^{T}\left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right|^N} \prod_{k=1}^{N} \frac{\left(\sum\limits_{t=1}^{T} q\left(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)}\right)\right)^{Tp}}{T^{Tp}\prod\limits_{t=1}^{T}\left(q\left(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)}\right)\right)^p}.$$

Since the covariance estimates are solution to fixed-point equations, we do not replace them and have the final form of the statistic.

II. GLRT FOR MARGINAL PROBLEM 1

For the marginal scheme, we have to compute the following GLRT:

$$\hat{\Lambda} = \frac{\max\limits_{\boldsymbol{\theta}_{01}, \boldsymbol{\theta}_T} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T}; \boldsymbol{\theta}_{01}, \boldsymbol{\theta}_T\right)}{\max\limits_{\boldsymbol{\theta}_0} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T}; \boldsymbol{\theta}_0\right)} \tag{10}$$

where $\boldsymbol{\theta}_0 = \left\{\tau_1, \dots, t_N, \boldsymbol{\Sigma}_0\right\}$, $\boldsymbol{\theta}_{01} = \left\{\tau_1^{(01)}, \dots, \tau_N^{(01)}, \boldsymbol{\Sigma}_{01}\right\}$ and $\boldsymbol{\theta}_T = \left\{\tau_1^{(T)}, \dots, \tau_N^{(T)}, \boldsymbol{\Sigma}_T\right\}$.

Using the assumption that all observations are independent, we can rewrite:

$$\hat{\Lambda} = \frac{\underset{\boldsymbol{\theta}_{01}, \boldsymbol{\theta}_T}{\max} \quad \prod_{k=1}^{k=N} \left(\prod_{t=1}^{t=T-1} p_{\mathbf{x}_k^{(t)}}^{\mathbb{C}\mathcal{N}} \left(\mathbf{x}_k^{(t)}; \boldsymbol{\theta}_{01} \right) \right) p_{\mathbf{x}_k^{(T)}}^{\mathbb{C}\mathcal{N}} \left(\mathbf{x}_k^{(T)}; \boldsymbol{\theta}_T \right)}{\underset{\boldsymbol{\theta}_0}{\max} \quad \prod_{\substack{k=1 \\ t=1}}^{\substack{k=N \\ t=T}} p_{\mathbf{x}_k^{(t)}}^{\mathbb{C}\mathcal{N}} \left(\mathbf{x}_k^{(t)}; \boldsymbol{\theta}_0 \right)}.$$

Just as for the omnibus problem in I, we optimise the numerator and denominator separately by plugging estimates in the likelihood functions:

$$\hat{\Lambda} = \frac{\mathcal{L}_1 \left(\hat{\boldsymbol{\theta}}_{01}, \hat{\boldsymbol{\theta}}_T \right)}{\mathcal{L}_0 \left(\hat{\boldsymbol{\theta}}_0 \right)}, \tag{11}$$

where

$$\begin{aligned}
\mathcal{L}_1(\boldsymbol{\theta}_{01}, \boldsymbol{\theta}_T) &= \prod_{k=1}^{k=N} \left(\prod_{t=1}^{t=T-1} p_{\mathbf{x}_k^{(t)}}^{\mathbb{C}\mathcal{N}} \left(\mathbf{x}_k^{(t)}; \boldsymbol{\theta}_{01} \right) \right) p_{\mathbf{x}_k^{(T)}}^{\mathbb{C}\mathcal{N}} \left(\mathbf{x}_k^{(T)}; \boldsymbol{\theta}_T \right), \\
\mathcal{L}_0(\boldsymbol{\theta}_0) &= \prod_{\substack{k=1 \\ t=1}}^{\substack{k=N \\ t=T}} p_{\mathbf{x}_k^{(t)}}^{\mathbb{C}\mathcal{N}} \left(\mathbf{x}_k^{(t)}; \boldsymbol{\theta}_0 \right), \\
\hat{\boldsymbol{\theta}}_0 &= \underset{\boldsymbol{\theta}_0}{\text{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0), \\
\hat{\boldsymbol{\theta}}_{01} &= \underset{\boldsymbol{\theta}_{01}}{\text{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_{01}, \boldsymbol{\theta}_T), \\
\hat{\boldsymbol{\theta}}_T &= \underset{\boldsymbol{\theta}_T}{\text{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_{01}, \boldsymbol{\theta}_T).
\end{aligned}$$

We consider optimising $\mathcal{L}_0$ and $\mathcal{L}_1$ separately:

- For $\mathcal{L}_0$, the problem is exactly the same as for the omnibus scheme presented in I which can be found previously.
- For $\mathcal{L}_1$, we optimize alternatively for $\tau_k^{(T)}$, $\tau_k^{(01)}$, $\boldsymbol{\Sigma}_T$ and $\boldsymbol{\Sigma}_{01}$. We have:

$$\begin{aligned}
\log \mathcal{L}_1 = &- \pi^{T\,N\,p} - (T-1)N \log |\boldsymbol{\Sigma}_{01}| - N \log |\boldsymbol{\Sigma}_T| - (T-1)p \sum_{\substack{k=1 \\ t=1}}^{\substack{k=N \\ t=T-1}} \log \left(\tau_k^{(t)} \right) \\
&- p \log \left(\tau_k^{(T)} \right) - \sum_{\substack{t=1 \\ k=1}}^{\substack{t=T-1 \\ k=N}} \frac{q\left(\boldsymbol{\Sigma}_{01}, \mathbf{x}_k^{(01)} \right)}{\tau_k^{(01)}} - \sum_{k=1}^{k=N} \frac{q\left(\boldsymbol{\Sigma}_T, \mathbf{x}_k^{(t)} \right)}{\tau_k^{(T)}}.
\end{aligned}$$

Using the same optimisation procedure as omnibus scheme (taking the derivative and equalling it to 0), we obtain:

$$
\begin{aligned}
\hat{\tau}_k^{(T)} &= \frac{1}{p}\, q\left(\mathbf{\Sigma}_t, \mathbf{x}_k^{(T)}\right),\\
\hat{\tau}_k^{(01)} &= \frac{1}{(T-1)p} \sum_{t=1}^{t=T-1} q\left(\mathbf{\Sigma}_{01}, \mathbf{x}_k^{(t)}\right),\\
\hat{\mathbf{\Sigma}}_T &= \frac{1}{N}\sum_{k=1}^{k=N} \frac{\mathbf{S}_k^{(T)}}{\tau_k^{(T)}},\\
\hat{\mathbf{\Sigma}}_{01} &= \frac{1}{(T-1)N} \sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T-1}} \frac{\mathbf{S}_k^{(t)}}{\tau_k^{(01)}}, \text{ that we denote } \hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}.
\end{aligned}
\tag{12}
$$

Here we remark that the estimate of $\hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}$ is basically the same as $\hat{\mathbf{\Sigma}}_{0}^{\mathrm{MT}}$ at eq. (5) with $T-1$ dates. $\hat{\mathbf{\Sigma}}_T$ can be given by eq. (8) as well.

Finally, we have:

$$
\hat{\Lambda} = \frac{\displaystyle\prod_{k=1}^{k=N}\left(\prod_{t=1}^{t=T-1} \frac{1}{\pi^p \left|\hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}\right| \left(\hat{\tau}_k^{(01)}\right)^p \exp\left\{-\frac{q(\hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k^{(01)}}\right\}}\right) \frac{1}{\pi^p \left|\hat{\mathbf{\Sigma}}_T^{\mathrm{TE}}\right| \left(\hat{\tau}_k^{(T)}\right)^p} \exp\left\{-\frac{q(\hat{\mathbf{\Sigma}}_T^{\mathrm{TE}}, \mathbf{x}_k^{(T)})}{\hat{\tau}_k^{(T)}}\right\}}{\displaystyle\prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{1}{\pi^p \left|\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}\right| (\hat{\tau}_k)^p} \exp\left\{-\frac{q(\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k}\right\}}
$$

$$
= \frac{\left|\hat{\mathbf{\Sigma}}_T^{\mathrm{MT}}\right|^{TN}}{\left|\hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}\right|^{(T-1)N} \left|\hat{\mathbf{\Sigma}}_T^{\mathrm{TE}}\right|^{N}} \prod_{k=1}^{k=N} \frac{\left(\hat{\tau}_k^{(0)}\right)^{Tp}}{\left(\hat{\tau}_k^{(01)}\right)^{(T-1)p} \left(\hat{\tau}_k^{(T)}\right)^{p}} \frac{\exp\left\{-\displaystyle\sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T-1}} \frac{q(\hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k^{(01)}} - \sum_{k=1}^{N} \frac{q(\hat{\mathbf{\Sigma}}_T^{\mathrm{TE}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k^{(t)}}\right\}}{\exp\left\{-\displaystyle\sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{q(\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k}\right\}}
$$

Now, if we replace the texture estimates by their expression at eq. (3) and eq. (21), we have:

$$
\hat{\Lambda} = \frac{\left|\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}\right|^{TN}}{\left|\hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}\right|^{(T-1)N} \left|\hat{\mathbf{\Sigma}}_T^{\mathrm{TE}}\right|^{N}} \frac{((T-1)p)^{(T-1)Np} p^{Np}}{(Tp)^{TNp}} \prod_{k=1}^{N} \frac{\left(\displaystyle\sum_{t=1}^{T} q\left(\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}, \mathbf{x}_k^{(t)}\right)\right)^{Tp}}{\left(\displaystyle\sum_{t=1}^{T-1} q\left(\hat{\mathbf{\Sigma}}_{01}^{\mathrm{MT}}, \mathbf{x}_k^{(t)}\right)\right)^{(T-1)p} \left(q\left(\hat{\mathbf{\Sigma}}_{\mathrm{TE}}^{T}, \mathbf{x}_k^{(T)}\right)\right)^{p}}.
$$

Since the covariance estimate are solution to fixed-point equations, we do not replace them and have the final form of the statistic.

III. GLRT FOR OMNIBUS PROBLEM 2

In this problem, we test a change in the covariance shape only. Thus, the GLRT for this problem has the following form:

$$\hat{\Lambda} = \frac{\underset{\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T}{\max} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T\right)}{\underset{\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T}{\max} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T\right)} \tag{13}$$

where $\boldsymbol{\theta}_0 = \{\boldsymbol{\Sigma}_0\}$, $\forall t \in [\![1,T]\!]$, $\boldsymbol{\theta}_t = \{\boldsymbol{\Sigma}_t\}$ and $\forall t \in [\![1,T]\!]$, $\boldsymbol{\Phi}_t = \left\{\tau_1^{(t)},\dots,\tau_N^{(t)}\right\}$.

Using the assumption that all observations are independent, we can rewrite:

$$\hat{\Lambda} = \frac{\underset{\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T}{\max} \quad \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_t,\boldsymbol{\Phi}_t\right)}{\underset{\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T}{\max} \quad \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_0,\boldsymbol{\Phi}_T\right)}.$$

This expression can be computed by optimising the numerator and denominator separately just as done in the previous derivations at sections I and II and compute:

$$\hat{\Lambda} = \frac{\mathcal{L}_1\left(\hat{\boldsymbol{\theta}}_1,\dots,\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\Phi}}_1^1,\dots,\hat{\boldsymbol{\Phi}}_T^1\right)}{\mathcal{L}_0\left(\hat{\boldsymbol{\theta}}_0,\hat{\boldsymbol{\Phi}}_1^0,\dots,\hat{\boldsymbol{\Phi}}_T^0\right)}, \tag{14}$$

where

$$\begin{aligned}
\mathcal{L}_1(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T) &= \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_t,\boldsymbol{\Phi}_t\right), \\
\mathcal{L}_0(\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T) &= \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_0,\boldsymbol{\Phi}_t\right), \\
\hat{\boldsymbol{\theta}}_0 &= \underset{\boldsymbol{\theta}_0}{\operatorname{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T), \\
\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\Phi}}_t^0 &= \underset{\boldsymbol{\Phi}_t}{\operatorname{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T), \\
\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\theta}}_t &= \underset{\boldsymbol{\theta}_t}{\operatorname{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T), \\
\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\Phi}}_t^1 &= \underset{\boldsymbol{\Phi}_t}{\operatorname{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\dots,\boldsymbol{\Phi}_T).
\end{aligned}$$

Here, the optimisation towards $\boldsymbol{\theta}_t$ and $\boldsymbol{\Phi}_t^1$ is exactly the same as done in section I where the parameters $\boldsymbol{\Phi}_t$ were compromised in the $\boldsymbol{\theta}_t$. Thus we will omit them here and only remind the results:

$$\forall t \in [\![1,T]\!],\ \hat{\tau}_k^{(t)} = \hat{\tau}_{1k}^{(t)} = \frac{1}{p}\, q\left(\boldsymbol{\Sigma}_t, \mathbf{x}_k^{(t)}\right),$$
$$\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\Sigma}}_t = \frac{p}{N}\sum_{k=1}^{N}\frac{\mathbf{S}_k^{(t)}}{q\left(\hat{\boldsymbol{\Sigma}}_t, \mathbf{x}_k^{(t)}\right)}.$$

Concerning the others estimation problems, we have:

$$\log \mathcal{L}_0 = -\pi^{T\,N\,p} - TN\log|\boldsymbol{\Sigma}_0| - p\sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}}\log\left(\tau_k^{(t)}\right) - \sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}}\frac{q\left(\boldsymbol{\Sigma}_0, \mathbf{x}_k^{(t)}\right)}{\tau_k^{(t)}}\,.$$

The optimisation towards each $\tau_k^{(t)}$ leads to:

$$\forall k \in [\![1,N]\!],\ \forall t \in [\![1,T]\!],\ \hat{\tau}_k^{(t)} = \hat{\tau}_{0k}^{(t)} = \frac{1}{p}\, q\left(\boldsymbol{\Sigma}_t, \mathbf{x}_k^{(t)}\right) \tag{15}$$

The optimisation towards $\boldsymbol{\Sigma}_0$ was solved using the same procedure that led to eq. (5) gives:

$$\hat{\boldsymbol{\Sigma}}_0 = \frac{1}{T\,N}\sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}}\frac{\mathbf{S}_k^{(t)}}{\tau_{0k}^{(t)}}\,. \tag{16}$$

And by plugging back eq. (15) in eq. (16), we obtain:

$$\hat{\boldsymbol{\Sigma}}_0 = \frac{p}{TN}\sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}}\frac{\mathbf{S}_k^{(t)}}{q\left(\hat{\boldsymbol{\Sigma}}_t, \mathbf{x}_k^{(t)}\right)}, \tag{17}$$

that we denote $\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}$.

Finally, we have to compute:

$$\begin{aligned}
\hat{\Lambda} &= \frac{\mathcal{L}_1\left(\hat{\boldsymbol{\theta}}_1,\dots,\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\Phi}}_1^1,\dots,\hat{\boldsymbol{\Phi}}_T^1\right)}{\mathcal{L}_0\left(\hat{\boldsymbol{\theta}}_0,\hat{\boldsymbol{\Phi}}_1^0,\dots,\hat{\boldsymbol{\Phi}}_T^0\right)},\\
&= \frac{\displaystyle\prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}}\frac{1}{\pi^p\left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right|\left(\hat{\tau}_{1k}^{(t)}\right)^p}\exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}},\mathbf{x}_k^{(t)})}{\hat{\tau}_{1k}^{(t)}}\right\}}{\displaystyle\prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}}\frac{1}{\pi^p\left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}\right|\left(\hat{\tau}_{0k}^{(t)}\right)^p}\exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}},\mathbf{x}_k^{(t)})}{\hat{\tau}_{0k}^{(t)}}\right\}}\\
&= \frac{\left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}\right|^{TN}}{\displaystyle\prod_{t=1}^{T}\left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right|^N}\prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}}\frac{\left(\hat{\tau}_{0k}^{(t)}\right)^p}{\left(\hat{\tau}_{1k}^{(t)}\right)^p}\,\frac{\exp\left\{-\displaystyle\sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}}\frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}},\mathbf{x}_k^{(t)})}{\hat{\tau}_{1k}^{(t)}}\right\}}{\exp\left\{-\displaystyle\sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}}\frac{q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}},\mathbf{x}_k^{(t)})}{\hat{\tau}_{0k}^{(t)}}\right\}}
\end{aligned}$$

When replacing the texture estimates by their expression, we obtain:

$$\hat{\Lambda} = \frac{\left|\hat{\mathbf{\Sigma}}_0^{\mathrm{Mat}}\right|^{TN}}{\prod_{t=1}^{T}\left|\hat{\mathbf{\Sigma}}_t^{\mathrm{TE}}\right|^{N}} \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{\left(q\left(\hat{\mathbf{\Sigma}}_0^{\mathrm{Mat}}, \mathbf{x}_k^{(t)}\right)\right)^p}{\left(q\left(\hat{\mathbf{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)}\right)\right)^p}. \tag{18}$$

IV. GLRT for marginal problem 2

For the marginal scheme, we have to compute the following GLRT:

$$\hat{\Lambda} = \frac{\max\limits_{\boldsymbol{\theta}_{01},\boldsymbol{\theta}_T,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_{01},\boldsymbol{\theta}_T,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T\right)}{\max\limits_{\boldsymbol{\theta}_0,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_0,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T\right)} \tag{19}$$

where $\boldsymbol{\theta}_0 = \{\mathbf{\Sigma}_0\}$, $\boldsymbol{\theta}_{01} = \{\mathbf{\Sigma}_{01}\}$, $\boldsymbol{\theta}_T = \{\mathbf{\Sigma}_T\}$ and $\forall t \in [\![1,T]\!]$, $\mathbf{\Phi}_t = \left\{\tau_1^{(t)},\dots,\tau_N^{(t)}\right\}$.

Using the assumption that all observations are independent, we can rewrite:

$$\hat{\Lambda} = \frac{\max\limits_{\boldsymbol{\theta}_{01},\boldsymbol{\theta}_T,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T} \quad \prod_{k=1}^{k=N}\left(\prod_{t=1}^{t=T-1} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_{01},\mathbf{\Phi}_t\right)\right) p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(T)}}\left(\mathbf{x}_k^{(T)};\boldsymbol{\theta}_T,\mathbf{\Phi}_t\right)}{\max\limits_{\boldsymbol{\theta}_0,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T} \quad \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_0,\mathbf{\Phi}_t\right)}.$$

This expression can be computed by optimising the numerator and denominator separately just as done in the previous derivations at sections I and II and compute:

$$\hat{\Lambda} = \frac{\mathcal{L}_1\left(\hat{\boldsymbol{\theta}}_{01},\hat{\boldsymbol{\theta}}_T,\hat{\mathbf{\Phi}}_1^1,\dots,\hat{\mathbf{\Phi}}_T^1\right)}{\mathcal{L}_0\left(\hat{\boldsymbol{\theta}}_0,\hat{\mathbf{\Phi}}_1^0,\dots,\hat{\mathbf{\Phi}}_T^0\right)}, \tag{20}$$

where

$$\begin{aligned}
\mathcal{L}_1(\boldsymbol{\theta}_{01},\boldsymbol{\theta}_T,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T) &= \prod_{k=1}^{k=N}\left(\prod_{t=1}^{t=T-1} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_{01},\mathbf{\Phi}_t\right)\right) p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(T)}}\left(\mathbf{x}_k^{(T)};\boldsymbol{\theta}_T,\mathbf{\Phi}_t\right), \\
\mathcal{L}_0(\boldsymbol{\theta}_0,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T) &= \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{C}\mathcal{N}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_0,\mathbf{\Phi}_t\right), \\
\hat{\boldsymbol{\theta}}_0 &= \underset{\boldsymbol{\theta}_0}{\mathrm{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T), \\
\forall t \in [\![1,T]\!],\ \hat{\mathbf{\Phi}}_t^0 &= \underset{\mathbf{\Phi}_t}{\mathrm{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T), \\
\hat{\boldsymbol{\theta}}_{01} &= \underset{\boldsymbol{\theta}_{01}}{\mathrm{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_{01},\boldsymbol{\theta}_T,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T), \\
\hat{\boldsymbol{\theta}}_T &= \underset{\boldsymbol{\theta}_T}{\mathrm{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_{01},\boldsymbol{\theta}_T,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T), \\
\forall t \in [\![1,T]\!],\ \hat{\mathbf{\Phi}}_t^1 &= \underset{\mathbf{\Phi}_t}{\mathrm{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_T,\mathbf{\Phi}_1,\dots,\mathbf{\Phi}_T).
\end{aligned}$$

Concerning the derivation of $\hat{\boldsymbol{\theta}}_0$ and $\hat{\boldsymbol{\Phi}}_t^0$, it has already been done in section III. We will denote the estimates of texture parameters $\hat{\tau}_k^{(t)}$ as $\hat{\tau}_k^{(t),0}$.

We consider here the case for $\log \mathcal{L}_1$:

$$\begin{aligned}\log \mathcal{L}_1 = &- \pi^{T\,N\,p} - (T-1)N \log|\boldsymbol{\Sigma}_{01}| - N\log|\boldsymbol{\Sigma}_T| - p \sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \log\left(\tau_k^{(t)}\right) \\ &- \sum_{\substack{t=1\\k=1}}^{\substack{t=T-1\\k=N}} \frac{q\left(\boldsymbol{\Sigma}_{01}, \mathbf{x}_k^{(01)}\right)}{\tau_k^{(t)}} - \sum_{k=1}^{k=N} \frac{q\left(\boldsymbol{\Sigma}_T, \mathbf{x}_k^{(t)}\right)}{\tau_k^{(T)}}\,.\end{aligned}$$

Optimising using the same methodologies as before, leads to:

$$\begin{aligned}\forall t \in [\![1, T-1]\!],\ \hat{\tau}_k^{(t)} &= \frac{1}{p}\, q\left(\boldsymbol{\Sigma}_{01}, \mathbf{x}_k^{(t)}\right), \text{ that we donote } \hat{\tau}_k^{(t),01}, \\ \hat{\tau}_k^{(T)} &= \frac{1}{p}\, q\left(\boldsymbol{\Sigma}_{01}, \mathbf{x}_k^{(T)}\right), \text{ that we donote } \hat{\tau}_k^{(T),01}, \\ \hat{\boldsymbol{\Sigma}}_T &= \frac{1}{N}\sum_{k=1}^{k=N} \frac{\mathbf{S}_k^{(T)}}{\tau_k^{(T)}}\,, \\ \hat{\boldsymbol{\Sigma}}_{01} &= \frac{1}{(T-1)N} \sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T-1}} \frac{\mathbf{S}_k^{(t)}}{\tau_k^{(t)}}\,, \text{ that we denote } \hat{\boldsymbol{\Sigma}}_{01}^{\mathrm{Mat}}.\end{aligned} \tag{21}$$

Here we remark that the estimate of $\hat{\boldsymbol{\Sigma}}_{01}^{\mathrm{Mat}}$ is basically the same as $\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}$ at eq. (16) with $T-1$ dates. $\hat{\boldsymbol{\Sigma}}_T$ can be given by eq. (8) as well.

Finally, we have:

$$\hat{\Lambda} = \frac{\displaystyle\prod_{k=1}^{k=N}\left(\prod_{t=1}^{t=T-1} \frac{1}{\pi^p \left|\hat{\boldsymbol{\Sigma}}_{01}^{\mathrm{Mat}}\right| \left(\hat{\tau}_k^{(t),01}\right)^p \exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_{01}^{\mathrm{Mat}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_k^{(t),01}}\right\}}\right) \frac{1}{\pi^p \left|\hat{\boldsymbol{\Sigma}}_T^{\mathrm{TE}}\right| \left(\hat{\tau}_k^{(T),01}\right)^p} \exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_T^{\mathrm{TE}}, \mathbf{x}_k^{(T)})}{\hat{\tau}_k^{(T),01}}\right\}}{\displaystyle\prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{1}{\pi^p \left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}\right| \left(\hat{\tau}_k^{(t),0}\right)^p} \exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}, \mathbf{x}_k^{(t),0})}{\hat{\tau}_k^{(t),0}}\right\}}$$

Now, if we replace the texture estimates by their expression at eq. (3) and eq. (21), we have:

$$\hat{\Lambda} = \frac{\left|\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}\right|^{TN}}{\left|\hat{\boldsymbol{\Sigma}}_{01}^{\mathrm{Mat}}\right|^{(T-1)N} \left|\hat{\boldsymbol{\Sigma}}_T^{\mathrm{TE}}\right|^N} \prod_{k=1}^{N} \frac{\displaystyle\prod_{t=1}^{T} q\left(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}, \mathbf{x}_k^{(t)}\right)^p}{\left(\displaystyle\prod_{t=1}^{T-1} q\left(\hat{\boldsymbol{\Sigma}}_{01}^{\mathrm{Mat}}, \mathbf{x}_k^{(t)}\right)^p\right)\left(q\left(\hat{\boldsymbol{\Sigma}}_{\mathrm{TE}}^{T}, \mathbf{x}_k^{(T)}\right)\right)^p}\,.$$

V. GLRT FOR OMNIBUS PROBLEM 3

In this problem, we test a change in the texture parameters only. Thus, the GLRT for this problem has the following form:

$$\hat{\Lambda} = \frac{\underset{\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T}{\max} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T\right)}{\underset{\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T}{\max} \quad p_{\boldsymbol{\mathcal{W}}_{1,T}}\left(\boldsymbol{\mathcal{W}}_{1,T};\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T\right)} \tag{22}$$

where $\boldsymbol{\theta}_0 = \{\tau_1,\ldots,\tau_N\}$, $\forall t \in [\![1,T]\!]$, $\boldsymbol{\theta}_t = \left\{\tau_1^{(t)},\ldots,\tau_N^{(t)}\right\}$ and $\forall t \in [\![1,T]\!]$, $\boldsymbol{\Phi}_t = \{\boldsymbol{\Sigma}_t\}$.

Using the assumption that all observations are independent, we can rewrite:

$$\hat{\Lambda} = \frac{\underset{\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T}{\max} \quad \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{CN}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_t,\boldsymbol{\Phi}_t\right)}{\underset{\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T}{\max} \quad \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{CN}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_0,\boldsymbol{\Phi}_T\right)}.$$

This expression can be computed by optimising the numerator and denominator separately just as done in the previous derivations at sections I and II and compute:

$$\hat{\Lambda} = \frac{\mathcal{L}_1\left(\hat{\boldsymbol{\theta}}_1,\ldots,\hat{\boldsymbol{\theta}}_T,\hat{\boldsymbol{\Phi}}_1^1,\ldots,\hat{\boldsymbol{\Phi}}_T^1\right)}{\mathcal{L}_0\left(\hat{\boldsymbol{\theta}}_0,\hat{\boldsymbol{\Phi}}_1^0,\ldots,\hat{\boldsymbol{\Phi}}_T^0\right)}, \tag{23}$$

where

$$\begin{aligned}
\mathcal{L}_1(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T) &= \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{CN}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_t,\boldsymbol{\Phi}_t\right),\\
\mathcal{L}_0(\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T) &= \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} p^{\mathbb{CN}}_{\mathbf{x}_k^{(t)}}\left(\mathbf{x}_k^{(t)};\boldsymbol{\theta}_0,\boldsymbol{\Phi}_t\right),\\
\hat{\boldsymbol{\theta}}_0 &= \underset{\boldsymbol{\theta}_0}{\operatorname{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T),\\
\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\Phi}}_t^0 &= \underset{\boldsymbol{\Phi}_t}{\operatorname{argmax}} \quad \mathcal{L}_0(\boldsymbol{\theta}_0,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T),\\
\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\theta}}_t &= \underset{\boldsymbol{\theta}_t}{\operatorname{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T),\\
\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\Phi}}_t^1 &= \underset{\boldsymbol{\Phi}_t}{\operatorname{argmax}} \quad \mathcal{L}_1(\boldsymbol{\theta}_1,\ldots,\boldsymbol{\theta}_T,\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_T).
\end{aligned}$$

Here, the optimisation towards $\boldsymbol{\theta}_t$ and $\boldsymbol{\Phi}_t^1$ is exactly the same as done in section I where the parameters $\boldsymbol{\Phi}_t$ were compromised in the $\boldsymbol{\theta}_t$. Thus we will omit them here and only remind the results:

$$\forall t \in [\![1,T]\!],\ \hat{\tau}_k^{(t)} = \hat{\tau}_{1k}^{(t)} = \frac{1}{p}\, q\left(\boldsymbol{\Sigma}_t, \mathbf{x}_k^{(t)}\right),$$

$$\forall t \in [\![1,T]\!],\ \hat{\boldsymbol{\Sigma}}_t = \frac{p}{N}\sum_{k=1}^{N} \frac{\mathbf{S}_k^{(t)}}{q\left(\hat{\boldsymbol{\Sigma}}_t, \mathbf{x}_k^{(t)}\right)}.$$

Concerning the others estimation problems, we have:

$$\log \mathcal{L}_0 = -\pi^{T\,N\,p} - N\sum_{t=1}^{t=T} \log|\boldsymbol{\Sigma}_t| - p\sum_{k=1}^{k=N}\log(\tau_k) - \sum_{\substack{t=1\\k=1}}^{\substack{t=T\\k=N}} \frac{q\left(\boldsymbol{\Sigma}_t, \mathbf{x}_k^{(t)}\right)}{\tau_k}.$$

The optimisation towards each $\tau_k^{(t)}$ leads to:

$$\forall k \in [\![1,N]\!],\ \hat{\tau}_k = \hat{\tau}_{0k} = \frac{1}{pT}\sum_{t=1}^{t=T} q\left(\boldsymbol{\Sigma}_t, \mathbf{x}_k^{(t)}\right). \tag{24}$$

The optimisation towards each $\boldsymbol{\Sigma}_t$ gives:

$$\hat{\boldsymbol{\Sigma}}_0 = \frac{1}{N}\sum_{k=1}^{k=N} \frac{\mathbf{S}_k^{(t)}}{\tau_{0k}}. \tag{25}$$

And by plugging back eq. (15) in eq. (16), we obtain:

$$\hat{\boldsymbol{\Sigma}}_t = \frac{Tp}{N}\sum_{t=1}^{t=T} \frac{\mathbf{S}_k^{(t)}}{\displaystyle\sum_{t'=1}^{t'=T} q\left(\hat{\boldsymbol{\Sigma}}_{t'}, \mathbf{x}_k^{(t')}\right)}, \tag{26}$$

that we denote $\hat{\boldsymbol{\Sigma}}_t^{\mathrm{Tex}}$.

Finally, we have to compute:

$$\begin{aligned}
\hat{\Lambda} &= \frac{\mathcal{L}_1\left(\hat{\boldsymbol{\theta}}_1, \dots, \hat{\boldsymbol{\theta}}_T, \hat{\boldsymbol{\Phi}}_1^1, \dots, \hat{\boldsymbol{\Phi}}_T^1\right)}{\mathcal{L}_0\left(\hat{\boldsymbol{\theta}}_0, \hat{\boldsymbol{\Phi}}_1^0, \dots, \hat{\boldsymbol{\Phi}}_T^0\right)},\\
&= \frac{\displaystyle\prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{1}{\pi^p \left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right| \left(\hat{\tau}_{1k}^{(t)}\right)^p} \exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_{1k}^{(t)}}\right\}}{\displaystyle\prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{1}{\pi^p \left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{Tex}}\right| \left(\hat{\tau}_{0k}^{(t)}\right)^p} \exp\left\{-\frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{Tex}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_{0k}^{(t)}}\right\}}\\
&= \prod_{t=1}^{T} \frac{\left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{Tex}}\right|^N}{\left|\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}\right|^N} \prod_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{\left(\hat{\tau}_{0k}^{(t)}\right)^p}{\left(\hat{\tau}_{1k}^{(t)}\right)^p} \frac{\exp\left\{-\displaystyle\sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{q(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{TE}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_{1k}^{(t)}}\right\}}{\exp\left\{-\displaystyle\sum_{\substack{k=1\\t=1}}^{\substack{k=N\\t=T}} \frac{q(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{Mat}}, \mathbf{x}_k^{(t)})}{\hat{\tau}_{0k}^{(t)}}\right\}}
\end{aligned}$$

When replacing the texture estimates by their expression, we obtain:

$$\hat{\Lambda}_{\text{Tex}} = \prod_{t=1}^{T} \frac{\left|\hat{\mathbf{\Sigma}}_t^{\text{Tex}}\right|^N}{\left|\hat{\mathbf{\Sigma}}_t^{\text{TE}}\right|^N} \prod_{k=1}^{N} \frac{\left(\sum_{t=1}^{T} q\left(\hat{\mathbf{\Sigma}}_t^{\text{Tex}}, \mathbf{x}_k^{(t)}\right)\right)^{Tp}}{T^{Tp} \prod_{t=1}^{T} \left(q\left(\hat{\mathbf{\Sigma}}_t^{\text{TE}}, \mathbf{x}_k^{(t)}\right)\right)^p} \underset{\text{H}_0}{\overset{\text{H}_1}{\gtrless}} \lambda\,, \tag{27}$$

VI. Convergence of $\hat{\mathbf{\Sigma}}_0^{\text{MT}}$

This section corresponds is aimed at detailing some considerations about the convergence of a fixed-point algorithm to obtain the estimate $\hat{\mathbf{\Sigma}}_0^{\text{MT}}$. We will hsow that, under some regularity conditions, the fixed-point algorithm will converge to a unique point up to a scale factor.

We have shown until now (in the paper) that $\hat{\mathbf{\Sigma}}_0^{\text{MT}}$ is a global maximum of the log-likelihood function and is unique. Thus it is the estimate that we were looking for. We additionally show here that the matrix solution to the fixed-point equation converges to an unique matrix up to a scale factor.

It still remains at this point to show that the fixed-point algorithm converges. In fact, $\mathbf{\Sigma}_0^{\text{MT}}$ is an alternate case of the well-known Tyler's fixed-point estimator whose convergence properties are well studied. For the Tyler fixed-point estimator, the convergence is ensured under some regularity conditions provided in [2]. The adaptation of this proof for the new estimate $\mathbf{\Sigma}_0^{\text{MT}}$ is straightforward but rather long. It is omitted in the paper since it is not novel.

The following theorem, summarises the regularity conditions to ensure convergence:

Theorem VI.1.
Let $\{\mathbf{x}_k^{(t)} | k \in [\![1, N]\!],\, t \in [\![1, T]\!]\}$ *be a set of observations.*
Let us define vectors $\mathbf{v}_i \in \mathbb{R}^p$ *such that* $\forall k, \forall t,\ \mathbf{v}_{(T-1)*N+k} = \left(\Re(\mathbf{x}_k^{(t)})^{\text{T}}, \Im(\mathbf{x}_k^{(t)})^{\text{T}}\right)^{\text{T}}$ *and* $\mathbf{v}_{(2T-1)*N+k} = \left(-\Im(\mathbf{x}_k^{(t)})^{\text{T}}, \Re(\mathbf{x}_k^{(t)})^{\text{T}}\right)^{\text{T}}$*, where* $\Re(\bullet)$ *and* $\Im(\bullet)$ *denotes the real and imaginary parts.*
Let $\mathbb{P}_{2TN}(\bullet)$ *be the empirical distribution of samples* $\{\mathbf{v}_i | i \in [\![1, 2TN]\!]\}$*. Then the fixed-point algorithm* $\left(\mathbf{\Sigma}_0^{\text{MT}}\right)_{k+1} = f_{N,T}^{\text{MT}}\left(\left(\mathbf{\Sigma}_0^{\text{MT}}\right)_{k+1}\right)$ *converges to a unique solution up to a scale factor if and only if the following condition is respected:*
(C1) $\mathbb{P}_{2TN}(\{\mathbf{0}\}) = 0$ *and for all linear subspaces* $V \subset \mathbb{R}^{2p}$*, we have* $\mathbb{P}_{2TN}(V) < \dim(V)/2p$*.*

Proof. The proof is done in three steps: we first go from the complex dataset to an equivalent real one. Then, we prove the sufficient implication and then the necessary one.

1) Let us define the mapping from $\mathbb{C}^{p\times p}$ to $\mathbb{R}^{2p\times 2p}$ as a function denoted$f_{\mathbb{CR}} : \mathbb{S}_{\mathbb{H}}^p \to \mathbb{S}_{++}^{2p}$, whose definition is:

$$f_{\mathbb{CR}}\left(\mathbf{\Sigma}\right) = \frac{1}{2}\begin{bmatrix} \Re(\mathbf{\Sigma}) & -\Im(\mathbf{\Sigma}) \\ \Im(\mathbf{\Sigma}) & \Re(\mathbf{\Sigma}) \end{bmatrix}. \tag{28}$$

Given the observations $\{\mathbf{x}_k^{(t)} | k \in [\![1, N]\!],\, t \in [\![1, T]\!]\}$, we define $\mathbf{v}_i \in \mathbb{R}^p$ such that $\forall k, \forall t,\ \mathbf{v}_{(T-1)*N+k} = \left(\Re(\mathbf{x}_k^{(t)})^{\text{T}}, \Im(\mathbf{x}_k^{(t)})^{\text{T}}\right)^{\text{T}}$ and $\mathbf{v}_{(2T-1)*N+k} = \left(-\Im(\mathbf{x}_k^{(t)})^{\text{T}}, \Re(\mathbf{x}_k^{(t)})^{\text{T}}\right)^{\text{T}}$.

Using identities of Theorem 1 in [2], we can show that $f_{\mathbb{CR}}\left(\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}\right)$ is solution to the following fixed-point equation:

$$\mathbf{\Sigma} = \frac{p}{2N}\sum_{k=1}^{2N}\frac{\sum_{t=1}^{T}\mathbf{v}_i^{(t)}\mathbf{v}_i^{(t)\mathrm{T}}}{\sum_{t=1}^{T}\mathbf{v}_i^{(t)\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{v}_i^{(t)}}\,. \tag{29}$$

Since there is an equivalence between $\mathbb{R}^{2p}$ and $\mathbb{C}^p$, we can consider the convergence of $f_{\mathbb{CR}}\left(\hat{\mathbf{\Sigma}}_0^{\mathrm{MT}}\right)$ and the result will apply to the complex case.

2) Now we prove that if the condition **C1** is respected, the fixed-point algorithm to compute the solution of eq. (29) converges. Since the transformation $\mathbf{x} \to \mathbf{M}\mathbf{x}$, for any non-singular matrix $\mathbf{M}$, leads to the transformation $\mathbf{\Sigma} \to \mathbf{M}\mathbf{\Sigma}\mathbf{M}^{\mathrm{T}}$ in eq. (29), we can assume $\mathbf{\Sigma} = \mathbf{I}_{2p}$ without loss of generality. We will show the convergence in three steps:

Step 1. Let $\lambda_1(\mathbf{\Sigma}_k), \ldots, \lambda_{2p}(\mathbf{\Sigma}_k)$, be the ordered eigenvalues of $\mathbf{\Sigma}_k$ the matrix at iteration k of the algorithm. We remark that for any $\mathbf{x} \in \mathbb{R}^{2p}$, $\mathbf{x}^{\mathrm{T}}\mathbf{\Sigma}_k^{-1}\mathbf{x} \geq \lambda_1(\mathbf{\Sigma}_k)^{-1}\mathbf{x}^{\mathrm{T}}\mathbf{x}$ so we can write the following inequality:

$$\mathbf{\Sigma}_{k+1} \leq \lambda_1(\mathbf{\Sigma}_k)(2N)^{-1}p\sum_{i=1}^{2N}\frac{\sum_{t=1}^{T}\mathbf{v}_i^{(t)}\mathbf{v}_i^{(t)\mathrm{T}}}{\sum_{t=1}^{T}\mathbf{v}_i^{(t)\mathrm{T}}\mathbf{v}_i^{(t)}} = \lambda_1(\mathbf{\Sigma}_k)\mathbf{I}_p,$$

where the ordering refers to the partial ordering of symmetric matrices. Similarly, we can write $\mathbf{\Sigma}_{k+1} \geq \lambda_{2p}(\mathbf{\Sigma}_k)\mathbf{I}_{2p}$. These two inequalities imply that

$$\lambda_1(\mathbf{\Sigma}_{k+1}) \leq \lambda_1(\mathbf{\Sigma}_k)\,\text{and}\,\lambda_{2p}(\mathbf{\Sigma}_{k+1}) \geq \lambda_{2p}(\mathbf{\Sigma}_k). \tag{30}$$

Step 2. Let $\mathbf{P}_k$ be the $2p \times 2p$ symmetric idempotent matrix which projects orthogonally into the eigenspace $\mathrm{E}_k = \{\mathbf{x} \in \mathbb{R}^{2p} | \mathbf{\Sigma}_k\mathbf{x} = \lambda_1(\mathbf{\Sigma}_k)\mathbf{x}\}$. We will show here that if $\lambda_1(\mathbf{\Sigma}_{k+1}) = \lambda_1(\mathbf{\Sigma}_k)$ then

$$\mathrm{E}_{k+1} \subset \mathrm{E}_k,\ \text{with equality only if}\,\mathrm{E}_k = \mathbb{R}^{2p}. \tag{31}$$

To this end, we use the fixed-point equation multiplied by $\mathbf{P}_{k+1}$, the inequality $\mathbf{v} \in \mathbb{R}^{2p}$, $\mathbf{v}^{\mathrm{T}}\mathbf{\Sigma}_k^{-1}\mathbf{v} \geq \lambda_1(\mathbf{\Sigma}_k)^{-1}\mathbf{v}^{\mathrm{T}}\mathbf{v}$ and the fact that $\mathbf{\Sigma} = \mathbf{I}_{2p}$ is a solution to the fixed point equation, to show that $\lambda_1(\mathbf{\Sigma}_{k+1})\mathbf{P}_{k+1} \leq \lambda_1(\mathbf{\Sigma}_k)\mathbf{P}_{k+1}$. Equality is obtained if and only if the following condition is respected:

$$\forall k, \forall t,\ \mathbf{P}_{k+1}\mathbf{v}_k^{(t)} = \mathbf{0}_{2p}\,\text{or}\,\mathbf{P}_k\mathbf{v}_k^{(t)} = \mathbf{v}_k^{(t)}. \tag{32}$$

Assume that this condition is respected and thus $\lambda_1(\mathbf{\Sigma}_{k+1}) = \lambda_1(\mathbf{\Sigma}_k)$. Then multiplying eq. (29) by $\mathbf{P}_{k+1}$ and $(\mathbf{I}_{2p} - \mathbf{P}_k)$ and using (32) leads to $\lambda_1(\mathbf{\Sigma}_{k+1})\mathbf{P}_{k+1}(\mathbf{I}_{2p} - \mathbf{P}_k) = \mathbf{0}_{2p,2p}$. Thus, we have: $\mathbf{P}_{k+1} = \mathbf{P}_k\mathbf{P}_{k+1}$. This means that $\mathrm{E}_{k+1} \subset \mathrm{E}_k$. If equality holds, then (32) implies $\mathbb{P}_{2TN}(\mathrm{E}_k) + \mathbb{P}_{2TN}(\mathrm{E}_k^{\perp}) = 1$. This contradicts condition **(C1)** unless $\mathrm{E}_k = \mathbb{R}^{2p}$.

Step 3. Statement (30) implies that $\lambda_1(\mathbf{\Sigma}_k) \to \lambda_1$ and $\lambda_{2p}(\mathbf{\Sigma}_k) \to \lambda_{2p}$ for some λ_1 and λ_2 such that $0 < \lambda_{2p} \leq \lambda_1 < \infty$. It just suffices to be shown at this point that $\lambda_1 = \lambda_{2p}$ which implies $\mathbf{\Sigma}_k \to \lambda_1 \mathbf{I}_{2p}$.

Since $\{\mathbf{\Sigma}_k\}$ has bounded eigenvalues, there exist a convergent subsequence, for example, $\mathbf{\Sigma}_{k(2p)} \to \mathbf{A}_0 \in \mathbb{S}^{2p}_{++}$, which implies

$$\mathbf{\Sigma}_{k(2p)+1} \to \mathbf{A}_1 = (2N)^{-1} p \sum_{k=1}^{N} \left(\sum_{t=1}^{T} \mathbf{v}_k^{(t)} \mathbf{v}_k^{(t)^{\mathrm{T}}} \right) / \left(\sum_{t=1}^{T} \mathbf{v}_k^{(t)^{\mathrm{T}}} \mathbf{A}_0^{-1} \mathbf{v}_k^{(t)} \right).$$

The largest and smallest eigenvalues of both $\mathbf{A}_0$ and $\mathbf{A}_1$ must be λ_1 and λ_{2p}. Let $\mathrm{E}_{\infty,k} = \{\mathbf{x} \in \mathbb{R}^{2p} | \mathbf{A}_k \mathbf{x} = \lambda_1 \mathbf{x}\}$. We can assume without loss of generality that $\dim(\mathrm{E}_{\infty,1}) \geq \dim(\mathrm{E}_{\infty,0})$. If this assumption is not respected, we can replace the subsequence $\{\mathbf{\Sigma}_{k(2p)}\}$ by $\{\mathbf{\Sigma}_{k(2p)+1}\}$ and so on until the assumption is met . This condition on the dimension together with (31), implies that $\mathrm{E}_{\infty,1} = \mathbb{R}^{2p}$. Thus, $\mathbf{\Sigma}_{k(2p)} \to \lambda_1 \mathbf{I}_{2p}$ which implies $\lambda_1 = \lambda_{2p}$.

3) Finally we show that if a solution to eq. (29) exist, then condition **(C1)** is respected.

Without loss of generality, we assume again that $\mathbf{\Sigma} = \mathbf{I}_{2p}$. Let S be a proper subspace and $\mathbf{Q}$ be the idempotent matrix which projects orthogonally into S. Letting $\mathbf{\Sigma} = \mathbf{I}_{2p}$, then multiplying both sides of eq. (29) by $(\mathbf{I}_{2p} - \mathbf{Q})$ and finally taking the trace yields:

$$2p - \dim(\mathrm{S}) = (2N)^{-1} p \sum_{k=1}^{N} \left(\sum_{t=1}^{T} \mathbf{v}_k^{(t)^{\mathrm{T}}} (\mathbf{I}_{2p} - \mathbf{Q}) \mathbf{v}_k^{(t)} \right) / \left(\sum_{t=1}^{T} \mathbf{v}_k^{(t)^{\mathrm{T}}} \mathbf{v}_k^{(t)} \right). \quad (33)$$

Since $\mathbf{x}^{\mathrm{T}}(\mathbf{I}_{2p} - \mathbf{Q})\mathbf{x} = 0$ for $\mathbf{x} \in \mathrm{S}$ and $\mathbf{x}^{\mathrm{T}}(\mathbf{I}_{2p} - \mathbf{Q})\mathbf{x} \leq \mathbf{x}^{\mathrm{T}}\mathbf{x}$ for $\mathbf{x} \notin \mathrm{S}$, it follows from eq. (33) that $2p - \dim(\mathrm{S}) \leq 2p(1 - \mathbb{P}_{2TN}(\mathrm{S}))$ which is equivalent to condition **(C1)**.

□

In fact a sufficient condition for the convergence is to have at least $p+1$ linearly independent observations $\mathbf{x}_k^{(t)}$, which is ensured in the data model we considered in the paper.

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