ROBUST STATISTICS FOR TESTING THE HOMOGENEITY OF COVARIANCE SCALE AND SHAPE

AN APPLICATION TO SAR CHANGE DETECTION

Seminar at NUS 10th April 2019

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ABOUT THIS TALK

We consider the problem of SAR change detection through covariance matrix equality testing.

Approach based on mutivariate statistical analysis. Pre-requisites:

- Basic algebra
- Basic statistics

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PLAN OF THE PRESENTATION

1. The Change detection problem
2. Covariance-based approaches
3. Statistics of decision under Gaussian assumption
4. Extension to Elliptical distributions
5. Results on SAR images

THE CHANGE DETECTION PROBLEM

WHAT WE WANT TO DO ?

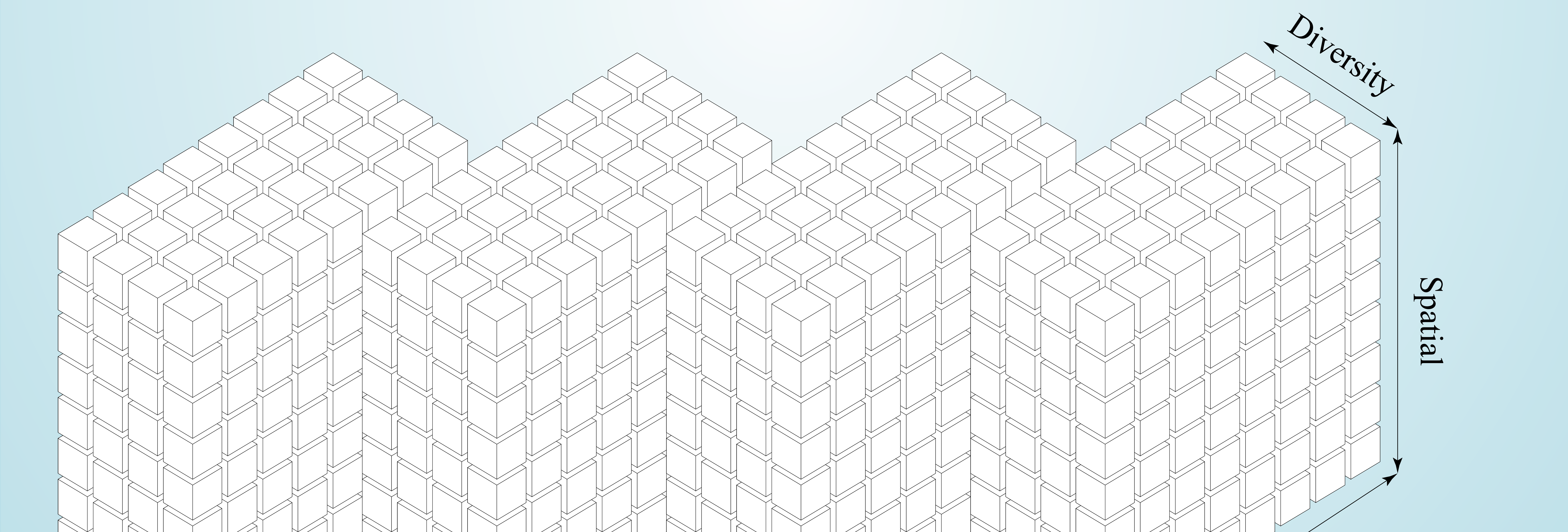
Considering a **time series** of SAR images, we want to **detect** spatial areas where notceable changes have occured.

Black RockCity before the Burning Man Festival, Nevada, USA - TerraSAR-X SpotLight Acquisition © 2011 Astrium Services / Infoterra GmbH

Black Rock City after the Burning Man Festival, Nevada, USA - TerraSAR-X SpotLight Acquisition © 2011 Astrium Services / Infoterra GmbH

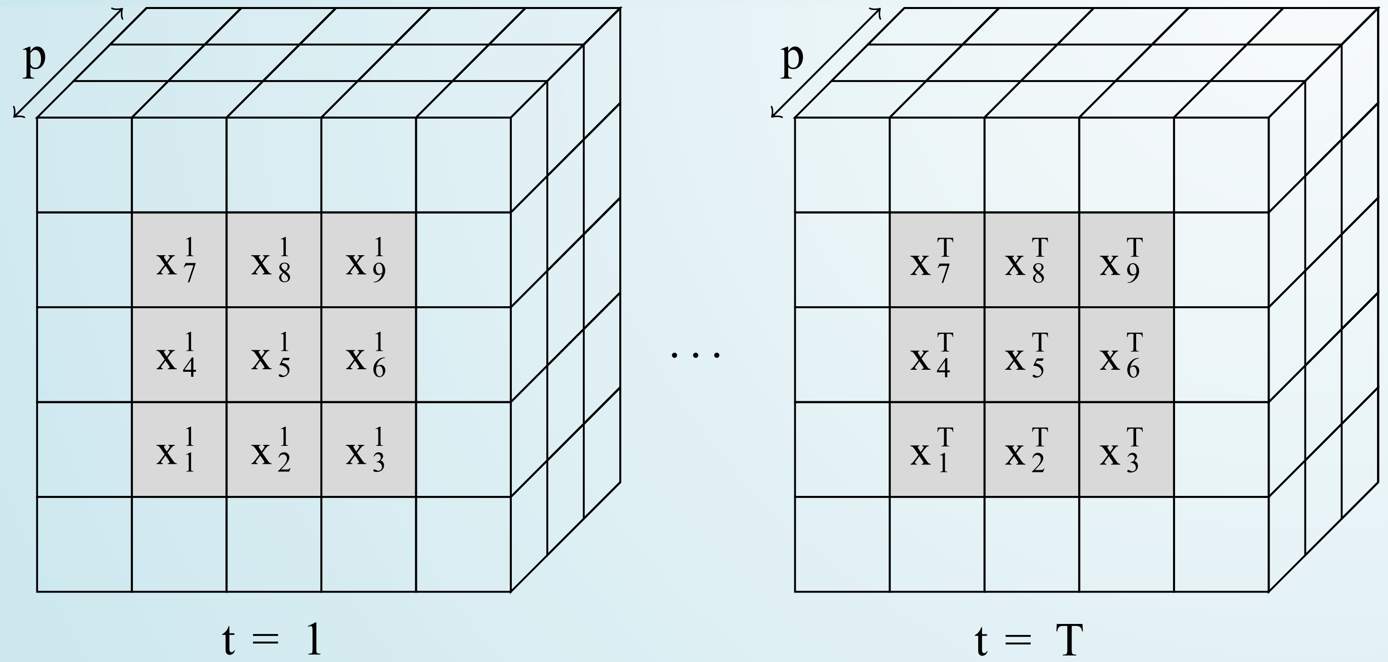
ASSUMPTIONS

- The images are co-registred
- Unsupervised methodology
- Data is complex and multivariate



COVARIANCE-BASED APPROACHES

DESCRIPTION OF THE APPROACH



We consider a sliding windows approach to consider spatially local data.

On this window, we assume i.i.d observations:

$$\forall (k,t)\, \mathbf{x}_k^t \sim \mathbb{C} \quad (\mathbf{0}_p, \boldsymbol{\Sigma}_t)$$

Idea: Compare the covariances $\boldsymbol{\Sigma}_t$ over time.

INTEREST OF THE APPROACH

- The SAR data is typically **noisy** (speckle noise) and can be **multivariate** (polarimetric for example).
- Covariance accounts for the local correlations between the observed noisy data. A change in the scene is likely to impact this matrix in a way.
- Using statistical detection theory allow to have theoretical results on false alarms or detection performance.

THE DETECTION PROBLEM

Denote by $\{ \quad _1,\ldots, \quad _T\}$ a collection of T mutually independent samples of i.i.d p-dimensional complex vectors: $\quad _t = [\mathbf{x}_1^t,\ldots,\mathbf{x}^t \quad]$.

We assume $\forall(k,t)$, $\mathbb{E}\{\mathbf{x}_k^t\} = \mathbf{0}_p$ and we denote $\boldsymbol{\Sigma}_t = \tau_t \boldsymbol{\xi}_t$ the shared covariance matrices among the elements of $\quad _t$. $\boldsymbol{\xi}_t$ is the shape matrix ($Tr(\boldsymbol{\xi}_t) = p$) and τ_t is the scale.

We want to choose between the following alternatives:

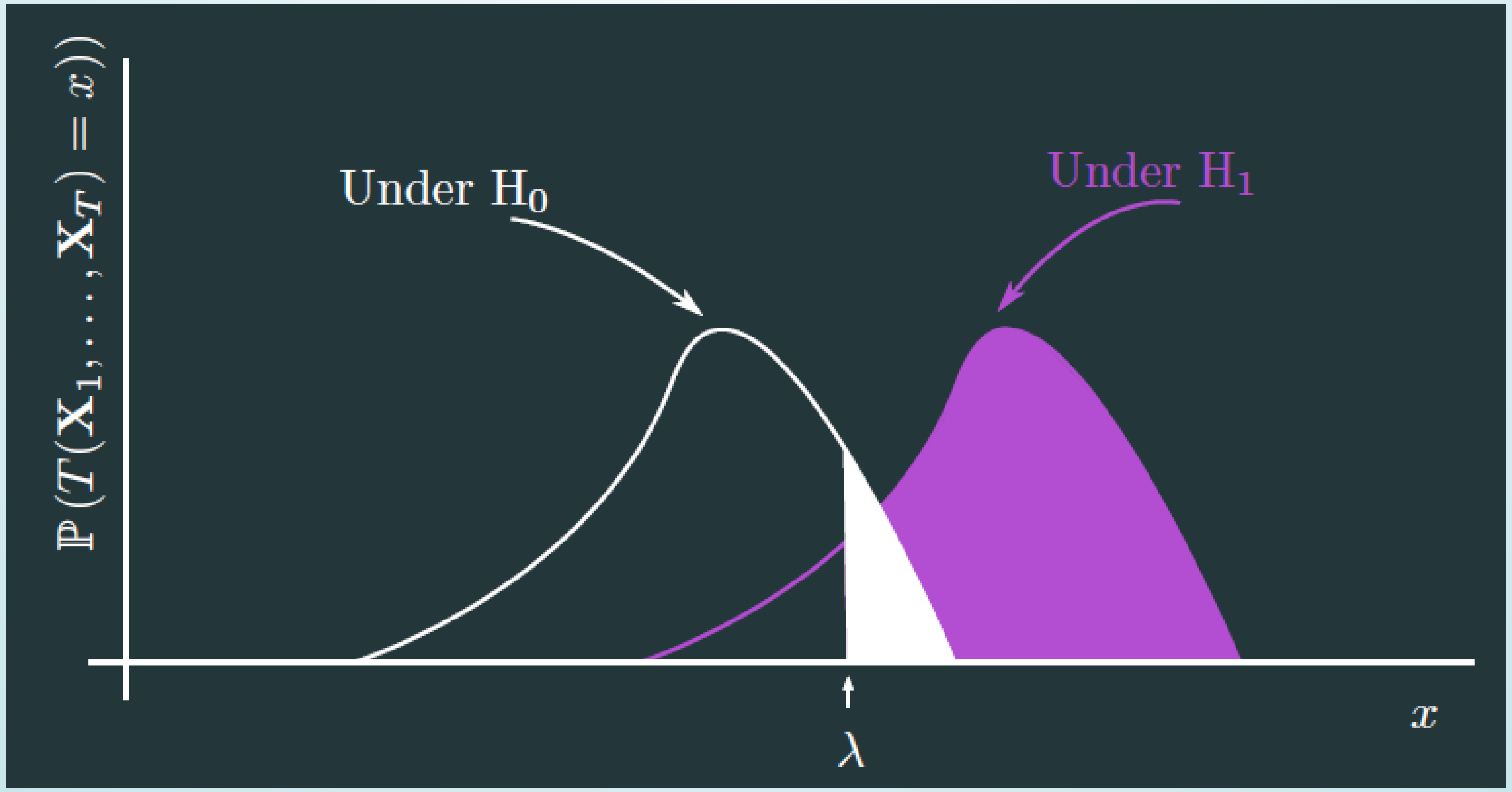
$$
\begin{aligned}
\mathrm{H}_0 &: \quad \boldsymbol{\Sigma}_1 = \ldots = \boldsymbol{\Sigma}_T = \boldsymbol{\Sigma}_0 \,, \\
\mathrm{H}_1 &: \quad \exists(t,t'),\ \boldsymbol{\Sigma}_t \quad \boldsymbol{\Sigma}_{t'}
\end{aligned}
$$

STATISTIC OF DECISION

We want to obtain:

- a statistic of decision T: $\begin{array}{rl} \mathbb{C}^{p\times} \times \cdots \times \mathbb{C}^{p\times} & \to \mathbb{R}^+ \\ {}_1, \ldots, {}_T & \to T({}_1, \ldots, {}_T) \end{array}$
- a threshold λ

So that $\mathbb{P}\,(T({}_1, \ldots, {}_T) > \lambda/\mathrm{H}_1)$ is high while $\mathbb{P}\,(T({}_1, \ldots, {}_T) > \lambda/\mathrm{H}_0)$ is low.



STATISTICS OF DECISION UNDER GAUSSIAN ASSUMPTION

STATISTICS OF DECISION UNDER GAUSSIAN ASSUMPTION (1/2)

Suppose $\forall t, \forall k, \mathbf{x}_k^t \sim \mathbb{C}\quad(\mathbf{0}_p, \mathbf{\Sigma}_t)$ so that $p_{\mathbf{x}_k^t;\mathbf{\Sigma}_t}(\mathbf{x}_k^t;\mathbf{\Sigma}_t) = \frac{1}{\pi^p|\mathbf{\Sigma}_t|}\mathrm{exp}\mathrm{tr}\left\{\mathbf{S}_k^t\mathbf{\Sigma}_t^{-1}\right\}$, where $\mathbf{S}_k^t = \mathbf{x}_k^t{\mathbf{x}_k^t}^{\mathrm{H}}$.

Many statistic exists but the options can be reduced to[1] :

- The Generalized Likelihood Ratio Test (GLRT) statistic:

$$\hat{\Lambda}_{\mathrm{G}} = \frac{\left|\hat{\mathbf{\Sigma}}_0^{\mathrm{SC}}\right|^T}{\quad_{t=1}^{T}\left|\hat{\mathbf{\Sigma}}_t^{\mathrm{SC}}\right|} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda, \text{ where :}$$

$$\forall t, \hat{\mathbf{\Sigma}}_t^{\mathrm{SC}} = \frac{1}{\quad}\quad_{k=1}\mathbf{S}_k^t \text{ and } \hat{\mathbf{\Sigma}}_0^{\mathrm{SC}} = \frac{1}{T}\quad_{t=1}^{T}\hat{\mathbf{\Sigma}}_t^{\mathrm{SC}}\quad.$$

[1] D. Ciuonzo, V. Carotenuto and A. De Maio, "On Multiple Covariance Equality Testing with Application

STATISTICS OF DECISION UNDER GAUSSIAN ASSUMPTION (2/2)

- The t_1 statistic obtained from either Terell or Rao tests:

$$\hat{\Lambda}_{\mathrm{t}_1} = \sum_{t=1}^{T} \mathrm{Tr}\left[\left(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{SC}}\left(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{SC}}\right)^{-1} - \mathbf{I}_p\right)^2\right] \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda.$$

- The Wald statistic:

$$\hat{\Lambda}_{\ \mathrm{ald}} = \sum_{t=2}^{T} \mathrm{Tr}\left[\left(\mathbf{I}_p - \hat{\boldsymbol{\Sigma}}_1^{\mathrm{SC}}\,(\hat{\boldsymbol{\Sigma}}_t^{\mathrm{SC}})^{-1}\right)^2\right]$$

$$- q \sum_{t=1}^{T} (\hat{\boldsymbol{\Sigma}}_t^{\mathrm{SC}})^{-T} \otimes (\hat{\boldsymbol{\Sigma}}_t^{\mathrm{SC}})^{-1}, \mathrm{vec} \sum_{t=2}^{T} \quad t \quad \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda,$$

SOME PROPERTIES OF THE STATISTICS

CFARness:

The GLRT, t_1 and Wald statistic have the CFAR property with regards to the covariance parameter.

*Proof:*The statistics are invariant for the group of transformation $\mathcal{G} = \left\{ \quad \mathbf{x}_k^t|, \quad \in \mathbb{S}_p^{\mathbb{H}} \right\}$.

Distribution under $\mathbb{H}_0$ (F-Approximation[1]):

Under null hypothesis, we have: $2(1-c)\ln(\hat{\Lambda}_{\mathrm{B}}) \sim \chi^2\left((T-1)p(p+1)\right)$ where

$c = \dfrac{T^2-1}{T(\quad -1)} \times \dfrac{2p^2+3p-1}{6(T-1)(p+1)}$ and $\hat{\Lambda}_{\mathrm{B}}$ is a modified version of $\hat{\Lambda}_{\mathrm{G}}$

There are similar results for others statistics.

[1] Box, G. E. P. “A General Distribution Theory for a Class of Likelihood Criteria.” Biometrika, vol. 36, no. 3/4, 1949, pp. 317–346.

FALSE ALARM/THRESHOLD RELATIONSHIP

Under H_0 we have [1]:

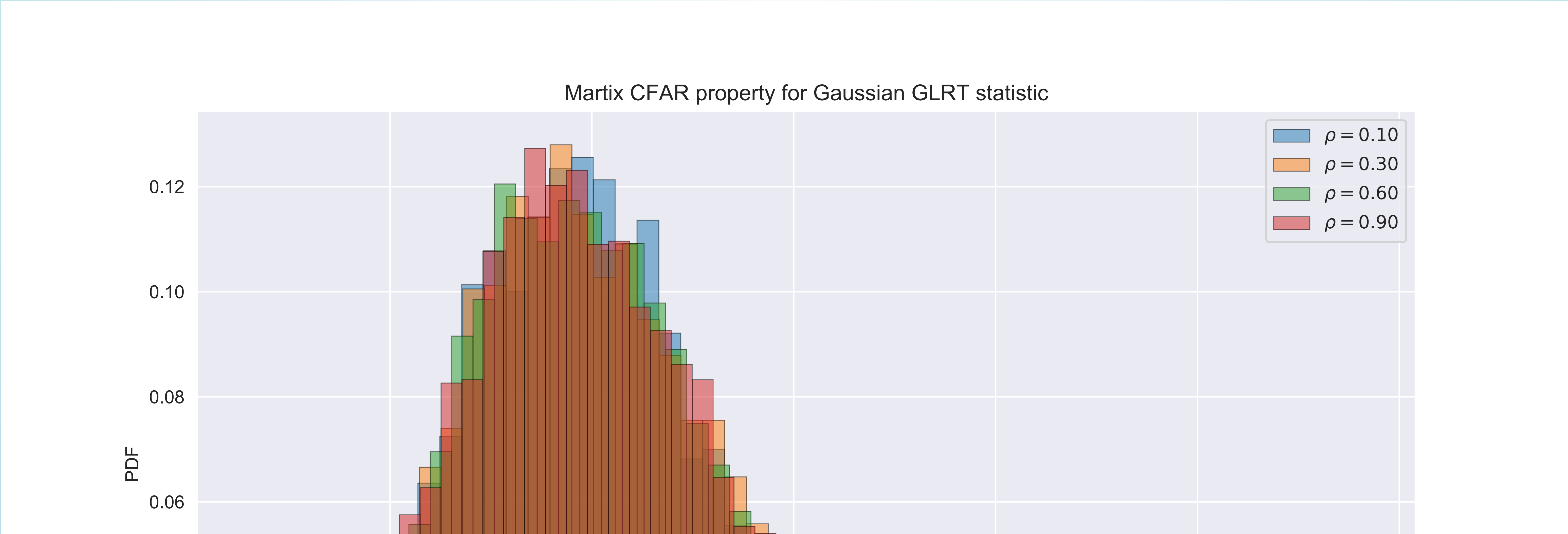
$$P \quad 2\rho\log(\hat{\Lambda}_{\mathrm{G}}) \leq z \quad \approx P\left\{\chi^2(f^2) \leq z\right\} + \omega_2 \left[P\left\{\chi^2(f^2+4) \leq z\right\} - P\left\{\chi^2(f^2) \leq z\right\}\right]$$

$$f = (T-1)p^2,\ \rho = 1 - \frac{(2p^2-1)}{6(T-1)p} \quad \frac{T}{\quad} - \frac{1}{\quad T} \quad ,$$

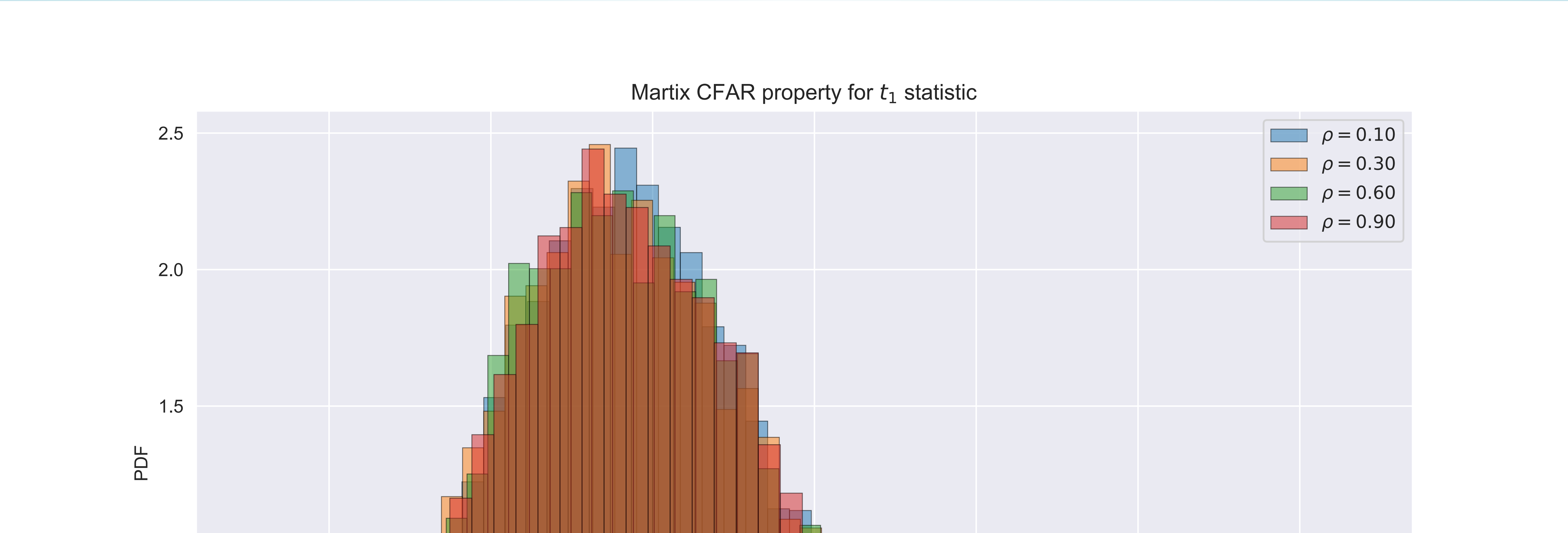
$$\omega_2 = \frac{p^2(p^2-1)}{24\rho^2} \quad \frac{T}{\quad 2} - \frac{1}{(\quad T)^2} \quad - \frac{p^2(T-1)}{4} \quad 1 - \frac{1}{\rho} \quad {}^2$$

[1] Anderson, T. W. (1962). An introduction to multivariate statistical analysis (No. 519.9 A53). New York: Wiley.

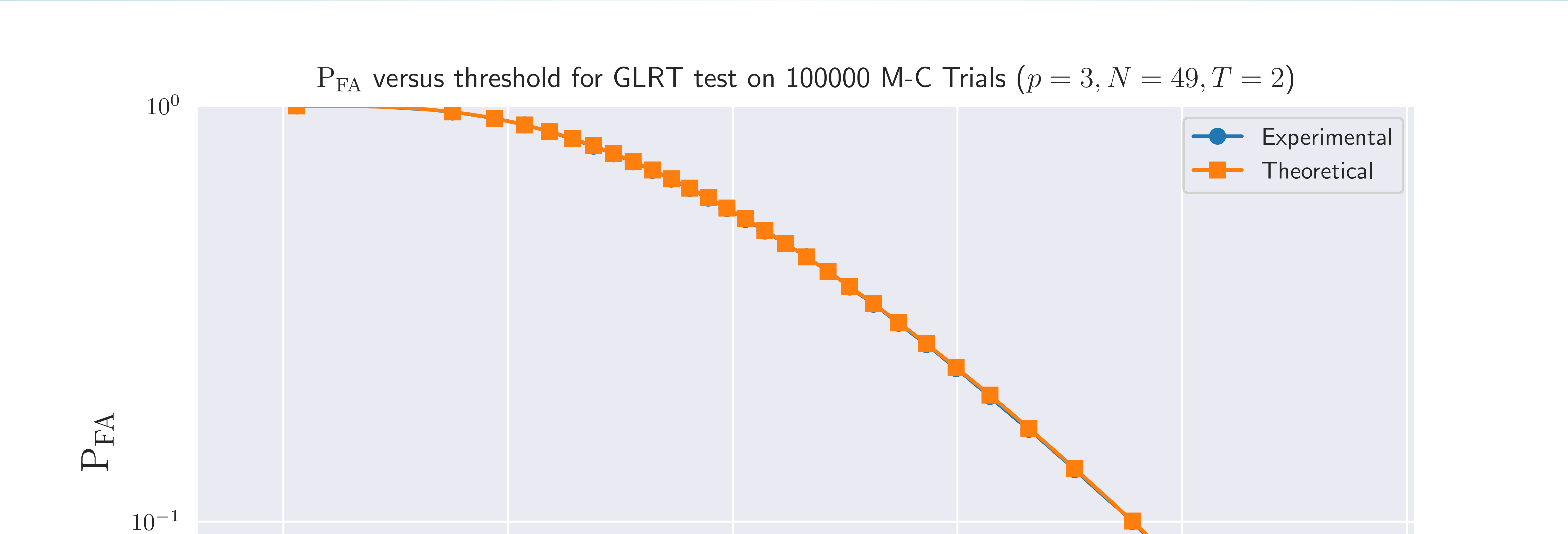
EXPERIMENTAL VALIDATION OF MATRIX CFAR PROPERTY (1/2)



EXPERIMENTAL VALIDATION OF MATRIX CFAR PROPERTY (2/2)



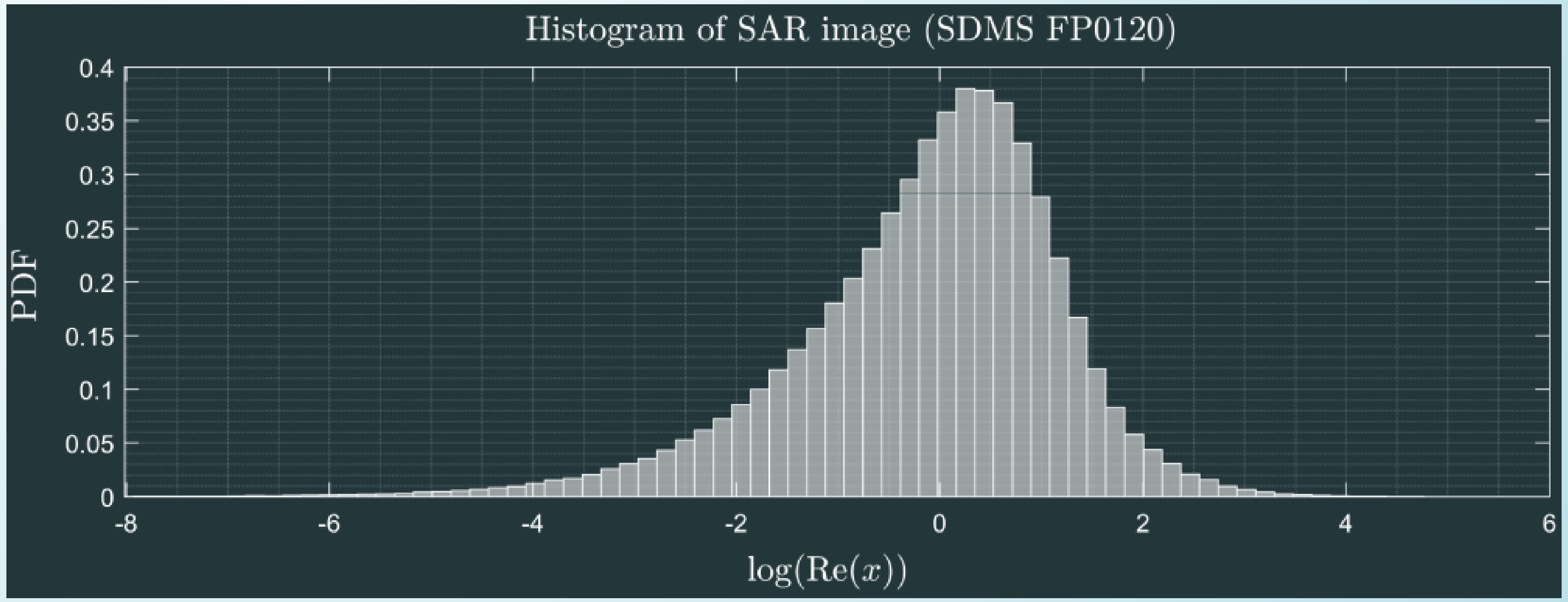
EXPERIMENTAL VALIDATION OF DISTRIBUTINO UNDER $\mathbf{H}_0$



EXTENSION TO ELLIPTICAL DISTRIBUTIONS

NON-GAUSSIANITY HIGH-RESOLUTION IMAGES

Sometimes the Gaussian hypothesis is not accurate !



To model this kind of distribution the family of elliptical distributions have been introduced[1]

$\mathbb{C}ES$ DISTRIBUTIONS

Probability distribution:

$$p_{\mathbf{x};g;\mathbf{\Sigma}}(\mathbf{x};g;\mathbf{\Sigma}) = \mathfrak{C}_{p,g}|\mathbf{\Sigma}|^{-1}g(\mathbf{x}^{\mathrm{H}}\mathbf{\Sigma}^{-1}\mathbf{x})$$

where g is a density generator function with some rgularity conditions, $\mathbf{\Sigma} \in \mathbb{S}_p^{\mathbb{H}}$ is the scatter matrix and $\mathfrak{C}_{p,g}$ is a normalisation constant.

Example:

Multivariate Student-t: $\dfrac{\Gamma \quad \frac{\nu+g}{2} \quad |\Sigma|^{-1/2}}{(\pi\nu)^{\frac{1}{2}}\Gamma \quad \frac{\nu}{2} \quad 1+\frac{\delta(\mathbf{y},\mu,\Sigma)}{\nu} \quad ^{\frac{1}{2}(\nu+g)}}$

Problem: How do we test the equality of scatter matrix in this model ? In peculiar, can we test scale and shape separately ?

APPROACH 1: BOOTSRAPPING THE GAUSSIAN GLRT

Since the distribution of Gaussian-derived detector is known[1] under $\mathbb{C}ES$ model it is possible to correct it in order to obtain one keeping the CFAR property in $\mathbb{C}ES$ context.

Hallin proposed to use the following test[2]:

$$= 2 \quad {}_{1\leq t\leq t'\leq T} \quad {}_{;t,t'} \geq \chi^2 \quad \frac{(T-1)p(p+1)}{2} \quad {}_{1-\mathrm{P_{FA}}}$$, where:

$$_{;t,t'} = \frac{1}{4(1+\hat{\kappa}_p)} \mathrm{Tr}\left[(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{SC}} \quad)^{-1} (\hat{\boldsymbol{\Sigma}}_t^{\mathrm{SC}} \quad - \hat{\boldsymbol{\Sigma}}_{t'}^{\mathrm{SC}} \quad)^2 \right] - \frac{\hat{\kappa}_p}{(p+2)\hat{\kappa}_p + 2} \mathrm{Tr}^2 \left[(\hat{\boldsymbol{\Sigma}}_0^{\mathrm{SC}} \quad)^{-1} (\hat{\boldsymbol{\Sigma}}_t^{\mathrm{SC}} \quad - \hat{\boldsymbol{\Sigma}}_{t'}^{\mathrm{SC}} \quad) \right] \quad ,$$

and $\hat{\kappa}_p = p(p+1)/2 \quad {}_{t=1}^{T} \quad {}_{k=1} d^4(\mathbf{x}_k^t, \hat{\boldsymbol{\Sigma}}_0^{\mathrm{SC}} \quad) - 1$and $d(\mathbf{x}, \boldsymbol{\Sigma}) = \|\boldsymbol{\Sigma}^{-1/2}\mathbf{x}\|$

[1] Yanagihara, H., Tonda, T., and Matsumoto, C. (2005).The effects of nonnormality on asymptotic distributions of some likelihood ratio criteria for testing covariance structures under normal assumption. Journal of Multivariate Analysis, 96(2):237{264.

[2] Marc Hallin, Davy Paindaveine, Optimal tests for homogeneity of covariance, scale, and shape,

APPROACH 2: GLRT UNDER $\mathbb{C}\mathcal{ES}$ MODEL

We compute the following statistic:

$$\hat{\Lambda} = \frac{\max_{\mathbf{\Sigma}_1,\dots,\mathbf{\Sigma}_T} \; p_{\ 1,\dots,\ T;\mathbf{\Sigma}_1,\dots,\mathbf{\Sigma}_T}(\ 1,\dots,\ T;\mathbf{\Sigma}_1,\dots,\mathbf{\Sigma}_T/\mathrm{H}_1)}{\max_{\mathbf{\Sigma}_0} \; p_{\ 1,\dots,\ T;\mathbf{\Sigma}_0}(\ 1,\dots,\ T;\mathbf{\Sigma}_0/\mathrm{H}_0)}$$

which yields:

$$\hat{\Lambda}^g = \frac{|\hat{\mathbf{\Sigma}}_0\ |^T}{\underset{t=1}{\overset{T}{}}\ |\hat{\mathbf{\Sigma}}_t\ |} \ \underset{t=1}{\overset{T}{}}\ \underset{k=1}{} \frac{g(\mathbf{x}_k^{t\ \mathrm{H}}\{\hat{\mathbf{\Sigma}}_0\ \}^{-1}\mathbf{x}_k^t)}{g(\mathbf{x}_k^{t\ \mathrm{H}}\{\hat{\mathbf{\Sigma}}_t\ \}^{-1}\mathbf{x}_k^t)} \ \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{}}\ \lambda,$$

where: $\hat{\mathbf{\Sigma}}_t \ = f_t(\hat{\mathbf{\Sigma}}_t\ )$, $\hat{\mathbf{\Sigma}}_0\ = \frac{1}{T}\underset{t=1}{\overset{T}{}} f_t(\hat{\mathbf{\Sigma}}_0\ )$ and $f_t(\mathbf{\Sigma}) = \frac{1}{\ }\underset{k=1}{} \frac{-g'(\mathbf{x}_k^{t\ \mathrm{H}}\{\hat{\mathbf{\Sigma}}\}^{-1}\mathbf{x}_k^t)}{g(\mathbf{x}_k^{t\ \mathrm{H}}\{\hat{\mathbf{\Sigma}}\}^{-1}\mathbf{x}_k^t)}\mathbf{x}_k^t\mathbf{x}_k^{t\ \mathrm{H}}$

Problem: We need to know g !!!

APPROACH 2BIS: SELECT $g = \frac{1}{x^p}$

The statistic reads:

$$\hat{\Lambda}^{1/x^p} = \frac{\left|\hat{\mathbf{\Sigma}}_0\right|^T}{\prod_{t=1}^{T} \left|\hat{\mathbf{\Sigma}}_t\right|} \prod_{t=1}^{T} \prod_{k=1} \frac{\left(\mathbf{x}_k^{t\,\mathrm{H}} \{\hat{\mathbf{\Sigma}}_t\}^{-1} \mathbf{x}_k^t\right)^p}{\left(\mathbf{x}_k^{t\,\mathrm{H}} \{\hat{\mathbf{\Sigma}}_0\}^{-1} \mathbf{x}_k^t\right)^p} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda,$$

Properties[1]: - This statistic is valid for testing the **Shape matrix** for any elliptical distribution.

- The distribution of $2\log(\hat{\Lambda}^g)$ under H_0 is assyptotically that of a $\chi^2\left((T-1)p(p+1)\right)$

Problem: We can't test a change in the scale !

[1] A. Mian, J. Ovarlez, G. Ginolhac and A. M. Atto, "A Robust Change Detector for Highly Heterogeneous Multivariate Images," 2018 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Calgary, AB, 2018, pp. 3429-3433.

APPROACH 3: COMPOUND-GAUSSIAN MODEL (1/2)

We consider the Compound-Gaussian model: $\mathbf{x}_k^t \sim \overline{\tau_k^t}\,\mathbb{C}\quad(\mathbf{0}_p, \boldsymbol{\xi}_t)$, where τ_k^t are assumed to be **deterministic** and $Tr(\boldsymbol{\xi}) = p$. $\rightarrow$ We can integrate these unknown parameters in the GLRT[1]:

$$\hat{\Lambda}_{\mathrm{T}} = \frac{|\hat{\mathbf{\Sigma}}_0^{\mathrm{T}}|^T}{\mathop{}_{t=1}^{T} |\hat{\mathbf{\Sigma}}_t|} \mathop{}_{k=1} \left(\frac{\mathop{}_{t=1}^{T} {\mathbf{x}_k^t}^{\mathrm{H}} \{\hat{\mathbf{\Sigma}}_t^{\mathrm{T}}\}^{-1} \mathbf{x}_k^t}{\mathop{}_{t=1}^{T} {\mathbf{x}_k^t}^{\mathrm{H}} \{\hat{\mathbf{\Sigma}}_0\}^{-1} \mathbf{x}_k^t} \right)^{Tp}_{p} \mathop{}^{\mathrm{H}_1}_{\mathrm{H}_0} \lambda, \text{ where :}$$

$$\hat{\mathbf{\Sigma}}_t = \frac{p}{} \mathop{}_{k=1} \frac{\mathbf{x}_k^t {\mathbf{x}_k^t}^{\mathrm{H}}}{{\mathbf{x}_k^t}^{\mathrm{H}} \{\hat{\mathbf{\Sigma}}\}^{-1} \mathbf{x}_k^t} \text{ and } \hat{\mathbf{\Sigma}}_0^{\mathrm{T}} = \frac{p}{} \mathop{}_{k=1} \frac{\mathop{}_{t=1}^{T} \mathbf{x}_k^t {\mathbf{x}_k^t}^{\mathrm{H}}}{\mathop{}_{t=1}^{T} {\mathbf{x}_k^t}^{\mathrm{H}} \{\hat{\mathbf{\Sigma}}^{\mathrm{T}}\}^{-1} \mathbf{x}_k^t}$$

APPROACH 3: COMPOUND-GAUSSIAN MODEL (2/2)

We can also test the scale parameters only and ignoring a change in the shape matrix.
$\rightarrow$ The GLRT[1] yields:

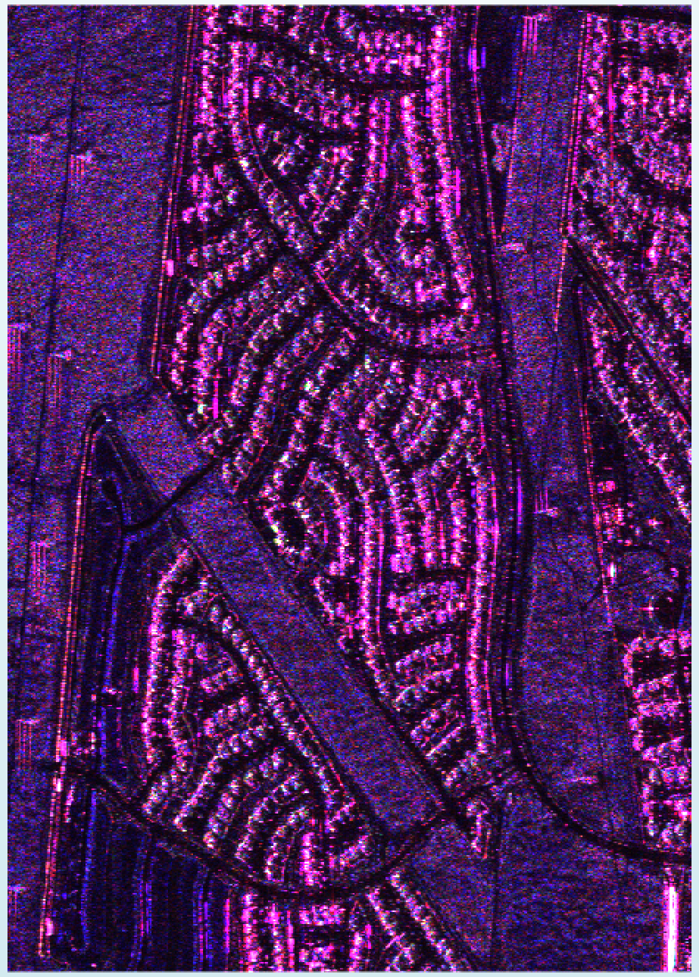
$$\hat{\Lambda}_{\text{Tex}} = \prod_{t=1}^{T} \frac{\left|\hat{\boldsymbol{\Sigma}}_t^{\text{Tex}}\right|}{\left|\hat{\boldsymbol{\Sigma}}_t\right|} \prod_{k=1} \frac{\left(\sum_{t=1}^{T} {\mathbf{x}_k^t}^{\text{H}} \{\hat{\boldsymbol{\Sigma}}_t^{\text{Tex}}\}^{-1} \mathbf{x}_k^t\right)^{Tp}}{\prod_{t=1}^{T} \left({\mathbf{x}_k^t}^{\text{H}} \{\hat{\boldsymbol{\Sigma}}_0\}^{-1} \mathbf{x}_k^t\right)^{p}} \underset{\text{H}_0}{\overset{\text{H}_1}{\gtrless}} \lambda, \text{ where :}$$

$$\hat{\boldsymbol{\Sigma}}_t = \frac{p}{} \sum_{k=1} \frac{\mathbf{x}_k^t {\mathbf{x}_k^t}^{\text{H}}}{{\mathbf{x}_k^t}^{\text{H}} \{\hat{\boldsymbol{\Sigma}}\}^{-1} \mathbf{x}_k^t} \text{ and } \hat{\boldsymbol{\Sigma}}_t^{\text{Tex}} = \frac{p}{} \sum_{k=1} \frac{\sum_{t=1}^{T} \mathbf{x}_k^t {\mathbf{x}_k^t}^{\text{H}}}{\sum_{t=1}^{T} {\mathbf{x}_k^t}^{\text{H}} \{\hat{\boldsymbol{\Sigma}}^{\text{Tex}}\}^{-1} \mathbf{x}_k^t}$$

RESULTS ON SAR IMAGES

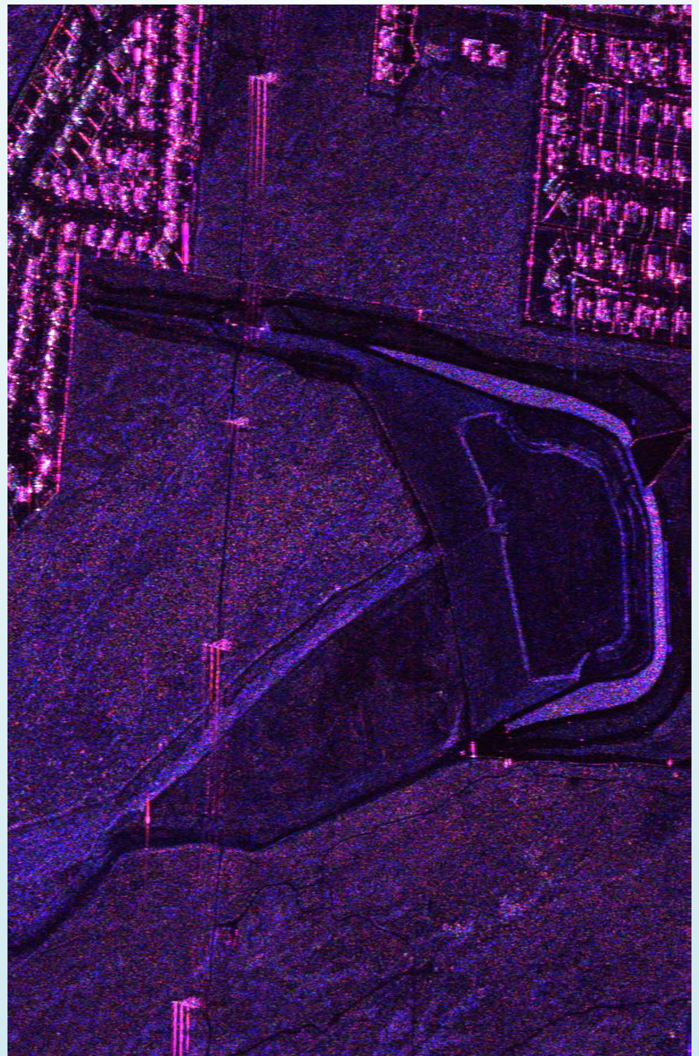
THE UAVSAR DATASET

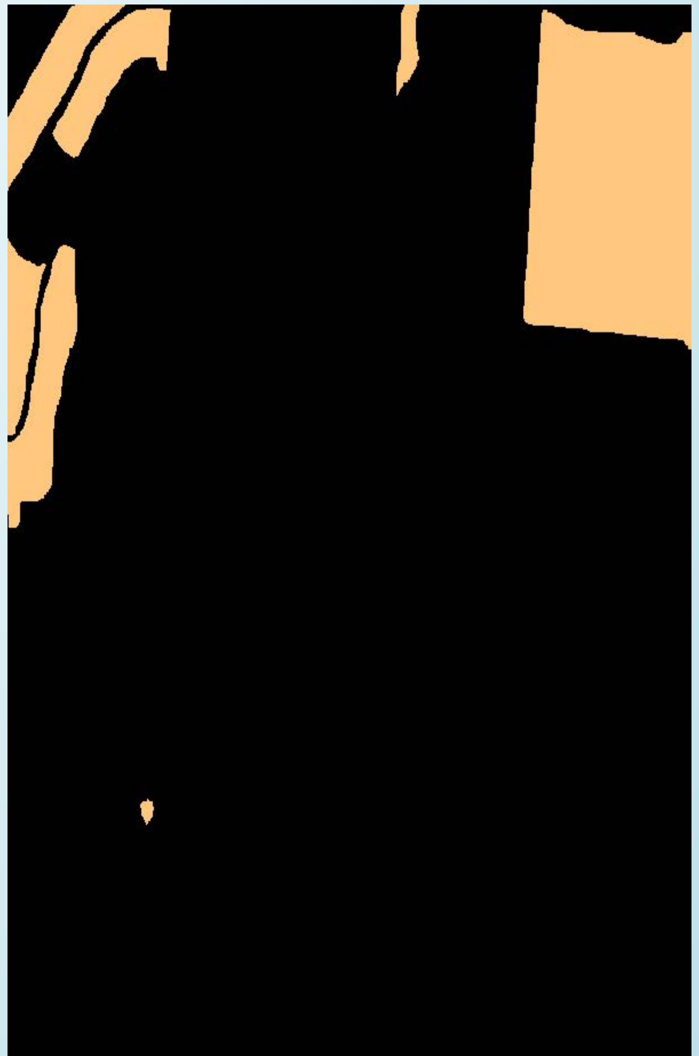
- Polarimetric data $(p = 3)$
- Dimensions: 2360px by 600 px
- Resolution: 1m67 (Range) and 1m (Azimuth)



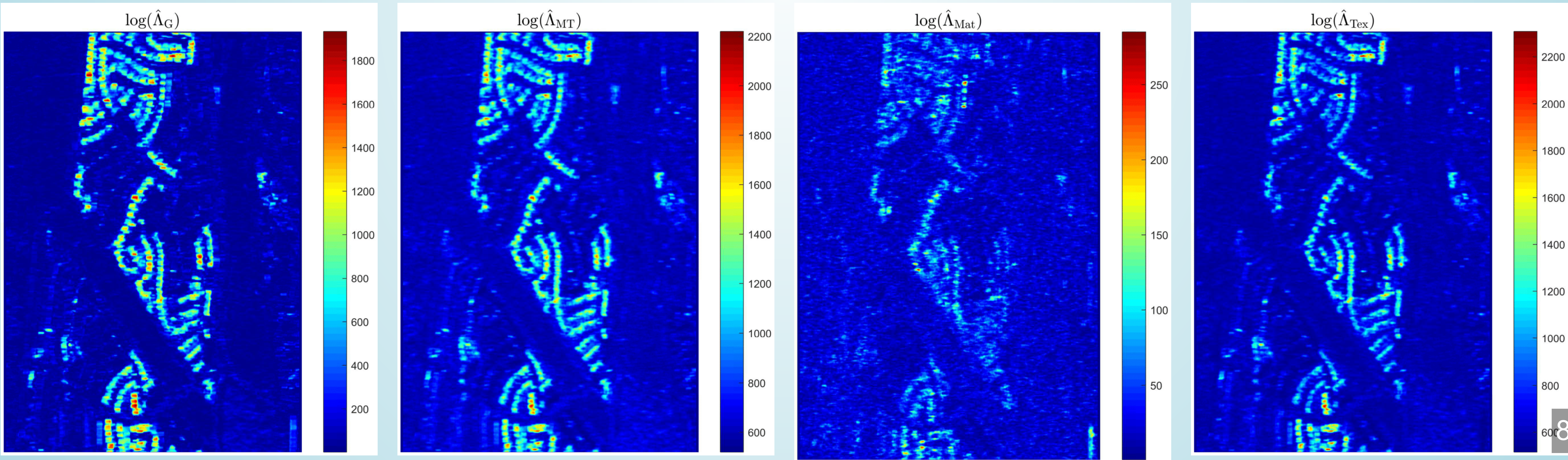
THE UAVSAR DATASET (2/2)

- Polarimetric data ($p = 3$)
- Dimensions: 2360px by 600 px
- Resolution: 1m67 (Range) and 1m (Azimuth)

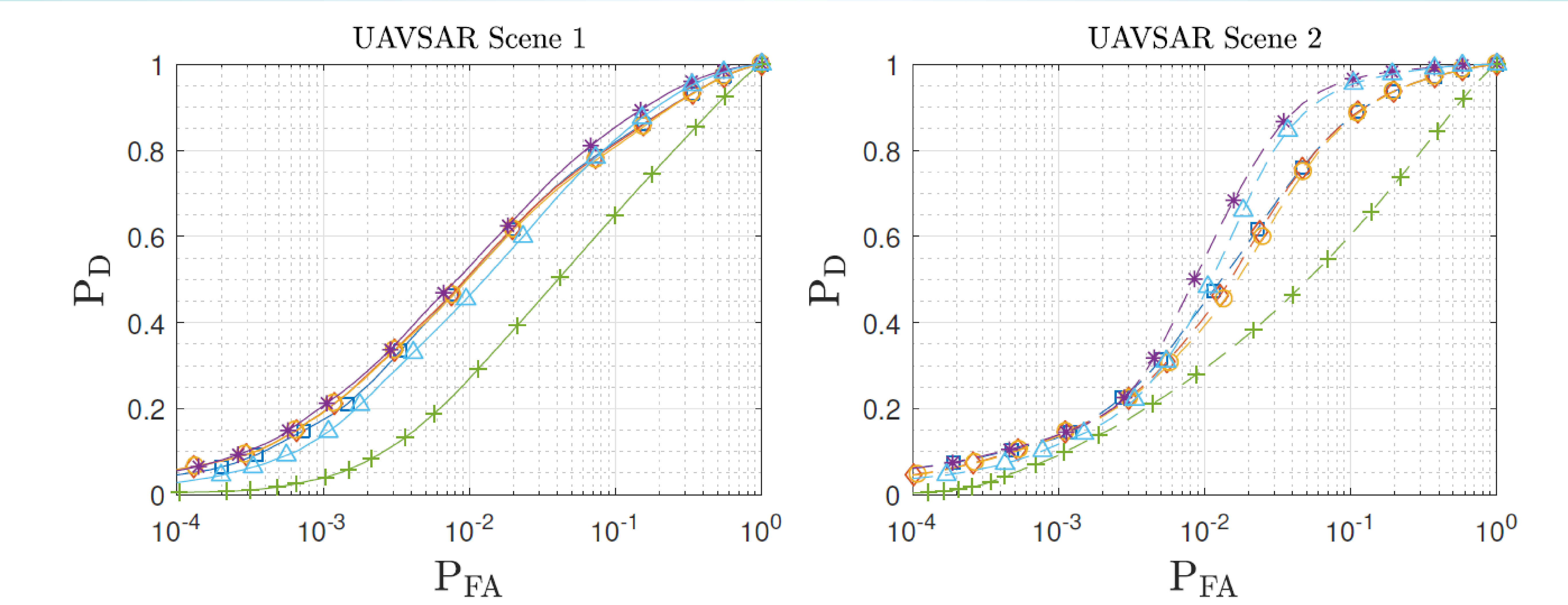




RESULTS (1/2)

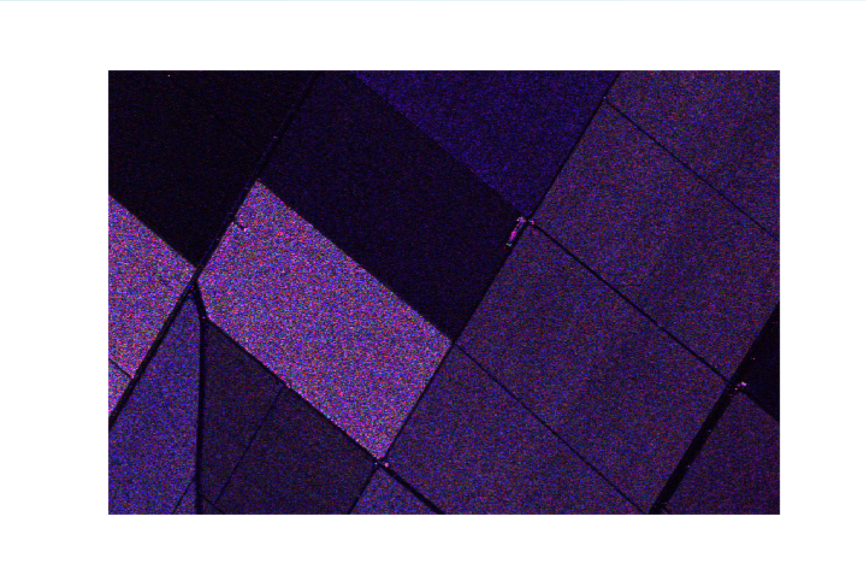

$\log(\hat{\Lambda}_{\mathrm{G}})$
1800
1600
1400
1200
1000
800
600
400
200
$\log(\hat{\Lambda}_{\mathrm{MT}})$
2200
2000
1800
1600
1400
1200
1000
800
600
$\log(\hat{\Lambda}_{\mathrm{Mat}})$
250
200
150
100
50
$\log(\hat{\Lambda}_{\mathrm{Tex}})$
2200
2000
1800
1600
1400
1200
1000
800

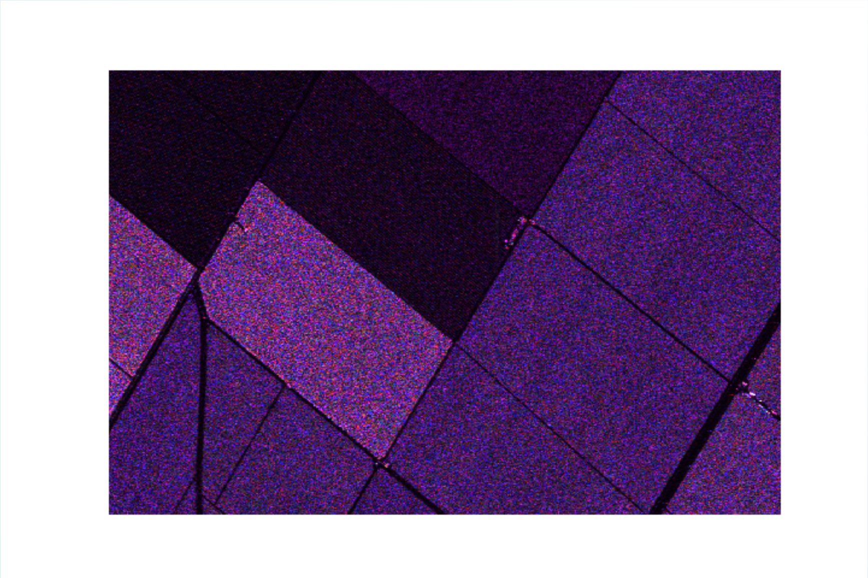
RESULTS (2/2)

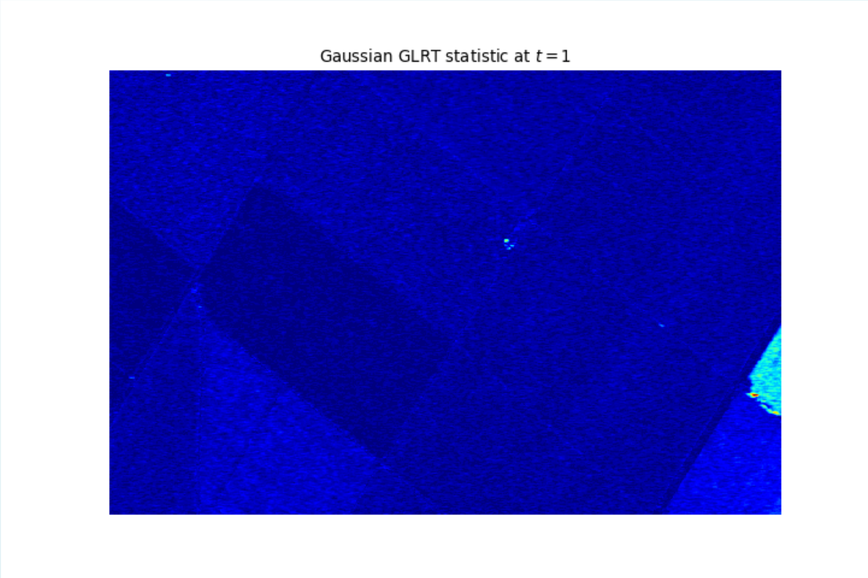


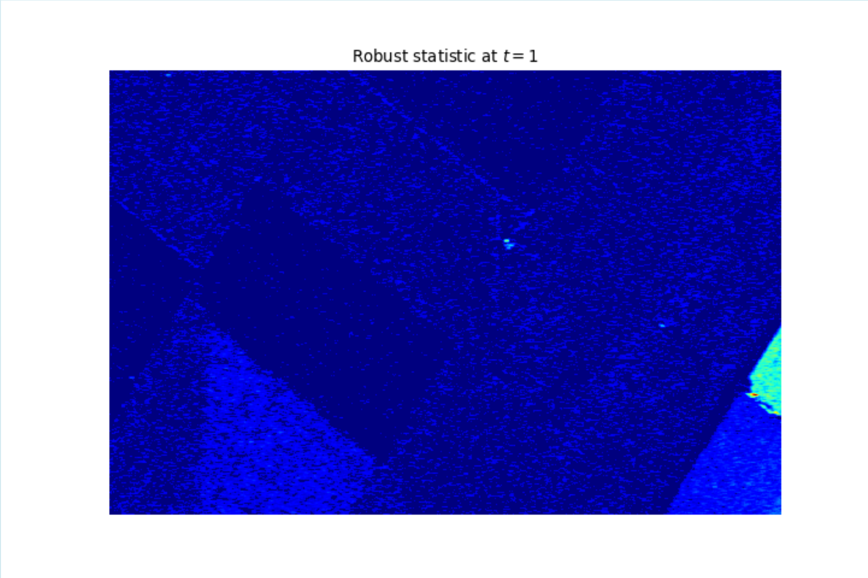
UAVSAR Scene 1
UAVSAR Scene 2
P_D
P_{FA}
1
0.8
0.6
0.4
0.2
0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}

RESULT ON ANOTHER DATASET

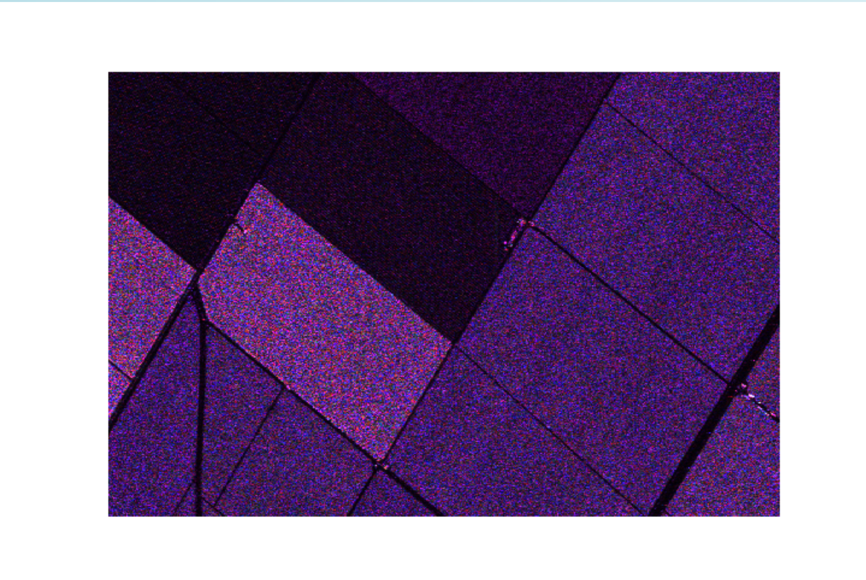


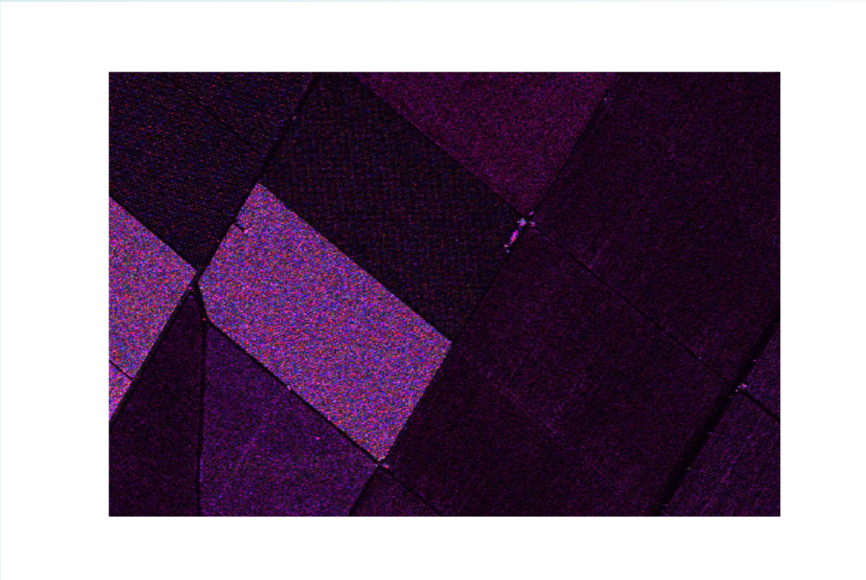


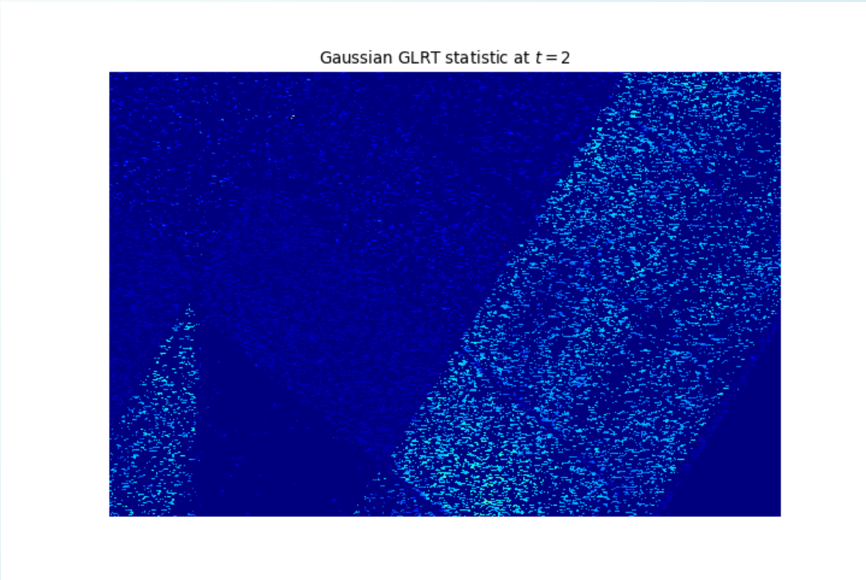


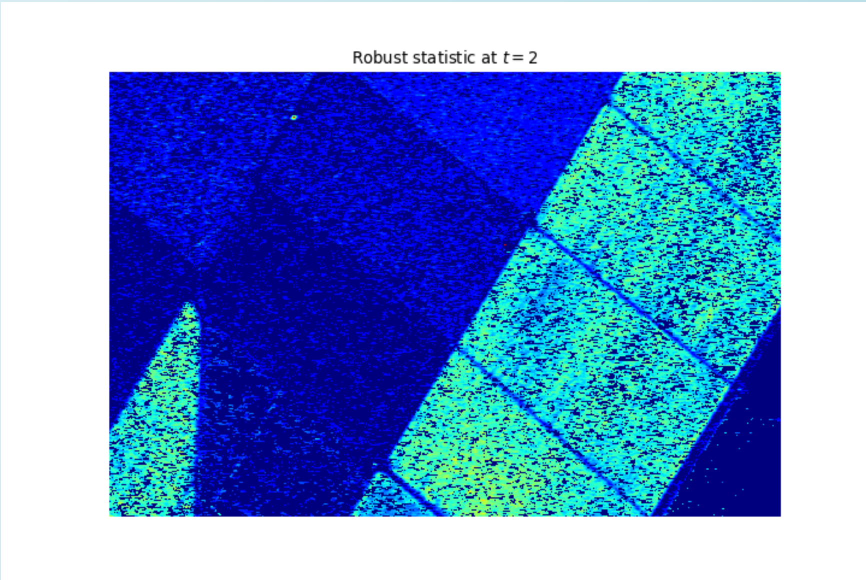


RESULT ON ANOTHER DATASET

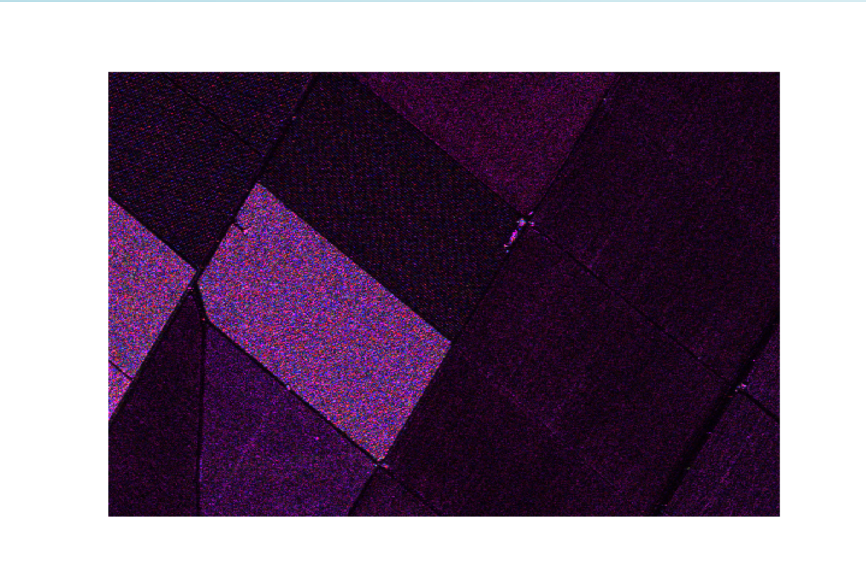


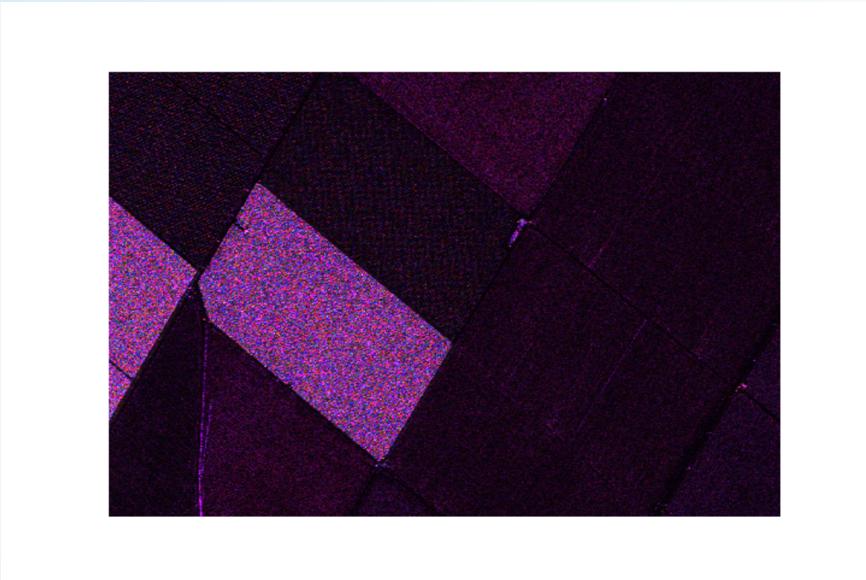


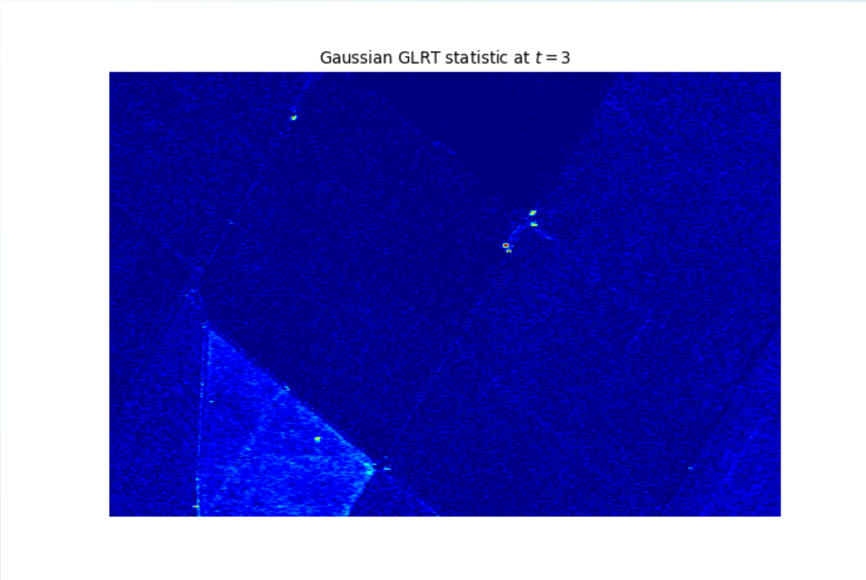


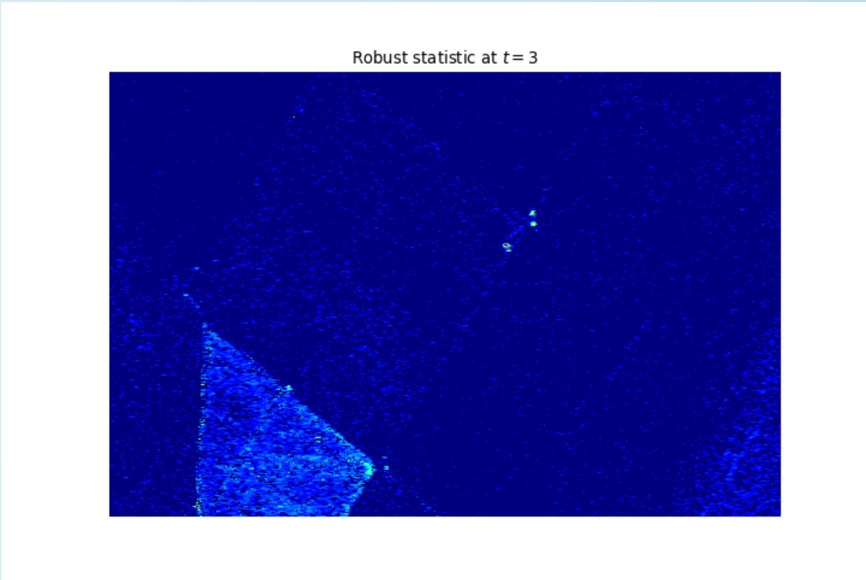


RESULT ON ANOTHER DATASET









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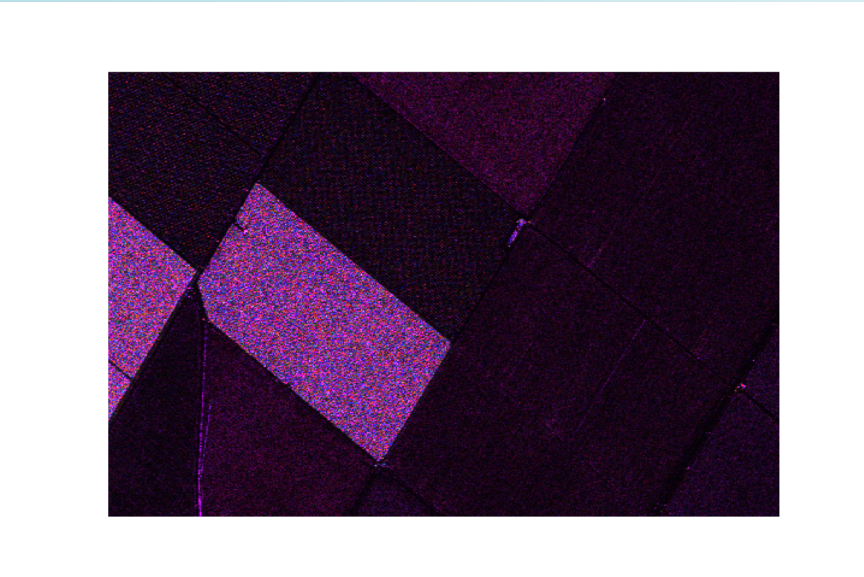

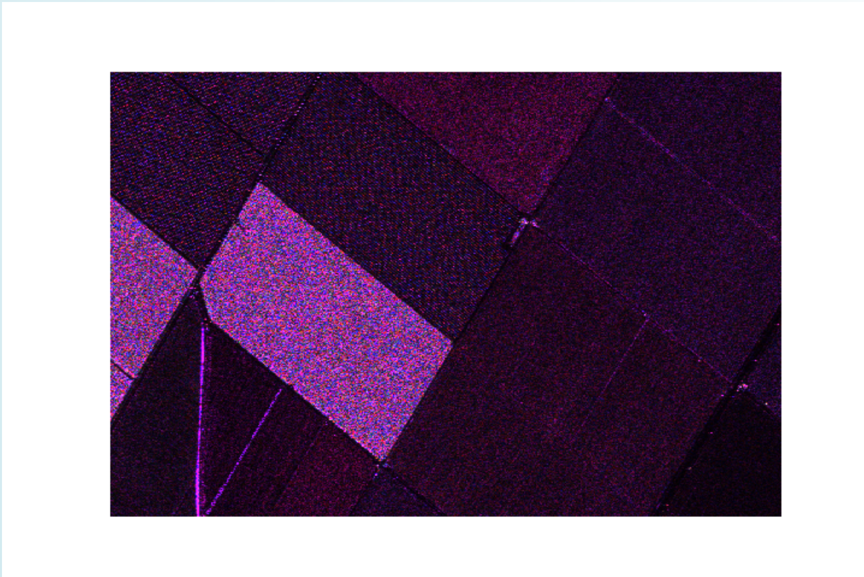

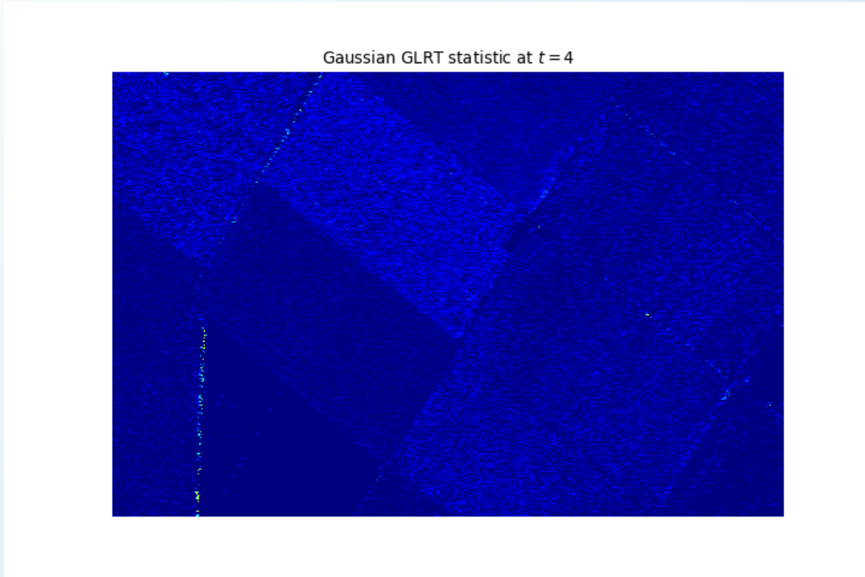

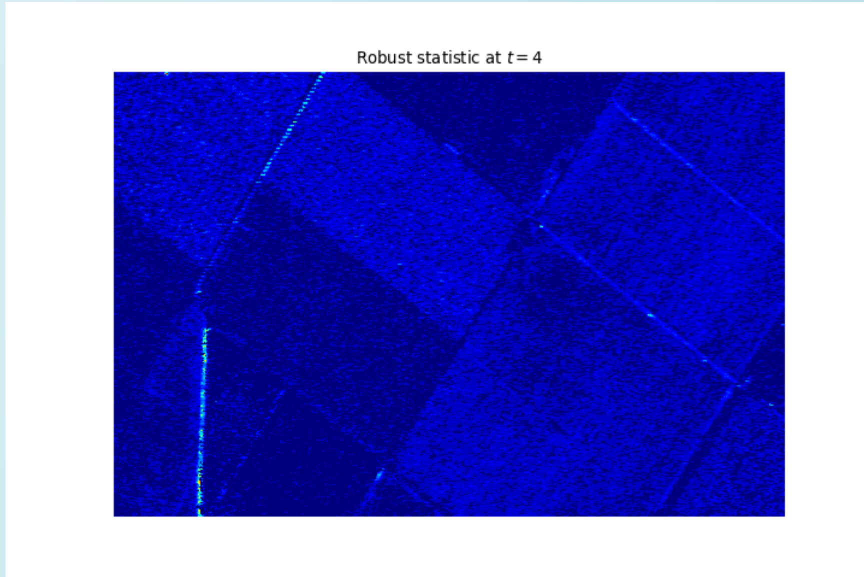

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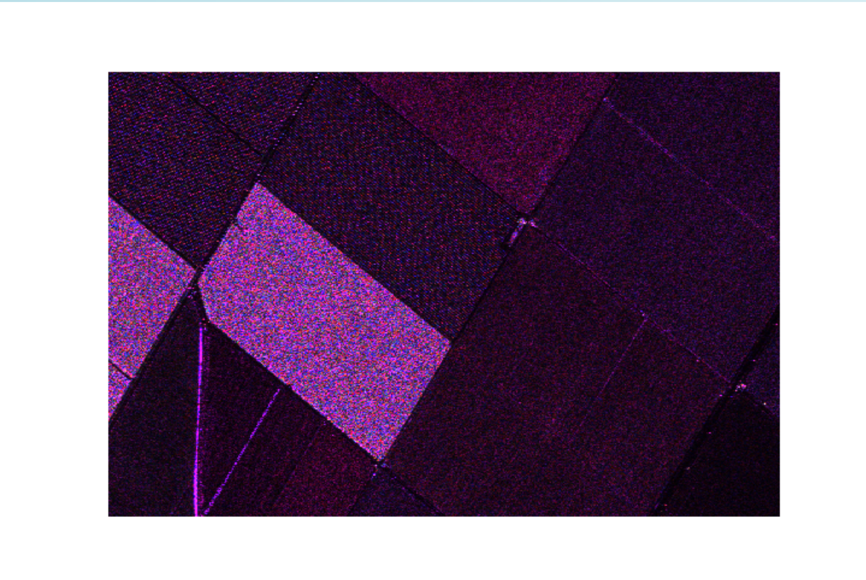

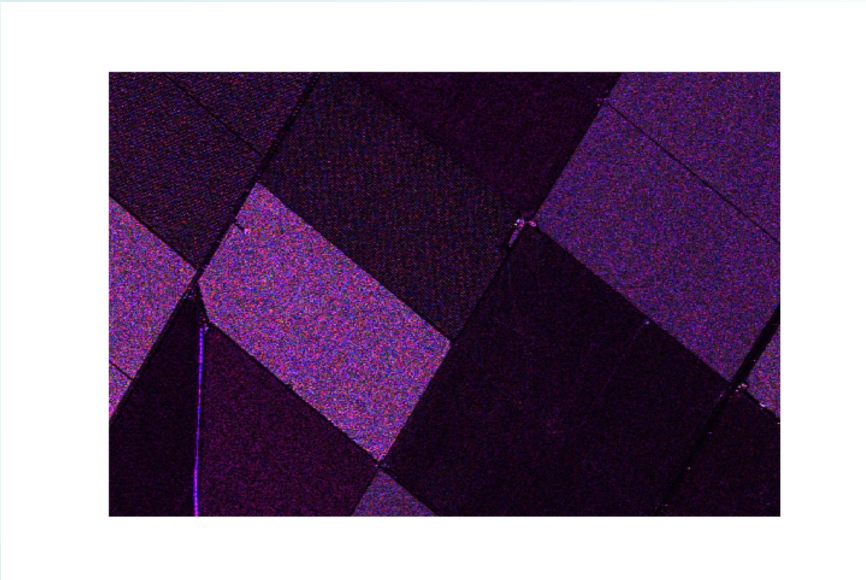

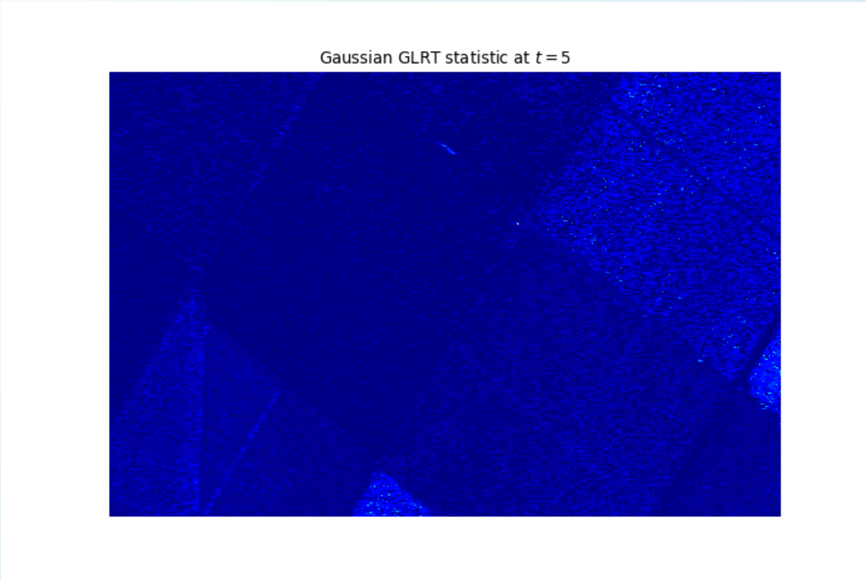

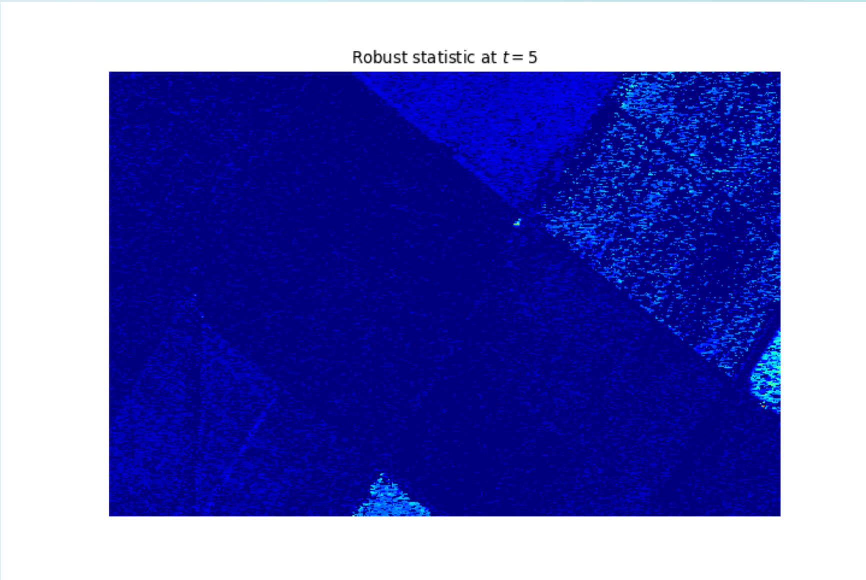

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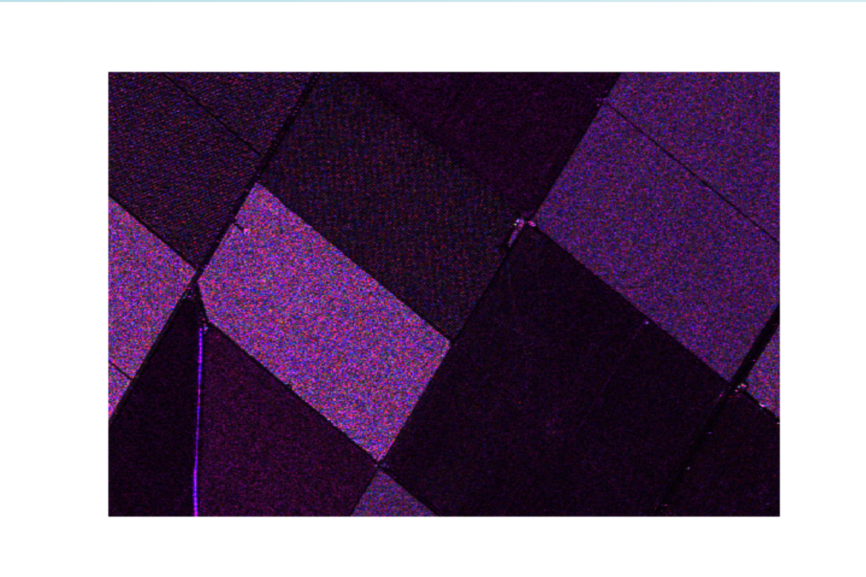

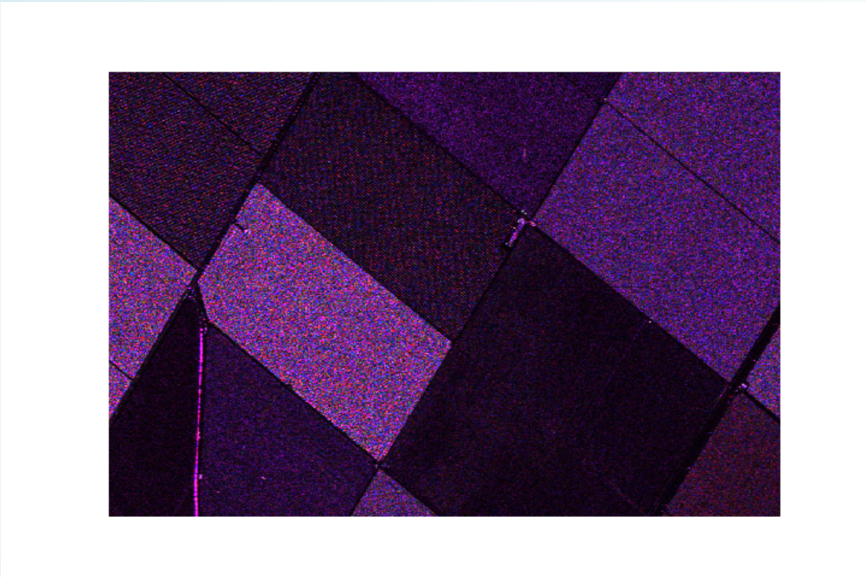

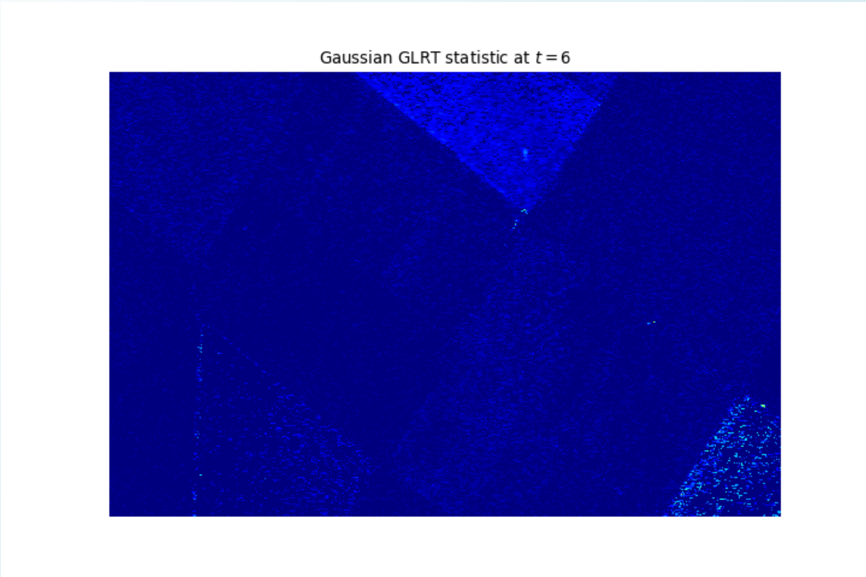

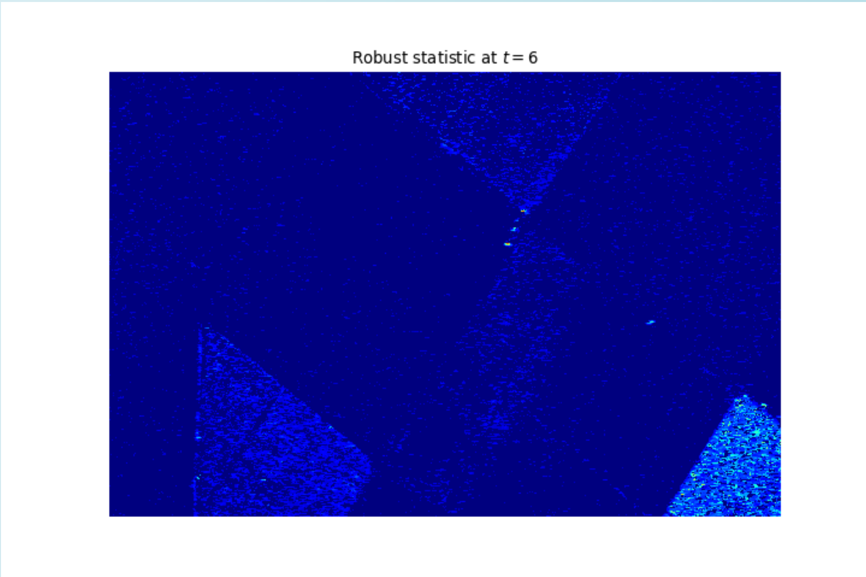

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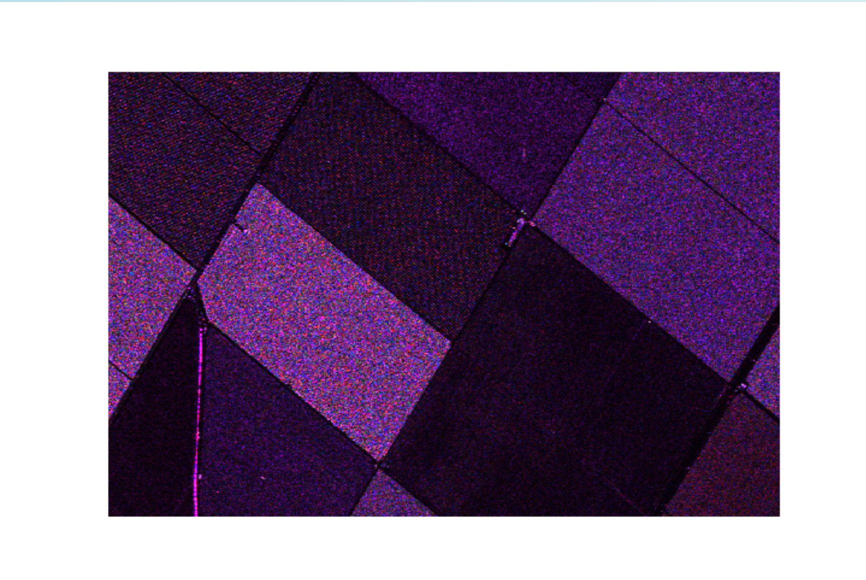

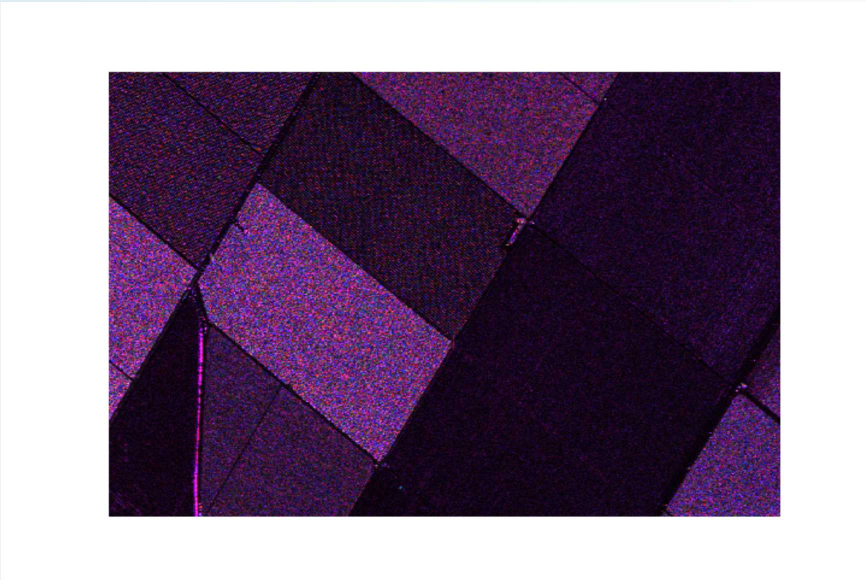

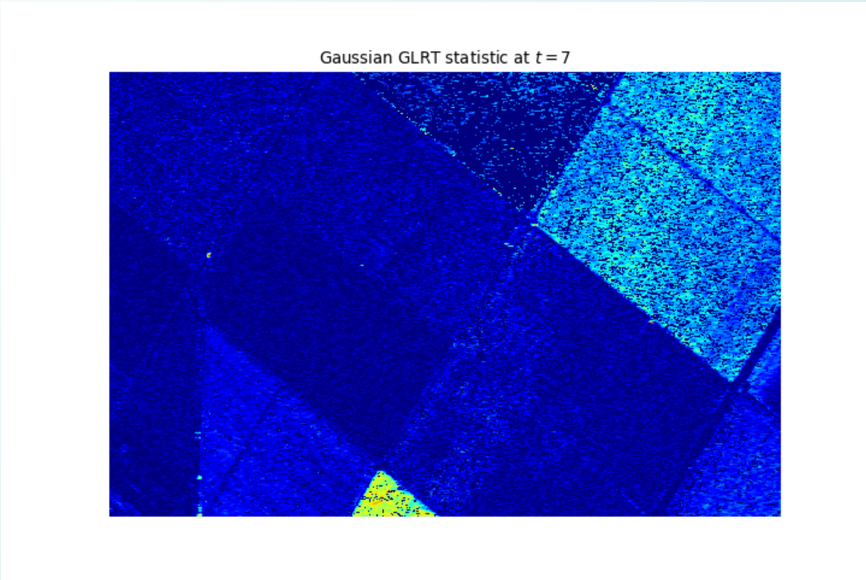

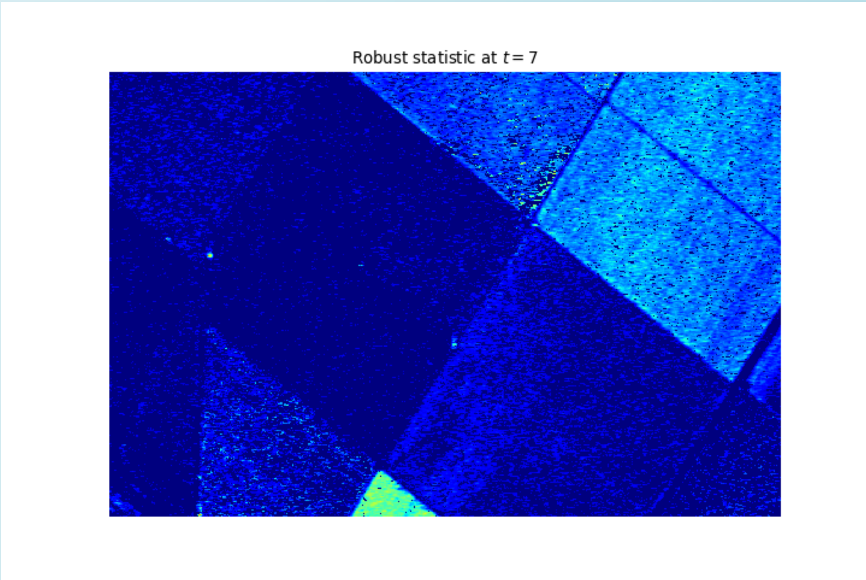

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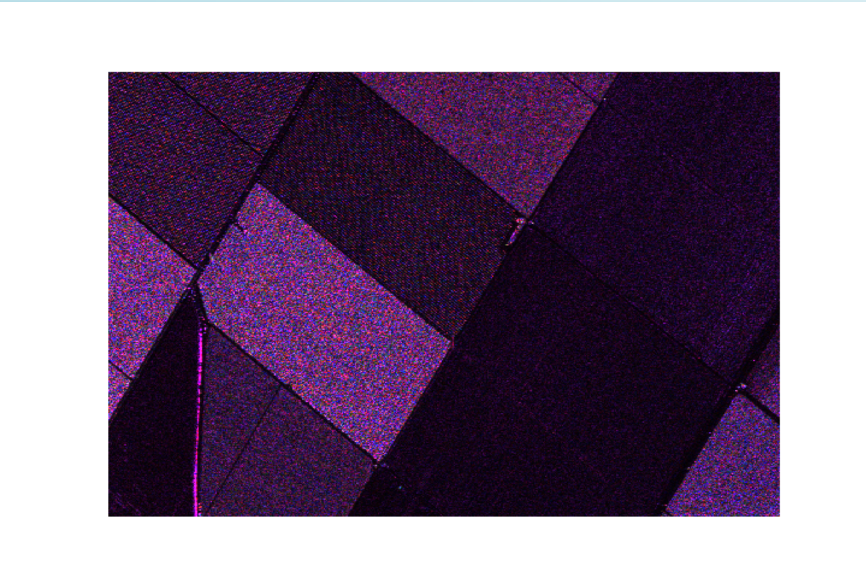

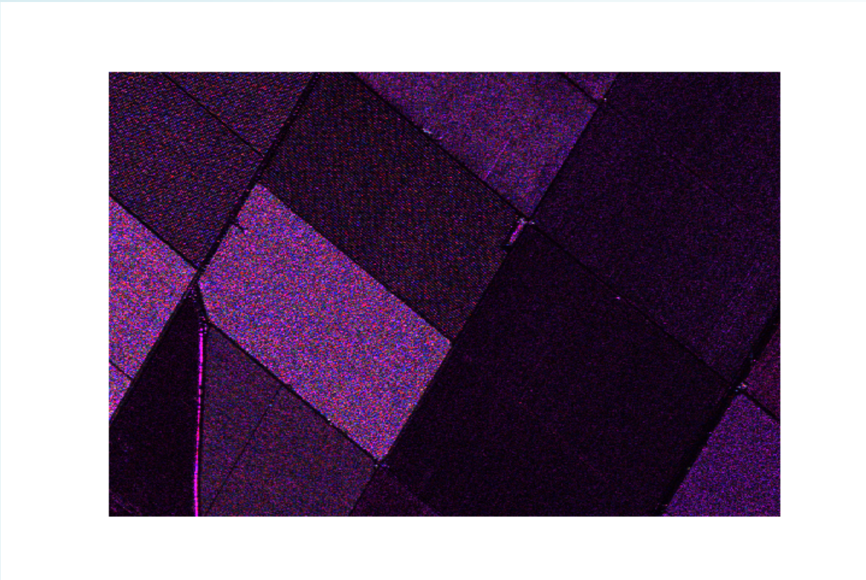

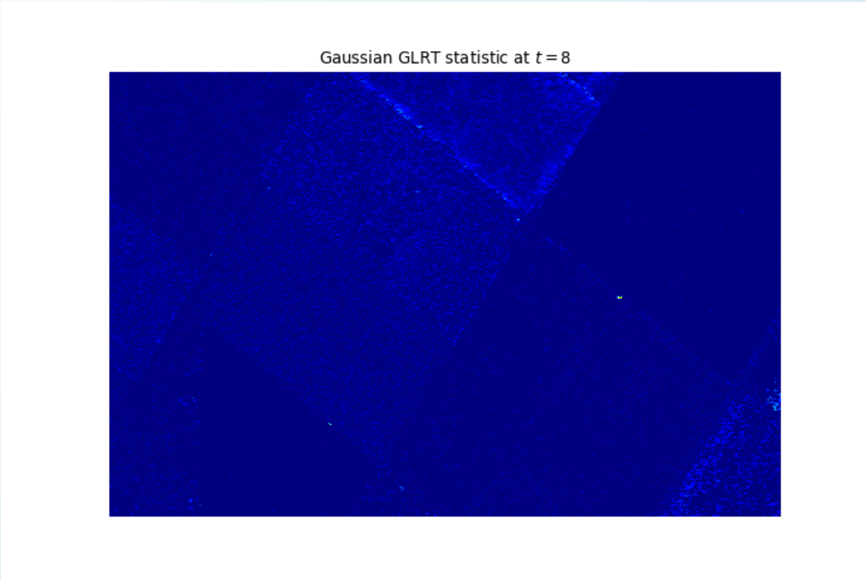

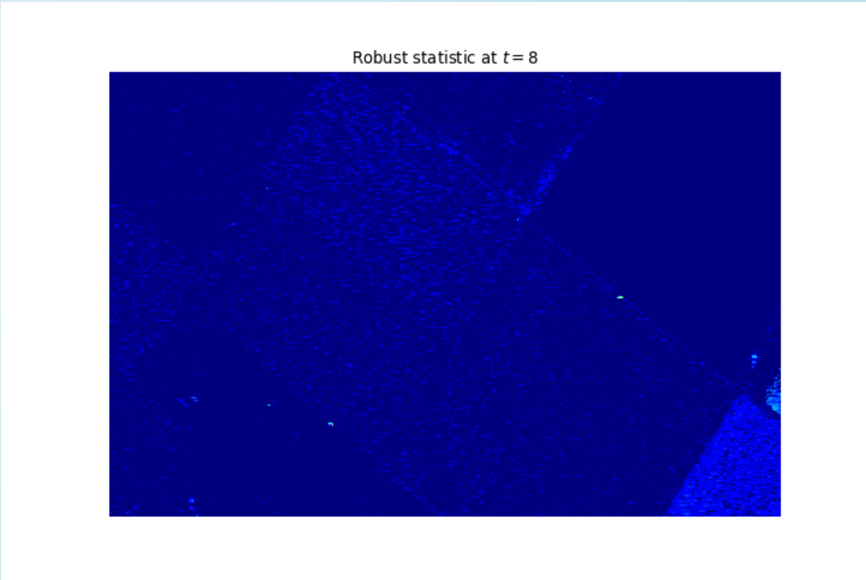

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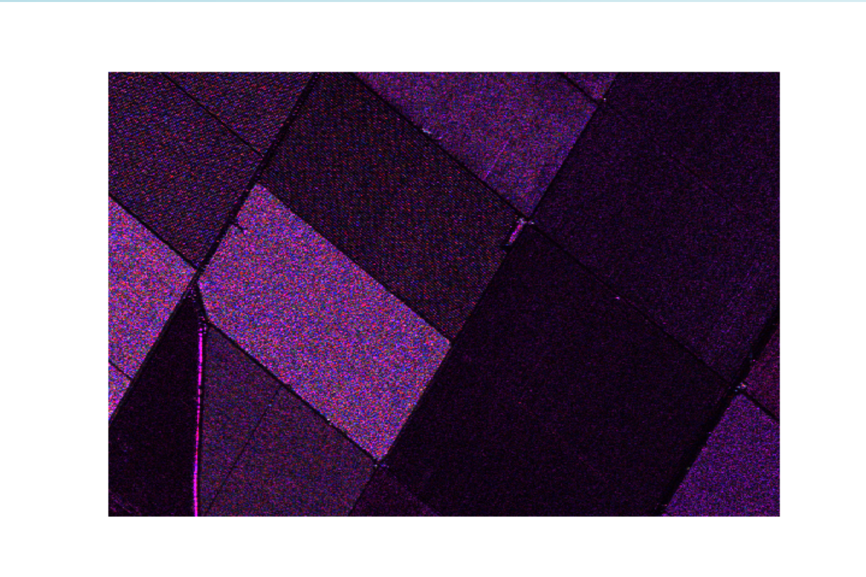

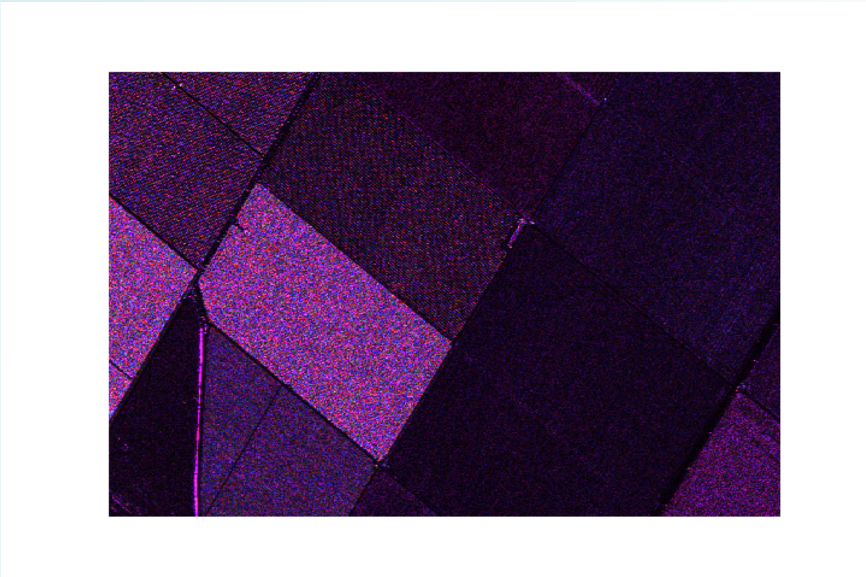

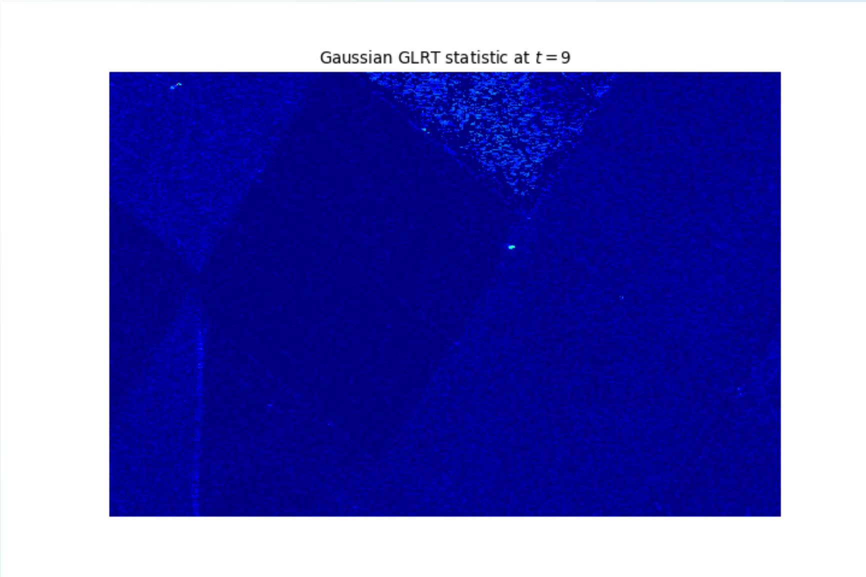

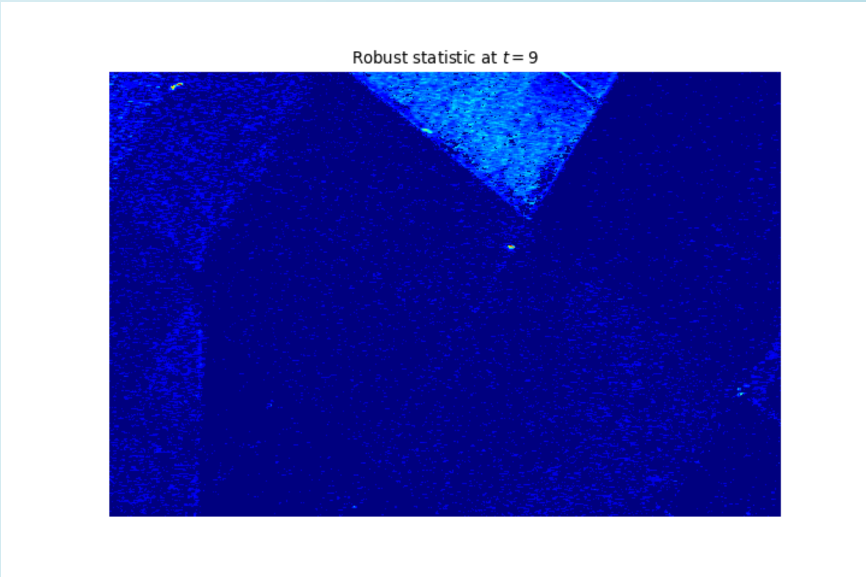

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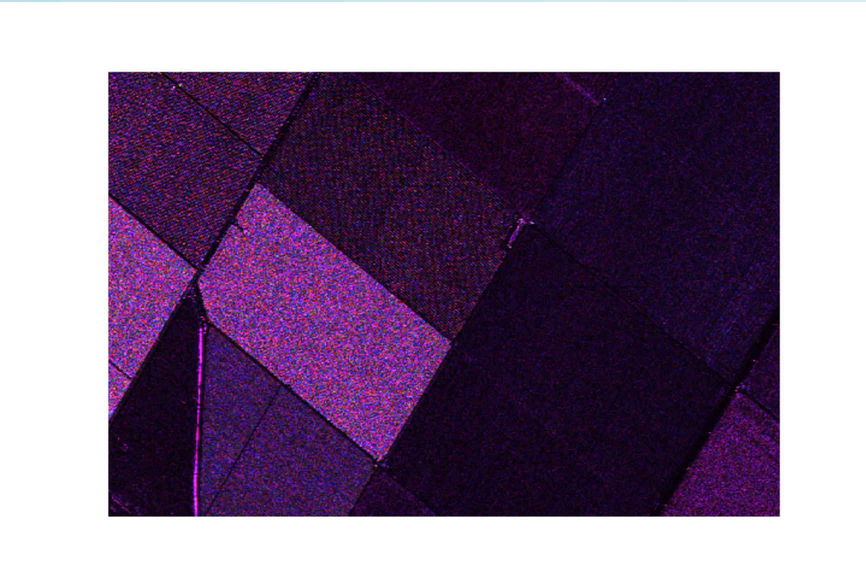

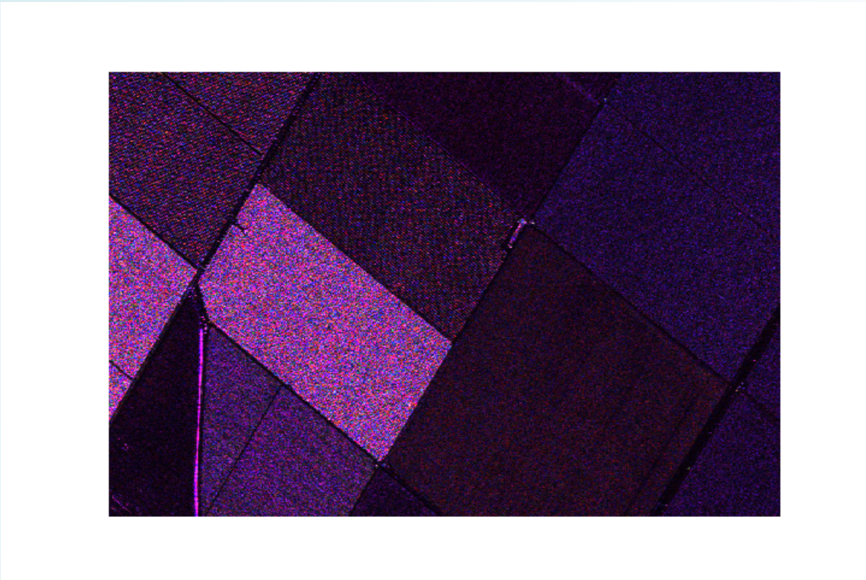

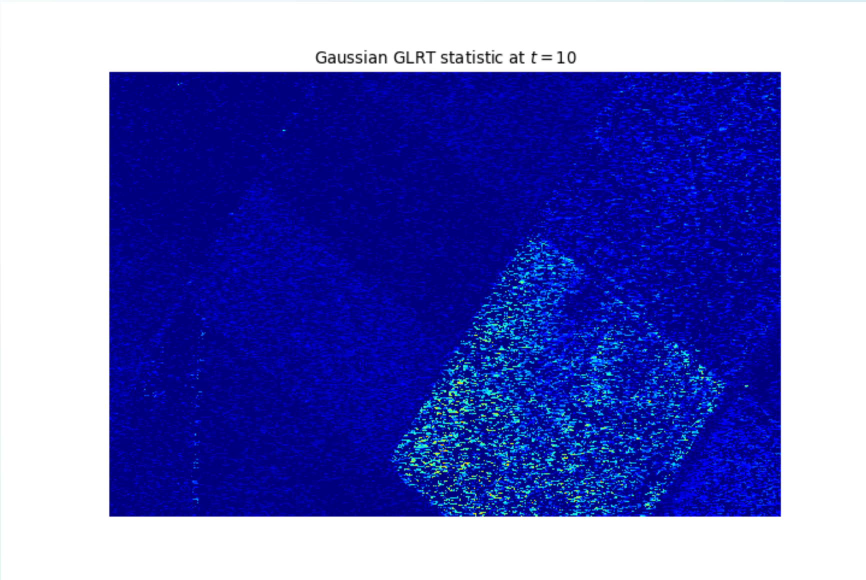

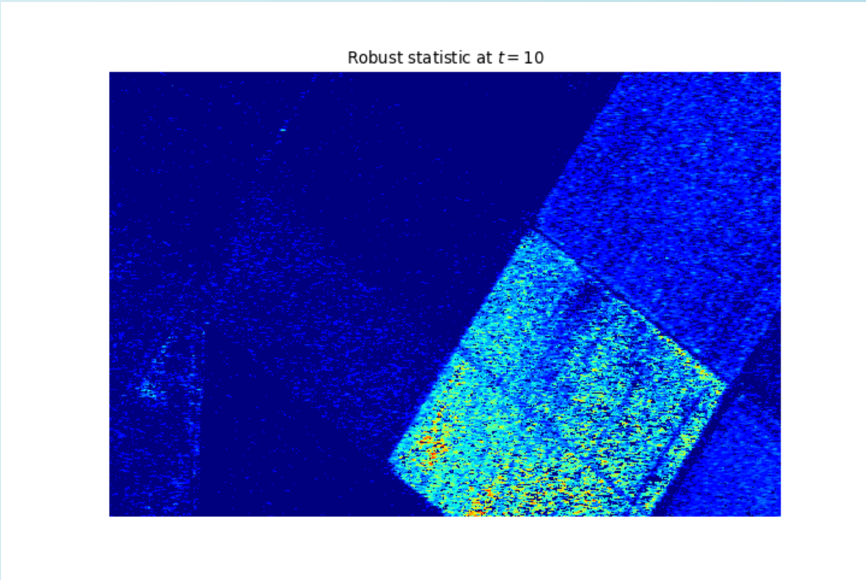

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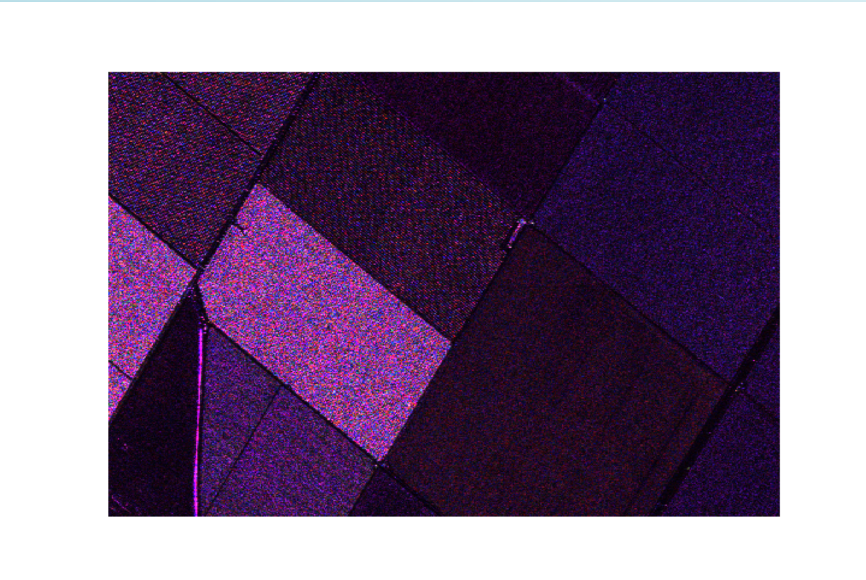

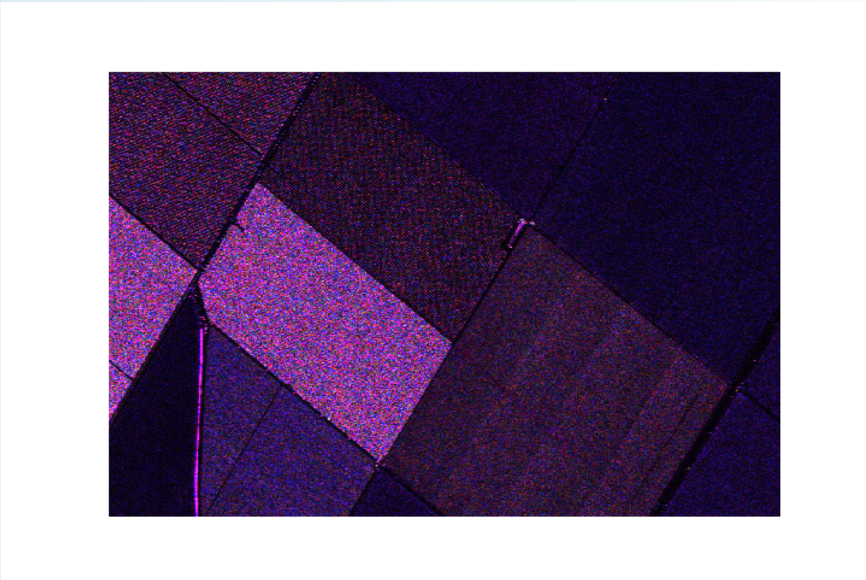

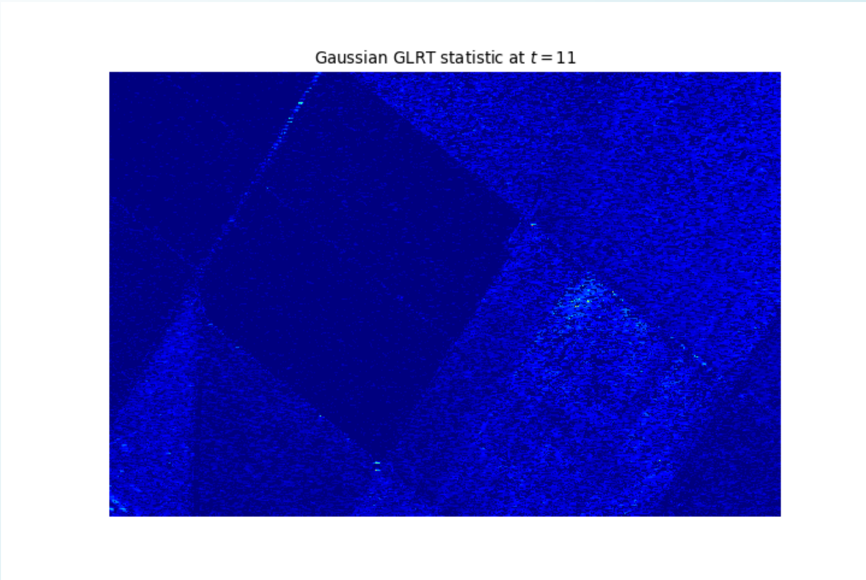

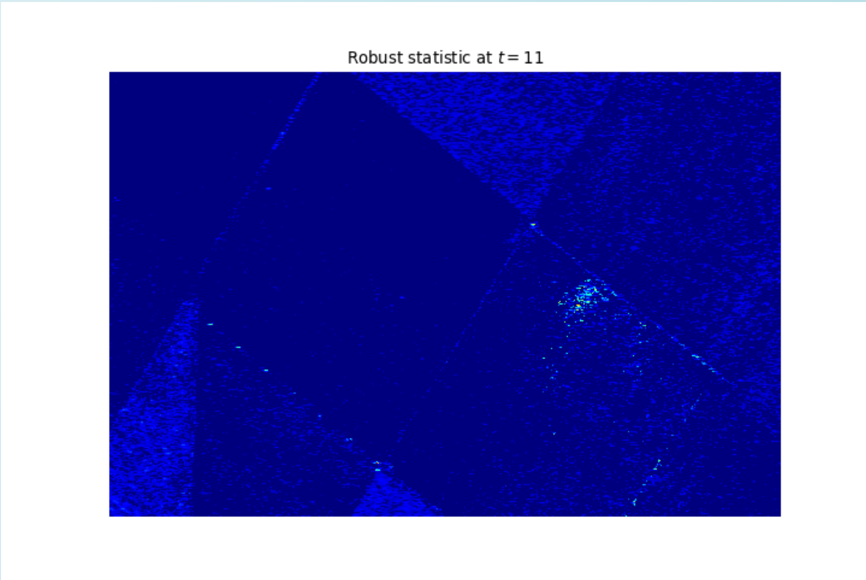

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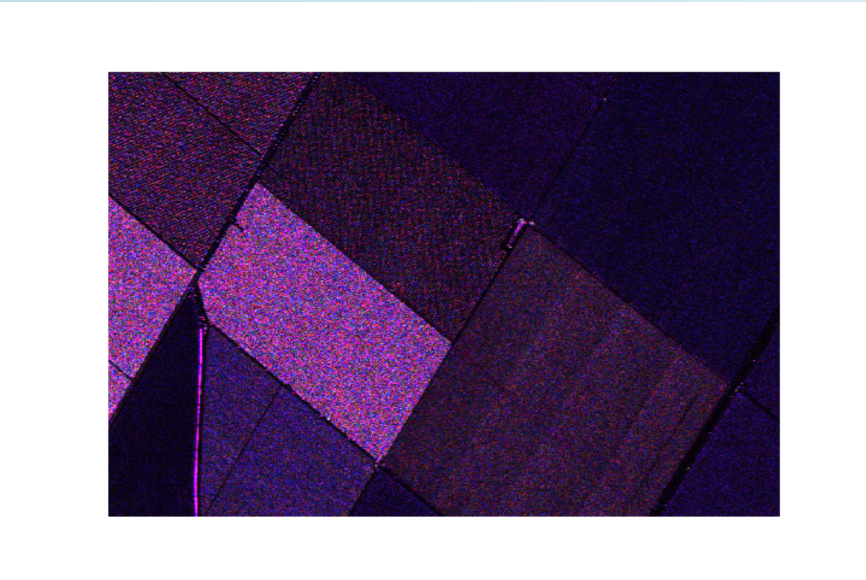

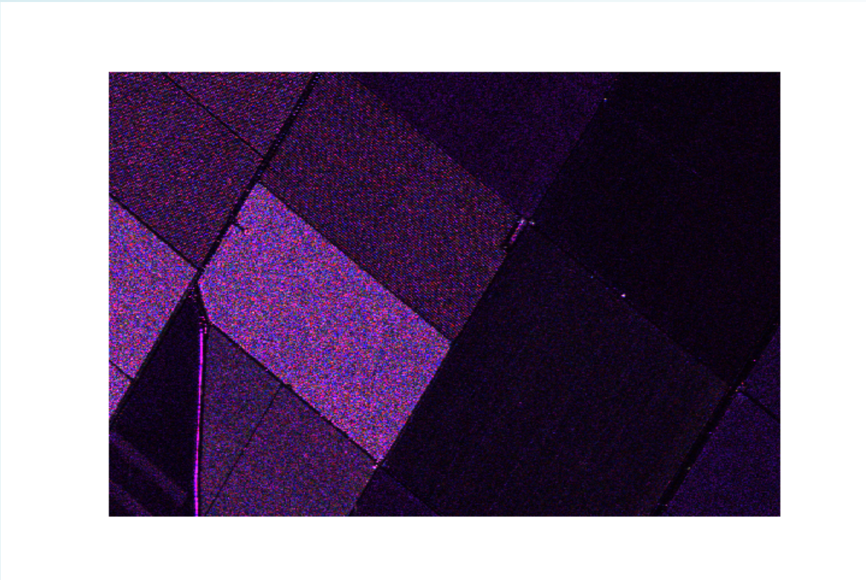

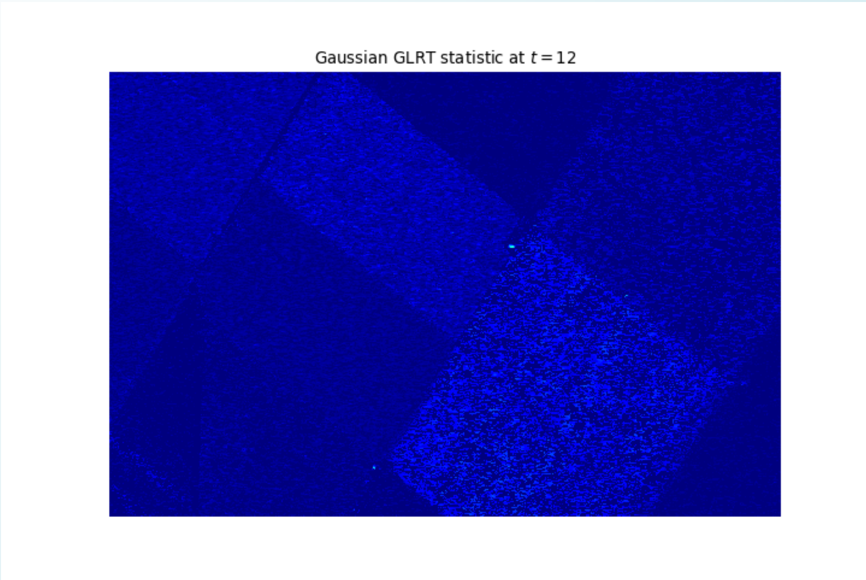

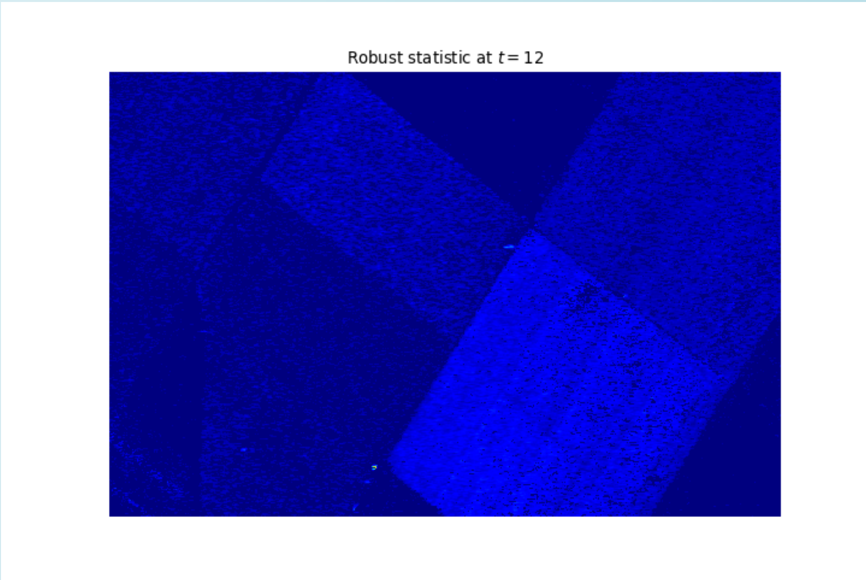

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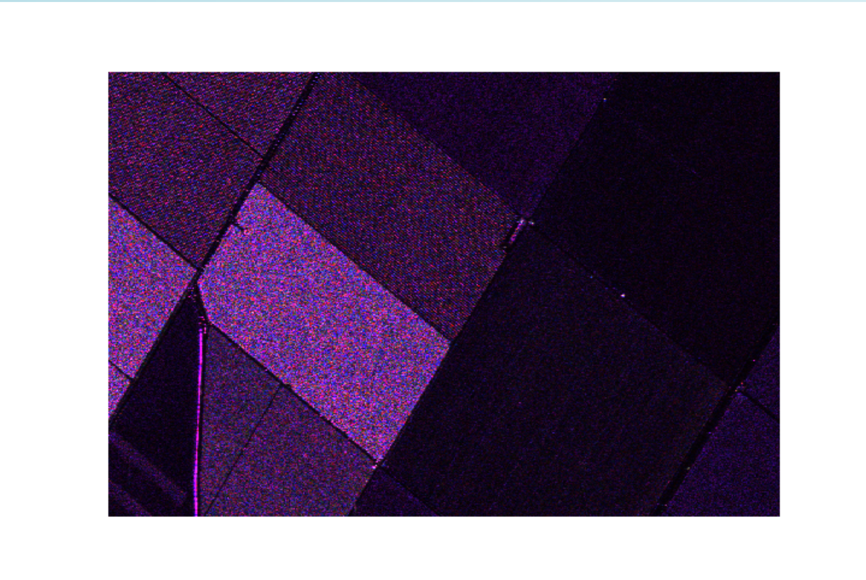

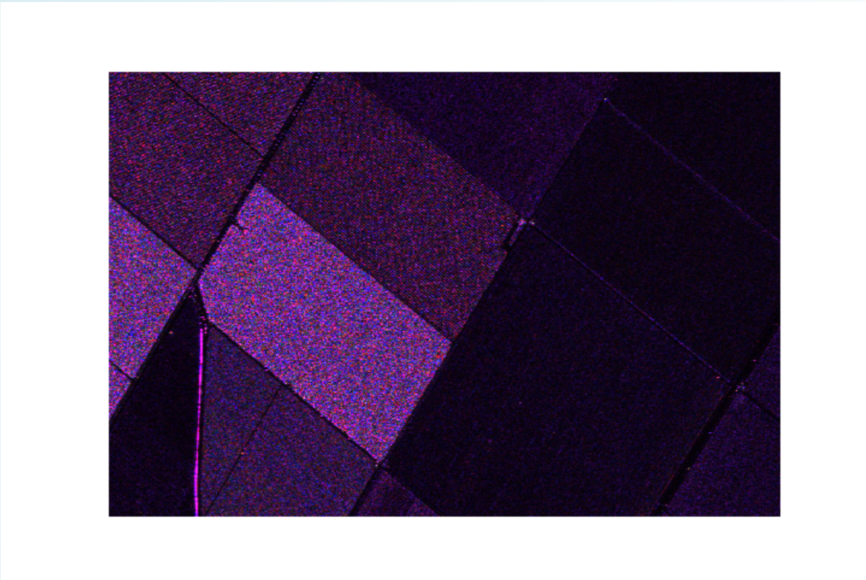

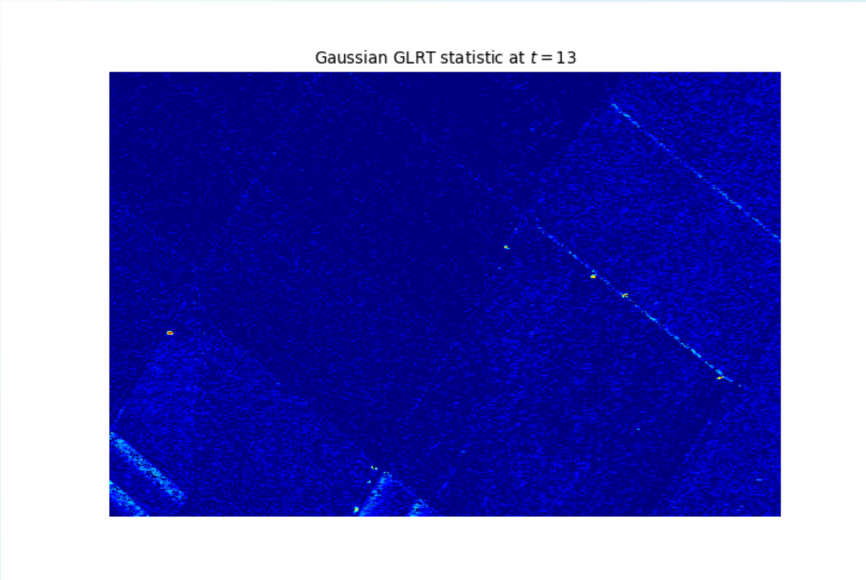

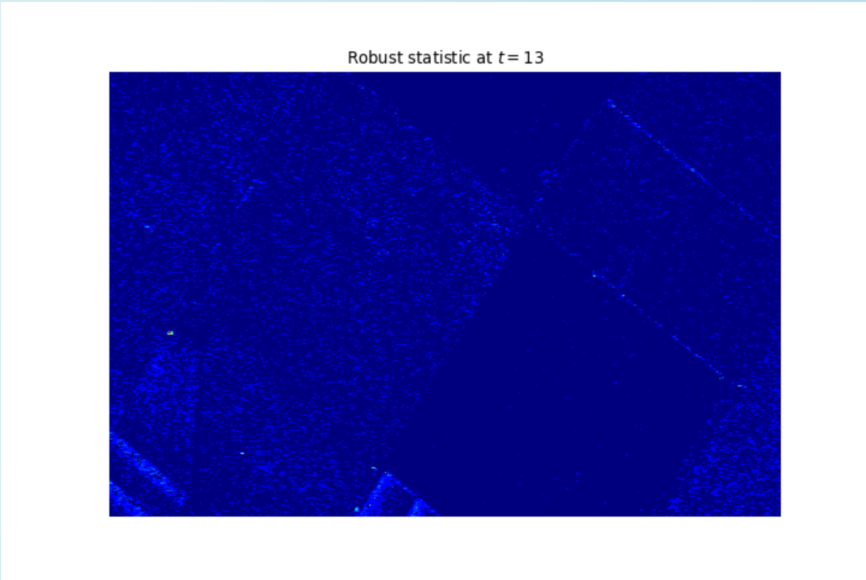

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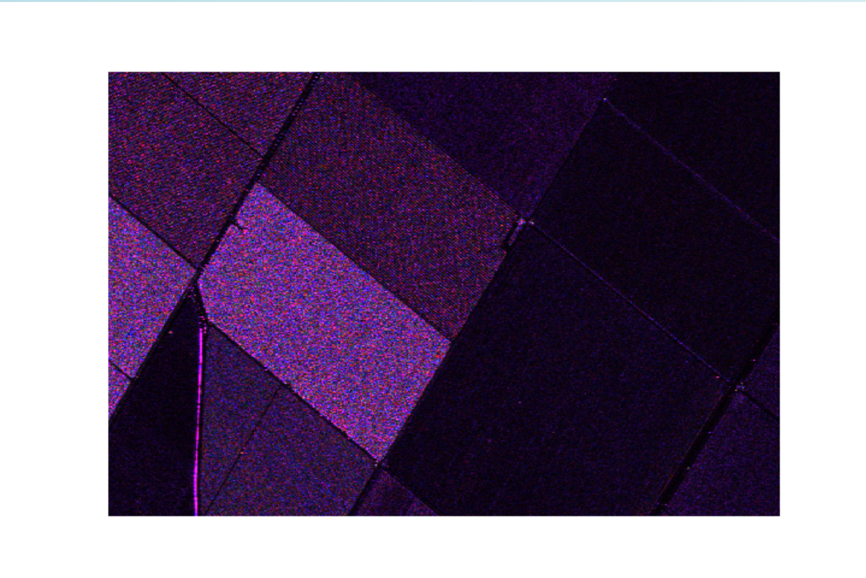

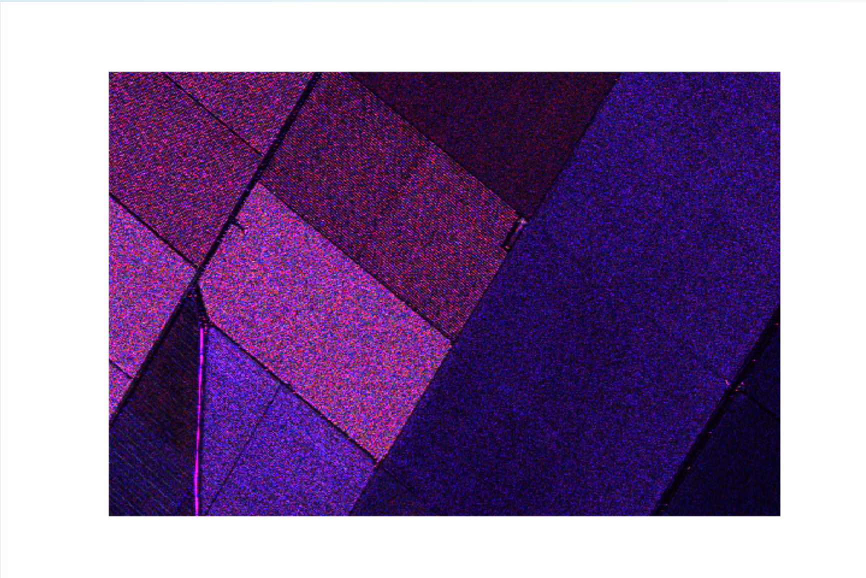

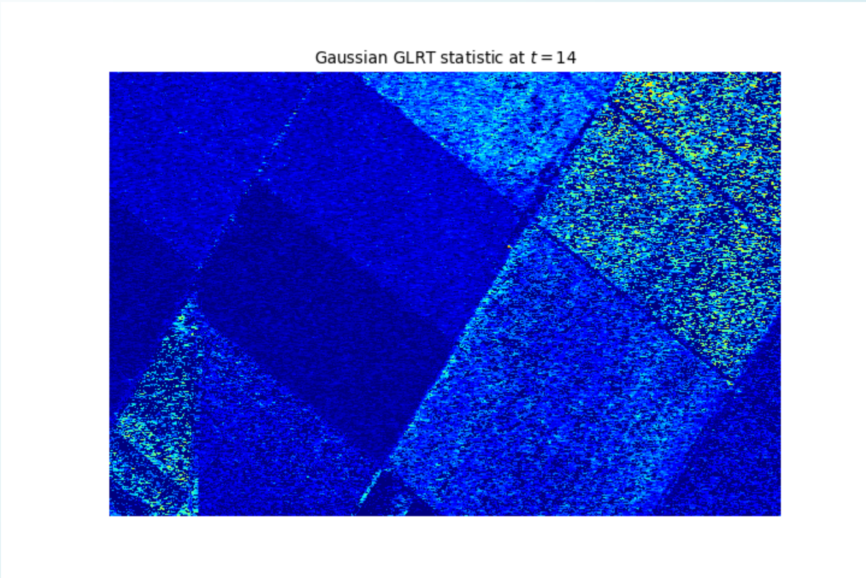

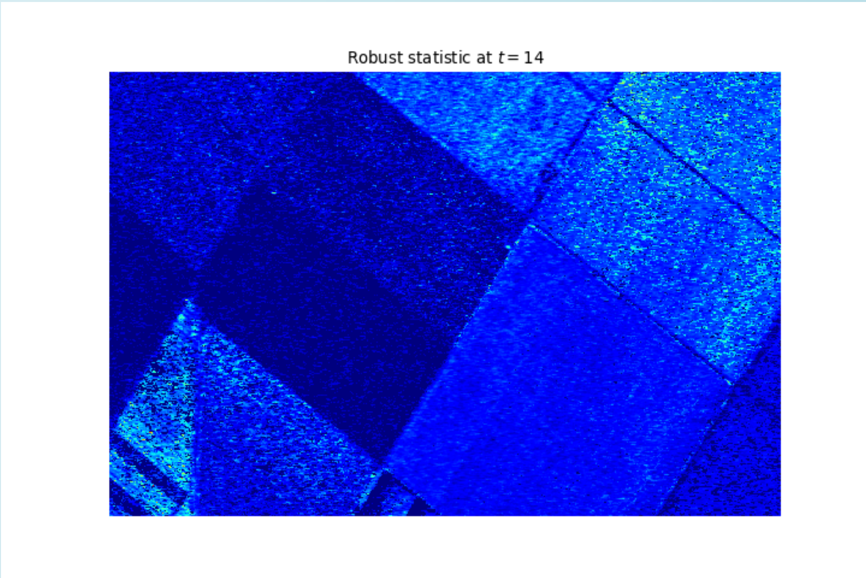

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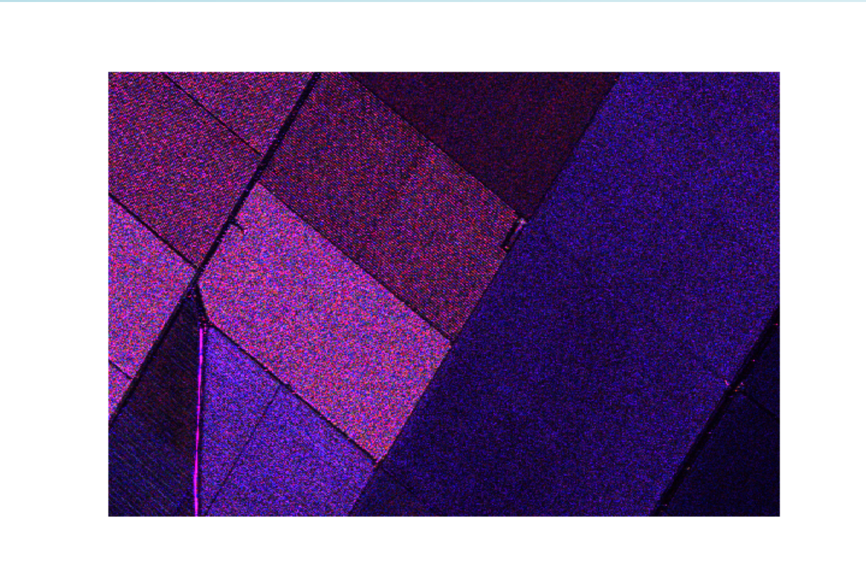

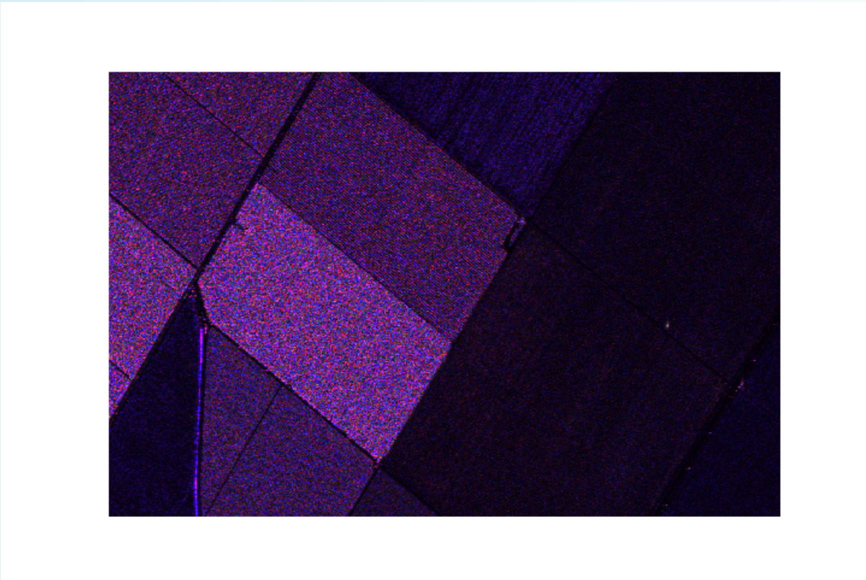

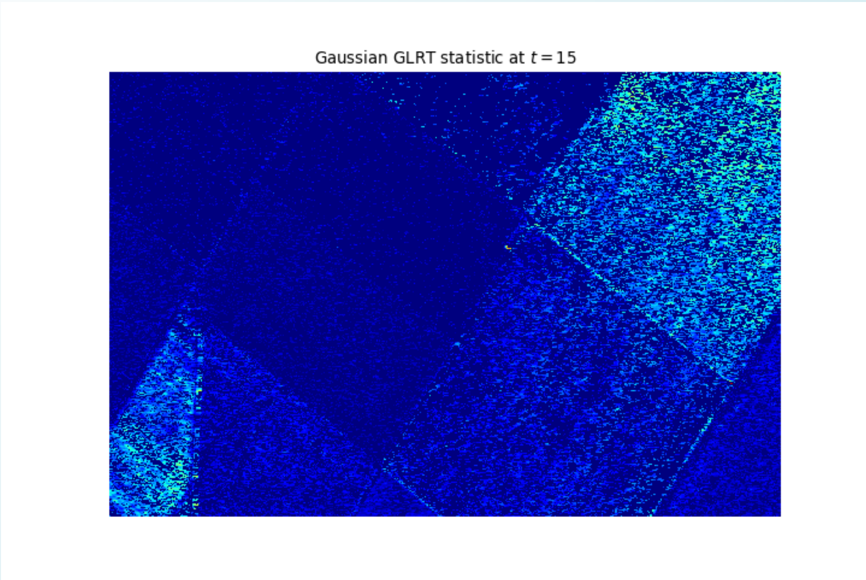

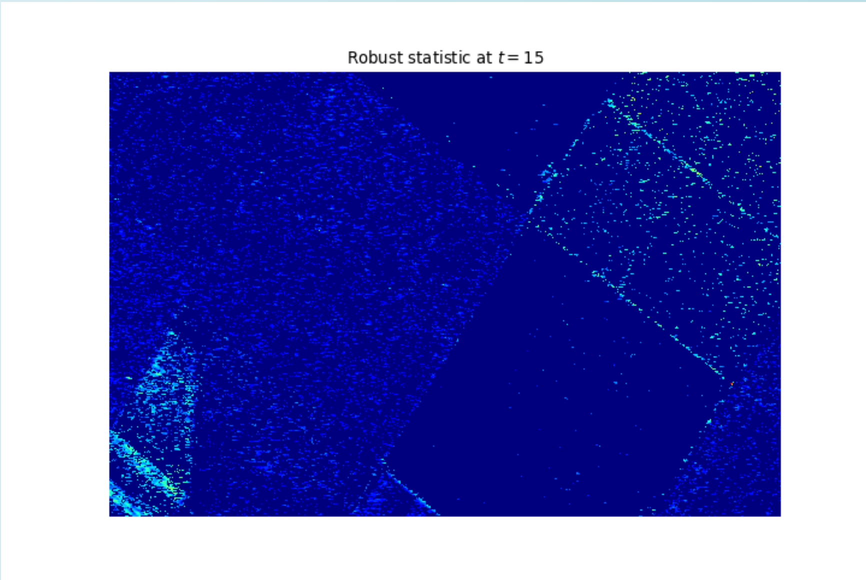

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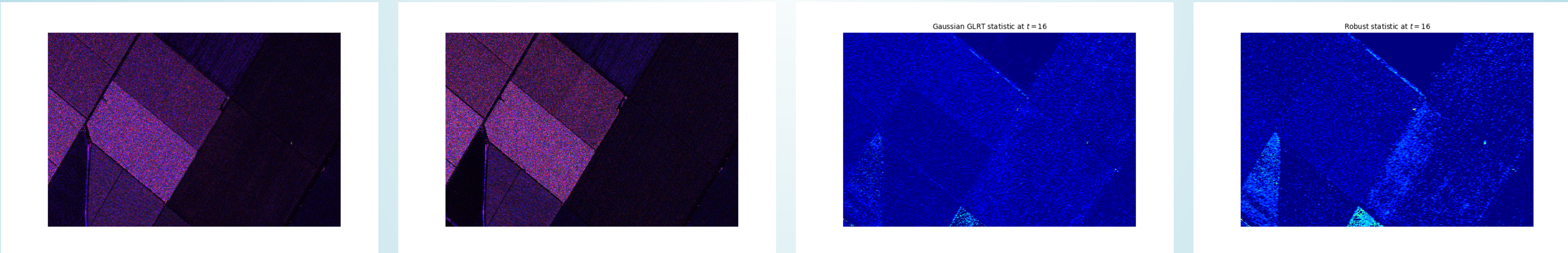

THANKS FOR YOUR ATTENTION !