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Second-order consensus of multiple non-identical agents with non-linear protocols

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Abstract: The second-order consensus of multiple interacting non-identical agents with non-linear protocols is studied in this article. Firstly, it is shown that all agents with different non-linear dynamics can achieve consensus without a leader. Secondly, an explicit expression of the consensus value is analytically developed for the group of all agents. Thirdly, for the consensus of multiple agents with a leader, it is proved that each agent can track the position and velocity of the leader, which are different from those of the follower agents. Finally, numerical simulations are given to illustrate the theoretical results.

1 Introduction

The problem of consensus of multiple agents has attracted great attention over the past decade. This is mainly because that multiple agent systems can be found in many applications, including cooperative and formation control of mobile vehicles, flocking of multiple agents, distributed multi-vehicles coordinated control, design of sensor network and autonomous underwater vehicles. Generally speaking, consensus means that all agents need to reach an agreement on certain quantities of interest (such as the position and velocity) under some control protocols. Vicsek *et al*. proposed a simple model for phase transition of a group of self-driven particles and numerically demonstrated complex dynamics of the model. In this model, each agent's motion is updated by using the local protocol based on the average of their own headings and neighbours. Jadbabaie *et al*. provided a theoretical explanation for the Vicsek model based on algebraic graph theory. Saber and Murry investigated a systematic framework of consensus problems with directed interconnection graphs or time-delays.

To the authors' best knowledge, most existing works concerned with the second-order consensus of multiple agents assumed that all agents have the same dynamics. As a matter of fact, the dynamics of multiple agents composed of a group can be different, for example, each agent can have different mass (inertia) and different velocity gain because of design restriction and inherent property (or the measuring instrument). In addition, non-linear factor inevitably exists as a non-linear function of a variable between the agents during the information exchange. For instance, because of the limitation of observational technique and the inaccuracy of model parameters, the velocity $v_i(t)$ of an agent may be unobservable in some cases. In such cases, instead, a non-linear function $f(v_i)$ of the velocity $v_i(t)$ can be observed (see). On the other hand, the linear parameters of an agent in the formation cannot be known exactly, and the agent is always subjected to external disturbances and some inevitable inaccurate data. Accordingly, in order to achieve high control performance and explore the collective dynamics of multiple agents, the consensus problem under non-linear protocols should be considered.

The main contribution of this paper is to provide an explicit expression of the consensus value, which is not only dependent on the initial positions and velocities, but also the masses (inertia) and velocity gains of all agents. For the case of agents with a leader, it will be shown that all agents can track the position and velocity of the leader, which are different from those of follower agents in the group. It will also be shown that the masses and velocity gains of all agents in the group have significant effects on the final consensus value and their influences are independent. The rest of the paper is organised as follows. Section 2 states the problem formulations and part of graph theory. Section 3 gives the main consensus results of multiple non-identical agents with and without a leader. Some numerical simulations are given in Section 4. Finally, Section 5 presents a brief conclusion to this paper.

2 Problem formulations

Some mathematical notations to be used throughout this paper are given below. Let R define a set of real numbers; R^n be the n-dimensional real vector; $R^{n\times n}$ be the set of $n\times n$ real matrices; and $\bar{n}=\{1,2,\ldots,n\}$ be an index set. To ensure the existence, uniqueness of the solution, and the cooperative property of the non-linear protocol, it is assumed that a set of continuously differentiable functions $f_l(z)$, $h_l(z)$ and $b_i(t)$ $(i\in\bar{N}, l\in\bar{n})$ satisfy the following properties (see and relevant references therein for details):

(i) $f_l(z)=0 \Leftrightarrow z=0;\ zf_l(z)>0,\ \forall z\neq 0;\ f_l'(z)$ denotes the differentiation with respect to z;
(ii) $h_l(-z)=-h_l(z),\ \forall z\in R; h_l(z)=0\Leftrightarrow z=0; (z_j-z_i)h_l(z_j-z_i)>0,\ \forall z_j\neq z_i\in R$;
(iii) $0<b\leq b_i(t)\leq\bar{b}<+\infty,\ \dot{b}_i(t)$ are bounded, where a dot denotes the differentiation with respect to time t.

Consider the system consisting of N agents, and the dynamics of each agent can be expressed as

$$\begin{cases}\dot{p}_i(t)=q_i(t),\\ m_i\dot{q}_i(t)=u_{i1}(t)+u_{i2}(t),\end{cases}\qquad i\in\bar{N}\tag{1}$$

where $p_i(t)=(p_{i1}(t),p_{i2}(t),\ldots,p_{in}(t))^\top\in R^n$ and $q_i(t)=(q_{i1}(t),q_{i2}(t),\ldots,q_{in}(t))^\top\in R^n$ are the position and velocity of agent i, respectively, m_i is the mass (inertia) of agent i, $u_{i1}(t)\in R^n$ and $u_{i2}(t)\in R^n$ are control inputs (control protocols). Here, the control protocols are assumed to be of the form

$$\begin{aligned}u_{i1}(t)&=-b_i(t)\,f(q_i),\\ u_{i2}(t)&=\sum_{j\in\mathcal{N}_i}c_{ij}h(p_j(t)-p_i(t)),\quad i\in\bar{N}\end{aligned}\tag{2}$$

where $b_i(t)$ is the velocity gain of agent i, $f(q_i) = (f_1(q_{i1}), f_2(q_{i2}), \ldots, f_n(q_{in}))^\top \in R^n$ and $h(p_j(t) - p_i(t)) = (h_1(p_{j1} - p_{i1}), h_2(p_{j2} - p_{i2}), \ldots, h_n(p_{jn} - p_{in}))^\top \in R^n$ are strictly increasing non-linear functions with $f_l(z)$, $h_l(z)$ and $b_i(t)$ satisfying the above assumptions (i)–(iii). The main aim of the paper is to determine $u_{i1}(t)$ and $u_{i2}(t)$ for all agents without a leader to satisfy

$$\lim_{t\to+\infty}(p_{jl}(t) - p_{il}(t)) = 0,$$
$$\lim_{t\to+\infty} q_{il}(t) = 0, \quad i, j \in \bar{N},\ l \in \bar{n}$$

and for all agents in the presence of a leader, to design the control protocols

$$u_{i1}(t) = -b_i(t) f(q_i),$$
$$u_{i2}(t) = \sum_{j\in\mathcal{N}_i} c_{ij} h(p_j(t) - p_i(t)) + c_{i\mathrm{L}} h(p_\mathrm{L}(t) - p_i(t)),\ \ i \in \bar{N} \tag{3}$$

such that

$$\lim_{t\to+\infty}(p_{il}(t) - p_{\mathrm{L}l}(t)) = 0,$$
$$\lim_{t\to+\infty}(q_{il}(t) - q_{\mathrm{L}l}(t)) = 0, \quad i \in \bar{N},\ l \in \bar{n}$$

where $c_{i\mathrm{L}} > 0$, if there is a connection from the leader to agent i, and $c_{i\mathrm{L}} = 0$, if no connection exists from the leader. $p_\mathrm{L}(t) \in R^n$ and $q_\mathrm{L}(t) \in R^n$ denote the position and velocity of the leader, respectively. The dynamics of the leader is governed by

$$\begin{cases} \dot{p}_\mathrm{L}(t) & = q_\mathrm{L}(t) \\ \dot{q}_\mathrm{L}(t) & = -b_\mathrm{L}(t) f(q_\mathrm{L}) \end{cases} \tag{4}$$

It should be noted that control input $u_{i2}(t)$ is a local control protocol with a neighbour-to-neighbour interaction between agents. Each agent (for example, agent i) can update its current velocity based on the position information of its neighbours. The protocol $u_{i2}(t)$ also includes some non-linear properties, such as uncertain linear model parameters and unpredictable external disturbances . Protocol $u_{i2}(t)$ requires neighbour's current positions only and does not need neighbour's other information.

In the analysis of the convergence conditions for multiple agents, the undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, C\}$ will be used to model the interaction among N agents. $\mathcal{V} = \{1, 2, \ldots, N\}$ is the node (agent) set; an edge of $\mathcal{G}$ is denoted by $e_{ij} = (i, j)$ which means nodes j and i can receive information from each other; a set of edges $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} | i \sim j\}$ containing unordered pairs of nodes represents communication links; a non-negative adjacency matrix C is defined as: $c_{ij} = c_{ji} > 0$ if $e_{ij} \in \mathcal{E}$ (there is a connection between agents i and j), while $c_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$ (there is no connection between agents i and j). The network neighbours of agent i are assumed to belong to a set $\mathcal{N}_i : \mathcal{N}_i = \{\, j \,|\, (i, j) \in \mathcal{E}\} \subseteq \mathcal{V}\backslash\{\, i \,\}$. The Laplacian matrix $\mathcal{L} = (l_{ij}) \in R^{N\times N}$ associated with $\mathcal{G}$ is defined as $l_{ii} = \sum_{j\in\mathcal{N}_i} c_{ij}$ and $l_{ij} = -c_{ij}$ $(i \neq j)$. A path of $\mathcal{G}$ is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_j \in \mathcal{V}$. An undirected (directed) graph is said to be connected (strongly connected), if there exists a path between any two distinct vertices of the graph. If there exists a path from agent i to agent j, then agent j is said to be reachable from agent i. For graph $\mathcal{G}$, if there is a path from every agent to agent r, then it can be said that agent r is globally reachable in $\mathcal{G}$. A graph $\mathcal{G}$ associated with the system consisting of N agents and one leader (in the leader case) will be considered in subsequent sections.

3 Consensus of non-identical agents with non-linear protocols

3.1 Consensus of non-identical agents without a leader

Theorem 1. *Consider system (1) with control protocol (2) and assume that the undirected graph $\mathcal{G}$ is connected. Then, $\lim_{t\to+\infty}(p_{jl}(t) - p_{il}(t)) = 0$, $\lim_{t\to+\infty} q_{il}(t) = 0$ $(i, j \in \bar{N}, l \in \bar{n})$.*

Proof: Select the Lyapunov function as

$$V(t, p_i(t), q_i(t)) = \frac{1}{2}\sum_{i=1}^{N} m_i q_i^\top(t) q_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} \times \int_0^{p_{jl}(t)-p_{il}(t)} h_l(s)\,\mathrm{d}s \tag{5}$$

Differentiating $V(t, p_i(t), q_i(t))$ with respect to time along the solution of (1) yields

$$\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} &= \sum_{i=1}^{N} m_i q_i^\top(t)\dot{q}_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} h_l(p_{jl}(t) \\
&\quad - p_{il}(t))(q_{jl}(t) - q_{il}(t)) \\
&= \sum_{i=1}^{N} m_i q_i^\top(t)\dot{q}_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} h_l(p_{jl}(t) \\
&\quad - p_{il}(t)) q_{jl}(t) \\
&\quad - \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} h_l(p_{jl}(t) - p_{il}(t)) q_{il}(t) \\
&= \sum_{i=1}^{N} m_i q_i^\top(t)\dot{q}_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ji} h_l(p_{il}(t) \\
&\quad - p_{jl}(t)) q_{il}(t) \\
&\quad - \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} h_l(p_{jl}(t) - p_{il}(t)) q_{il}(t) \\
&= \sum_{i=1}^{N} m_i q_i^\top(t)\dot{q}_i(t) - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} h_l(p_{jl}(t) \\
&\quad - p_{il}(t)) q_{il}(t) \\
&= -\sum_{i=1}^{N} b_i(t) q_i^\top(t)\, f(q_i) + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} h_l(p_{jl}(t) \\
&\quad - p_{il}(t)) q_{il}(t) \\
&\quad - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij} h_l(p_{jl}(t) - p_{il}(t)) q_{il}(t) \\
&\leq -b\sum_{i=1}^{N} q_i^\top(t)\, f(q_i) \leq 0
\end{aligned} \tag{6}$$

then $\dot{V}(t, p_i(t), q_i(t)) \leq 0$ and $\int_{t_0}^{t} g(s)\,ds \leq V(t_0, p_i(t_0), q_i(t_0)) - V(t, p_i(t), q_i(t))$, where $g(t) = b\sum_{i=1}^{N}\sum_{l=1}^{n} q_{il}(t) f_l(q_{il})$. Furthermore, $\lim_{t\to+\infty}\int_{t_0}^{t} g(s)\,\mathrm{d}s \leq V(t_0, p_i(t_0), q_i(t_0)) < +\infty$ exists and is finite. By considering Lyapunov function (5) and equality (6) as well as the properties of $f_l(z)$, $h_l(z)$ and $b_i(t)$, it can be seen that $\dot{g}(t) = b\sum_{i=1}^{N}\sum_{l=1}^{n} 1/m_i[-b_i(t) f_l(q_{il}) + \sum_{j\in\mathcal{N}_i} c_{ij} h_l(p_{jl}(t) - p_{il}(t))][f_l(q_{il}) + q_{il}(t) f_l'(q_{il})]$ is bounded.

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Therefore $g(t)$ is uniformly continuous in t and

$$m_i\dot{q}_{il}(t) = -b_i(t)f_l(q_{il}) + \sum_{j\in\mathcal{N}_i} c_{ij}h_l(p_{jl}(t) - p_{il}(t)),$$
$$i, j \in \bar{N},\ l \in \bar{n} \tag{7}$$

thus $\lim_{t\to+\infty} g(t) = 0$ by applying the Barbalat lemma , which further gives $\lim_{t\to+\infty} q_{il}(t) = 0$. On the other hand, $\lim_{t\to+\infty} \dot{q}_{il}(t) = 0$ since $\dot{q}_{il}(t)$ is uniformly continuous in t. Taking the limit as $t \to +\infty$ on both sides of (7) gives rise to the expression $\lim_{t\to+\infty} \dot{q}_{il}(t) = \lim_{t\to+\infty}\{\sum_{j\in\mathcal{N}_i} c_{ij}h_l(p_{jl}(t) - p_{il}(t))\} = 0$, then $\lim_{t\to+\infty}(p_{jl}(t) - p_{il}(t)) = 0$. □

Corollary 1. *Assume $f(q_i) = q_i(t)$, $b_i(t) = b_i$ and suppose that the undirected graph $\mathcal{G}$ is connected. Then*

$$p_i(t) \to \frac{1}{\beta}\sum_{i=1}^{N}(b_i p_i(0) + m_i q_i(0)), \quad i \in \bar{N}$$

as $t \to +\infty$, where $\beta = \sum_{i=1}^{N} b_i$, $p_i(0)$ and $q_i(0)$ are the initial position and velocity of $p_i(t)$ and $q_i(t)$, respectively.

Proof: Note that $\alpha(t) = \sum_{i=1}^{N}(b_i p_i(t) + m_i q_i(t))$ is an invariant quantity and its first-order derivative $\dot{\alpha}(t) = 0$ because of the symmetry of matrix $C = (c_{ij})^{N\times N}$. Applying $\alpha(0) = \alpha(+\infty)$ and the above results yields $p_i(t) \to 1/\beta\sum_{i=1}^{N}(b_i p_i(0) + m_i q_i(0))$ as $t \to +\infty$. □

Remark 1. *Lyapunov function $V(t, p_i(t), q_i(t))$ is positively definite since the properties of $h_l(z)$, and $V(t, p_i(t), q_i(t)) = 0$ if and only if $q_{il}(t) = 0$ and $p_{jl}(t) - p_{il}(t) = 0$ ($i, j \in \bar{N},\ l \in \bar{n}$).*

Remark 2. *In proving Theorem 1, the Barbalat lemma has been used instead of Lasalle Invariance Principle since system (1) is a time-varying system. Theorem 1 indicates that the asymptotic consensus is reachable even if all agents are non-identical. Corollary 1 presents an explicit expression of the consensus value, which depends on the masses (inertial) and velocity gains, the initial positions and velocities of all agents. For given initial conditions, suitable b_i and m_i can be easily selected such that all agents achieve a desired consensus value. Moreover, the effects of m_i and b_i on the final consensus value are independent because of symmetric interaction of topology C.*

By following a similar analysis to the one developed in Theorems 1, system (1) can be extended to the directed interconnected systems satisfying detailed balance conditions .

3.2 Consensus of non-identical agents with a leader

In some cases, all agents are required to achieve a desired consensus value, which is irrelevant to agents' initial conditions. This situation is usually referred to as the leader-following consensus in the literature. The dynamics of the leader to be considered here is non-linear and different from that of any other agents. All other agents need to reach the same dynamics. To address the consensus of non-identical agents with a leader, system (1) with a virtual leader labelled as 'L' with dynamics (4) will be investigated in this subsection ($m_i = 1, i \in \bar{N}$).

Theorem 2. *Consider system (1) with control protocol (3). Suppose that $b_{\mathrm{L}}(t)$ has the same properties with $b_i(t) (i \in \bar{N})$, $f_l(z)$ satisfying $k \le f_l(z)/z \le \bar{k}$ ($\bar{k} \ge k > 0, l \in \bar{n}$). Then, $\lim_{t\to+\infty}(p_{il}(t) - p_{\mathrm{L}\,l}(t)) = 0$, $\lim_{t\to+\infty}(q_{il}(t) - q_{\mathrm{L}\,l}(t)) = 0$ provided that the leader has a path to all agents in graph $\mathcal{G}$.*

Proof: Denote the position and velocity error functions between agent i and the leader 'L' as $\hat{p}_i(t) = p_i(t) - p_{\mathrm{L}}(t)$, $\hat{q}_i(t) = q_i(t) - q_{\mathrm{L}}(t)$ ($i \in \bar{N}$), respectively. Then the error systems obtained from (1), (3) and (4) can be written as

$$\begin{cases} \dot{\hat{p}}_i(t) = \hat{q}_i(t), \\ \dot{\hat{q}}_i(t) = -b_i(t)f(\hat{q}_i(t) + q_{\mathrm{L}}(t)) + b_{\mathrm{L}}(t)\,f(q_{\mathrm{L}}) \\ \quad + \sum_{j\in\mathcal{N}_i} c_{ij}\,h(\hat{p}_j(t) - \hat{p}_i(t)) - c_{i\mathrm{L}}\,h(\hat{p}_i), \end{cases} \quad i \in \bar{N} \tag{8}$$

Select the Lyapunov function as

$$\begin{aligned} &V(t, q_{\mathrm{L}}(t), \hat{p}_i(t), \hat{q}_i(t)) \\ &\quad = \frac{M}{2bk}q_{\mathrm{L}}^\top(t)q_{\mathrm{L}}(t) + \frac{1}{bk}\sum_{i=1}^{N}\hat{q}_i^\top(t)\hat{q}_i(t) \\ &\qquad + \frac{2}{bk}\sum_{i=1}^{N}\sum_{l=1}^{n} c_{i\mathrm{L}}\int_0^{\hat{p}_{il}(t)} h_l(s)\mathrm{d}s \\ &\qquad + \frac{1}{bk}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij}\int_0^{\hat{p}_{jl}(t)-\hat{p}_{il}(t)} h_l(s)\mathrm{d}s \end{aligned}$$

Differentiation of $V(t, q_{\mathrm{L}}(t), \hat{p}_i(t), \hat{q}_i(t))$ with respect to time along the trajectory of (8) results in

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(8)} &= \frac{M}{bk}q_{\mathrm{L}}^\top(t)\dot{q}_{\mathrm{L}}(t) + \frac{2}{bk}\sum_{i=1}^{N}\hat{q}_i^\top(t)\dot{\hat{q}}_i(t) \\ &\quad + \frac{2}{bk}\sum_{i=1}^{N}\sum_{l=1}^{n} c_{i\mathrm{L}}h_l(\hat{p}_{il})\hat{q}_{il}(t) \\ &\quad + \frac{1}{bk}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sum_{l=1}^{n} c_{ij}h_l(\hat{p}_{jl}(t) \\ &\quad - \hat{p}_{il}(t))(\hat{q}_{jl}(t) - \hat{q}_{il}(t)) \\ &= -\frac{M}{bk}q_{\mathrm{L}}^\top(t)b_{\mathrm{L}}(t)f(q_{\mathrm{L}}) + \frac{2}{bk}\sum_{i=1}^{N}\hat{q}_i^\top(t) \\ &\quad \times \Bigg\{ -b_i(t)f(\hat{q}_i(t) + q_{\mathrm{L}}(t)) + b_{\mathrm{L}}(t)f(q_{\mathrm{L}}) \\ &\quad + \sum_{j\in\mathcal{N}_i} c_{ij}h(\hat{p}_j(t) - \hat{p}_i(t)) - c_{i\mathrm{L}}h(\hat{p}_i)\Bigg\} \\ &\quad + \frac{2}{bk}\sum_{i=1}^{N} c_{i\mathrm{L}}\hat{q}_i^\top(t)h(\hat{p}_i) \\ &\quad - \frac{2}{bk}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{ij}\hat{q}_i^\top(t)h(\hat{p}_j(t) - \hat{p}_i(t)) \\ &= -\frac{M}{bk}q_{\mathrm{L}}^\top(t)b_{\mathrm{L}}(t)f(q_{\mathrm{L}}) - \frac{2}{bk}\sum_{i=1}^{N}\hat{q}_i^\top(t)b_i(t)f(\hat{q}_i(t) \\ &\quad + q_{\mathrm{L}}(t)) + \frac{2}{bk}\sum_{i=1}^{N}\hat{q}_i^\top(t)b_{\mathrm{L}}(t)f(q_{\mathrm{L}}) \\ &\le -Mq_{\mathrm{L}}^\top(t)q_{\mathrm{L}}(t) - 2\sum_{i=1}^{N}\sum_{l=1}^{n}(\hat{q}_{il}(t) + q_{\mathrm{L}l}(t))^2 \end{aligned}$$

Table 1 Mobile robot parameters

Parameters	Pioneer 3DX	Pioneer 2DX	Pioneer 2DX with load (4 Kg)	Units
ϑ_1	0.24089	0.3037	0.1992	s
ϑ_2	0.2424	0.2768	0.13736	s
ϑ_3	-9.3603e^{-4}	-4.018e^{-4}	-1.954e^{-3}	s.m/rad^2
ϑ_4	0.99629	0.9835	0.9907	
ϑ_5	-3.7256e^{-3}	-3.818e^{-3}	-1.554e^{-2}	s/m
ϑ_6	1.0915	1.0725	0.9866	

$$
\begin{aligned}
&+\sum_{i=1}^{N}\hat{q}_i^{\top}(t)\hat{q}_i(t)+\frac{4\bar{b}^2\bar{k}^2+2bk\bar{b}\bar{k}}{b^2k^2}\sum_{i=1}^{N}q_{\mathrm{L}}^{\top}(t)q_{\mathrm{L}}(t)\\
&\leq -\frac{1}{2}\sum_{i=1}^{N}\hat{q}_i^{\top}(t)\hat{q}_i(t)\\
&\quad-\left(M-\frac{2N(bk\bar{b}\bar{k}+3b^2k^2+2\bar{b}^2\bar{k}^2)}{b^2k^2}\right)q_{\mathrm{L}}^{\top}(t)q_{\mathrm{L}}(t)
\end{aligned}
$$

so $\dot{V}(t,q_{\mathrm{L}}(t),\hat{p}_i(t),\hat{q}_i(t))\leq 0$ if choosing $M>2N(bk\bar{b}\bar{k}+3b^2k^2+2\bar{b}^2\bar{k}^2)/b^2k^2>0$. The last term in the first inequality is obtained based on the inequality $|ab|\leq 1/2(a^2+b^2)$. Then, by using the same analysis procedure developed in Theorem 1, it is easy to obtain that $\lim_{t\to+\infty}\hat{q}_i^{\top}(t)\hat{q}_i(t)=0$, $\lim_{t\to+\infty}q_{\mathrm{L}}^{\top}(t)q_{\mathrm{L}}(t)=0$ and $\lim_{t\to+\infty}\hat{p}_{il}(t)=0$, which imply that $q_i(t)\to q_{\mathrm{L}}(t)$, $p_i(t)\to p_{\mathrm{L}}(t)$ and $q_{\mathrm{L}}(t)\to 0$ as $t\to+\infty$. □

Corollary 2. *Assume $b_{\mathrm{L}}(t)=b_{\mathrm{L}}$, $f(q_{\mathrm{L}})=q_{\mathrm{L}}(t)$ and suppose that the leader has a path to all agents in graph $\mathcal{G}$. Then, $p_i(t)\to p_{\mathrm{L}}(0)+q_{\mathrm{L}}(0)/b_{\mathrm{L}}$ as $t\to+\infty$, where $p_{\mathrm{L}}(0)$ and $q_{\mathrm{L}}(0)$ are the initial position and velocity of the leader, respectively.*

Proof: If letting $b_{\mathrm{L}}(t)=b_{\mathrm{L}}$, $f(q_{\mathrm{L}})=q_{\mathrm{L}}(t)$, then the solution of (4) is given by

$$
\begin{aligned}
p_{\mathrm{L}}(t)&=p_{\mathrm{L}}(0)+\frac{q_{\mathrm{L}}(0)}{b_{\mathrm{L}}}-\frac{q_{\mathrm{L}}(0)}{b_{\mathrm{L}}}\exp(-b_{\mathrm{L}}t)\\
q_{\mathrm{L}}(t)&=q_{\mathrm{L}}(0)\exp(-b_{\mathrm{L}}t)
\end{aligned}
$$

so $p_i(t)\to p_{\mathrm{L}}(0)+q_{\mathrm{L}}(0)/b_{\mathrm{L}}$ as $t\to+\infty$. □

Remark 3. *Protocol (3) contains a leader, which provides the information to other agents and does not receive any information from other agents. In protocol (3), the second term on the right hand side of $u_{i2}(t)$ suggests that the leader has relationships with some of the other agents in the group. In particular, if for any $i\in\bar{N}$, $c_{i\mathrm{L}}>0$, then the leader has connections with all other agents. If for a specific agent r and $c_{rL}>0$, $c_{i\mathrm{L}}=0$ ($i\in\bar{N}, i\neq r$), then the leader has a connection with one agent in the group. Since the leader has a path to all other agents in graph $\mathcal{G}$, at least one agent can directly receive information from the leader, and the other agents receive information from the others either directly or indirectly. Theorem 2 also confirms that control of one agent in the group can ensure all agents to achieve a desired consensus value provided that the graph $\mathcal{G}$ is connected .*

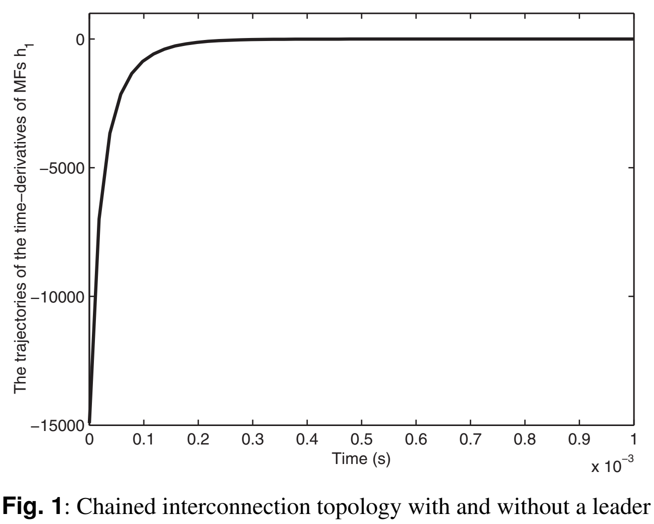

Fig. 1: Chained interconnection topology with and without a leader

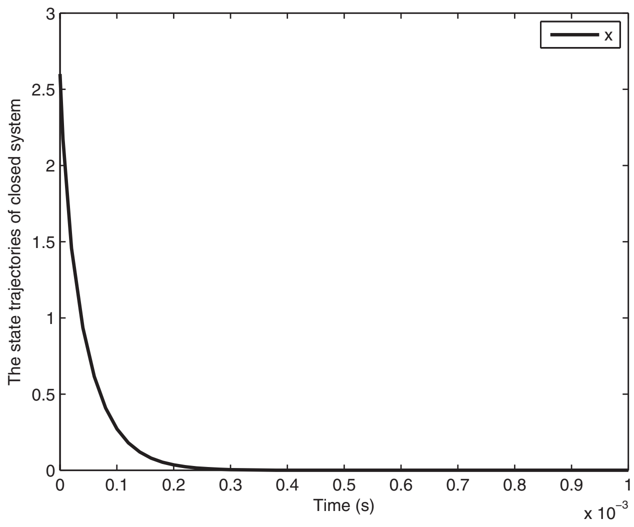

Fig. 2: Consensus of trajectories of agents without a leader
a $\cos(t)\omega_1=0.5$
b $\omega_1=0$ Initial conditions and other parameters are chosen as $(p_i(0),q_i(0))=(0.2i,0.3i)$, $c_{ij}=c_{ji}=0.2(i+j)$ and $m_i=0.1i(i,j=1,2,\ldots,6)$.

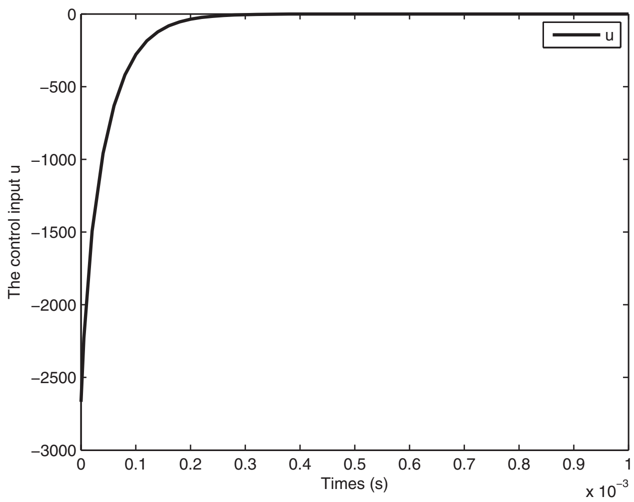

Fig. 3: Consensus of trajectories of agents with a leader
a $d_L(t)=0.6+0.15\cos(t),\omega_2=0.5$
b $b_L=0.6,\omega_2=0$
Initial conditions and other parameters are chosen as $(p_i(0),q_i(0))=(-0.3i,0.4i)$, $(p_L(0),q_L(0))=(1,0.3)$, $c_{ij}=c_{ji}=0.3(i+j)$, $b_i(t)=0.2i+0.15\cos(t),\omega_1=0.5$, $c_{1L}=1$ and $m_i=m_L=1$ $(i,j=1,2,\ldots,5)$.

4 Numerical simulations

In order to illustrate the theoretical results, numerical simulations were carried out for the system consisting of six ($N=6$, $n=1$) non-identical agents with and without a leader, as shown in Fig. 1. In the leaderless case, the non-linear control protocols can be selected as

$$u_{i1}(t) = -b_i(t)f(q_i),$$
$$u_{i2}(t) = \sum_{j\in\mathcal{N}_i} c_{ij}\,(p_j(t) - p_i(t)) + \sum_{j\in\mathcal{N}_i} c_{ij}(p_j(t) - p_i(t))^3 \tag{9}$$

and in the presence of a leader, the control protocols can be chosen as

$$\begin{aligned} u_{i1}(t) &= -b_i(t)f(q_i), \\ u_{i2}(t) &= \sum_{j\in\mathcal{N}_i} c_{ij}(p_j(t) - p_i(t) + (p_j(t) - p_i(t))^3) \\ &\quad + c_{i\mathrm{L}}\,(p_{\mathrm{L}}(t) - p_i(t) + (p_{\mathrm{L}}(t) - p_i(t))^3) \end{aligned} \tag{10}$$

where $f(q_i) = q_i(t) + \omega_1 \sin(q_i)$ and $f(q_{\mathrm{L}}) = q_{\mathrm{L}}(t) + \omega_2 \sin(q_{\mathrm{L}})$ $(\omega_1 \geq 0, \omega_2 \geq 0)$.

Fig. 2 shows the consensus process of positions and velocities of all agents in system (1) with protocol (9). It indicates that all non-identical agents can achieve consensus within a short period of time. From the result of Corollary 1, the consensus value is found to be $1/4.2\sum_{i=1}^{6}(b_ip_i(0) + m_iq_i(0)) \approx 1.5167$ as $t \to +\infty$ when $b_i(t) = 0.2i$, $m_i = 0.1i$ and $f(q_i) = q_i(t)$. Fig. 3 illustrates that all positions and velocities of system (1) with protocol (10) can track the position and velocity of the leader. When $b_{\mathrm{L}} = 0.6$, $f(q_{\mathrm{L}}) = q_{\mathrm{L}}(t)$, the exact position of the leader is $p_{\mathrm{L}}(0) + q_{\mathrm{L}}(0)/b_{\mathrm{L}} = 1.5$ as $t \to +\infty$ from Corollary 2. It can be concluded from Figs. 2 and 3 that the theoretical results are in good agreement with numerical simulations.

5 Conclusion

The consensus of multiple non-identical agents with non-linear control protocols has been studied for the agents with and without a leader. It was shown that all different agents in the group can achieve consensus, and an explicit consensus value for the group was obtained for some cases. For the agents with a leader, it was found that all agents can track the position and velocity of the leader, which are distinct from those of any follower agents. Numerical simulations were used to illustrate the theoretical results.

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7 References

1 Vicsek T., Czirók A., Ben-Jacob E., Cohen I.: 'Novel type of phase transition in a system of self-driven particles', *Phys. Rev. Lett.*, 1995, **75**, pp. 1226–1229
2 Jadbabaie A., Lin J., Morse A.S.: 'Coordination of groups of mobile autonomous agents using nearest neighbor rules', *IEEE Trans. Autom. Control*, 2003, **48**, pp. 988–1001
3 Olfati-Saber R., Murray R.M.: 'Consensus problems in networks of agents with switching topology and time-delays', *IEEE Trans. Autom. Control*, 2004, **49**, pp. 1520–1533
4 Ren W., Atkins E.: 'Distributed multi-vehicle coordinated control via local information exchange', *Int. J. Robust Nonlinear Control*, 2007, **17**, pp. 1002–1033
5 Ren W.: 'On consensus algorithms for double-integrator dynamics', *IEEE Trans. Autom. Control*, 2008, **53**, pp. 1503–1509
6 Xie G., Wang L.: 'Consensus control for a class of networks of dynamic agents', *Int. J. Robust Nonlinear Control*, 2007, **17**, pp. 941–959
7 Gao Y., Wang L.: 'Consensus of multiple dynamic agents with sampled information', *IET Control Theory Appl.*, 2010, **4**, pp. 945–956
8 Tanner H.G., Jadbabaie A., Pappas G.J.: 'Flocking in fixed and switching networks', *IEEE Trans. Autom. Control*, 2007, **52**, pp. 863–868
9 Lee D.J., Spong M.W.: 'Stable flocking of multiple inertial agents on balanced graphs', *IEEE Trans. Autom. Control*, 2007, **52**, pp. 1469–1475
10 Hu J., Lin Y.S.: 'Consensus control for multi-agent systems with double-integrator dynamics and time delays', *IET Control Theory Appl.*, 2010, **4**, pp. 109–118
11 Lin P., Jia Y.: 'Further results on decentralised coordination in networks of agents with second-order dynamics', *IET Control Theory Appl.*, 2009, **3**, pp. 957–970
12 Lin P., Jia Y.: 'Robust H_∞ consensus analysis of a class of second-order multi-agent systems with uncertainty', *IET Control Theory Appl.*, 2010, **4**, pp. 487–498
13 He W., Cao J.: 'Consensus control for high-order multi-agent systems', *IET Control Theory Appl.*, 2011, **1**, pp. 231–238
14 Hui Q., Haddad W.M.: 'Distributed nonlinear control algorithms for network consensus', *Automatica*, 2008, **44**, pp. 2375–2381
15 Liu X., Chen T., Lu W.: 'Consensus problem in directed networks of multi-agents via nonlinear protocols', *Phys. Lett. A*, 2009, **373**, pp. 3122–3127
16 Zheng Y., Zhu Y., Wang L.: 'Consensus of heterogeneous multi-agent systems', *IET Control Theory Appl.*, 2011, **5**, pp. 1881–1888
17 Zhang W., Zeng D., Qu S.: 'Dynamic feedback consensus control of a class of high-order multi-agent systems', *IET Control Theory Appl.*, 2010, **4**, pp. 2219–2222
18 Su H.S., Wang X.F., Lin Z.L.: 'Flocking of multi-agents with a virtual leader', *IEEE Trans. Autom. Control*, 2009, **54**, pp. 293–307
19 Song Q., Cao J., Yu W.: 'Second-order leader-following consensus of nonlinear multi-agent system via pinning control', *Syst. Control Lett.*, 2010, **59**, pp. 553–562
20 Hong Y.G., Chen G.R., Bushnell L.D.: 'Distributed observers design for leader-following control of multi-agent networks', *Automatica*, 2008, **44**, pp. 846–850
21 Hu J., Hong Y.: 'Leader-following coordination of multi-agent systems with coupling time delays', *Physica A*, 2007, **374**, pp. 853–863
22 Ren W.: 'Consensus strategies for cooperative control of vehicle formations', *IET Control Theory Appl.*, 2007, **1**, pp. 505–512
23 Khalil H.K.: 'Nonlinear systems' (Prentice-Hall, New Jersey, 2002, 3rd edn.)