MAE 376: HW-3 ; Due: 3/7/2022

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(1) Evaluate the determinant

$$\begin{vmatrix} s+1 & 1 & -1 \\ 0 & s+2 & 2 \\ -1 & 2 & s \end{vmatrix}, \quad s = \text{parameter}$$

$$\text{Det} = (s+1)(s+2)\cdot s + 1\cdot 2\cdot(-1) + -1\cdot 0\cdot 2 - \left((1\cdot 0\cdot s)+(s+1)\cdot 2\cdot 2+(-1)(s+2)(-1)\right)$$

$$= s^3+3s^2+2s-2-(4s+4+s+2)$$

$$= \boxed{s^3+3s^2-3s-8} \quad (ii)$$

(2) Find A^{-1}

$$A = \begin{bmatrix} a & 0 & -1 \\ 0 & a+1 & 2 \\ 1 & 0 & a+2 \end{bmatrix} \rightarrow$$

$$C_{11} = \begin{vmatrix} a+1 & 2 \\ 0 & a+2 \end{vmatrix} = a^2+3a+2 \qquad C_{21} = -\begin{vmatrix} 0 & -1 \\ 0 & a+2 \end{vmatrix} = 0 \qquad C_{31} = \begin{vmatrix} 0 & -1 \\ a+1 & 2 \end{vmatrix} = +a+1$$

$$C_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & a+2 \end{vmatrix} = +2 \qquad C_{22} = \begin{vmatrix} a & -1 \\ 1 & a+2 \end{vmatrix} = a^2+2a+1 \qquad C_{32} = -\begin{vmatrix} a & -1 \\ 0 & 2 \end{vmatrix} = -2a$$

$$C_{13} = \begin{vmatrix} 0 & a+1 \\ 1 & 0 \end{vmatrix} = -a-1 \qquad C_{23} = -\begin{vmatrix} a & 0 \\ 1 & 0 \end{vmatrix} = 0 \qquad C_{33} = \begin{vmatrix} a & 0 \\ 0 & a+1 \end{vmatrix} = a^2+a$$

$$\text{adj}(A) = \begin{bmatrix} a^2+3a+2 & 0 & a+1 \\ +2 & a^2+2a+1 & -2a \\ -a-1 & 0 & a^2+a \end{bmatrix}$$

$$\det(A) = [a\cdot(a+1)(a+2)+0+0]-[0+0+1\cdot(a+1)\cdot(-1)]$$

$$= a^3+3a^2+2a+a+1$$

$$= a^3+3a^2+3a+1 \rightarrow (a+1)^3$$

$$\therefore A^{-1} = \frac{1}{\det(A)}\cdot \text{adj}(A) \rightarrow (iii)$$

(3)

$$A = \begin{bmatrix} a & 1 & -2 \\ -1 & 2a & 1 \\ 0 & 1 & 3a \end{bmatrix}, \quad b = \begin{Bmatrix} a \\ 4a \\ 3a+2 \end{Bmatrix} \rightarrow$$

$$C_{11} = (-1)^{(1+1)}\begin{vmatrix} 2a & 1 \\ 1 & 3a \end{vmatrix} = 6a^2-1 \qquad C_{21} = (-1)^{(2+1)}\begin{vmatrix} 1 & -2 \\ 1 & 3a \end{vmatrix} = -3a-2$$

$$C_{12} = (-1)^{(1+2)}\begin{vmatrix} -1 & 1 \\ 0 & 3a \end{vmatrix} = 3a \qquad C_{22} = (-1)^{(2+2)}\begin{vmatrix} a & -2 \\ 0 & 3a \end{vmatrix} = 3a^2$$

$$C_{13} = (-1)^{(1+3)}\begin{vmatrix} -1 & 2a \\ 0 & 1 \end{vmatrix} = -1 \qquad C_{23} = (-1)^{(2+3)}\begin{vmatrix} a & 1 \\ 0 & 1 \end{vmatrix} = -a$$

$$C_{31} = (-1)^{(3+1)}\begin{vmatrix} 1 & -2 \\ 2a & 1 \end{vmatrix} = 1+4a$$

$$C_{32} = (-1)^{(3+2)}\begin{vmatrix} a & -2 \\ -1 & 1 \end{vmatrix} = -a+2$$

$$C_{33} = (-1)^{(3+3)}\begin{vmatrix} a & 1 \\ -1 & 2a \end{vmatrix} = 2a^2+1$$

$$|A| = [6a^3+0+2]-[-3a+a+0]$$

$$|A| = 6a^3+2a+2$$

$$\therefore x = A^{-1}b = \frac{1}{6a^3+2a+2}\begin{bmatrix} 6a^2-1 & -3a-2 & 1+4a \\ 3a & 3a^2 & -a+2 \\ -1 & -a & 2a^2+1 \end{bmatrix}\begin{Bmatrix} a \\ 4a \\ 3a+2 \end{Bmatrix} \quad (ii)$$

$$x = \begin{Bmatrix} 6a^3-a-12a^2-8a+3a+12a^2+2+8a \\ 3a^2+12a^3-3a^2+6a-2a+4 \\ -a-4a^2+6a^3+3a+4a^2+2 \end{Bmatrix}\cdot\frac{1}{6a^3+2a+2}$$

$$x = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

Matlab Code

```
syms a
A=[a,1,-2;-1,2*a,1;0,1,3*a];
b=[a;4*a;3*a+2];
x=A\b; → disp(x);
```

4)

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 1 & 0 \\ 1 & 4 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \rightarrow A_x = \begin{bmatrix} 3 & 2 & -3 \\ 5 & 1 & 0 \\ 4 & 4 & -2 \end{bmatrix}, \quad A_y = \begin{bmatrix} 1 & 3 & -3 \\ -2 & 5 & 0 \\ 1 & 4 & -2 \end{bmatrix}, \quad A_z = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 5 \\ 1 & 4 & 4 \end{bmatrix}$$

$$|A| = [-2+0+24] - [8+0-3] = 17$$

$$|A_x| = [-6+0+60] - [-20+0-12] = -34$$

$$|A_y| = [-10+0+24] - [12+0-15] = 17$$

$$|A_z| = [4+10-24] - [-16+20+3] = -17$$

$$x = \begin{Bmatrix} \frac{-34}{17} \\ \frac{17}{17} \\ \frac{-17}{17} \end{Bmatrix} = \begin{Bmatrix} -2 \\ 1 \\ -1 \end{Bmatrix}$$

ii

```
MATLAB: A=[1,2,-3;-2,1,0;1,4,-2]; b=[3,5,4];
Ax=[3,2,-3;5,1,0;4,4,-2]; x=det(Ax)/det(A);
Ay=[1,3,-3;-2,5,0;1,4,-2]; y=det(Ay)/det(A);
Az=[1,2,3;-2,1,5;1,4,4]; z=det(Az)/det(A);
disp([x;y;z]);
```

5) $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$,

$\det(A - \lambda I) = 0$

$$\begin{bmatrix} 0-\lambda & a \\ a & 0-\lambda \end{bmatrix} = \lambda^2 - a^2 = 0$$

$(\lambda + a)(\lambda - a) = 0$

$\lambda_1 = -a, \lambda_2 = a$

$(A + a_1 I)z = 0$

$\begin{pmatrix} a & a \\ a & a \end{pmatrix} z = 0$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$\downarrow$

$z_1 + z_2 = 0$

$z_1 = -z_2$

$z^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A - a_2 I)z = 0$

$\begin{pmatrix} -a & a \\ a & -a \end{pmatrix} z = 0$

$\left(\begin{array}{cc|c} -a & a & 0 \\ 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$-z_1 + z_2 = 0$

$z_1 = z_2$

$z^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = V$

or

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = V$ iii

6) Express system model
$$\begin{cases} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = F_1(t) \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = F_2(t) \end{cases}$$

a) Configuration Form:
$$\ddot{x}_1 = \frac{1}{m_1}\left(-c_1\dot{x}_1 - k_1x_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + F_1(t)\right) = \ddot{q}_1$$

$$\ddot{x}_2 = \frac{1}{m_2}\left(-k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) + F_2(t)\right) = \ddot{q}_2$$

i

b) Standard, 2[nd] order Matrix Form:
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

$$\ddot{q}_1 = \frac{1}{m_1}\left((-k_1 - k_2)q_1 + k_2q_2 + (-c_1 - c_2)\dot{q}_1 + c_2\dot{q}_2 + F_1(t)\right)$$

$$\ddot{q}_2 = \frac{1}{m_2}\left(k_2q_1 - k_2q_2 + c_2\dot{q}_1 - c_2\dot{q}_2 + F_2(t)\right)$$

$$\Rightarrow \begin{cases} m_1\ddot{q} + (k_1+k_2)q_1 - k_2q_2 + (c_1+c_2)\dot{q}_1 - c_2\dot{q}_2 = F_1(t) \\ m_2\ddot{q} - k_2q_1 + k_2q_2 - c_2\dot{q}_1 + c_2\dot{q}_2 = F_2(t) \end{cases}$$

7) Find SVE's, SV's, & SEq

$$\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2x_1 - \dot{x}_2 - 3x_2 = f_1(t) \\ 2\ddot{x}_2 - \dot{x}_1 - 2x_1 + \dot{x}_2 + 3x_2 = f_2(t) \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 = -2x_1 - \dot{x}_1 + 3x_2 + \dot{x}_2 + f_1(t) \\ \ddot{x}_2 = x_1 + \frac{1}{2}\dot{x}_1 - \frac{3}{2}x_2 - \frac{1}{2}\dot{x}_2 + \frac{1}{2}f_2(t) \end{cases} \rightarrow$$

$$\begin{cases} x_1 = x_1 \rightarrow \dot{x}_1 = x_3 \rightarrow 0x_1 + 0x_2 + 1x_3 + 0x_4 + 0f_1(t) + 0f_2(t) \\ x_2 = x_2 \rightarrow \dot{x}_2 = x_4 \rightarrow 0x_1 + 0x_2 + 0x_3 + 1x_4 + 0f_1(t) + 0f_2(t) \\ x_3 = \dot{x}_1 \rightarrow \dot{x}_3 = \ddot{x}_1 \rightarrow -2x_1 + 3x_2 - 1x_3 + 1x_4 + 1f_1(t) + 0f_2(t) \\ x_4 = \dot{x}_2 \rightarrow \dot{x}_4 = \ddot{x}_2 \rightarrow 1x_1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 + 0f_1(t) + \frac{1}{2}f_2(t) \end{cases}$$

$$\therefore A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3 & -1 & 1 \\ 1 & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

ii

8) SS-form, outputs x_2 & $\dot{x}_2$

$$\begin{cases} \ddot{x}_1 + 2(x_1 - x_3) - 2(\dot{x}_2 - \dot{x}_1) - \frac{1}{2}(x_2 - x_1) = f(t) \\ \ddot{x}_2 + 2(\dot{x}_2 - \dot{x}_1) + \frac{1}{2}(x_2 - x_1) = 0 \\ x_3 - 2(x_1 - x_3) = 0 \rightarrow 3x_3 - 2x_1 = 0 \end{cases}$$

$\rightarrow \ddot{x}_1 + 2\left(\frac{1}{3}x_1\right) - 2\dot{x}_2 + 2\dot{x}_1 - \frac{1}{2}x_2 + \frac{1}{2}x_1 = f(t)$; $\ddot{x}_1 = -\frac{7}{6}x_1 + \frac{1}{2}x_2 - 2\dot{x}_1 + 2\dot{x}_2 + f(t)$

$\rightarrow \ddot{x}_2 = \frac{1}{2}x_1 - \frac{1}{2}x_2 + 2\dot{x}_1 - 2\dot{x}_2$.

$x_3 = \frac{2}{3}x_1$ (Plug into ① & ②)

$x_1 = x_1 \rightarrow \dot{x}_1 = x_3 \rightarrow 0x_1 + 0x_2 + 1x_3 + 0x_4 + 0f(t)$

$x_2 = x_2 \rightarrow \dot{x}_2 = x_4 \rightarrow 0x_1 + 0x_2 + 0x_3 + 1x_4 + 0f(t)$

$x_3 = \dot{x}_1 \rightarrow \dot{x}_3 = \ddot{x}_1 \rightarrow -\frac{7}{6}x_1 + \frac{1}{2}x_2 - 2x_3 + 2x_4 + 1f(t)$

$x_4 = \dot{x}_2 \rightarrow \dot{x}_4 = \ddot{x}_4 \rightarrow \frac{1}{2}x_1 - \frac{1}{2}x_2 + 2x_3 - 2x_4 + 0f(t)$

$$\Rightarrow A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{7}{6} & \frac{1}{2} & -2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 2 & -2 \end{bmatrix}$$ (i)

9) $\ddot{x} + 4\dot{x} + 3x = f(t)$, x is output $\rightarrow \ddot{x} = -4\dot{x} - 3x + f(t)$

a) Find SS-form

$x = x_1 \rightarrow \dot{x}_1 = \dot{x} = x_2 \rightarrow 0x_1 + 1x_2 + 0f(t)$

$\dot{x} = x_2 \rightarrow \dot{x}_2 = \ddot{x} \rightarrow -3x_1 - 4x_2 + 1f(t)$

$$\Rightarrow \boxed{\underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{f(t)}_{u} = \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}}}$$

b) Decouple & obtain ~~transformed~~ SS-form

$$y = x = x_1 = [1 \;\; 0] \underbrace{\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}}_{x} + 0u = Cx + Du$$

$$|A - \lambda I| = \left| \begin{bmatrix} 0-\lambda & 1 \\ -3 & -4-\lambda \end{bmatrix} \right| = 4\lambda + \lambda^2 + 3 = 0$$

$(\lambda + 3)(\lambda + 1) = 0$

$\lambda_1 = -3, \lambda_2 = -1$

largest first

$$\tilde{D} = \text{diag}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

Eigenvectors

$$A - \lambda_1 I = \begin{bmatrix} 0-(-3) & 1 \\ -3 & -4-(-3) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} R_1 + R_2 = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow 3z_1 + z_2 = 0$$

$z_2 = -3z_1$

$$\rightarrow z^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 0-(-1) & 1 \\ -3 & -4-(-1) \end{bmatrix} = \begin{bmatrix} +1 & 1 \\ -3 & -3 \end{bmatrix} R_1 + \tfrac{1}{3}R_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow z_1 + z_2 = 0$$

$z_1 = -z_2$

$$z^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$ (iii)

10) Find (I/O)E

$$\begin{cases} \ddot{x}_1 + \dot{x}_1 + 2(x_1 - x_2) = 0 \\ \ddot{x}_2 + \dot{x}_2 - 2(x_1 - x_2) = f(t) \end{cases}$$

x_1 = output

$f(t)$ = input

$\rightarrow (s^2 + s + 2)X_1(s) - 2X_2(s) = 0$

$(s^2 + s + 2)X_2(s) - 2X_1(s) = F_2(s)$

$$\rightarrow \begin{bmatrix} s^2+s+2 & -2 \\ -2 & s^2+s+2 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}$$

$X_1(s)$

1st $\rightarrow F_1(s) = 0 \rightarrow X_1(s) = \frac{2F(s)}{(s^2+s+2)^2 - 4} \Rightarrow x_1^{(4)} + 2x_1^{(3)} + 5\ddot{x}_1 + 4\dot{x}_1 = 2f_2(t)$ (ii)

2nd $\rightarrow F_2(s) = 0 \rightarrow X_1(s) = \frac{0}{(s^2+s+2)^2 - 4} = 0$

$X_2(s)$

3rd $\rightarrow F_1(s) = 0 \rightarrow X_2(s) = \frac{(s^2+s+2) \cdot F(s)}{(s^2+s+2)^2 - 4} \Rightarrow x_1^{(4)} + 2x_1^{(3)} + 5\ddot{x}_1 + 4\dot{x}_1 = (\ddot{x}_1 + \dot{x}_1 + 2x_1) f(t)$

4th $\rightarrow F_2(s) = 0 \rightarrow X_2(s) = \frac{(s^2+s+2) \cdot 0}{(s^2+s+2)^2 - 4} = 0$

Cramers Rule

$$X_2(S) = \frac{\begin{vmatrix} s^2+s+2 & 0 \\ -2 & F(s) \end{vmatrix}}{\begin{vmatrix} s^2+s+2 & -2 \\ -2 & s^2+s+2 \end{vmatrix}} = \frac{(s^2+s+2) \cdot F(s)}{(s^2+s+2)^2 - 4}$$

$$X_1(s) = \frac{\begin{vmatrix} 0 & -2 \\ F(s) & s^2+s+2 \end{vmatrix}}{\begin{vmatrix} s^2+s+2 & -2 \\ -2 & s^2+s+2 \end{vmatrix}} = \frac{+2F(s)}{(s^2+s+2)^2 \ldots}$$

(back to left side)