1) Data Head of Disk

MAE 476: Midterm I

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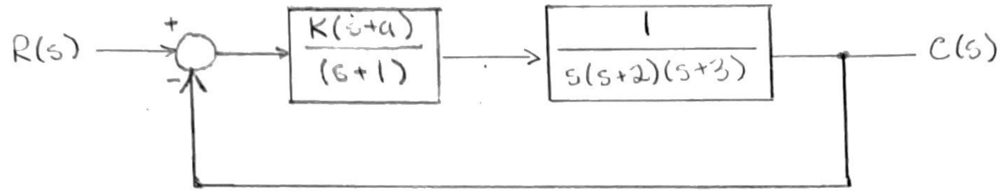
MAE 476 - Sec 03
Due: 11/9/2022



a) Find type of system (0, 1, 2) based on value of K (for which system is stable)

# of poles at origin from transfer function = system type

$$G(s) = G_1(s)\cdot G_2(s) = \frac{K(s+a)}{s(s+1)(s+2)(s+3)}$$

$$T_{CL} = \frac{\dfrac{K(s+a)}{s(s+1)(s+2)(s+3)}}{1-\left(-\dfrac{K(s+a)}{s(s+1)(s+2)(s+3)}\right)} = \frac{K(s+a)}{s(s+1)(s+2)(s+3)+K(s+a)} = \frac{K(s+a)}{(s^2+s)(s^2+5s+6)+Ks+Ka} \Rightarrow$$

$$T_{CL} = \frac{K(s+a)}{s^4+5s^3+6s^2+s^3+5s^2+6s+Ks+Ka} = \frac{K(s+a)}{s^4+6s^3+11s^2+(6+K)s+Ka}$$ $K > -6$

Check Stability: (coeff. are > 0 & all elements in 1st column are > 0)

s^4:	1	11	Ka
s^3:	6	$6+K$	0
s^2:	$\frac{60-K}{6}$	Ka	0
s^1:	c_1	0	
s^0:	Ka		

(c_1 too large see variables to the right)

$$b_1 = \frac{-\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} = \frac{-\begin{vmatrix} 1 & 11 \\ 6 & 6+K \end{vmatrix}}{6} = \frac{-(6+K-66)}{6} = \frac{60-K}{6} > 0 \Rightarrow \boxed{K < 60}$$

$$b_2 = \frac{-\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{-\begin{vmatrix} 1 & Ka \\ 6 & 0 \end{vmatrix}}{6} = \frac{6Ka}{6} = Ka > 0 \Rightarrow K > 0 \text{ or } a<0 \;\&\; K<0$$

Note $K > -6$

$$c_1 = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{-\begin{vmatrix} 6 & 6+K \\ \frac{60-K}{6} & Ka \end{vmatrix}}{\left(\frac{60-K}{6}\right)} = \frac{6Ka-(60-K)-\left(\frac{60K-K^2}{6}\right)}{\frac{60-K}{6}}$$

(cont.)

$$\Rightarrow c_1 = \frac{36Ka-(360-6K)-(60K-K^2)}{60-K} = \boxed{\frac{K^2+(36a-54)K-360}{60-K}} > 0$$

$$K^2+(36a-54)K-360 > 0$$

Solve by quad. formula
- K is dependent on a
- Ex: if $a=1 \Rightarrow K_1 < -12$, $K_2 > 30$

∴ Need to find range of a

$$d_1 = \frac{-\begin{vmatrix} \frac{60-K}{6} & Ka \\ c_1 & 0 \end{vmatrix}}{c_1} = \frac{c_1 Ka}{c_1} = Ka > 0 \Rightarrow K > 0$$

$K < 60$ & $K > -6$. apply extrema

$K = -6$

$(-6)^2+(36a-54)(-6)-360>0$

$(36a-54)(-6) > 324$

$36a-54 < -54$

$a < 0$

$(60)^2-(36a-54)\cdot 60-360>0$

$3240-(36a-54)\cdot 60 > 0$

$(36a-54)60 > -3240$

$36a-54 > -9$

$36a > 45 \Rightarrow a > \frac{5}{4}$

From testing in MATLAB I have determined that a is able to reach all values if K is selected correctly

∴ Range of a can be between these values as well

b) Let $K=30$ & $a=0.5$

Poles: $0 = s_1 \rightarrow$ Type 0

$= s_2$

$-0.8455 \pm 1.73i = s_{3,4}$

$$e_{ss} = \lim_{s\to 0}\left(\frac{s^n}{s^n+K_v}\right)\frac{1}{s^k}$$

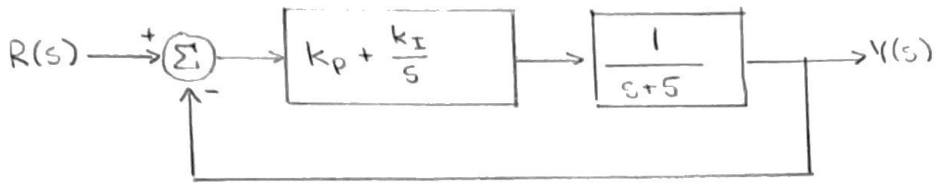
$$K_p = \lim_{s\to 0}\left(s^0\cdot G D_{CL}(s)\right)$$

$$K_p = \lim_{s\to 0}\left(1\cdot\frac{30(s+a5)}{s^4+6s^3+11s^2+36s+15}\right)$$

$$\boxed{K_p = \frac{15}{15} = 1}$$

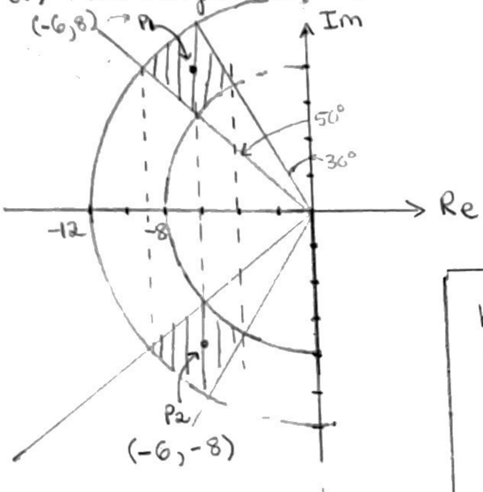
∴ For $K=30$ & $a=0.5$, the system is stable and the positioning error $K_p = 1$

2) Feedback control system



$$\frac{\dot{\Theta}_m}{V_a(s)} \rightarrow T.F = \frac{k_t}{R_a} \cdot \frac{1}{J_m s + b + \frac{k_t k_3}{R_a}}$$

a) Find ranges of ω_n & ζ within shaded region



From sketch $\rightarrow$ $8 \le \omega_n \le 12$ (rad/s)

Vertical lines (dashed):

$$\zeta\omega_n = \begin{Bmatrix} \omega_n \sin 30° \\ \omega_n \sin 50° \end{Bmatrix}$$

$$\zeta = \begin{Bmatrix} \sin 30° \\ \sin 50° \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.766 \end{Bmatrix} \Rightarrow 0.5 \le \zeta \le 0.766$$

b) Design PI-controller by choosing P_1, P_2 in shaded region (and are CL poles)

Let $-\zeta\omega_n = -6$ & $\omega_n = 10$ $\Rightarrow \zeta = \frac{6}{10} = \frac{3}{5} = 0.6$

$P_{1,2} = -6 \pm 8j$ $\Rightarrow$ Char eqn: $(s+6+8j)(s+6-8j) = s^2 + 12s + 100$

$$\therefore \text{T.F.} \; (\mathcal{Y}(s)) = \frac{G(s)}{s^2+12s+100} = \frac{\left(k_p + \frac{k_I}{s}\right)\left(\frac{1}{s+5}\right)}{1 - \left(-\left(k_p + \frac{k_I}{s}\right)\left(\frac{1}{s+5}\right)\right)}$$

$\downarrow$ Simplify

$$\frac{\frac{k_p \cdot s + k_I}{s(s+5)}}{1 + \frac{k_p \cdot s + k_I}{s(s+5)}} = \frac{k_p \cdot s + k_I}{s^2 + 5s + k_p \cdot s + k_I}$$

$\therefore 12s = (5 + k_p)s \Rightarrow \boxed{k_p = 7}$

$\boxed{100 = k_I}$

$$\mathcal{Y}(s) = \frac{7s+100}{s^2+12s+100} \qquad G(s) = \left(7 + \frac{100}{s}\right)\left(\frac{1}{s+5}\right) = \frac{7s+100}{s^2+5s}$$

c) Estimate M_p, t_r, & t_s

$$M_p = e^{(-\pi\zeta)/(\sqrt{1-\zeta^2})}$$

$$= e^{(-\pi(0.6)/\sqrt{1-0.6^2})} = 0.09478$$

$\boxed{M_p \approx 9.48\%}$

$\boxed{t_r = \frac{1.8}{\omega_n} = \frac{1.8}{10} = 0.18 \text{ s}}$

$\boxed{t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{6} = 0.767 \text{ s}}$

d) $e_{ss} = \lim_{s\to 0}\left(\frac{1 - \mathcal{Y}(s)}{s^{k+1}} s\right)$

with a unit-step input : $k = 0$

$$e_{ss} = \lim_{s\to 0}\left(\frac{1 - \frac{7s+100}{s^2+12s+100}}{s^0} \cdot s\right)$$

$$= \lim_{s\to 0}\left(\frac{s^2+12s+100-(7s+100)}{s^2+12s^2+100}\right)$$

$$= \lim_{s\to 0}\left(\frac{s^2+5s^2}{s^2+12s^2+100}\right) \rightarrow \frac{0}{100} = 0$$

e) $e_{ss} = \lim_{s\to 0}\left(\frac{s^2+5s^2}{s(s^2+12s+100)}\right)$

$$e_{ss} = \lim_{s\to 0}\left(\frac{s+5}{s^2+12s+100}\right) = \frac{5}{100} = \boxed{\frac{1}{20}}$$

(ramp response $k = 1$)

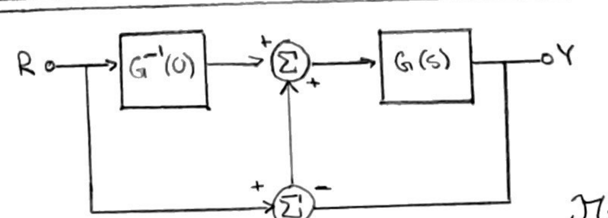
f) $K_I = 0$ & $k_p = 7$, find e_{ss} for unit-step $R(s)$ $\{k=0\}$

$$e_{ss} = \lim_{s\to 0}\left(\frac{1 - \frac{\frac{7+0}{s+5}}{1+\frac{7}{s+5}}}{s^{0+1}} \cdot s\right) = \lim_{s\to 0}\left(\left(1 - \frac{7}{s+12}\right)\right) = \lim_{s\to 0}\left(\frac{s+5}{s+12}\right)$$

$\boxed{e_{ss} = \frac{5}{12}}$

PART H $\rightarrow$ SEE MATLAB

g)



where $K_I = 0$

$\therefore G(s) = (k_p + 0)\left(\frac{1}{s+5}\right) = \frac{7}{s+5}$

$G^{-1}(0) = \frac{0+5}{7} = \frac{5}{7}$

$$\mathcal{Y}(s) = \frac{G^{-1}(0)G + G}{1 + G}$$

$$= \frac{\frac{5}{7} \cdot \frac{7}{s+5} + \frac{7}{s+5}}{1 + \frac{7}{s+5}} = \frac{\frac{12}{s+5}}{\frac{s+12}{s+5}}$$

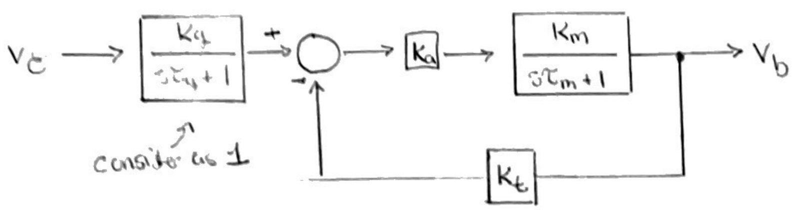
$$\mathcal{Y}(s) = \frac{12}{s+12}$$

$$e_{ss} = \lim_{s\to 0}\left(\frac{1 - \frac{12}{s+12}}{s^{0+1}} \cdot s\right)$$

$\boxed{e_{ss} = \lim_{s\to 0}\left(\frac{s}{s+12}\right) = \frac{0}{12} = 0}$

PART H $\rightarrow$ SEE MATLAB

3) $V_{c_{max}} = \frac{30°}{s} = \frac{\pi}{6}$ rad/s , Assume $K_t = 1$, $\dot{\psi} = r = 1$ $\left(\frac{k_g}{s\tau_g+1} = 1\right)$



$$T.F. = \mathcal{T}(s) = \frac{K_a\left(\frac{K_m}{s\tau_m+1}\right)K_t}{1-\left(-K_a\left(\frac{K_m}{s\tau_m+1}\right)K_t\right)} = \frac{K_aK_mK_t}{s\tau_m+1+K_aK_mK_t}$$

$$\mathcal{T}(s) = \frac{V_b(s)}{V_c(s)}$$

a) Find $E(s)$

$$\frac{E}{R} = \frac{1}{1+GD_{cl}} \Rightarrow \frac{E(s)}{V_c(s)} = \frac{1}{1+\frac{K_aK_m}{s\tau_m+1}} \Rightarrow \boxed{E(s) = \frac{s\tau_m+1}{s\tau_m+1+K_aK_m}\cdot V_c(s)}$$

b) $e_{ss} = \frac{1°}{s} = \frac{\pi/180 \text{ rad}}{s}$; $V_c = max = \frac{30°}{s} = \frac{\pi}{6}$ rad/s

Determine necessary loop gain $K_aK_mK_t$ for step input

$$e_{ss} = \lim_{s\to 0}\left(\frac{1-\mathcal{T}(s)}{s^{k+1}}\cdot R(s)\right)\cdot s \quad ; \text{where } R(s) = \frac{\pi}{6}\cdot\frac{1}{s^{k+1}}$$

$$\frac{\pi}{180} = \lim_{s\to 0}\left(\frac{1-\frac{K_aK_mK_t}{s\tau_m+1+K_aK_mK_t}}{s^{0+1}}\cdot s\cdot\frac{\pi}{6}\right)$$

$$\frac{\pi}{180} = \lim_{s\to 0}\left(\frac{s\tau_m+1}{s\tau_m+1+K_aK_mK_t}\cdot\frac{\pi}{6}\right)$$

$$\frac{\pi}{180} = \frac{1}{1+K_aK_mK_t}\cdot\frac{\pi}{6}$$

$$\frac{6}{180} = \frac{1}{1+K_aK_mK_t}$$

$$\frac{1}{30} = \frac{1}{1+K_aK_mK_t}$$

$$1+K_aK_mK_t = 30$$

$\boxed{K_aK_mK_t = 29}$ is the necessary loop gain

If $K_t = 1 \Rightarrow \boxed{K_aK_m = 29}$

c) $\tau_m = 0.5$ s . Determine necessary $H(s)$ s.t. $t_s \le 0.005\cdot V_{b,final} \Rightarrow t_s \le 0.04$ s

$$0.005\cdot V_{b,final} = 0.04$$

$$V_{b,final} = 8$$

Assume deg/s

Substitute $K_t = 1$ & $\tau_m = 0.5$ into $\mathcal{T}(s)$

$$\mathcal{T}(s) = \frac{K_aK_m}{0.5s+1+K_aK_m} = \frac{V_b(s)}{V_c(s)}$$

Assume $V_c(s)$ is max at 30°/s

$$\frac{K_aK_m}{0.5s+1+K_aK_m} = \frac{8\ \frac{deg}{s}}{30\ \frac{deg}{s}}$$

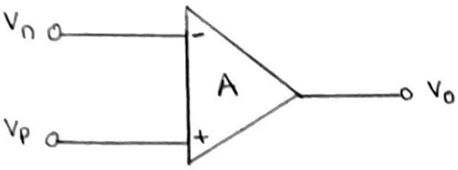
$$\Rightarrow \quad 30K_aK_m = 4s+8+8K_aK_m$$

$$\Rightarrow \quad 22K_aK_m = 4s+8$$

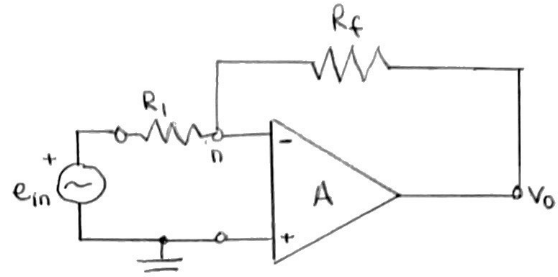
$$11K_aK_m = 2s+4$$

$$\boxed{K_aK_m = \frac{2s+4}{11}}$$

Extra Credit



Open Loop



Closed loop

Node n: $i_{in} = i_{out}$

Current eqn: $\dfrac{e_{in} - V_n}{R_1} + \dfrac{V_0 - V_n}{R_f} = 0$, $V_n = \dfrac{V_0}{G}$ $(G = \text{op-amp gain})$

$$\Downarrow$$

$$V_0 = \frac{G\left(\frac{R_f}{R_1}\right)e_{in}}{\frac{R_f}{R_1} - G}$$

Let $K = \dfrac{R_1}{R_f}$:

$$T.F. = \mathcal{T}(s) = T = \frac{V_0}{e_{in}} = \frac{G}{1 - GK}$$

Closed Loop Sensitivity, S_G^T

$$S_G^{T_{cl}} = \frac{G}{T} \cdot \frac{dT}{dG} = \frac{G}{T} \cdot \frac{d}{dG}\left(\frac{G}{1-GK}\right)$$

where,

$$\frac{d}{dG}\left(\frac{G}{1-GK}\right) = \frac{1 \cdot (1-GK) - G(-K)}{(1-GK)^2} = \frac{1}{(1-GK)^2}$$

$$\boxed{\therefore S_{G\,(cl)}^{T^{cl}} = \frac{G}{\frac{G}{1-GK}} \cdot \frac{1}{(1-GK)^2} = \frac{1}{1-GK}}$$

Open Loop Sensitivity = Unity = 1

$(S_G^{T_{ol}} = 1)$

$S_G^{T^{cl}} = ?$ when $G = 10^4$ & $K = 0.2$

$$S_G^{T^{cl}} = \frac{1}{1 - 10^4 \cdot 0.2} = -5 \times 10^{-4}$$

$\boxed{\therefore \text{Closed loop sensitivity is much smaller than that of the open loop.}}$

Find sensitivity due to changes in R_1, S_K^T. Comment on sensitivity, especially when $GK >> 1$.

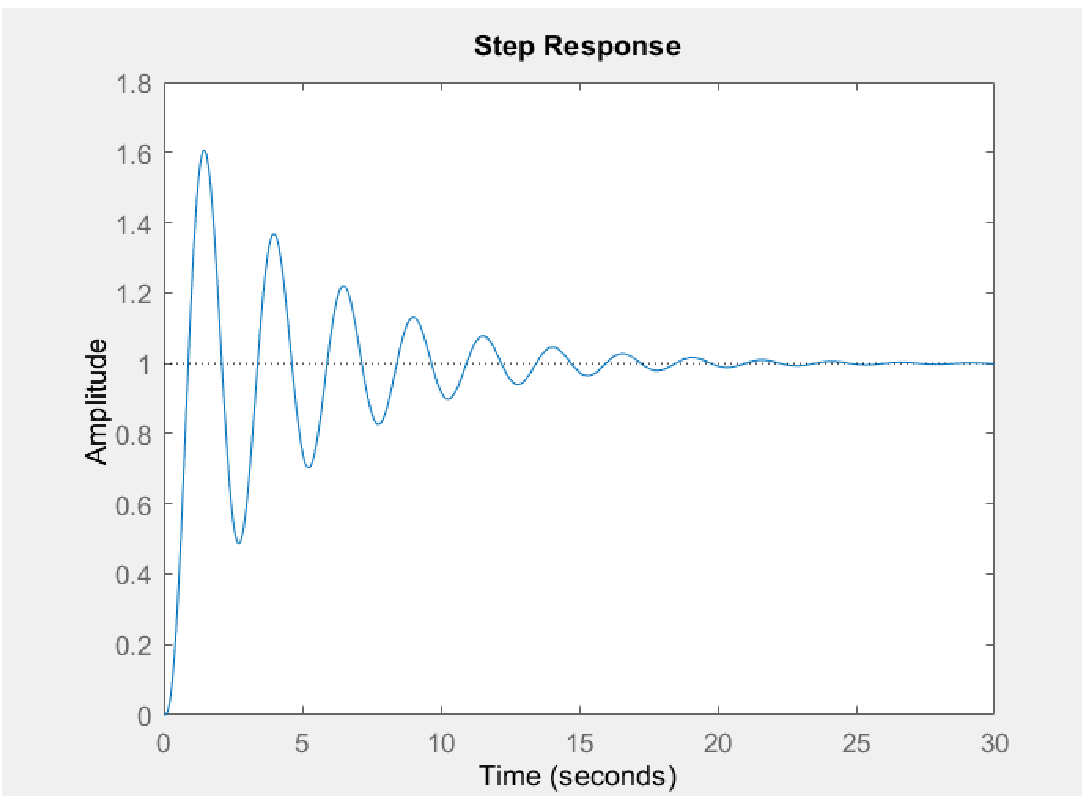
$$S_G^{T_{cl}} = \frac{1}{1 - G\left(\frac{R_1}{R_f}\right)} = \frac{1}{1 - \frac{V_0}{V_n}\left(\frac{R_1}{R_f}\right)}$$

$$G = \frac{V_0}{V_n}$$

$\therefore$ As R_1 increases, sensitivity decreases.
When $GK >> 1$, sensitivity is near 0.

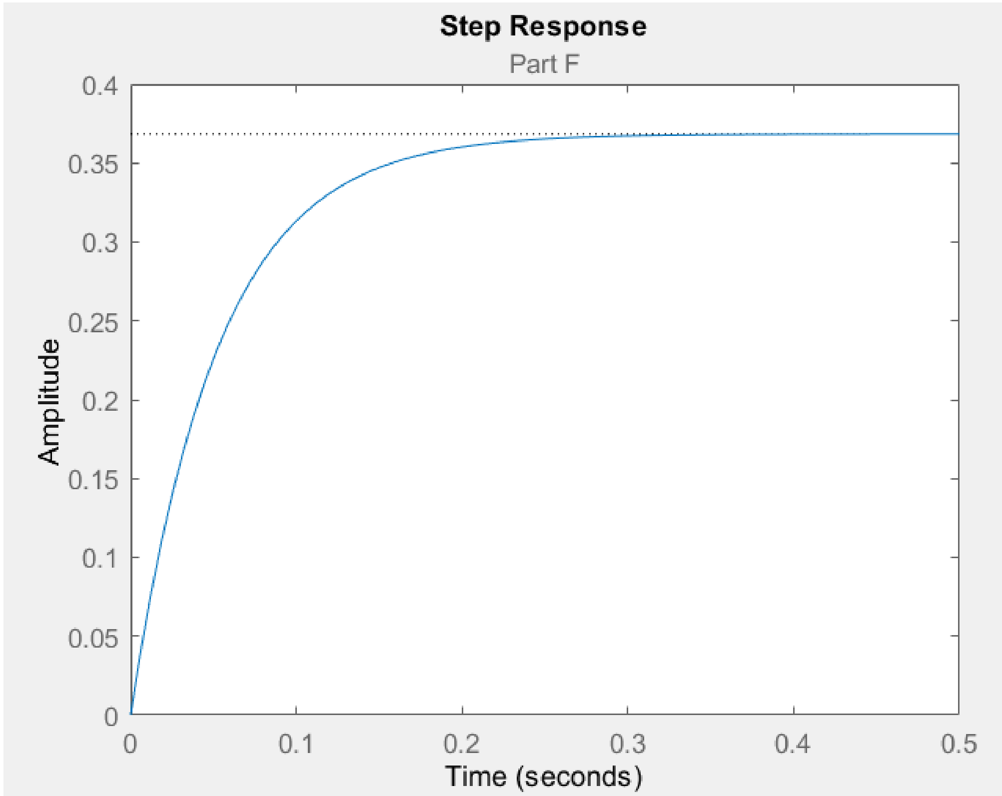
MidtermP1.m * | MidtermP2.m

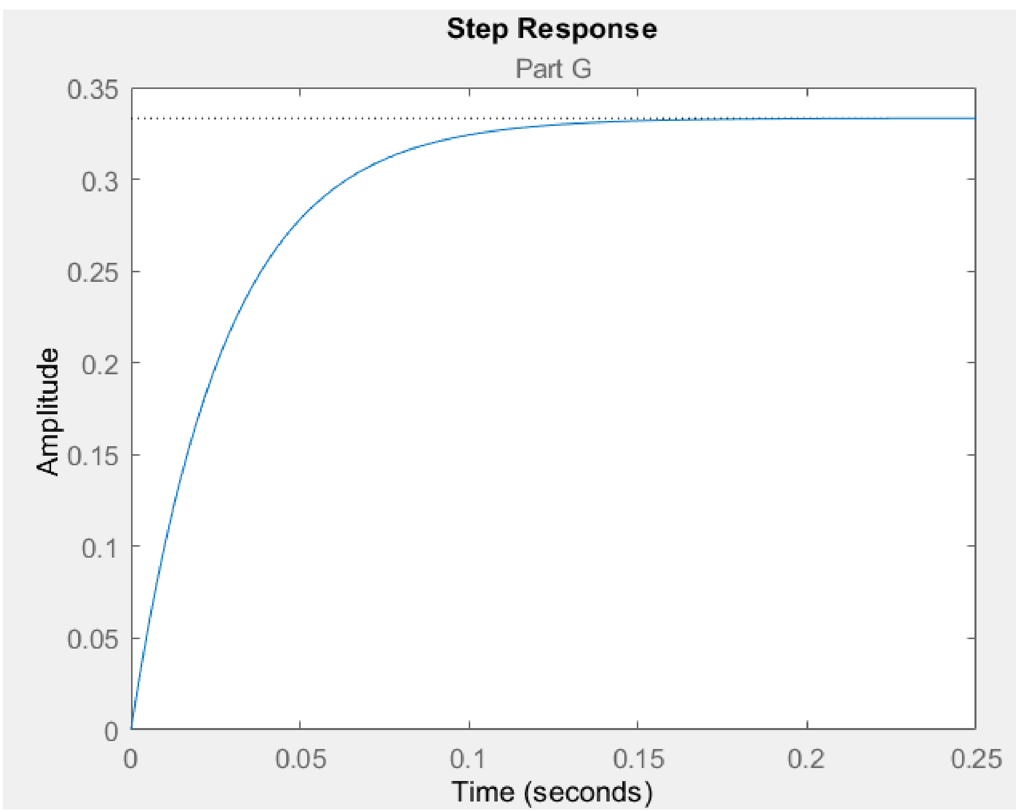
```matlab
%
% Aramis Kelkelyan
% MAE 476: Midterm
% Problem 1
% Date: 11/8/2022
%---------------------------------------
clear;clc;

%Type 0
K = 30;
a = 0.5;

num = [K K*a];
den = [1 6 11 6+K K*a];
sys = tf(num,den);
poles_are=pole(sys);
roots_char_eqn = roots(den);
disp(roots_char_eqn);

step(sys);
%if there are any POSITIVE roots, then the system is unstable

```



```
%
% Aramis Kelkelyan
% MAE 476: Midterm
% Problem 2
% Date: 11/8/2022
%----------------------------------------
clear;clc;

% part f)
kp = 7;
sys = tf([kp],[1 5+kp]);
clp = feedback(sys,1);
figure(1);
step(clp);

% part g)
kI = 0; kp2 = 7;
sys = tf(12, [1 12]);
clp = feedback(sys,2);
figure(2);
step(clp);

```

Graphs on the next page


Step Response
Part F
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Amplitude
0
0.1
0.2
0.3
0.4
0.5
Time (seconds)


Step Response
Part G
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Amplitude
0
0.05
0.1
0.15
0.2
0.25
Time (seconds)