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Problem 1 (3.1.4). *Prove that in the quotient group G/N, $(gN)^\alpha = g^\alpha N$ for all $\alpha \in \mathbb{Z}$.*

Proof. First take $\alpha > 0$. Since G/N is a group we have $(gN)^\alpha = gN \cdot gN \cdot \dots \cdot gN$ where there are α gNs. From the generalized associative property and the fact that $gN \cdot gN = (g \cdot g)N$, this reduces to $(g \cdot g \cdot \dots \cdot g)N = g^\alpha N$. For $\alpha = 0$ we get $(gN)^0 = N = 1N = g^0 N$. Finally, if $\alpha < 0$ then again since G/N is a group we have $(gN)^\alpha = \left((gN)^{-\alpha}\right)^{-1} = (g^{-\alpha}N)^{-1} = (g^{-\alpha})^{-1}N = g^\alpha N$. $\square$

Problem 2 (3.1.5). *Use the preceding exercise to prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.*

Proof. Let n be as defined. We know $g^n \in N$ which means $g^n N = N$. Then using Problem 1, $N = g^n N = (gN)^n$. Since n is the smallest positive integer such that this is true, we must have $|gN| = n$. If no such n exists, then $g^n \notin N$ for all positive n. Therefore $g^n N \neq N$ for all positive n and thus $|gN| = \infty$.

As an example, let $G = D_8$ and $N = \langle r \rangle$. Then $|r| = 4$ in G, and $|rN| = 1$. $\square$

Problem 3 (3.1.16). *Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that if $G = \langle x, y \rangle$ then $\overline{G} = \langle \overline{x}, \overline{y} \rangle$. Prove more generally that if $G = \langle S \rangle$ for any subset S of G, then $\overline{G} = \langle \overline{S} \rangle$.*

Proof. Let $S = \{a_1, \dots, a_n\}$ be a subset of G such that $G = \langle S \rangle$. Let $\overline{x} \in \overline{G}$. Since $G = \langle S \rangle$ we can write $\overline{x} = \overline{a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}}$. Using the generalized associative principle this reduces to $\overline{a_1}^{i_1} \overline{a_2}^{i_2} \dots \overline{a_n}^{i_n}$. Thus any element of $\overline{G}$ can be written as a product of powers of elements in $\overline{S}$. Therefore $\overline{G} = \langle \overline{S} \rangle$. In particular, if $S = \{x, y\}$ then $G = \langle x, y \rangle$ and $\overline{G} = \langle \overline{x}, \overline{y} \rangle$. $\square$

Problem 4 (3.1.17). *Let G be the dihedral group of order 16:*

$$G = \langle r, s \mid r^8 = s^2 = 1, rs = sr^{-1} \rangle$$

and let $\overline{G} = G/\langle r^4 \rangle$ be the quotient of G by the subgroup generated by r^4 (this subgroup is the center of G, hence is normal).
(a) Show that the order of $\overline{G}$ is 8.
(b) Exhibit each element of $\overline{G}$ in the form $\overline{s}^a \overline{r}^b$, for some integers a and b.
(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).
(d) Write each of the following elements of $\overline{G}$ in the form $\overline{s}^a \overline{r}^b$, for some integers a and b as in (b): $\overline{rs}$, $\overline{sr^{-2}s}$, $\overline{s^{-1}r^{-1}sr}$.
(e) Prove that $\overline{H} = \langle \overline{s}, \overline{r}^2 \rangle$ is a normal subgroup of $\overline{G}$ and $\overline{H}$ is isomorphic to the Klein 4-group. Describe the isomorphism type of the complete preimage of $\overline{H}$ in G.
(f) Find the center of $\overline{G}$ and describe the isomorphism type of $\overline{G}/Z(\overline{G})$.

Proof. (a) From Lagrange's Theorem, $|\overline{G}| = |G|/|\langle r \rangle| = 16/2 = 8$.

(b) We have

$$\begin{aligned}
\overline{G} &= \{x\langle r^4 \rangle \mid x \in G\} \\
&= \{\langle r^4 \rangle, r\langle r^4 \rangle, r^2\langle r^4 \rangle, r^3\langle r^4 \rangle, s\langle r^4 \rangle, sr\langle r^4 \rangle, sr^2\langle r^4 \rangle, sr^3\langle r^4 \rangle\} \\
&= \{\langle r^4 \rangle, r\langle r^4 \rangle, (r\langle r^4 \rangle)^2, (r\langle r^4 \rangle)^3, s\langle r^4 \rangle, s\langle r^4 \rangle r\langle r^4 \rangle, s\langle r^4 \rangle (r\langle r^4 \rangle)^2, s\langle r^4 \rangle (r\langle r^4 \rangle)^3\} \\
&= \{\overline{1}, \overline{r}, \overline{r}^2, \overline{r}^3, \overline{s}, \overline{sr}, \overline{sr}^2, \overline{sr}^3\}
\end{aligned}$$

(c) Using Problem 1 and the fact that $\langle r^4 \rangle = \{1, r^4\}$, we have $|\overline{1}| = 1$, $|\overline{r}| = 4$, $|\overline{r}^2| = 2$, $|\overline{r}^3| = 4$, $|\overline{s}| = 2$, $|\overline{sr}| = 2$, $|\overline{sr}^2| = 2$, $|\overline{sr}^3| = 2$.

(d) We have $\overline{rs} = \overline{rs}$, $\overline{sr^{-2}s} = \overline{r^2 s^2} = \overline{r}^2$ and $\overline{s^{-1}r^{-1}sr} = \overline{sr^{-1}sr} = \overline{rssr} = \overline{r}^2$.

(e) Note that since $\langle r^4\rangle \subseteq H$ we have $\langle r^4\rangle \leq H$ and so the Third Isomorphism Theorem applies. That is, $H/\langle r\rangle \trianglelefteq G/\langle r\rangle$. Now note that $\overline{H} = \{\overline{s}, \overline{r}^2, \overline{sr}^2, \overline{1}\}$. From part (c) we know that each of the nonidentity elements has order 2. Furthermore, $\overline{s}\cdot\overline{r}^2 = \overline{sr}^2$, $\overline{r}^2\cdot\overline{s} = \overline{sr}^{-2} = \overline{sr}^6 = \overline{sr}^2$, $\overline{s}\cdot\overline{sr}^2 = \overline{r}^2$, $\overline{sr^2}\cdot\overline{s} = \overline{r}^{-2}\overline{s}^2 = \overline{r}^2$, $\overline{r}^2\cdot\overline{sr}^2 = \overline{sr}^{-2}\overline{r}^2 = \overline{s}$ and $\overline{sr}^2\cdot\overline{r}^2 = \overline{sr}^4 = \overline{s}$. We've shown that all the relations hold for $\overline{H}$ being isomorphic the the Klein 4-group.

The preimage of $\overline{H}$ in G is $\{1, r^4, s, sr^4, r^2, r^6, sr^2, sr^6\}$. Renaming r^2 as r we see that $r^4 = s^2 = 1$ and $rs = sr^{-1}$. Thus the preimage of $\overline{H}$ in G is isomorphic to D_8.

(f) We know $\overline{r}, \overline{s} \notin Z(\overline{G})$ since $\overline{sr} = \overline{r}^{-1}\overline{s}$. The same applies to $\overline{r}^3$. Multiplying s by sr^i results in r^i and so these elements are not in $Z(\overline{G})$ either. This only leaves $\overline{r}^2$, which obviously commutes with $\overline{r}$ and $\overline{r}^3$. Now consider $\overline{r}^2(\overline{sr}^i) = \overline{sr}^{-2+i} = (\overline{sr}^i)\overline{r}^2$. Thus $Z(\overline{G}) = \{\overline{1}, \overline{r}^2\}$. We also have $\overline{G}/Z(\overline{G}) \cong V_4$. This can be seen by noticing $\overline{\overline{r}}^2 = \overline{\overline{r}^2} = \overline{\overline{1}}$, $\overline{\overline{s}}^2 = \overline{\overline{1}}$ and $\overline{\overline{sr}}^2 = \overline{\overline{sr}^2} = \overline{\overline{1}}$, and all the elements commute with each other. $\square$

Problem 5 (3.1.21)**.** *Let $G = Z_4 \times Z_4$ be given in terms of the following generators and relations:*

$$G = \langle x, y \mid x^4 = y^4 = 1, xy = yx\rangle.$$

Let $\overline{G} = G/\langle x^2y^2\rangle$ (note that every subgroup of the abelian group G is normal).
(a) Show that the order of $\overline{G}$ is 8.
(b) Exhibit each element of $\overline{G}$ in the form $\overline{x}^a\overline{y}^b$, for some integers a and b.
(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).
(d) Prove that $\overline{G} \cong Z_4 \times Z_2$.

Proof. (a) Let $N = \langle x^2y^2\rangle$. From Lagrange's Theorem we know $|\overline{G}| = |G|/|N| = 16/2 = 8$.

(b) We have

$$\overline{G} = \{N, xN, x^2N, x^3N, yN, yxN, yx^2N, yx^3N\} = \{\overline{1}, \overline{x}, \overline{x}^2, \overline{x}^3, \overline{y}, \overline{yx}, \overline{yx}^2, \overline{yx}^3\}.$$

(c) We have $|\overline{1}| = 1$, $|\overline{x}| = 4$, $|\overline{x}^2| = 2$, $|\overline{x}^3| = 4$, $|\overline{y}| = 4$, $|\overline{yx}| = 2$, $|\overline{yx}^2| = 4$, $|\overline{yx}^3| = 4$.

(d) Let $Z_4 \times Z_2 = \langle a, b \mid a^2 = b^4 = 1, ab = ba\rangle$. Let $\phi : \overline{G} \to Z_4 \times Z_2$ be a function such that $\phi(\overline{x}) = b$ and $\phi(\overline{yx}^3) = a$. Note that $|\overline{x}| = |b| = 4$ and $|\overline{yx}^3| = |a| = 2$. Furthermore, $\overline{G} = \langle\overline{x}, \overline{yx}^3\rangle$. To see this, note that $\overline{x}^i\overline{y}^j = (\overline{yx}^3)^j(\overline{x})^{-3j+i}$. Since the generators of $\overline{G}$ are mapped to the generators of $Z_4 \times Z_2$ and these groups have the same order, we see ϕ preserves the group structure. Injectivity and surjectivity also follow from this fact and we see that $\overline{G} \cong Z_4 \times Z_2$. $\square$

Problem 6 (3.1.24)**.** *Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.*

Proof. Let $h \in H$ and let $x \in N \cap H$. Then we have $hxh^{-1} \in N$ since $N \trianglelefteq G$ and $hxh^{-1} \in H$ since $H \leq G$. Therefore $h(N \cap H)h^{-1} \subseteq N \cap H$ for all $h \in H$. Thus $(N \cap H) \trianglelefteq H$. $\square$

Problem 7 (3.1.31)**.** *Prove that if $H \leq G$ and N is a normal subgroup of H then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of G in which N is normal (i.e., is the join of all subgroups H for which $N \trianglelefteq H$).*

Proof. Since $N \trianglelefteq H$, for all $h \in H$ we have $hNh^{-1} = N$. But then $H \leq \{g \in G \mid gNg^{-1} = N\} = N_G(N)$. Since this fact is true for any subgroup H for which $N \trianglelefteq H$, we see that $N_G(N)$ is the join of all such subgroups. $\square$

Problem 8 (3.1.36)**.** *Prove that if $G/Z(G)$ is cyclic then G is abelian.*

Proof. Assume that $G/Z(G)$ is cyclic with generator $xZ(G)$. The left cosets of $G/Z(G)$ partition G, so for $u \in G$, we know $u \in (xZ(G))^a = x^aZ(G)$ for some integer a. But this means we can write $u = x^az$ for $z \in Z(G)$. Now take $u, v \in G$. Since $Z(G)$ is the set of elements of G which commute with every element of G, we can write $uv = (x^az_1)(x^bz_2) = x^ax^bz_2z_1 = x^{a+b}z_2z_1 = x^bz_2x^az_1 = vu$. $\square$

Problem 9 (3.1.37). *Let A and B be groups. Show that $\{(a,1) \mid a \in A\}$ is a normal subgroup of $A \times B$ and the quotient of $A \times B$ by the is subgroup is isomorphic to B.*

Proof. Let $N = \{(a,1) \mid a \in A\}$. Let $(x,y) \in A \times B$ and consider $(x,y)(a,1)(x^{-1},y^{-1}) = (xax^{-1}, yy^{-1}) = (xax^{-1}, 1)$. Thus $(x,y)N(x,y)^{-1} \subseteq N$ for all $(x,y) \in A \times B$. Thus $N \trianglelefteq A \times B$. Now consider the function $\varphi : A \times B/N \to B$ such that $\phi((a,b)N) = b$. This function is injective, since $(a_1,b_1)N \neq (a_2,b_2)N$ implies $(a_2^{-1}a_1, b_2^{-1}b_1) \notin N$. Thus $b_2^{-1}b_1 \neq 1$ and $b_1 \neq b_2$. The map is clearly surjective since given $b \in B$ any element of the form $(a,b)N$ will map to it. Suppose we have $(a_1,b_1)N = (a_2,b_2)N$. Then $(a_2^{-1}a_1, b_2^{-1}b_1) \in N$. But this means $b_2^{-1}b_1 = 1$ and $b_1 = b_2$. Thus φ is well defined. Finally, note $\varphi((a_1,b_1)N(a_2,b_2)N) = \varphi((a_1a_2, b_1b_2)N) = b_1b_2 = \varphi((a_1,b_1)N)\varphi((a_2,b_2)N)$. Thus φ is an isomorphism and $A \times B/N \cong B$. □

Problem 10 (3.1.41). *Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy \mid x,y \in G\rangle$ is a normal subgroup of G and G/N is abelian (N is called the* commutator subgroup *of G).*

Proof. First note that $(x^{-1}y^{-1}xy){-}1 = y^{-1}x^{-1}yx$. For $g \in G$ we have $g(x^{-1}y^{-1}xy)g^{-1} = gx^{-1}g^{-1}gy^{-1}g^{-1}gxg^{-1}gyg^{-1} = (gxg^{-1})^{-1}(gyg^{-1})^{-1}(gxg^{-1})(gyg^{-1})$. Thus, conjugation of a single commutator results in another commutator. Now suppose $z = (x_1^{-1}y_1^{-1}x_1y_1)\dots(x_n^{-1}y_n^{-1}x_ny_n)$ is the product of commutators. Then we have $gzg^{-1} = g(x_1^{-1}y_1^{-1}x_1y_1)g^{-1}\dots g(x_n^{-1}y_n^{-1}x_ny_n)g^{-1}$ and from the above result, we know this is then the product of commutators. This is then extended to the case where each commutator is raised to a power. For positive powers, the exact same argument holds. For negative powers, first separate the power into an inverse taken to a positive power, then use the first result of the proof. The conjugation is then a product of commutators. Therefore $gNg^{-1} \subseteq N$ for all $g \in G$ and thus $N \trianglelefteq G$.

Now let $aN, bN \in G/N$. Then $aNbN = abN = \{abx \mid x \in N\}$. Now consider some element of abN, $ab(x^{-1}y^{-1}xy)$. We can write this as $ba(a^{-1}b^{-1}ab)(x^{-1}y^{-1}xy)$. Thus for each element of abN we can find an equivalent element in baN and vice versa. Therefore $abN = baN$ which means $aNbN = bNaN$ and G/N is abelian. □

Problem 11 (3.2.4). *Show that if $|G| = pq$ for primes p and q (not necessarily distinct) then either G is abelian or $Z(G) = 1$.*

Proof. We know $|Z(G)| \mid |G|$. Assuming that $Z(G) \neq 1$, without loss of generality we either have $|Z(G)| = p$ or $|Z(G)| = pq$. In the later case we're done since $G = Z(G)$ which is abelian. In the former case, let $x \in G \backslash Z(G)$. Then $|y| = q$ and so $G = Z(G) \cup \langle y\rangle$. Since $Z(G)$ commutes with everything and $\langle y\rangle$ is abelian, G must be abelian. □

Problem 12 (3.2.11). *Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$.*

Proof. Note that $|K : H|$ is the number of left cosets of H in K. Also, $|G : K|$ is the number of left cosets of K in G. That is, for each coset K in G, we can further partition this coset into $|K : H|$ cosets of H in G. Since there are $|G : K|$ of these partitions, and this gives all left cosets of H in G, this gives $|G : K| \cdot |K : H| = |G : H|$. □

Problem 13 (3.2.19). *Prove that if N is a normal subgroup of the finite group G and $(|N|, |G : N|) = 1$ then N is the unique subgroup of G of order $|N|$.*

Proof. Note that G/N partitions G into $|G : N|$ left cosets each with $|N|$ elements. But since $|N|$ and $|G : N| = |G|/|N|$ are relatively prime, there's only one way to do this. □

Problem 14 (4.1.1). *Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$ (G_a is the stabilizer of a). Deduce that if G acts transitively on A then the kernel of the action is $\bigcap_{g \in G} gG_ag^{-1}$.*

Proof. Let $x \in G_b$. Then $x \cdot b = b = g \cdot a$. Therefore $g \cdot a = x \cdot b = x(g \cdot a)$ and so $a = g^{-1}g \cdot a = g^{-1}xg \cdot a$. Thus $x \in gG_ag^{-1}$. If G acts transitively on A then $b = g \cdot a$ for all $b \in A$ and some $g \in G$. Then for all $b \in A$ we have $G_b = gG_ag^{-1}$ for some $g \in G$. But we know the kernel of the action is $\bigcap_{b \in A} G_b = \bigcap_{g \in G} gG_ag^{-1}$. □

Problem 15 (4.2.8)**.** *Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.*

Proof. Let G act on the set A of left cosets of H by left multiplication. Let π_H be the permutation representation afforded by this action. Then we know $\ker \pi_H = \bigcap_{x \in G} xHx^{-1}$. Let $K = \ker \pi_H$. Then we know K is normal and contained in H. Furthermore, since $|G : H| = n$, $\pi_H(G) \leq S_n$ and by the first isomorphism theorem $G/K \leq S_n$. Therefore $|G : K| \leq n!$. □

Problem 16 (4.2.9)**.** *Prove that if p is prime and G is a group of order p^α for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index p is normal in G. Deduce that every group of order p^2 has a normal subgroup of order p.*

Proof. We know that if q is the smallest prime dividing $|G|$ then any subgroup of order q is normal in G. But since the only prime which divides $|G|$ is p, we know that p is the smallest prime dividing $|G|$ and hence every subgroup of order p is normal in G. Suppose $|G| = p^2$ then there exists $x \in G$ such that $x \neq 1$. Since $\langle x \rangle \mid |G|$ we know $\langle x \rangle = p$. But then $|G : \langle x \rangle| = p$ as well as so $\langle x \rangle \trianglelefteq G$. □

Problem 17 (4.3.7)**.** *For $n = 3$, 4, 6 and 7 make lists of the partitions of n and give representatives for the corresponding conjugacy classes of S_n.*

For $n = 3$ the partitions are $(1, , 1, 1)$, $(1, 2)$ and (3) and they have representatives (1), (1 2) and (1 2 3).

For $n = 4$ the partitions are $(1, 1, 1, 1)$, $(1, 1, 2)$, $(1, 3)$, (4) and $(2, 2)$ and they have representatives (1), (1 2), (1 2 3), (1 2 3 4) and (1 2)(3 4).

For $n = 6$ the partitions are $(1, 1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 2)$, $(1, 1, 1, 3)$, $(1, 1, 4)$, $(1, 5)$, (6), $(1, 1, 2, 2)$, $(1, 2, 3)$, $(2, 2, 2)$, $(2, 4)$ and $(3, 3)$ and they have representatives (1), (1 2), (1 2 3), (1 2 3 4), (1 2 3 4 5), (1 2 3 4 5 6), (1 2)(3 4), (1 2)(3 4)(5 6), (1 2)(3 4 5 6) and (1 2 3)(4 5 6).

For $n = 7$ the partitions are $(1, 1, 1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 1, 2)$, $(1, 1, 1, 1, 3)$, $(1, 1, 1, 4)$, $(1, 1, 5)$, $(1, 6)$, (7), $(1, 1, 1, 2, 2)$, $(1, 1, 2, 3)$, $(1, 2, 2, 2)$, $(1, 3, 3)$, $(1, 2, 4)$, $(2, 5)$, and $(3, 4)$ and they have representatives (1), (1 2), (1 2 3), (1 2 3 4), (1 2 3 4 5), (1 2 3 4 5 6), (1 2 3 4 5 6 7), (1 2)(3 4), (1 2)(3 4 5), (1 2)(3 4)(5 6), (1 2 3)(4 5 6), (1 2)(3 4 5 6), (1 2)(3 4 5 6 7) and (1 2 3)(4 5 6 7).

Problem 18 (4.3.26)**.** *Let G be a transitive permutation group on the finite set A with $|A| > 1$. Show that there is some $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$ (such an element σ is called* fixed point free*).*

Proof. Let $a \in A$. We know that $|\{\sigma(a) \mid \sigma \in G\}| = |G : G_a| = |G|/|G_a|$. But since G acts transitively on A we know that $A = \{\sigma(a) \mid \sigma \in G\}$. Therefore $|G|/|A| = |G_a|$. This is true for all $a \in A$. Furthermore, since $|A| > 1$ we know that $|G_a| < |G|$ for each $a \in A$. Thus there exists $\sigma \in G$ for each $a \in A$ such that $\sigma(a) \neq a$. Then taking the union $U = \bigcup_{a \in A} G_a$ we can find $\sigma \in G \backslash U$ which doesn't fix any $a \in A$. □

Problem 19 (4.3.29)**.** *Let p be a prime and let G be a group of order p^α. Prove that G has a subgroup of order p^β, for every β with $0 \leq \beta \leq \alpha$.*

Proof. For the base case $\alpha = 0$ the problem is trivial since $|G| = 1$. Assume the statement is true for groups of order α and suppose that $|G| = p^{\alpha+1}$. Then we know $Z(G) \neq 1$ which means $|Z(G)| = p^\gamma$ where $1 < \gamma \leq \alpha + 1$. Since $Z(G)$ is abelian, we know there exists $x \in Z(G)$ such that $|x| = p$. Now consider $\overline{G} = |G|/\langle x \rangle$. Note that $\overline{G} \cong H$ for some $H \leq G$. But also $|\overline{G}| = |H| = p^\alpha$. Therefore H has subgroups of order p^β for each $0 \leq \beta \leq \alpha$ and therefore G does as well. □

Problem 20. *Write the class equation for A_4.*

Proof. From Problem 17 we know that representatives of the cycles types of even permutations of S_4 can be taken to be (1), (1 2 3) and (1 2)(3 4). Furthermore we know that

$$C_{S_4}((1\ 2\ 3)) = \{(1\ 2\ 3)^i \tau \mid i = 0,\ 1 \text{ or } 2 \text{ and } \tau \in S_{4-3}\} = \langle (1\ 2\ 3) \rangle$$

which directly implies $C_{A_4}((1\ 2\ 3)) = \langle (1\ 2\ 3) \rangle$. This group has order 3 and index 4. Since there are 8 3-cycles in A_4, and four of them are in the conjugacy class of (1 2 3), there must be another 3-cycle not in this class. Using the same logic as above, we see that the centralizer of this 3-cycle is also of order 3 and so it has index 4. Finally, note that $\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle \cong V_4$ and so all of these elements commute with each other. Since these are the only elements of A_4 with this cycle type, we see that $|C_{A_4}((1\ 2)(3\ 4))| = 4$ and has index 3. Therefore, the class equation for A_4 is

$$|A_4| = |Z(A_4)| + |A_4 : C_{A_4}((1\ 2\ 3))| + |A_4 : C_{A_4}((1\ 3\ 2))| + |A_4 : C_{A_4}((1\ 2)(3\ 4))| = 1 + 4 + 4 + 3 = 12.$$

$\square$