Kris Harper
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Homework 10

Problem 1. *If (X, d) is a complete metric space, show that it is isometric to its completion, $(\widetilde{X}, \widetilde{d})$.*

Proof. From the completion of X we know there exists a function $\phi : X \to \widetilde{X}$ such that $\phi : X \to \phi(X)$ is an isometry. It is clear that $\phi(X) \subseteq \widetilde{X}$. It remains to be shown that $\widetilde{X} \subseteq \phi(X)$. Note that $\phi(x) = \overline{(x_k)}$ where $x_k = x$ for all $k \in \mathbb{N}$. Consider $\widetilde{x} \in \widetilde{X}$ such that $\widetilde{x} = \overline{(x_n)}$. Since X is complete, $\lim_{n\to\infty} x_n$ exists in X. Call this limit x. But if this is the case, then $\lim_{n\to\infty} d(x, x_n) = 0$ and so $\lim_{n\to\infty} d(x_k, x_n) = 0$, where $x_k = x$ for all $k \in \mathbb{N}$. Thus $\phi(x) = \overline{(x_n)} = \widetilde{x}$ and so $\widetilde{x} \in \phi(X)$. Therefore $\widetilde{X} \subseteq \phi(X)$ and so $\phi(X) = \widetilde{X}$. Then ϕ is a isometry between X and $\widetilde{X}$. □

Problem 2. *For a metric space (X, d) and it's completion $(\widetilde{X}, \widetilde{d})$, prove that if (X', d') is a complete metric space such that X is isometric to a dense subset of X', then $(\widetilde{X}, \widetilde{d})$ and (X', d') are isometric.*

Proof. Since X is isometric to a dense subset of both $\widetilde{X}$ and X', we know there exist functions $\phi : X \to \widetilde{X}$ and $\varphi : X \to X'$ such that $\phi(X)$ is dense in $\widetilde{X}$ and $\varphi(X)$ is dense in X'. Define a function $f : \widetilde{X} \to X'$ such that $f(\phi(x)) = \varphi(x)$. Note that for $x, y \in X$ we have

$$\widetilde{d}(\phi(x), \phi(y)) = d(x, y) = d'(\varphi(x), \varphi(y))$$

and so

$$d'(f(\phi(x)), f(\phi(y))) = d'(\varphi(x), \varphi(y).$$

Then it must be the case that f preserves distances between elements of $\phi(X) and \varphi(X)$. Consider an element $x \in \widetilde{X}$ such that $x \notin \phi(X)$. Then x can be identified with a Cauchy sequence, (x_n), of points in X such that $(\phi(x_n))$ is then a Cauchy sequence in $\widetilde{X}$ that converges to x. Then since f preserves distance between $\phi(X)$ and $\varphi(X)$ we see that $(f(\phi(x_n)))$ is a Cauchy sequence in X'. Since X' is complete, this sequence converges. Call its limit y and define $f(x) = y$. If $x, y \in \widetilde{X}$ such that $x = \overline{(x_n)}$ and $y = \overline{(y_n)}$ then we defined $d(x, y) = \lim_{n\to\infty} d(x_n, y_n)$. Then $d(f(x), f(y)) = \lim_{n\to\infty} d'(f(x_n), f(y_n)) = d'(f(x), f(y))$ since we know f preserves distances between $\phi(X)$ and $\varphi(X)$. Additionally, for an element $\varphi(x) \in \varphi(X)$ we can say that $f^{-1}(\varphi(x)) = \phi(x)$. A similar argument can then be used to show that f^{-1} extends to elements of X' which are not in $\varphi(X)$. Thus f has an inverse $f^{-1} : X' \to \widetilde{X}$. Since f is a bijection which preserves distances, it is an isometry between $\widetilde{X}$ and X'. □

Problem 3. *Let (X, d) be a metric space, and for any $x, y \in X$, let $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$.*
1) Show that d' defines a metric on X.
2) Show that U is open in (X, d) if and only if U is open in (X, d').
3) If a set A is compact in (X, d), is it necessarily compact in (X, d')?
4) If (X, d') is complete, is (X, d) necessarily complete?

Proof. 1) It is clear that $d'(x, y) \geq 0$. Supposing that $d'(x, y) = 0$, then it must be the case that $d(x, y) = 0$ and so $x = y$. Conversely, if $x = y$ then $d(x, y) = 0$ and so $d'(x, y) = 0/1 = 0$. Also note that

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = d'(y, x).$$

Finally, for $z \in Z$ note that $d(x,z) \leq d(x,y) + d(y,z)$ and so

$$\begin{aligned}
d'(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\
&\leq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \\
&\leq \frac{d(x,y)+d(y,z)+d(x,y)d(y,z)}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)} \\
&\leq \frac{d(x,y)+d(y,z)+2d(x,y)d(y,z)}{1+d(x,y)+d(y,z)+d(x,y)d(y,z)} \\
&= \frac{d(x,y)(1+d(y,z))+d(y,z)(1+d(x,y))}{(1+d(x,y))(1+d(y,z))} \\
&= \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\
&= d'(x,y) + d'(y,z).
\end{aligned}$$

2) Suppose that U is open in (X,d). Then for all $x \in U$ there exists $r \in \mathbb{R}$ such that $B_r(x) \subseteq U$. Consider some element $y \in B_r(x)$. Then $d(x,y) < r$. Choose $r' \in \mathbb{R}$ such that $r' < r/(1+d(x,y))$. Then the set $\{y \in X \mid d(x,y) < (1+d(x,y))r'\} \subseteq B_r(x)$. But since $(1+d(x,y)) \geq 1$, it must be the case that $d(x,y) < d(x,y)/(1+d(x,y)) < r'$. Then $\{y \mid d'(x,y) < r'\} \subseteq B_r(x)$ and so U is also open in (X,d').

Conversely, suppose that U is open in (X,d'). Then for all $x \in U$ there exists $r' \in \mathbb{R}$ such that $\{y \in X \mid d'(x,y) < r'\} \subseteq U$. Consider some element $y \in \{y \in X \mid d'(x,y) < r'\}$ and choose $r \in \mathbb{R}$ such that $r < (1+d(x,y))r'$. Then $d(x,y) < r < (1+d(x,y))r'$. But then

$$\{y \in X \mid d(x,y) < r\} \subseteq \{y \in X \mid d'(x,y) < r'\} \subseteq U$$

and so U is open in (X,d).

3) No. Consider a compact space (X,d) such that for $x,y \in X$ we have $d(x,y) \geq 1$. Then $d'(x,y) \leq 1$. But then there must exist an open cover for X which has a finite subcover in (X,d) but not in (X,d') since all the distances are shorter.

4) Yes. Consider a Cauchy sequence (a_n) in (X,d). Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n,m > N$ we have $d(a_n,a_m) < \varepsilon$. But then using the same argument as in Part 1) we can find $N' \in \mathbb{N}$ such that for all $n,m > N'$ we have $d'(a_n,a_m) < \varepsilon$. □

**** Problem 1.** *On $\mathbb{R}^2$ with the usual metric, find all isometries $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(0) = 0$.*

Proof. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Note that since $T(0) = 0$, we know that a T is not a translation. Suppose that T is a rotation about 0 or a reflection across a line passing through 0. Then it's clear that each of these has an inverse, namely rotating in the opposite direction and reflecting across the same line again. Furthermore, we see that T must preserve distance so that for two points $x,y \in \mathbb{R}^2$ we have $d(x,y) = d(T(x),T(y))$ where d is the usual metric. As shown earlier, since T is bijective and preserves distance, it must be a homeomorphism and therefore an isometry. Note that these are the only isometries with the given conditions because they are the only ones that will preserve distance. □

**** Problem 2.** *For $f \in \mathcal{C}([0,1],\mathbb{R})$ define*

$$||f|| = \int_0^1 |f(x)|dx$$

as a norm on this space. Is this space complete with respect to the metric defined by this norm?

No.

Proof. Consider the sequence $f_n = x^n$ and let $\varepsilon > 0$. Then choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$ and choose $n, m > N$ such that $m < n$. Note that

$$||f_n - f_m|| = \int_0^1 |x^n - x^m| dx = \int_0^1 (x^n - x^n) dx = \frac{x^{n+1}}{n+1} - \frac{x^{m+1}}{m+1}\bigg|_{x=0}^{x=1} = \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{N} < \varepsilon$$

and so (f_n) is a Cauchy sequence in $\mathcal{C}([0,1], \mathbb{R})$. Then suppose that (f_n) converges to some function $f \in \mathcal{C}([0,1], \mathbb{R})$. Then for all $\varepsilon > 0$ there exists N such that for all $n > N$ we have $||f - f_n|| < \varepsilon$. Note that

$$|\int_0^1 f(x) dx|$$

is a constant and so choose

$$0 < \varepsilon < \left| \int_0^1 f(x) dx \right|.$$

But note that

$$\varepsilon > ||f - f_n|| = \int_0^1 |f(x) - x^n| dx \geq \left| \int_0^1 (f(x) - x^n) dx \right| = \left| \int_0^1 f(x) dx - \int_0^1 x^n dx \right| = \left| \int_0^1 f(x) dx - \frac{1}{n+1} \right|.$$

Then for large enough n, the last term will be larger than ε. This is a contradiction and so no function f can exist. Thus, $\mathcal{C}([0,1], \mathbb{R})$ is not complete with respect to this metric. □

** **Problem 3.** *For a metric space (X, d) let X' be the set of all Cauchy sequences on X and define a relation on X' where $(a_n) \sim (b_n)$ if $\lim_{n\to\infty} d(a_n, b_n) = 0$. Show that this is an equivalence relation.*

Proof. Since $d(a_n, a_n) = 0$ for all $n \in \mathbb{N}$ it's clear that $\lim_{n\to\infty} d(a_n, a_n) = 0$ and so $(a_n) \sim (a_n)$. Thus, $\sim$ is reflexive. Suppose that $\lim_{n\to\infty} d(a_n, b_n) = 0$. Then since d is symmetric, we have $\lim_{n\to\infty} d(b_n, a_n) = 0$. Thus $\sim$ is symmetric. Finally, suppose that for a Cauchy sequence $(c_n) \in X'$ we have $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Then we have $d(a_n, c_n) \leq d(a_n, b_n) + d(b_n, c_n)$. But then

$$\lim_{n\to\infty} d(a_n, c_n) \leq \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} (b_n, c_n) = 0 + 0 = 0.$$

Thus $(a_n) \sim (c_n)$ and $\sim$ is transitive. Therefore $\sim$ is an equivalence relation on X'. □

** **Problem 4.** *Show that $\widetilde{d}$ is well-defined.*

Proof. Let $\overline{(a_n)}, \overline{(b_n)}, \overline{(c_n)}, \overline{(d_n)} \in \widetilde{X}$ such that $(a_n) \sim (c_n)$ and $(b_n) \sim (d_n)$. Suppose that $\widetilde{d}(\overline{(a_n)}, \overline{(c_n)}) = d$. Then we know that

$$\lim_{n\to\infty} d(a_n, c_n) = d$$

and also that

$$\lim_{n\to\infty} d(a_n, b_n) = \lim_{n\to\infty} d(c_n, d_n).$$

Then for all $\varepsilon > 0$ there exist $N_1, N_2, N_3 \in \mathbb{N}$ such that for all $n > N_1$ we have $|d - d(a_n, c_n)| < \varepsilon/3$, for all $n > N_2$ we have $|d(a_n, b_n)| < \varepsilon/3$ and for all $n > N_3$ we have $|d(c_n, d_n)| < \varepsilon/3$. Let $N = \max(N_1, N_2, N_3)$ so that all three statements are true for all $n > N$. Then note that

$$\begin{aligned}
|d - d(b_n, d_n)| &\leq |d - d(b_n, c_n) + d(c_n, d_n)| \\
&\leq |d - d(a_n, c_n) + d(a_n, b_n) + d(c_n, d_n)| \\
&\leq |d - d(a_n, c_n)| + |d(a_n, b_n)| + |d(c_n, d_n)| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

for all $n > N$. Thus $\lim_{n\to\infty} d(b_n, d_n) = \lim_{n\to\infty} d(a_n, c_n)$ and so $\widetilde{d}$ is well-defined. □

** **Problem 5.** *Show that $\widetilde{X}$ is complete.*

Proof. Consider a Cauchy sequence $(\widetilde{x}_n) \in \widetilde{X}$. Since $\phi(X)$ is dense in $\widetilde{X}$ we can choose $\widetilde{z}_n \in \phi(X)$ such that $\widetilde{d}(\widetilde{z}_n, \widetilde{x}_n) < \frac{1}{n}$ for every $n \in \mathbb{N}$. Then we have

$$\widetilde{d}(\widetilde{z}_n, \widetilde{z}_m) \leq \widetilde{d}(\widetilde{z}_n, \widetilde{y}_n) + \widetilde{d}(\widetilde{y}_n, \widetilde{y}_m) + \widetilde{d}(\widetilde{y}_m, \widetilde{z}_m) < \frac{1}{n} + \varepsilon + \frac{1}{m}$$

since $(\widetilde{y}_n)$ is Cauchy. This implies that $(\widetilde{z}_n)$ is Cauchy in $\widetilde{X}$. Since $\widetilde{z}_n \in \phi(X)$ for all n, let $x_n = \phi^{-1}(\widetilde{z}_n)$. Then (x_n) is Cauchy in X since ϕ is an isometry. Call $\widetilde{y}$ the element of $\widetilde{X}$ defined by the equivalence class containing (x_n). Then

$$\widetilde{d}(\widetilde{y}_n, \widetilde{y}) \leq \widetilde{d}(\widetilde{y}_n, \widetilde{z}_n) + \widetilde{d}(\widetilde{z}_n, \widetilde{y}) \leq \frac{1}{n} + \widetilde{d}(\widetilde{z}_n, \widetilde{y})$$

and note that $\widetilde{d}(\widetilde{z}_n, \widetilde{y}) = \lim_{k\to\infty}(x_n, x_k)$. Since (x_n) is Cauchy in X, for n and k large, $d(x_n, x_k)$ is arbitrarily small. □

** **Problem 6.** *For primes p_1 and p_2 with $p_1 \neq p_2$ show that $\mathbb{Q}_{p_1}$ is not isomorphic to $\mathbb{Q}_{p_2}$.*

Proof. Suppose that there exists a isomorphism, $f : \mathbb{Q}_{p_1} \to \mathbb{Q}_{p_2}$. From Problem 6 Part 9) we know that for $x, y \in \mathbb{Q}_{p_1}$ we have $x \in p_1^n U_{p_1}$ for some $n \in \mathbb{Z}$ and likewise for y. Suppose that $x, y \in p_1^n U_{p_1}$ for some $n \in \mathbb{Z}$. Then we have $|x|_{p_1} = |y|_{p_1} = p_1^{-n}$. Also, this implies that $|x + y|_{p_1} < p^{-n}$. Suppose that we chose x and y such that $|f(x)|_{p_2} \neq |f(y)|_{p_2}$. It is certainly possible to do this for $x, y \in \mathbb{Q}$. Then $|x|_{p_2} = p_2^{-j}$ and $|y|_{p_2} = p_2^{-k}$ for some $j \neq k$. Note $|f(x + y)|_{p_2}$ must be strictly less than each of these. But then this is a contradiction because $|f(x)|_{p_2} \neq |f(y)|_{p_2}$ and so $|f(x) + f(y)|_{p_2} = \max(|f(x)|_{p_2}, |f(y)|_{p_2}) \neq |f(x + y)|_{p_2}$. □

Problem 4. *1) Show that addition, multiplication and $|\cdot|_p$ are well-defined in $\mathbb{Q}_p$.*
2) Show that $\mathbb{Q}_p$ is a field with the operations given above.
3) Show that $|\cdot|_p$ on $\mathbb{Q}_p$ satisfies the same properties as it does in $\mathbb{Q}$.
4) Show that the image of $\mathbb{Q}_p$ under $|\cdot|_p$ is the same as that of $\mathbb{Q}$ under $|\cdot|_p$, that is, $\{p^k \mid k \in \mathbb{Z}\} \cup \{0\}$.
5) Show that $\mathbb{Q}_p$ cannot be made into an ordered field.

Proof. Let $\overline{(a_n)}, \overline{(b_n)}, \overline{(c_n)}, \overline{(d_n)} \in \mathbb{Q}_p$ such that $(a_n) \sim (b_n)$ and $(c_n) \sim (d_n)$. Then

$$\lim_{n\to\infty} |a_n - b_n|_p = \lim_{n\to\infty} |c_n - d_n|_p = 0.$$

Then for all $\varepsilon > 0$ there exists N such that for all $n > N$ we have

$$|(a_n + c_n) - (b_n + d_n)|_p = |(a_n - b_n) + (c_n - d_n)|_p \leq |a_n - b_n|_p + |c_n - d_n|_p \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\lim_{n\to\infty} |(a_n + c_n) - (b_n + d_n)|_p = 0$ which means $(a_n + c_n) \sim (b_n + d_n)$. Likewise, we have

$$|a_n c_n - b_n d_n|_p \leq |a_n c_n - a_n d_n - b_n c_n + b_n d_n|_p = |(a_n - b_n)(c_n - d_n)|_p = |a_n - b_n|_p |c_n - d_n|_p \leq \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon.$$

Thus $\lim_{n\to\infty} |a_n c_n - b_n d_n|_p = 0$ which means $(a_n c_n) \sim (b_n d_n)$. Finally,

$$||a_n|_p - |b_n|_p|_p \leq |a_n - b_n|_p < \varepsilon$$

which means that $\lim_{n\to\infty} ||a_n|_p - |b_n|_p|_p = 0$ and so $(|a_n|_p) \sim (|b_n|_p)$. This shows that $+$, $\cdot$ and $|\cdot|_p$ are well-defined on $\mathbb{Q}_p$.

2) Let $\overline{(a_n)}, \overline{(b_n)} \in \mathbb{Q}_p$. Then for all $\varepsilon > 0$ there exists N_1 such that for all $n, m > N_1$ we have $|a_n - a_m|_p < \varepsilon/2$ and there exists N_2 such that for all $n, m > N_2$ we have $|b_n - b_m|_p < \varepsilon/2$. Let $N = \max(N_1, N_2)$ so that both statements are true for all $n, m > N$. But then

$$|(a_n + b_n) - (a_m + b_m)|_p = |(a_n - a_m) + (b_n - b_m)|_p \leq |a_n - a_m|_p + |b_n - b_m|_p \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n, m > N$. Thus $(a_n + b_n)$ is Cauchy and $\mathbb{Q}_p$ is closed under addition. Likewise for $\sqrt{\varepsilon}$

$$|a_nb_n - a_mb_m|_p \leq |a_nb_n - a_nb_m - a_mb_n + a_mb_m|_p = |(a_n - a_m)(b_n - b_m)|_p = |a_n - a_m|_p|b_n - b_m|_p \leq \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$$

for all $n, m > N$. Thus (a_nb_m) is Cauchy and $\mathbb{Q}_p$ is closed under multiplication. Associativity and commutativity of addition and multiplication as well as the distributive property all follow from their counterparts in $\mathbb{Q}$ and the fact that addition and multiplication are term based operations. Also note that if $\overline{(0)} = 0$ is the constant zero sequence then

$$\overline{(0)} + \overline{(a_n)} = \overline{(0 + a_n)} = \overline{(a_n)}$$

and so 0 is the additive identity. A similar proof holds to show that $\overline{(1)} = 1$ is the multiplicative identity. It follows that $-\overline{(a_n)} = \overline{(-a_n)}$ is the additive inverse of $\overline{(a_n)}$ since

$$\overline{(-a_n)} + \overline{(a_n)} = \overline{(-a_n + a_n)} = \overline{(0)} = 0.$$

Finally, since $a_n \in \mathbb{Q}$ for all n, it follows that $1/a_n \in \mathbb{Q}$ for all n. Thus $\overline{(a_n)}^{-1} = \overline{(1/a_n)}$ is the multiplicative inverse of $\overline{(a_n)}$ since

$$\overline{(1/a_n)} \cdot \overline{(a_n)} = \overline{(1/a_n \cdot a_n)} = \overline{(1)} = 1.$$

Since all the axioms are met, it follows that $\mathbb{Q}_p$ is a field.

3) Let $\overline{(a_n)}, \overline{(b_n)} \in \mathbb{Q}_p$ such that $a = \overline{(a_n)}$ and $b = \overline{(b_n)}$. Note that $|a|_p = |\overline{(a_n)}|_p = \lim_{n\to\infty} |a_n|_p$. Since each term in $(|a_n|_p)$ is greater than or equal to 0, it follows that the limit is greater than or equal to zero. If $a = 0$ then we know $|a|_p = 0$ since $0 \in \mathbb{Q}$. If $|a|_p = 0$ then $\lim_{n\to\infty} |a_n|_p = 0$. But then $(a_n) \sim (0)$ and so $a = 0$. Next consider

$$|ab|_p = \lim_{n\to\infty} |a_nb_n|_p = \lim_{n\to\infty} |a_n|_p|b_n|_p = \lim_{n\to\infty} |a_n|_p \lim_{n\to\infty} |b_n|_p = |a|_p \cdot |b|_p.$$

Thirdly,

$$|a + b|_p = \lim_{n\to\infty} |a_n + b_n|_p \leq \lim_{n\to\infty} \max(|a_n|_p, |b_n|_p) = \max\left(\lim_{n\to\infty} |a_n|_p, \lim_{n\to\infty} |b_n|_p\right) = \max(|a|_p, |b|_p).$$

Finally, suppose that $|a|_p \neq |b|_p$. Then

$$\lim_{n\to\infty} |a_n|_p \neq \lim_{n\to\infty} |b_n|_p$$

which implies that $|a_n + b_n|_p = \max(|a_n|_p, |b_n|_p)$ for all n. But then

$$|a + b|_p = \lim_{n\to\infty} |a_n + b_n|_p = \lim_{n\to\infty} \max(|a_n|_p, |b_n|_p) = \max\left(\lim_{n\to\infty} |a_n|_p, \lim_{n\to\infty} |b_n|_p\right) = \max(|a|_p, |b|_p).$$

4) Note that if $|a|_p = \lim_{n\to\infty} |a_n|_p \neq 0$ then $(|a_n|_p)$ is eventually constant and so it will converge to the eventual constant. Since this is the p-adic absolute value of a rational number, it must be the case that the image of $\mathbb{Q}_p$ under $|\cdot|_p$ is the same as that of $\mathbb{Q}$ under $|\cdot|_p$.

5) $\mathbb{Q}_p$ cannot be an ordered field because a square root of -7 exists in $\mathbb{Q}_2$ and a square root of $1 - p$ exists in $\mathbb{Q}_p$ for $p > 2$. We can see this because the square root algorithm for $x \in \mathbb{Q}_p$, $a_n = 1/2(a_{n-1} + x/a_{n-1})$, converges for these values. □

Problem 5. *Show that R_p is a commutative ring with 1.*

Proof. Let $x, y \in R_p$. Then $|x|_p \leq 1$ and $|y|_p \leq 1$. But then $|x+y|_p \leq \max(|x|_p, |y|_p) \leq 1$ and so R_p is closed under addition. Likewise $|xy|_p = |x|_p \cdot |y|_p \leq 1$ and so R_p is closed under multiplication. Note that associativity and commutativity of addition and multiplication as well as the distributive property hold as they do in $\mathbb{Q}_p$. Also, $|0|_p = 0 \leq 1$ and so $0 \in R_p$ and then $|-x|_p = |x|_p \leq 1$ and so R_p has additive inverses. Finally $|1|_p = 1 \leq 1$ and so $1 \in R_p$ which shows that R_p is a commutative ring with 1. $\square$

Problem 6. *1) Show that U_p is in fact the set of units in R_p.*
2) Show that U_p is a group under multiplication.
3) Show that $\mathcal{P}$ is an ideal in R_p.
4) Show that $\mathcal{P}$ is a maximal ideal in R_p.
5) For $n \in \mathbb{Z}$, define $\mathcal{P}^n = p^n R_p = \{p^n x \mid x \in R_p\} = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{-n}\}$. Show that $\mathcal{P}^n$ is a subgroup of $(\mathbb{Q}_p, +)$.
6) Show that $\mathcal{P}^n \backslash \mathcal{P}^{n+1} = p^n U_p$.
7) Show that, if $n > 0$, $\mathcal{P}^n$ is an ideal in R_p.
8) Show that $\mathbb{Q}_p = \bigcup_{n \in \mathbb{Z}} \mathcal{P}^n$.
9) Show that $\mathbb{Q}_p^\times = \bigcup_{n \in \mathbb{Z}} p^n U_p$.

Proof. 1) Let $x \in U_p$ such that $x = \overline{(x_n)}$. Then $|x|_p = 1$ which means that $(|x_n|_p)$ is eventually constant. Thus there exists N such that for all $n > N$ we have $|x_n|_p = 1$. But then since p is not a factor of the numerator or denominator for all x_n with $n > N$, we have $|1/x_n|_p = 1$ as well. Thus $(|1/x_n|_p)$ is eventually constant and converges to 1 which means $|1/x|_p = 1$. Therefore $1/x \in U_p$ and so U_p is a subset of the units of R_p. Now consider an element, $x \in R_p$ with $x = \overline{(x_n)}$ such that $x^{-1} \in R_p$. Since $x^{-1} \in R_p$ we have $|1/x|_p \leq 1$. Then the sequence $(|1/x_n|_p)$ is eventually constant and converges to p^{-n} for some $n \in \mathbb{N}$. But this implies that $(|x|_p)$ is eventually constant and converges to p^n. Note that we also have $x \in R_p$ and so $|x|_p \leq 1$. Then the sequence $(|1/x_n|_p)$ is eventually constant and converges to p^{-n} for some $n \in \mathbb{N}$. The only way this can happen is if $n = 0$ and so both x and $1/x$ are in U_p. Therefore the set of units of in R_p is a subset of U_p.

2) Let $x, y \in U_p$. Then $|x|_p = |y|_p = 1$ and $|xy|_p = |x|_p|y|_p = 1 \cdot 1 = 1$. Thus U_p is closed under multiplication. Obviously $1 \in U_p$ and Part 1) Shows that multiplicative inverses are in U_p. Associativity and commutativity of multiplication are the same as in $\mathbb{Q}_p$.

3) Let $x \in R_p$ and let $a \in \mathcal{P}$. Then $|x|_p \leq 1$ and $|a|_p \leq 1/p$ which means $|ax|_p = |a|_p|x|_p \leq 1/p$. Thus $ax \in \mathcal{P}$.

4) Suppose that $x \in U_p$ and suppose there exists $A \subsetneq R_p$ such that A is an ideal of R_p containing x and $\mathcal{P}$. Then consider some element $y \in R_p$ such that $y \notin A$. Since $y \notin \mathcal{P}$ we know that $|y|_p = 1$. But then choose some element $a \in \mathcal{P}$. Then $|ay|_p = |a|_p|y|_p \leq 1/p$ which means $ay \in \mathcal{P}$. Therefore A is not an ideal in R_p.

5) Consider $x, y \in \mathcal{P}^n$. Then $|x|_p \leq p^{-n}$ and $|y|_p \leq p^{-n}$. But then $|x+y|_p \leq \max(|x|_p, |y|_p) \leq p^{-n}$. Thus $\mathcal{P}^n$ is closed under addition. Commutativity and associativity of addition hold as they do in $\mathbb{Q}_p$. Certainly $|0|_p = 0 \leq p^{-n}$ and so $0 \in \mathcal{P}^n$. Finally, additive inverses are in $\mathcal{P}^n$ since $|-x|_p = |x|_p$.

6) Note that

$$\mathcal{P}^n \backslash \mathcal{P}^{n+1} = \{x \in \mathbb{Q}_p \mid p^{-(n+1)} < |x|_p \leq p^{-n}\} = \{x \in \mathbb{Q}_p \mid |x|_p = p^{-n}\} = \{p^n x \mid x \in U_p\} = p^n U_p.$$

7) Let $a \in \mathcal{P}^n$ and let $x \in R_p$. Then $|x|_p \leq 1$ and $|a|_p \leq p^{-n}$. Then $|ax|_p = |a|_p|x|_p \leq 1 \cdot p^{-n} = p^{-n}$. Thus $ax \in \mathcal{P}^n$.

8) Let $x \in \mathbb{Q}_p$. We know that the image of $\mathbb{Q}_p$ under $|\cdot|_p$ is $\{p^k \mid k \in \mathbb{Z}\} \cup \{0\}$ from Problem 4 Part 4). Thus $|x|_p = p^k$ for some $k \in \mathbb{Z}$. But then $x \in \mathcal{P}^{-k} = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^k\}$ and $\mathcal{P}^{-k} \subseteq \bigcup_{n\in\mathbb{Z}} \mathcal{P}^n$. The converse is certainly true since $\mathcal{P}^n = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{-n}\}$ is clearly a subset of $\mathbb{Q}_p$.

9) This is a similar proof as Part 8). Since the image of $\mathbb{Q}_p$ under $|\cdot|_p$ is $\{p^k \mid k \in \mathbb{Z}\}$, it must be the case that for $x \in \mathbb{Q}_p^\times$ we have $|x|_p = p^k$ for some $k \in \mathbb{Z}$. But then $x \in p^{-k}U_p$. As in Part 8), we know that $p^n U_p \subseteq \mathbb{Q}_p^\times$ for all $n \in \mathbb{N}$. $\square$