Kris Harper
MATH 20700
November 3, 2008

Homework 5

** **Problem 1.** *Find $f \in \mathrm{Aut}(\mathbb{C})$ such that f is not the identity or the conjugate function.*

Proof. Consider some element $a \in \mathbb{C}$ such that a is the root of a polynomial in $\mathbb{Q}[x]$. Let the polynomial of least degree with this property be p. Let f be an automorphism with domain $\mathbb{Q}$. Then there exists an isomorphism extending f to $\mathbb{Q}(a)$ and sending a to b if and only if b is the root of a polynomial obtained by applying f to the coefficients of p. Here, $\mathbb{Q}(a)$ denotes the extension of $\mathbb{Q}$ generated by a and is the intersection of all subfields of $\mathbb{C}$ which contain $\mathbb{Q}$ and a. It is now possible to use Zorn's Lemma to show that any isomorphism, f, with domain $\mathbb{Q}$ can be extended to an isomorphism of $\mathbb{A}$. Let F be the set of isomorphisms extending f to a subfield of $\mathbb{A}$. F is nonempty since f extends itself to $\mathbb{Q}$. Consider isomorphisms of subfields of $\mathbb{C}$ as sets of ordered paris. Let C be a chain of sets from F and let D be the union of all the isomorphisms in C. Let $(a, b), (c, d) \in D$. Then $(a, b) \in f_1$ and $(c, d) \in f_2$ for isomorphisms in F. Since C is a chain it follows that (a, b) and (c, d) are in the same isomorphism since $f_1 \subseteq f_2$ or $f_2 \subseteq f_1$. From here it follows that D is an isomorphism in F. Use Zorn's Lemma to choose g as a maximal element of F. Suppose that the domain of g is not $\mathbb{A}$. Then there exists $a \in \mathbb{A}$ not in the domain. But we've already shown that we can extend g to include this element. This isomorphism will be in F as well and g is not the maximal element. This is a contradiction and so the domain of g is $\mathbb{A}$. A similar proof using Zorn's Lemma shows that for any isomorphism on a finite extension of $\mathbb{Q}$ we can create an automorphism of $\mathbb{C}$. □

** **Problem 2.** *Show that $\mathbb{A}$ and $\mathbb{A}_{\mathbb{R}}$ are fields.*

Proof. We have that $\mathbb{A}$ is the set of numbers which are roots of elements in $\mathbb{Z}[x]$. Note that we can equivalently replace $\mathbb{Z}[x]$ with $\mathbb{Q}[x]$ by taking any element of $\mathbb{Q}[x]$ and multiplying by the least common denominator of each of the coeffiecients. We first need to show that $\mathbb{A}$ is closed under addition and multiplication. Let $x, y \in \mathbb{A}$ such that $u(x) = \sum_{i=0}^{n} a_i x^i = 0$ and $v(y) = \sum_{i=0}^{m} b_i y^i = 0$. Suppose that u and v are the polynomials of least degree with coefficients in $\mathbb{Q}$ and x and y as roots. Then we can say that the sets

$$A = \{1, x, x^2, \ldots, x^{n-1}\}$$

and

$$B = \{1, y, y^2, \ldots, y^{m-1}\}$$

are linearly independent. Note that we can create x^n from a linear combination of elements from A. To see this note that

$$x^n = -\frac{1}{a_n} \sum_{i=0}^{n-1} a_i x^i.$$

Additionally, if we multiply both sides of this equation by x we have

$$x^{n+1} = -\frac{1}{a_n} \sum_{i=0}^{n-1} a_i x^{i+1}$$

where the sum is a linear combination of elements of A. Inductively this shows that $x^k \in \langle A \rangle$ where $k \in \mathbb{N} \cup \{0\}$ and $\langle A \rangle$ denotes the span of A over $\mathbb{Q}$. Similarly, $y^l \in \langle B \rangle$ where $l \in \mathbb{N} \cup \{0\}$. Also note that the set

$$C = \{1, x, x^2, \ldots, x^{n-1}, xy, xy^2, \ldots, x^{n-1}y^{m-2}, x^{n-1}y^{m-1}\}$$

is finite and that $x^k y^l \in \langle C \rangle$ for all $k, l \in \mathbb{N} \cup \{0\}$ Thus there exists some finite basis C' for the space spanned by C. Now suppose that there exists no polynomial with coefficients in $\mathbb{Q}$ where

$$w(x+y) = \sum_{i=0}^{p} c_i (x+y)^i = \sum_{i=0}^{p} c_i \sum_{j=0}^{i} \binom{i}{j} x^i y^{i-j} = 0$$

where we've used the binomial theorem to expand each term. But if this is the case for all $p \in \mathbb{N}$, we will never have a set of products of powers of x and y which is linearly independent. Thus, a basis for all products of powers of x and y must be infinite. But C' is a finite basis. This is a contradiction and so there must exist a polynomial in $\mathbb{Q}[x]$ such that $w(x+y) = 0$. The same proof holds for a polynomial $w'(xy) = 0$. Thus, $\mathbb{A}$ and therefore $\mathbb{A}_{\mathbb{R}}$ are closed under addition and multiplication.

At this point, the axioms for commutativity and associativity of multiplication and distributivity follow from the fact that $\mathbb{C}$ and $\mathbb{R}$ are fields. Note that 0 and 1 are algebraic numbers from the polynomials $x = 0$ and $x - 1 = 0$. Thus the additive and multiplicative identities for $\mathbb{C}$ and $\mathbb{R}$ are in $\mathbb{A}$ and $\mathbb{A}_{\mathbb{R}}$. Note also that $r(x) = x + 1 = 0$ shows that $-1 \in \mathbb{A}$ and since $\mathbb{A}$ is closed under multiplication, if $x \in \mathbb{A}$ then $-1 \cdot x = -x$ so $-x \in \mathbb{A}$. The same is true for $\mathbb{A}_{\mathbb{R}}$. Thus, additive inverses are in $\mathbb{A}$ and $\mathbb{A}_{\mathbb{R}}$. We are left with multiplicative inverses. Let $x \in \mathbb{A}_{\mathbb{R}}$ such that

$$p(x) = \sum_{i=0}^{n} a_i x^i = 0.$$

Then multiply both sides by $1/x^n$ so that we have

$$0 = \sum_{i=0}^{n} a_i x^{i-n} = \sum_{i=0}^{n} a_i \left(\frac{1}{x}\right)^{n-i}.$$

Thus, there exists a polynomial with $1/x$ as a root and so $1/x \in \mathbb{A}_{\mathbb{R}}$. Finally, let $z \in \mathbb{A}$ such that $z = a + bi$. Then there exists a polynomial in $\mathbb{Q}[x]$ such that

$$p(z) = \sum_{i=0}^{n} a_i z^i = 0.$$

Then

$$\sum_{i=0}^{n} a_i \overline{z}^i = \sum_{i=0}^{n} a_i \overline{z^i} = \sum_{i=0}^{n} \overline{a_i z^i} = \overline{\sum_{i=0}^{n} a_i z^i} = \overline{0} = 0$$

and so $\overline{z}$ is a root of p as well. Note also that $z + \overline{z} \in \mathbb{A}$ and $z + \overline{z} = 2a$ and $z - \overline{z} \in \mathbb{A}$ and $z - \overline{z} = bi$. Since $i \in \mathbb{A}$ from the equation $x^2 + 1 = 0$, we see that $a, b \in \mathbb{A}$ if $a + bi \in \mathbb{A}$. Thus $|z| \in \mathbb{A}$ and so $1/|z| \in \mathbb{A}$ and finally $\overline{z}/|z| \in \mathbb{A}$ from closure under multiplication. Thus the multiplicative inverse for z is algebraic. Since all the axioms are met for both $\mathbb{A}$ and $\mathbb{A}_{\mathbb{R}}$, they are both fields. □

** **Problem 3.** *Find* $\mathrm{Aut}(\mathbb{A}_{\mathbb{R}})$ *and* $\mathrm{Aut}(\mathbb{A})$.

Proof. Let $a \in \mathbb{A}_{\mathbb{R}}$ such that $a > 0$. Then $a = b^2$ for some $b \in \mathbb{A}_{\mathbb{R}}$. To see this note that a is a root of some polynomial $p(x) = \sum_{i=0}^{n} a_n x^n$ such that $p(a) = p(b^2) = 0$. Then consider $p'(x) = \sum_{i=0}^{n} a_n (x^2)^n$ and $p'(b) = 0$. Thus $b \in \mathbb{A}_{\mathbb{R}}$. Now let f be an automorphism of $\mathbb{A}_{\mathbb{R}}$. Then

$$f(a) = f(b^2) = f(b \cdot b) = f(b) \cdot f(b) = (f(b))^2 > 0$$

since $\mathbb{A}_{\mathbb{R}}$ is a field. Thus if $a > 0$ then $f(a) > 0$ and so automorphisms of $\mathbb{A}_{\mathbb{R}}$ preserve order. Note that $\mathbb{Q} \subseteq \mathbb{A}_{\mathbb{R}}$ because for any rational $q \in \mathbb{Q}$, q is a root of $x - q$. Then the rationals are fixed under f. Suppose that $a < f(a)$. Then there exists $r \in \mathbb{Q}$ such that $a < r < f(a)$. But then $f(a) < f(r) = r$. Hence $a \geq f(a)$.

A similar proof shows that $a \leq f(a)$ and thus $a = f(a)$. Thus $\mathrm{Aut}(\mathbb{A}_{\mathbb{R}}) = \{I\}$ where I is the identity function.

Certainly the identity and complex conjugation are in $\mathrm{Aut}(\mathbb{A})$. From ** Problem 1 we see that there are other elements which arise from finite extensions of $\mathbb{Q}$ which are generated by algebraic numbers. $\square$

** **Problem 4.** *Complete Project 10.2 for Chapter 3.*

** **Problem 4.1** *Determine which of the following converge:*
1) $a_n = 1$ *for all* n.
2) $a_n = 1/n$.
3) $a_n 1/2^n$.
4) $a_n = (-1)^{n+1}$.
5) $a_n = (-i)^{n+1}/(n^2+1)$.
6) $a_n = e^{in\theta}/n$ *for a fixed* $0 \leq \theta \leq 2\pi$.
7) $a_n = \sin(n\pi)/n^2$.

Proof. 1) We see that $\sum_{n=1}^{\infty} a_n$ diverges. To show this, suppose it converges to L. Let $\varepsilon = 1/2$. Then for all $N \in \mathbb{N}$, use the Archimedean Property choose n such that $n > |L+1|$. Then $|S_n - L| \geq 1/2 = \varepsilon$.

2) Group the terms of (a_n) to make a new sequence (b_k) such that

$$b_k = \sum_{i=n_{k-1}+1}^{n_k} \frac{1}{n}$$

where $n_k = 2^{k-1}$ for $k \in \mathbb{N}$ and $n_0 = 0$. Note that for $k \geq 2$, b_k has $2^{k-1} - 2^{k-2} = 2^{k-2}$ terms, the smallest of which is $1/2^{k-1}$. Thus, for all $k \geq 2$, $b_k \geq 2^{k-2}/2^{k-1} = 1/2$. Also $b_1 = \sum_{n=1}^{1} 1/n = 1$. So for all $k \in \mathbb{N}$ we have $b_k \geq 1/2$. But then there are no terms of (b_k) in $(-1/2;1/2)$ so $\lim_{k\to\infty} b_k \neq 0$. Thus, $\sum_{k=1}^{\infty} b_k$ is not convergent and therefore $\sum_{n=1}^{\infty} 1/n$ is not convergent.

3) Let $\varepsilon > 0$ and choose N such that $1/N < \varepsilon$. Then for $n > N$ we have $1/2^n < 1/N$ since $2^n > N$. Thus for all $n > N$ we have $|1 - S_N| = |1 - 1 + 1/2^n| = 1/2^n < 1/N < \varepsilon$. Thus $\sum_{n=1}^{\infty} a_n = 1$.

4) This series diverges since the partial sums are either 1 or 0. Thus, for any value $a \in \mathbb{C}$, there exists some ball $B_r(a)$ such that $|a| < r$ and there are infinitely many terms of (S_N) which are not in $B_r(a)$.

5) This sequence can be broken up into a real sequence

$$a_n' = \frac{(-i)^{2n}}{n^2+1}$$

and an imaginary sequence

$$a_n'' = \frac{(-i)^{2n-1}}{n^2+1}.$$

Each of these series converges using the comparison test and the fact that $\sum_{n=1}^{\infty} 1/n^2$ converges. Thus, the original sequence must also converge.

6) This series will converge for particular values of θ. For example, if $\theta = \pi$ then $e^{in\pi}/n = (\cos(n\pi) + i\sin(n\pi))/n = (-1)^n/n$. Then we have $\sum_{n=1}^{\infty}(-1)^n/n$ converges by the alternating series test.

7) Note that $\sin(n\pi) = 0$ for all $n \in \mathbb{N}$ so that we have $a_n = 0$ for all n. Then $\sum_{n=1}^{\infty} a_n = 0$. $\square$

**** Problem 4.2** *Suppose that a series $\sum_{n=1}^{\infty} a_n$ converges. Show that $\lim_{n\to\infty} a_n = 0$.*

Proof. Let $\sum_{n=1}^{\infty} a_n = S$. Then the sequence of partial sums (S_N) converges to S and (S_N) is a Cauchy sequence. Thus for all $\varepsilon > 0$ there exists $N' \in \mathbb{N}$ such that for all $n, m > N'$ we have $|S_n - S_m| < \varepsilon$. But note that $S_{n+1} - S_n = a_n$ so for $n > N' + 1$ we have $|a_n| < \varepsilon$ which means $\lim_{n\to\infty} a_n = 0$. □

**** Problem 4.3** *1) If $N \in \mathbb{N}$ and $z \neq 1$ show that $S_N = \sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}$*
2) If $|z| < 1$, show that $\lim_{n\to\infty} z^n = 0$.
3) If $|z| > 1$, show that $\lim_{n\to\infty} z^n$ does not exist.

Proof. 1) Note that

$$\sum_{n=0}^{N} z^n = 1 + z + z^2 + \cdots + z^N.$$

Multiply both sides of this equality by $1 - z$. Then we have

$$(1-z)\sum_{n=0}^{N} z^n = (1-z)(1 + z + z^2 + \cdots + z^N) = 1 - z^{N+1}$$

and since $z \neq 1$ we have $1 - z \neq 0$ so we can multiply by $1/(1-z)$ to obtain

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}.$$

2) Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ such that $N \geq 2$ and $1/N < \varepsilon$. Then for all $n > N$ we have $|z^n| < 1/N$ since $|z| < 1$. Thus, $\lim_{n\to\infty} z^n = 0$.

3) Note that since $|z| > 1$, it follows that z^n is unbounded. Then for any complex number w there exists a ball $B_r(w)$ with infinitely many points of z^n outside of it. Thus, z^n cannot converge to w. □

**** Problem 4.4** *What can you say if $|z| = 1$?*

Proof. If $z \in \mathbb{R}$ then $\lim_{n\to\infty} z^n = 1$. If z is purely imaginary, then z^n will not converge as i^n will cycle through four different values. □

**** Problem 4.5** *Show that by removing an infinite number of terms from the series $\sum_{n=1}^{\infty} 1/n$, the remaining subseries can be made to converge to any real number.*

Proof. Let $c \in \mathbb{R}$ and suppose that it is not possible to remove infinitely many terms of $a_n = 1/n$ so that the subseries, $\sum_{n=1}^{\infty} b_n$, converges to c. Consider the partial sums $S_N = \sum_{n=1}^{N} b_n$. Then there exists some $\varepsilon > 0$ such that for all N there exists an $n > N$ such that $|S_n - c| \geq \varepsilon$. But then we can add or remove terms of (a_n) until the inequality is satisfied. □

**** Problem 4.6** *If $p \in \mathbb{R}$ show that $\sum_{n=1}^{\infty} 1/n^p$ diverges for $p < 1$ and converges for $p > 1$.*

Proof. 1) Let S_n be the nth partial sum. Then

$$S_{2n} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(2n)^p} = 1 + \left(\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right) + \left(\frac{1}{3^p} + \frac{1}{5^p} + \cdots + \frac{1}{(2n-1)^p}\right).$$

If $p > 1$ then we have

$$S_{2n} > 1 + \frac{1}{2^p} S_n + \left(\frac{1}{4^p} + \frac{1}{6^p} + \cdots + \frac{1}{(2n)^p}\right)$$

which means

$$S_{2n} > 1 + \frac{1}{2^p}S_n - \frac{1}{2^p} + \frac{1}{2^p}S_n = 1 - \frac{1}{2^p} + \frac{2}{2^p}S_n$$

and

$$S_{2n} < 1 + \frac{2}{2^p}S_n.$$

Thus

$$\frac{2^p - 1}{2^p} + \frac{2}{2^p}S_n < S_{2n} < 1 + \frac{2}{2^p}S_n.$$

A similar proof shows that for $p < 1$ we have

$$1 + \frac{2}{2^p}S_n < S_{2n} < \frac{2^p - 1}{2^p} + \frac{2}{2^p}S_n.$$

For $p < 0$ we see that $1/n^p > 1$ for large enough values of n and so the series will eventually diverge and for $p = 0$ we have the constant sequence 1 which will diverge. Assume that $\lim_{n\to\infty} S_n = S$. Let $0 < p \leq 1$. Then from the second inequality we have

$$1 < S - \frac{2}{2^p}S < 1 - \frac{1}{2^p}$$

which is a contradiction. Thus, $\sum_{n=1}^{\infty} 1/n^p$ diverges for $p \leq 1$. Now consider $p > 1$. Then from the first inequality we have

$$S - \frac{2}{2^p}S = \frac{2^p - 2}{2^p}S < 1$$

which means

$$S < \frac{2^p}{2^p - 2}.$$

So S_n is a bounded and increasing sequence. Thus it must converge. □

**** Problem 4.7** *1) Suppose $a_n > 0$ for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges. If $b_n \in \mathbb{C}$ satisfies $|b_n| \leq a_n$ for all n, then the series $\sum_{n=1}^{\infty} b_n$ converges absolutely and thus converges.*
2) If the series $\sum_{n=1}^{\infty} a_n$ converges to s and c is any constant show that the series $\sum_{n=1}^{\infty} ca_n$ converges to cs.
3) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are infinite series. Suppose that $a_n > 0$ and $b_n > 0$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n/b_n = c > 0$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Proof. 1) Note that $a_n > 0$ for all n and so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} |a_n|$ is an absolutely convergent sequence. Then the sequence of partial sums, (S_n) is convergent and therefore bounded. Thus there exists $C \in \mathbb{R}$ such that $S_n \leq C$ for all $n \in \mathbb{N}$. But then since $|b_n| \leq a_n = |a_n|$ for all n we have

$$\sum_{n=1}^{N} |b_n| \leq \sum_{n=1}^{N} a_n \leq C.$$

Thus, the sequence of partial sums, (T_n), for $\sum_{n=1}^{\infty} |b_n|$ is bounded. But also $|b_n| \geq 0$ and so

$$T_{N+1} = \sum_{n=1}^{N} |b_n| + |b_{N+1}| = T_n + |b_{N+1}| \geq T_n.$$

Thus (T_n) is a bounded increasing sequence and therefore it is convergent. Thus $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

2) Suppose that $\sum_{n=1}^{\infty} a_n = s$. Then note that

$$cS_N = c\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} ca_n.$$

We know that $\lim_{n\to\infty} S_n = s$ so let $\varepsilon > 0$ and consider $\varepsilon/|c|$. There exists N such that for all $n > N$ we have $|S_n - s| < \varepsilon/|c|$. Then $|c||S_n - s| = |cS_n - cs| < \varepsilon$. Thus, $\lim_{n\to\infty} cS_n = cs$ and so $\sum_{n=1}^{\infty} ca_n = cs$.

3) Assume that $\sum_{n=1}^{\infty} a_n = s$. Then note that

$$cs = c\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty} \frac{a_n}{b_n} s_n.$$

From we see that $\sum_{n=1}^{\infty} b_n$ must converge by the Comparison Test. A similar proof holds for the converse with the fact that $c > 0$. □

**** Problem 4.8** *Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero numbers. Give examples to show that if $\lim_{n\to\infty} |a_{n+1}/a_n| = r = 1$, the series may converge or diverge.*

Proof. Consider $a_n = 1/n$ then $\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty}(n+1)/n = 1$, but $\sum_{n=1}^{\infty} a_n$ diverges. Similarly, if $b_n = 1/n^2$ then $\lim_{n\to\infty} |b_{n+1}/b_n| = \lim_{n\to\infty}(n+1)^2/n^2 = 1$, and $\sum_{n=1}^{\infty} b_n$ converges. □

**** Problem 4.9** *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of non-negative real numbers and let $x_0 = \limsup_{n\to\infty} x_n$. For any $\varepsilon > 0$, show that there are only finitely many terms of the sequence greater than $x_0 + \varepsilon$, whereas there are infinitely many terms less than $x_0 + \varepsilon$.*

Proof. We know that x_0 is the limit of a sequence (y_n) where

$$y_n = \sup\{a_k \mid k \geq n\}.$$

Thus (y_n) is a decreasing sequence. Let $\varepsilon > 0$ and choose n such that $|x_0 - y_n| < \varepsilon$. Since (y_n) is decreasing we have $x_0 < y_n < \varepsilon$. By definition, y_n is greater than or equal to every term of (x_n) except for those with indices less than n. The fact that $y_n < \varepsilon$ gives us the strict inequality for finitely many terms greater than $x_0 + \varepsilon$ and infinitely many less than $x_0 + \varepsilon$. □

**** Problem 4.10** *Let $\sum_{n=1}^{\infty} a_n$ be a series. Give examples to show that if $\limsup_{n\to\infty} |a_n|^{1/n} = r = 1$ then the series may converge or diverge.*

Proof. Consider $a_n = 1^n$ then $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} (1^n)^{1/n} = 1$, but $\sum_{n=1}^{\infty} a_n$ diverges. Similarly, if $b_n = 1/n^2$ then $\lim_{n\to\infty} |b_n|^{1/n} = \lim_{n\to\infty}(1/n^2)^{1/n} = 1$, and $\sum_{n=1}^{\infty} b_n$ converges. □

**** Problem 4.11** *Let $\sum_{n=1}^{\infty} a_n$ is a series such that $r = \lim_{n\to\infty} |a_{n+1}|/|a_n|$ exists. Show that $\limsup_{n\to\infty} |a_n|^{1/n} = r$ as well.*

Proof. This follows from the fact that $|a_n|^{1/n} \leq |a_{n+1}|/|a_n|$ for large enough values of n. Then $||a_n|^{1/n} - r| < ||a_{n+1}|/|a_n| - r| < \varepsilon$ if given $\varepsilon > 0$. □

**** Problem 4.12** *Show that if a complex power series around z_0 converges absolutely for a complex number z then it also converges for any complex number w such that $|w - z_0| \leq |z - z_0|$, that is, the series converges on the disk $\{w \in \mathbb{C} \mid |w - z_0| \leq |z - z_0|\}$.*

Proof. Let $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ be absolutely convergent. Then note that

$$|w-z_0|^n \leq |z-z_0|^n$$

if $|w-z_0| \leq |z-z_0|$. But then

$$|a_n(w-z_0)^n| = |a_n||(w-z_0)^n| = |a_n||w-z_0|^n \leq |a_n||z-z_0|^n = |a_n||(z-z_0)^n| = |a_n(z-z_0)^n|.$$

Then by the comparison test, the power series will converge on the disk $\{w \in \mathbb{C} \mid |w-z_0| \leq |z-z_0|\}$. $\square$

**** Problem 4.13** *Determine the radius of convergence for the following power series:*
1)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

2)

$$\sum_{n=2}^{\infty} \frac{z^n}{\ln(n)}.$$

3)

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n.$$

Proof. 1) The sequence $a_n = 1/n!$ satisfies the ratio test so that $\lim_{n\to\infty} |a_{n+1}|/|a_n| = \lim_{n\to\infty} 1/(n+1) = 0$. The result of the root test must be the same and so the radius of convergence is infinity.

2) The sequence $a_n = 1/\ln(n)$ satisfies the ratio test so that $\lim_{n\to\infty} |a_{n+1}|/|a_n| = \lim_{n\to\infty} \ln(n+1)/\ln(n) = 1$. The result of the root test must be the same and so the radius of convergence is 1.

3) The sequence $a_n = n^n/n!$ satisfies the root test so that $\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} n/(n!)^{1/n}$ diverges. The radius of convergence must then be 0. $\square$

**** Problem 5.** *For $x, y \in \mathbb{R}^n$ Let*

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

and $d_p(x,y) = ||x-y||_p$. Show that d_p is a metric.

Proof. Let $x, y \in \mathbb{R}^n$. We have $|x_i - y_i| \geq 0$ and thus $|x_i - y_i|^p \geq 0$ for each $1 \leq i \leq n$. Then $\sum_{i=1}^{n} |x_i - y_i|^p \geq 0$ and raising this to $1/p$ we have

$$d_p(x,y) = ||x-y||_p = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} \geq 0.$$

Now suppose that $x = y$. Then $x_i = y_i$ for all $1 \leq i \leq n$ and so $|x_i - y_i| = 0$ for all $1 \leq i \leq n$. It follows that $d(x,y) = 0$. Conversely, suppose that $d(x,y) = 0$. Then

$$\left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}} = 0$$

and raising both sides to the pth power we have $\sum_{i=1}^{n}|x_i-y_i|^p=0$. But since $p>1$ we know that $|x_i-y_i|^p\geq 0$ for all $1\leq i\leq n$. Thus $|x_i-y_i|=0$ and so $x_i=y_i$ for all $1\leq i\leq n$. Therefore $x=y$.

Note that since $|a-b|=|-1||a-b|=|-(a-b)|=|b-a|$ for all $a,b\in\mathbb{R}$ we have

$$d_p(x,y)=||x-y||_p=\left(\sum_{i=1}^{n}|x_i-y_i|^p\right)^{\frac{1}{p}}=\left(\sum_{i=1}^{n}|y_i-x_i|^p\right)^{\frac{1}{p}}=||y-x||_p=d(y,x).$$

Now let $z\in\mathbb{R}^n$ as well. Note that

$$||x-z||_p^p=\sum_{i=1}^{n}|x_i-z_i|^p\leq\sum_{i=1}^{n}|x_i-z_i|^{p-1}|x_i|+\sum_{i=1}^{n}|x_i-z_i|^{p-1}|z_i|.$$

If we now assume that $q=p/(p-1)$, then we can apply Hölder's Inequality to both terms on the right so we have

$$||x-z||_p^p\leq\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|x_i-z_i|^{(p-1)q}\right)^{\frac{1}{q}}+\left(\sum_{i=1}^{n}|z_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|x_i-z_i|^{(p-1)q}\right)^{\frac{1}{q}}.$$

Now multiply both sides by

$$\left(\sum_{i=1}^{n}|x_i-z_i|^{(p-1)q}\right)^{-\frac{1}{q}}$$

and note that $1-1/q=1/p$ so that we have

$$||x-z||_p^p=\left(\sum_{i=1}^{n}|x_i-z_i|^p\right)^{\frac{1}{p}}\leq\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|z_i|^p\right)^{\frac{1}{p}}\leq\left(\sum_{i=1}^{n}|x_i-y_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|y_i-z_i|^p\right)^{\frac{1}{p}}$$

Thus $||x-z||_p\leq||x-y||_p+||y-z||_p$. □

**** Problem 6.** *Define $l_n^p(\mathbb{C})$.*

Proof. The norm $l_n^p(\mathbb{C}$ is defined as

$$||z||_p=\left(\sum_{j=1}^{n}|z_i|^p\right)^{\frac{1}{p}}.$$

The proof that this is a metric is the same as the proof that $l_n^p(\mathbb{R})$ is a metric because the properties of absolute value apply in the same way. □

**** Problem 7.** *Show the following for $r,s\in\mathbb{Q}$ such that $r=a/b=p^k(a'/b')$ and $s=c/d=p^l(c'/d')$:*
1) $|r|_p\geq 0$ and $|r|_p=0$ if and only if $r=0$.
2) $|rs|_p=|r|_p|s|_p$.
3) $|r+s|_p\leq\max(|r|_p,|s|_p)$ and $|r+s|_p=\max(|r|_p,|s|_p)$ if and only if $|r|_p\neq|s|_p$.

Proof. 1) Note that $|r|_p=p^{-k}\geq 0$. Also, by definition $|0|_p=0$.

2) Note that

$$rs=\left(p^k\frac{a'}{b'}\right)\left(p^l\frac{c'}{d'}\right)=p^{k+l}\frac{a'}{b'}\frac{c'}{d'}$$

and so $|rs|_p = p^{-(k+l)} = p^{-k}p^{-l} = |r|_p|s|_p$.

3) Note that

$$r + s = p^k\frac{a'}{b'} + p^l\frac{c'}{d'} = \frac{p^ka'd' + p^lb'c'}{b'd'} = p^m\frac{p^{k-m}a'd' + p^{l-m}b'c'}{b'd'}$$

where $m = \min(k, l)$. Then $|r + s|_p = p^{-m} \leq \max(p^{-k}, p^{-l}) = \max(|r|_p, |s|_p)$. Note that if $|r|_p \neq |s|_p$ then $p^{-k} \neq p^{-l}$ and so m must be the $\min(k, l)$ which makes $|r + s|_p = \max(|r|_p, |s|_p)$. □

** **Problem 8.** *Let $d_p(r, s) = |r - s|_p$ for $r, s \in \mathbb{Q}$. Show d_p is a metric on $\mathbb{Q}$.*

Proof. Let $r, s, t \in \mathbb{Q}$ such that $r = a/b = p^k(a'/b')$ and $s = c/d = p^l(c'/d')$. ** Problem 7 Part 1) shows that $d_p(x, y) \geq 0$ and that $d(x, y) = 0$ if and only if $x - y = 0$ which means $x = y$. Now consider

$$r - s = p^k\frac{a'}{b'} - p^l\frac{c'}{d'} = \frac{p^ka'd' - p^lb'c'}{b'd'} = p^m\frac{p^{k-m}a'd' - p^{l-m}b'c'}{b'd'}$$

and

$$s - r = p^l\frac{c'}{d'} - p^k\frac{a'}{b'} = \frac{p^lb'c' - p^ka'd'}{b'd'} = p^m\frac{p^{l-m}b'c' - p^{k-m}a'd'}{b'd'}$$

where $m = \min(k, l)$. Then $|r - s|_p = p^{-m} = |s - r|_p$. Finally, note that

$$|r - t| \leq \max(|r|_p, |t|_p) \leq \max(|r|_p, |s|_p) + \max(|s|_p, |t|_p) = |r - s|_p + |s - t|_p$$

since $|x|_p \geq 0$ for all $x \in \mathbb{Q}$. □