Sheet 26: Log and Exp

Definition 1 *For $x > 0$ let*
$$\log x = \int_1^x \frac{1}{t} dt.$$

Theorem 2 *If $x, y > 0$ then*
$$\log(xy) = \log x + \log y.$$

Proof. Let $f(x) = \log(xy)$. Then $f'(x) = \log'(xy) = (1/xy)y = 1/x = \log' x$ (21.16, 22.17). Then $\log(xy) = \log(x) + c$ for some constant c. Letting $x = 1$ we have $\log y = 0 + c$ so we must have $\log(xy) = \log x + \log y$. □

Corollary 3 *For a natural number n and $x > 0$ we have*
$$\log(x^n) = n \log x.$$

Proof. Note that for $n = 1$ we have $\log x = \log x$. Induct on n and assume that for some $n \in \mathbb{N}$ we have $\log(x^n) = n \log x$. Consider
$$\log(x^{n+1}) = \log(x \cdot x^n) = \log x + \log(x^n) = \log x + n \log x = (n + 1) \log x$$
by Theorem 2 (26.2). Then by mathematical induction the statement must be true for all $n \in \mathbb{N}$. □

Corollary 4 *For $x, y > 0$ we have*
$$\log\left(\frac{x}{y}\right) = \log x - \log y.$$

Proof. Note that
$$\log(y) + \log\left(y^{-1}\right) = \log(yy^{-1}) = \log 1 = \int_1^1 \frac{1}{t} dt = 0$$
and thus $\log(y^{-1}) = -\log y$. We have
$$\log\left(\frac{x}{y}\right) = \log\left(xy^{-1}\right) = \log x + \log\left(y^{-1}\right) = \log x + (-\log y) = \log x - \log y$$
from Theorem 2 and Corollary 3 (26.2, 26.3). □

Theorem 5 *The function* log *is increasing, unbounded and takes on every real value exactly once.*

Proof. Let $x, y \in \mathbb{R}$ such that $0 < x < y$. Then we have
$$\log y - \log x = \int_1^y \frac{1}{t} dt - \int_1^x \frac{1}{t} dt = \int_x^y \frac{1}{t} dt$$
for $x, y > 0$ (22.10). Note that for all $x > 0$, $1/x > 0$. Then for a partition $P = \{t_0, \ldots, t_n\}$ with $x = t_0$ and $y = t_n$ we have the lower sum
$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$$

where $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Note that for all values of $x \in [t_{i-1}, t_i]$, $x \leq t_i$ so $1/t_i \leq 1/x$. Then $m_i = 1/t_i > 0$ for all $1 \leq i \leq n$ which means

$$\int_x^y \frac{1}{t} dt \geq L(f, P) > 0.$$

Thus $\log y > \log x$ and so log is increasing. The function log must be unbounded above because the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent, which means the partial sums are unbounded (15.7). That is, if the partial sums of this series were bounded, they would form a bounded increasing sequence and it would converge. But because

$$\int_1^x \frac{1}{t} dt$$

is defined by lower and upper sums, we can always choose a partial made of natural numbers which will correspond to the series $\sum_{n=1}^{\infty} 1/n$. Thus log must be unbounded above. To show log is unbounded below choose a partition $P = \{1, 1/2, 1/3, \ldots, 1/n\}$. Then

$$U(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} \frac{1}{t_{i-1}}(t_i - t_{i-1}) = \sum_{i=1}^{n}(i+1)\left(\frac{1}{i+1} - \frac{1}{i}\right) = \sum_{i=1}^{n}(i+1)\left(\frac{1}{i(i+1)}\right) = \sum_{i=1}^{n} \frac{1}{i}$$

which is divergent and so the partial sums are unbounded. Since $1/x$ is integrable we know that log is continuous (22.16). Then we have an unbounded continuous function so log must take on every real value. But we also know that log is strictly increasing so it's impossible that log take on one value twice. □

Definition 6 *The exponential function*

$$\exp = \log^{-1}.$$

Theorem 7 *For all x we have $\exp'(x) = \exp(x)$.*

Proof. Note that $\log(\exp(x)) = x$ and taking the derivatives of both sides we have $\log'(\exp(x)) \exp'(x) = 1$ and so $1/\exp(x) = 1/\exp'(x)$ which means $\exp(x) = \exp'(x)$. □

Theorem 8 *For all x, y we have*

$$\exp(x + y) = \exp(x) \exp(y).$$

Proof. We have $\log(\exp(x + y)) = x + y = \log(\exp(x)) + \log(\exp(y)) = \log(\exp(x) \exp(y))$ (26.2). But since log takes on every real number exactly once, $\exp(x + y) = \exp(x) \exp(y)$ (26.5). □

Definition 9 *Let*

$$e = \exp(1).$$

Exercise 10 *Show that $2 < e < 4$.*

Proof. Let $P = \{1, 3/2, 2\}$ be a partition of $[1; 2]$. We have

$$\log 2 = \int_1^2 \frac{1}{x} dx \leq U(f, P) = \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{5}{6} < 1 = \log e$$

and so $2 < e$. Now let $P = \{1, 3/2, 2, 5/2, 3, 7/2, 4\}$ be a partition of $[1; 4]$. We have

$$\log e = 1 < \frac{341}{280} = \frac{1}{2}\left(\frac{3}{2} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \frac{1}{4}\right) = L(f, P) \leq \int_1^4 \frac{1}{x} dx = \log 4$$

and so $e < 4$. □

Definition 11 *For a real number x let*

$$e^x = \exp(x).$$

Definition 12 *For a real number x and $a > 0$ let*

$$a^x = e^{x \log a}.$$

Theorem 13 *For $a > 0$ the following hold:*
1) $(a^b)^c = a^{bc}$;
2) $a^1 = a$;
3) $a^{b+c} = a^b a^c$.

Proof. We have

$$a^{bc} = e^{bc \log a} = e^{c \log\left(a^b\right)} = (a^b)^c,$$
$$a^1 = e^{\log a} = \exp(\log(a)) = a,$$

and

$$a^{b+c} = e^{(b+c) \log a} = \exp((b+c) \log a) = \exp(b \log a + c \log a) = \exp(b \log a) \exp(c \log a) = e^{b \log a} e^{c \log a} = a^b a^c$$

from Corollary 3 and Theorem 8 (26.3, 26.8). □

Exercise 14 *Analyze the functions* $\log x$ *and* a^x.

Proof. Note that $\log' x = 1/x > 0$ for all $x > 0$. Thus $\log x$ is increasing for all $x > 0$. Also $\log'' x = -1/x^2 < 0$ for all $x > 0$ which means $\log x$ is concave down for all $x > 0$. Furthermore $1/x > 0$ for all $x > 0$ and so it's never the case that $\log' x = 0$ which means log has no maximum or minimum values.

Note that $(a^x)' = (e^{x \log a})' = \log a e^{x \log a}$. Since $e^{x \log a} > 0$ for all x we have $(a^x)' > 0$ for $a > 1$ and $(a^x)' < 0$ for $a < 1$. Then for $a > 1$ we have a^x is increasing and for $a < 1$ we have a^x is decreasing. Also $(a^x)'' = (\log a)^2 e^{x \log a} \geq 0$ for all x. Thus a^x is concave up for all x. □

Theorem 15 *Let f be a differentiable real function such that for all x we have*

$$f'(x) = f(x).$$

Then there exists c such that

$$f(x) = ce^x.$$

Proof. Consider

$$\left(\frac{f(x)}{e^x}\right)' = \frac{e^x f'(x) - f(x) e^x}{e^{2x}} = 0.$$

Thus the function $f(x)/e^x$ must equal a constant, c. Therefore $f(x) = ce^x$. □

Theorem 16 *If $x > 0$ the*

$$\frac{x}{1+x} < \log(1+x) < x.$$