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Homework 3

Problem 1. *Let A, B, C be wffs and T a consistent set of wffs.*
(a) $\{A, C\} \vdash B$ if and only if $A \vdash C \to B$.
(b) Suppose $T_0 \vdash A$, where $T_0 \subseteq T$ is finite. Then there exists a wff Y such that $T_0 \vdash Y$, $Y \vdash T_0$ and $Y \vdash A$.

Proof. (a) Suppose $A \vdash C \to B$. Then clearly $\{A, C\} \vdash C \to B$ and of course $\{A, C\} \vdash C$. Therefore we have the deduction C, $C \to B$, B. Thus $\{A, C\} \vdash B$. Conversely, suppose that $\{A, C\} \vdash B$. Then since B is not necessarily a tautology, we must have a chain of wffs which either ends with $A \to B$, B or $C \to B$, B. Without loss of generality suppose it is the latter. Then we have $\{A, C\} \vdash C \to B$. But this is equivalent to saying $A \vdash C \to B$.

(b) Let $T_0 = \{B_1 \dots B_n\}$. Now let $Y = \bigwedge_{i=1}^{n} B_i$. Now note that for each i we have $T_0 \vdash B_i$ and therefore $T_0 \vdash Y$. Furthermore, for $B_i \in T_0$, we know that $Y \to B_i$ is a tautology as well. This follows because Y is only true if B_i is true for all i. Finally, note that since we can deduce all $B_i \in T_0$ from Y, we can further deduce all elements in the deduction of A from T_0 from Y. From here, we can deduce A and so we also have $Y \vdash A$. $\square$

Problem 2. *Let M be a model, let A, B, C be wffs, and let T be a maximal consistent theory.*
(a) For each wff A, either $A \in T$ or $\neg A \in T$. $A \wedge B$ is in T if and only if A, B are both in T.
(b) Suppose $M \models C$, $M \models C \to B$. Then $M \models B$.

Proof. (a) Suppose that $A \wedge B \in T$. Then $A \wedge B \to A$ is a tautology. Therefore $T \vdash A$ which means that $T \cup \{A\}$ is consistent. Since T is maximal, $T \cup \{A\} = T$ and $A \in T$. The same can be said for B. Conversely, given that $A, B \in T$, we can deduce $A \wedge B$. Once again, since T is maximal, we must have $T = T \cup \{A \wedge B\}$ and so $A \wedge B \in T$.

Now suppose that $A \in T$ and $\neg A \in T$. We can inductively deduce any sentence symbol or it's negation which is used in A. That is, if $S_i \in A$ then $T \vdash S_i$ using the previous statement and the fact that A and it's negation are both in T. But since both S_i and $\neg S_i$ are in T, from the previous statement we also have $S_i \wedge \neg S_i \in T$. Thus T is not consistent which is contradiction. Therefore exactly one of A or $\neg A$ is in T.

(b) Note that $C \to B \equiv \neg(C \wedge \neg B)$. Thus $M \models \neg(C \wedge \neg B)$ which means that M does not model C or M does not model $\neg B$. We've assumed that $M \models C$ which implies that $M \models B$. $\square$

Problem 3. *Let A, B be wffs which are not tautologies. Show that if $A \models B$ there is some C with $\sup(C) \subseteq \sup(A) \cap \sup(B)$ such that $A \models C$ and $C \models B$.*

Proof. Let Y be a wff which is always true and Z be a wff which is always false. For each wff A and sentence symbol S, let A_S^+ be the wff formed by replacing S in A with Y and A_S^- be the wff formed by replacing S in A with Z. Define $A_S = A_S^+ \vee A_S^-$. We claim that $A \models A_S$. Suppose that M is a model for A. We can inductively deconstruct A into it's component wffs, keeping track of whether or not M is a model for each wff. Eventually we will deduce either that M is a model of S or M is not a model of S. In the former case we must have $M \models A_S^+$ and in the later case $M \models A_S^-$. Therefore $M \models A_S$ and $A \models A_S$.

Now suppose that $A \models B$ and let $S \notin \sup(B)$. Note that since $A \models A_S$ for any model M with $M \models A$ we have $M \models A_S$ and $M \models B$. Therefore, given a model M of A_S, we know $M \models B$ provided that $M \models A$. But since $B \notin S$, if M is a model such that $S \notin M$, then $M \models A_S$. Again, since $S \notin M$ then we must also have $M \models A$ and thus $M \models B$. Therefore $A_S \models B$.

Let $X \in \sup(A)$. Now let C be the wff formed by replacing each $S \in \sup(A) \setminus \sup(B)$ as above with $Y = X \vee \neg X$ and $Z = X \wedge \neg X$. Then $\sup(C) \subseteq \sup(A) \cap \sup(B)$ and from the above statements we see that $A \models C$ and $C \models B$. $\square$

Problem 4. *Let $A, B_1, B_2, \ldots$ be wffs and $S_1, S_2 \ldots$ be elements of* $\sup(A)$*. Let C be the wff obtained by replacing every instance of S_i in A by B_i for each i. Let M, N be models such that, for each i, $M \models S_i$ if and only if $N \models B_i$. Show by induction that $M \models A$ if and only if $N \models C$.*

Proof. Suppose that $M \models A$ we induct on i. For the case $i = 1$ we have $A = S_1$ and therefore since $M \models S_1$ if and only if $N \models B_1$, we know that $N \models C$. Now suppose that $|sup(A)| = i$ for some i and that $N \models C$. Let A' be the wff which contains $S_1, \ldots, S_{i+1}$ and define C' accordingly. Note that we must either have $A' = A \wedge S_{i+1}$ or $A' \wedge \neg S_{i+1}$. In the former case we know that if $M \models A'$ then $M \models A$ and $M \models S_{i+1}$. Thus $N \models B_{i+1}$ and since $N \models C$ and $N \models B_{i+1}$ we know $N \models C'$. The case where $A' = A \wedge \neg S_{i+1}$ follows similarly.

Conversely, we consider the case where $N \models C$. The base case $i = 1$ is identical to above. Suppose that for some i with $|\sup(A)| = i$ we have $M \models A$. Let A' and C' be as above and note that $C' = C \wedge B_{i+1}$ or $C' = C \wedge \neg B_{i+1}$. For the former case we know $N \models C$ and $N \models B_{i+1}$. Therefore $M \models S_{i+1}$ which along with our inductive hypothesis that $M \models A$ gives us $M \models A \wedge S_{i+1}$. Therefore $M \models A'$. □

Problem 5. *State the special case of the Recursion Theorem which says that given a model M, the function which maps each wff A into $\{0, 1\}$ according to whether or not $M \models A$ exists. That is, state explicitly what each of the sets and functions mentioned should be in this case.*

Proof. Let $f_\wedge$ and $f_\neg$ be the sentence building functions. Let U be the set of all expressions. Note that $f_\wedge : U \times U \to U$ and $f_\neg : U \to U$. Let B be the set of basic sentence symbols. Let C be the set of wffs which are generated by $f_\wedge$ and $f_\neg$ from B. That is, C is the set of elements of U which arise by applying $f_\wedge$ and $f_\neg$ finitely many times to elements of B.

To see that $f_\wedge$ is injective, take $(A_1, B_1), (A_2, B_2) \in U \times U$. Then without loss of generality, assume that some truth assignment gives different values for A_1 and A_2. Then we see that $A_1 \wedge B_1 \neq A_2 \wedge B_2$. This shows that $f_\wedge$ is injective. Similarly, if we take $A, B \in U$ with $A \neq B$ then we have $f_\neg(A) = \neg A \neq \neg B = f_\neg(B)$ because of opposite truth assignments under negation. Therefore $f_\wedge$ and $f_\neg$ are both injective. It's easy to see that B is disjoint from $f_\wedge(C)$ and $f_\neg(C)$ because elements of B have no connectives. We also see that $f_\wedge(C) \cap f_\neg(C) = \emptyset$ since elements of these sets use different connectives. This shows that C is freely generated from B by $f_\wedge$ and $f_\neg$.

Now let $h : B \to \{0, 1\}$ be the function such that $h(S) = 0$ if $S \notin M$ and $h(S) = 1$ if $S \in M$. Let $F : V \times V \to V$ be the function such that $F(x) = 1$ if and only if $x = (1, 1)$. Let $G : V \to V$ be the function such that $G(x) = 1 - x$. Since C is freely generated from B by $f_\wedge$ and $f_\neg$ we now know from the Recursion Theorem that there exists $\overline{h}$ such that $\overline{h}(S) = h(S)$ for $S \in B$, and for $A, B \in C$ we have $\overline{h}(A \wedge B) = \overline{h}(f_\wedge(A, B)) = F(\overline{h}(A), \overline{h}(B)) = 1$ only if $\overline{h}(A) = \overline{h}(B) = 1$. Thus $M \models A \wedge B$ if and only if $M \models A$ and $M \models B$. Likewise, $\overline{h}(\neg A) = \overline{h}(g(A)) = G(\overline{h}(A)) = 1$ only if $\overline{h}(A) = 0$. Therefore, $M \models \neg A$ if and only if M does not model A. □