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Homework 4

** **Problem 1.** *Find a Borel set in $\mathbb{R}^n$ which is neither open nor closed.*

Proof. Note that the Borel sets contain every open set and every closed set, and are closed under countable intersection, by the properties of σ-algebras. Thus, if we take the set

$$(0,1) \times (0,1) \times \cdots \times (0,1) \cap [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \times \cdots \times [\frac{1}{2}, 1]$$

we have a half open rectangle which is neither open nor closed. $\square$

** **Problem 2.** *Suppose that $A \subseteq \mathbb{R}^n$ such that for all $\varepsilon > 0$, there exists a finite union of rectangles, P, such that $m(P \triangle A) < \varepsilon$. Then A is Lebesgue measurable.*

Proof. Let $E \subseteq \mathbb{R}^n$. Note that $P \triangle A = (A \cup P) \backslash (A \cap P)$. Clearly P is measurable so we have $m^*(E \backslash P) + m^*(E \cap P) = m^*(E)$. We have

$$\begin{aligned}(m^*(E \backslash A) + m^*(E \cap A)) - (m^*(E \backslash P) + m^*(E \cap P)) &= (m^*(E \backslash A) - m^*(E \backslash P)) + (m^*(E \cap A) - m^*(E \cap P)) \\ &\leq m^*((E \backslash A) \backslash (E \backslash P)) + m^*((E \cap A) \backslash (E \cap P)) \\ &= m^*((E \cap P) \backslash (P \cap A)) + m^*((E \cap (A \backslash P)) \cup (E \cap (P \backslash A)) \\ &= m^*((E \cap P) \backslash (A \cap P)) + m^*((E \cap ((A \cup P) \backslash (A \cap P))).\end{aligned}$$

Note that the last two sets are subsets of $(A \cup P) \backslash (A \cap P) = P \triangle A$. Thus the last equality evaluates to less than 2ε. But ε is arbitrary and so

$$m^*(E \backslash A) + m^*(E \cap A) = (m^*(E \backslash P) + m^*(E \cap P)) = m^*(E).$$

Thus A is Lebesgue measurable. $\square$

** **Problem 3.** *Suppose X and Y are metric spaces such that X has the Borel measure on it. If $f : X \to Y$ is continuous then f is measurable.*

Proof. Suppose that f is continuous, then its preimage maps open sets to open sets. If f is measurable its preimage maps every open set to a measurable set. Thus it suffices to show that open sets are measurable sets. But since X has the Borel measure on it, every open set is measurable and so f is measurable. $\square$

** **Problem 4.** *The Lebesgue measure is inner and outer regular.*

Proof. Let A be a Lebesgue measurable set. For each $\varepsilon > 0$ there exists a sequence of open rectangles I_j such that $A \subseteq \bigcup_j I_j$ and $\sum_j Vol(I_j) < m(A) + \varepsilon$. Then if $O = \bigcup_j I_j$ we have

$$m(A) \leq m(O) \leq \sum_j m(I_j) = \sum_j Vol(I_j) < m(A) + \varepsilon.$$

Since this is true for every $\varepsilon > 0$ we have

$$m(A) = \inf\{m(O) \mid O \subseteq \mathbb{R}^n, A \subseteq O\}$$

where O is open. Now let B be a Lebesgue measurable set which is bounded. Then there exists a compact set C such that $B \subseteq C$. Then for every $\varepsilon > 0$ there exists an open set O such that $C \backslash B \subseteq O$ and

$m(O) \leq m(C \backslash B) + \varepsilon$. Since $m(B) < \infty$, we have $m(O) < m(C) - m(B) + \varepsilon$. For the compact set $K = C \backslash O$ we have $K \subseteq B$ and $C \subseteq K \cup O$. Then

$$m(C) \leq m(K \cup O) \leq m(K) + m(O) \leq m(K) + m(C) - m(B) + \varepsilon$$

and so $m(B) - \varepsilon < m(K)$. Therefore

$$m(B) = \sup\{m(K) \mid K \subseteq \mathbb{R}^n, K \subseteq B\}$$

where K is compact. The result follows for arbitrary Lebesgue measurable sets using the fact that the Lebesgue measure is continuos from below. $\square$

** **Problem 5.** *Show Lebesgue measure in $\mathbb{R}^n$ is invariant under rotations.*

Proof. We prove the following. Let T be an invertible $n \times n$ matrix and let $J = [0,1)^n$ be the half open unit n-cube. Let $a \in \mathbb{R}$ be a number such that $m(TJ) = am(J)$. Then if A is measurable we have TA is measurable and $m(TA) = am(A)$.

We know that J is a countable union of compact sets and since T maps compact sets to compact sets, we know TJ is the union of countably many compact sets. Therefore TJ is measurable which shows that a must exist. We wish to show that $m(TU) = am(U)$ for some open set U in $\mathbb{R}^n$. We know that we can write $G = \bigcup_{k=1}^{\infty} J_k$ where J_k are pairwise disjoint dilations and translations of J. Let $J_k = z_k + t_k J$. Then we have $m(J_k) = t_k^n m(J)$ and

$$m(TJ_k) = t_k^n (TJ) = t_k^n am(J) = t_k^n a t_k^{-n} m(J_k).$$

Thus $m(TJ_k) = am(J_k)$ and $TG = \bigcup_{k=1}^{\infty} TJ_k$ which is a pairwise disjoint collection of measurable sets. Therefore

$$m(TG) = \sum_{k=1}^{\infty} m(TJ_k) = \sum_{k=1}^{\infty} am(J_k) = am(G).$$

We have shown through examples that $a = |\det(T)|$. A rotation matrix is one such that $\det(T) = \pm 1$. Therefore, Lebesgue measurable sets are invariant under rotations. $\square$