Kris Harper
MATH 16200
Miklós Abért
February 5, 2008

Homework 4

Exercise 1 *Is it true that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it maps compact sets to compact sets?*

No. Use the counterexample of the floor function, $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \lfloor x \rfloor$ or $f(x)$ is the least integer less than or equal to x. Let C be a compact set and let $C \subseteq [a; b]$. Then because $[a; b]$ is bounded there will be a least and greatest element of $f([a; b])$ and f only outputs integers so we have $f([a; b])$ is a finite set of integers. Then $f(C) \subseteq f([a; b])$ and so $f(C)$ is finite set of integers as well. So $f(C)$ is compact because it's closed and bounded, but f is discontinuous because the left and right hand limits as x approaches some integer are different.

Exercise 2 *Let the real function f be defined as follows.*

$$f(x) \begin{cases} 0 & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x \text{ is rational of reduced form } \frac{p}{q}. \end{cases}$$

Then f is not continuous at nonzero rational numbers but for all $a \in \mathbb{R}$ we have

$$\lim_{x \to a} f(x) = 0.$$

Proof. Note that $0 \leq f(x) \leq |x|$ for all $x \in \mathbb{R}$. Consider $a \leq 0$ and for all $\varepsilon > 0$ let $\delta = \varepsilon$. Then if $0 < |a - x| < \delta$ we have $x \in (a - \delta; a + \delta)$ and so $f(x) \in (a - \delta; a + \delta) = (a - \varepsilon; a + \varepsilon)$. Thus $|a - f(x)| < \varepsilon$, but $a \leq 0$ and so $|-(-a + f(x))| < \varepsilon$ which means $0 \leq |f(x)| \leq |-a + f(x)| < \varepsilon$. Now consider $a > 0$ and let $\delta = \varepsilon + a$. Then if $0 < |a - x| < \delta$ we have $f(x) \in (a - \delta; a + \delta) = (-\varepsilon; \varepsilon)$ and so $|f(x)| < \varepsilon$. Thus for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ when $0 < |a - x| < \delta$ we have $|f(x)| < \varepsilon$ and so $\lim_{x \to a} f(x) = 0$ for all $x \in \mathbb{R}$. But we know that for nonzero rationals, $f(x) \neq 0$ because of how f is defined and since a function is only continuous at a if $\lim_{x \to a} f(x) = f(a)$ we have f is discontinuous at all nonzero rationals. □

Exercise 3 *Let $a \in \mathbb{R}$ such that $0 \leq a$. Show that there exists $x \in \mathbb{R}$ such that $x^2 = a$.*

Proof. If $a = 0$ then $0^2 = a$ so we can assume that $0 < a$. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ be a function. Consider the function $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x$. Let $O \subseteq \mathbb{R}$ be an open set. Clearly $g^{-1}(O)$ is open and so g is continuous. But then $f = g \cdot g$ is continuous as well since the product of two continuous functions is continuous. Since $0 < a$ we have $f(0) = 0 < a$ and $a < a^2 + 2a + 1 = f(a + 1)$. But since f is continuous, by the Intermediate Value Theorem f takes on every value between $0 and (a + 1)^2$ on the interval $[0; a + 1]$ which means there exists some $x \in \mathbb{R}$ such that $x^2 = a$. □

Exercise 4 *Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes on every real number exactly twice?*

Proof. Suppose to the contrary that there exists such a function f. Then we have $f(a) = f(b) = 0$ for some $a, b \in \mathbb{R}$ such that $a \neq b$. Without loss of generality suppose that $a < b$. There exists $c \in [a; b]$ such that $f(c) \neq 0$. Suppose first that $f(c) > 0$. Then by the Intermediate Value Theorem f takes on every value between 0 and $f(c)$ on $[a; c]$ and on $[c; b]$. But then f takes on every value between 0 and $f(c)$ exactly twice on $[a; b]$. We know there exists some $d \in \mathbb{R}$ such that $f(d) = f(c)$ and $d \neq c$. Note that also $d \neq a$ and $d \neq b$. Consider the case where $d < a$. Then f takes on every value between 0 and $f(d) = f(c)$ on $[d; a]$. But this is a contradiction because f has already taken on these values twice. A similar proof holds for $d > b$. If

$d \in (a;b)$ and $d < c$ then f takes on every value between 0 and $f(d)$ on $[a;d]$ but this is also a contradiction because f has already taken on these values twice. So for all $d \in \mathbb{R}$ we have f taking on values of $\mathbb{R}$ more than two times. A similar proof holds for $f(c) < 0$. Since we have a contradiction, f cannot exist. $\square$

Exercise 5 *Define* $\lim_{x\to\infty} f(x) = l$ *and* $\lim_{x\to-\infty} f(x) = l$.

Let f be a real function. We say that f approaches l as x goes to infinity, or

$$\lim_{x\to\infty} f(x) = l$$

if for all $\varepsilon > 0$ there exists $n > 0$ such that if $x > n$ then we have $|f(x) - l| < \varepsilon$. We say that f approaches l as x goes to negative infinity, or

$$\lim_{x\to-\infty} f(x) = l$$

if for all $\varepsilon > 0$ there exists $n > 0$ such that if $x < n$ then we have $|f(x) - l| < \varepsilon$.

Exercise 6 *We have*

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h).$$

Proof. Assume that $\lim_{x\to a} f(x) = l$. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ when $0 < |a - x| < \delta$ we have $|f(x) - l| < \varepsilon$. Now let $h = x - a$. So then we have for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $a + h \in \mathbb{R}$ when $0 < |0 - h| < \delta$ we have $|f(a+h) - l| < \varepsilon$. So we have $\lim_{h\to 0} f(a+h) = l = \lim_{x\to a} f(x)$.

Likewise, assume that $\lim_{h\to 0} f(a+h) = l$. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $a + h \in \mathbb{R}$ when $0 < |-h| < \delta$ we have $|f(a+h) - l| < \varepsilon$. Now let $x = a + h$. So then we have for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ when $0 < |a - x| < \delta$ we have $|f(x) - l| < \varepsilon$. So we have $\lim_{x\to a} f(x) = l = \lim_{h\to 0} f(a+h)$.

We can rename $x = h + a$ because the definition is still true for all $x \in \mathbb{R}$ if we do this. $\square$