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Sheet 22: Integrals

Definition 1 *Let $a < b$. A partition of the interval $[a;b]$ is a finite collection of points in $[a,b]$, one of which is a and one of which is b.*

Definition 2 *Suppose f is bounded on $[a;b]$ and $P = \{t_0, \dots, t_n\}$ is a partition of $[a;b]$. Let*

$$m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

$$M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}.$$

The lower sum of f for P, denoted by $L(f,P)$, is defined as

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of f for P, denoted by $U(f,P)$, is defined as

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Theorem 3 *Let P_1 and P_2 be partitions of $[a;b]$, and let f be a function which is bounded on $[a;b]$. Then*

$$L(f,P_1) \leq U(f,P_2).$$

Proof. Consider some partition $Q = \{t_0, \dots, t_n\}$ and some other partition Q' such that $Q \subset Q'$. First consider the case where Q' has only one more point than Q. Then $Q' = \{t_0, t_1, \dots, t_{k-1}, q, t_k, \dots t_n\}$. Let $m_1 = \inf\{f(x) \mid t_{k-1} \leq x \leq q\}$ and $m_2 = \inf\{f(x) \mid q \leq x \leq t_k\}$. Then

$$L(f,Q) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

and

$$U(f,Q') = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m_1(q - t_{k-1}) + m_2(t_k - q) + \sum_{i=k+1}^{n} m_i(t_i - t_{i-1}.$$

Note that

$$\{f(x) \mid t_{k-1} \leq x \leq q\} \subseteq \{f(x) \mid t_{k-1} \leq x \leq t_k\}$$

and

$$\{f(x) \mid q \leq x \leq t_k\} \subseteq \{f(x) \mid t_{k-1} \leq x \leq t_k\}$$

so $m_k \leq m_1$ and $m_k \leq m_2$. Thus

$$m_k(t_k - t_{k-1}) = m_k(q - t_{k-1}) + m_k(t_k - q) \leq m_1(q - t_{k-1}) + m_2(t_k - q)$$

and so $L(f, Q) \leq U(f, Q')$. Now consider the case where Q' contains n more points than Q. Then we can make a sequence of partitions which each contain one more point than the one before it $Q, Q_1, Q_2, \ldots, Q_{n-1}, Q'$. Then

$$L(f, Q) \leq L(f, Q_1) \leq \cdots \leq L(f, Q_{n-1}) \leq L(f, Q').$$

A similar proof holds to show for two partitions $Q \subseteq Q'$ that $U(f, Q) \geq U(f, Q')$. Now consider two partitions P_1 and P_2 of $[a; b]$ and let P be a partition which contains both P_1 and P_2. Then $L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$. □

Definition 4 *A function f which is bounded on $[a; b]$ is integrable on $[a; b]$ if*

$$\sup\{L(f, P) \mid P \text{ is a partition of } [a; b]\} = \inf\{U(f, P) \mid P \text{ is a partition of } [a; b]\}.$$

In this case, this common number is called the integral of f on $[a; b]$ and is denoted by

$$\int_a^b f = \int_a^b f(x)dx.$$

When $f(x) \geq 0$ for all $x \in [a; b]$, the integral is also called the area of the region defined by f, $x = a$, $x = b$ and $f(x) = 0$.

Exercise 5 *Show that for $c \in \mathbb{R}$, the function $f(x) = c$ is integrable on the interval $[a; b]$.*

Proof. Let $P = \{t_0, \ldots, t_n\}$ be some partition of $[a; b]$. Then note that $m_i = M_i = c$ for all $0 \leq i \leq n$. Thus

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = U(f, P)$$

for all partitions P. Thus

$$\sup\{L(f, P) \mid P \text{ is a partition of } [a; b]\} = \inf\{U(f, P) \mid P \text{ is a partition of } [a; b]\}.$$

and f is integrable on $[a; b]$. □

Exercise 6 *Let f be defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

Show that f is not integrable on the closed interval $[a; b]$.

Proof. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a; b]$. Then note that for all $0 \leq i \leq n$ we have $m_i = 0$ because there exists an irrational in $[t_{i-1}; t_i]$ and $M_i = 1$ because there exists a rational in $[t_{i-1}; t_i]$. Then $L(f, P) = 0$ and $U(f, P) = b - a$ for all partitions and so it's not the case that

$$\sup\{L(f, P) \mid P \text{ is a partition of } [a; b]\} = \inf\{U(f, P) \mid P \text{ is a partition of } [a; b]\}.$$

Thus f is not integrable on $[a; b]$. □

Theorem 7 *If f is bounded on $[a; b]$, then f is integrable on $[a; b]$ if and only if for every $\varepsilon > 0$ there exists a partition, P, of $[a; b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof. Suppose that for all $\varepsilon > 0$ there exists a partition, P, of $[a; b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Note that $\inf\{U(f, P)\} \leq U(f, P)$ and $\sup\{L(f, P)\} \geq L(f, P)$ so we have

$$\inf\{U(f, P)\} - \sup\{L(f, P)\} < \varepsilon.$$

Note that it's never the case that $\inf\{U(f, P)\} < \sup\{L(f, P)\}$ and if $\inf\{U(f, P)\} > \sup\{L(f, P)\}$ then we have $\inf\{U(f, P)\} - \sup\{L(f, P)\} > 0$. Then there exists $c \in \mathbb{R}$ such that

$$\inf\{U(f, P)\} - \sup\{L(f, P)\} > c > 0$$

and letting $c = \varepsilon$ we have a contradiction. Thus $\inf\{U(f, P)\} = \sup\{L(f, P)\}$ which shows that f is integrable. Conversely, assume that $\inf\{U(f, P)\} = \sup\{L(f, P)\}$. Then for all $\varepsilon > 0$ there exists partitions P_1 and P_2 such that $U(f, P_1) - L(f, P_2) < \varepsilon$. Then if P is a partition such that $P_1 \subseteq P$ and $P_2 \subseteq P$, we have $U(f, P) \leq U(f, P_1)$ and $L(f, P) \geq L(f, P_2)$ (22.3). Thus

$$U(f, P) - L(f, P) \leq U(f, P_1) - U(f, P_2) < \varepsilon.$$

$\square$

Exercise 8 *Show that $y = x$ is integrable on the closed interval $[a; b]$.*

Proof. Let $f(x) = x$ and let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a; b]$ of equal division so that $t_i - t_{i-1} = (b - a)/n$. Then $t_i = a + ((b - a)i)/n = (an + (b - a)i)/n$. Then note that $m_i = t_{i-1}$ and $M_i = t_i$ for all $0 \leq i \leq n$. Then

$$L(f, P) = \sum_{i=1}^{n} t_{i-1}(t_i - t_{i-1}) = \sum_{i=1}^{n} \left(\frac{(an + (b - a)(i - 1)}{n}\right)\left(\frac{b - a}{n}\right) = \sum_{i=1}^{n} \frac{an(b - a) + (b - a)^2(i - 1)}{n^2}$$

and likewise

$$U(f, P) = \sum_{i=1}^{n} \frac{an(b - a) + (b - a)^2 i}{n^2}.$$

Note that for all $\varepsilon > 0$ there exist n such that $1/n < \varepsilon/(b - a)^2$ by the Archimedean Property. Then $(b - a)^2/n^2 < \varepsilon$. Thus

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \frac{an(b - a) + (b - a)^2 i}{n^2} - \sum_{i=1}^{n} \frac{an(b - a) + (b - a)^2(i - 1)}{n^2} = \left(\frac{b - a}{n}\right)^2 < \varepsilon$$

which means that f is integrable on $[a; b]$ (22.7). $\square$

Theorem 9 *If f is continuous on $[a; b]$, then f is integrable on $[a; b]$.*

Proof. Note that since f is continuous on $[a; b]$, we know that f is uniformly continuous on $[a; b]$. Thus for all $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x, y \in [a; b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)|\varepsilon/(b - a)$. Now choose a partition $P = \{t_0, \ldots, t_n\}$ of $[a; b]$ such that $|t_i - t_{i-1}| < \delta$ for all $0 \leq i \leq n$. Then for all $0 \leq i \leq n$ with $x, y \in [t_{i-1}; t_i]$ we have

$$|f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Since f is continuous on $[a; b]$ we know that it takes on m_i and M_i for each i. Thus for all $0 \leq i \leq n$ we have

$$M_i - m_i < \frac{\varepsilon}{b - a}$$

which means

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \frac{\varepsilon}{b - a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{\varepsilon}{b - a}(b - a) = \varepsilon$$

and so f is integrable on $[a; b]$ (22.7). $\square$

Theorem 10 *Let $a < c < b$ for $a, b, c \in \mathbb{R}$. Then f is integrable on $[a; b]$ if and only if f is integrable on $[a; c]$ and on $[c; b]$. Also, if f is integrable on $[a; b]$, then*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Let f be integrable on $[a; b]$. Then there exists some partition $P = \{t_0, \ldots, t_n\}$ such that $U(f, P) - L(f, P) < \varepsilon$ for all $\varepsilon > 0$. In the case that P doesn't include the point c let P' be a partition which includes every point in P as well as c. Then $L(f, P) \leq L(f, P')$ and $U(f, P) \geq U(f, P')$ so

$$U(f, P') - L(f, P') \leq U(f, P) - L(f, P) < \varepsilon$$

which means we can assume that P contains c. Then we let $P_1 = \{t_0, \ldots, c\}$ and $P_2 = \{c, \ldots, t_n\}$. We have $P = P_1 \cup P_2$ and so

$$L(f, P) = L(f, P_1) + L(f, P_2)$$

and

$$U(f, P) = U(f, P_1) + U(f, P_2).$$

Then

$$(U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) = U(f, P) - L(f, P) < \varepsilon$$

and since each of the terms on the left is greater than 0, each must be less than ε. Thus there exists partitions P_1 and P_2 such that $U(f, P_1) - L(f, P_1) < \varepsilon$ and $U(f, P_2) - L(f, P_2) < \varepsilon$ which means that f is integrable on $[a; c]$ and on $[c; b]$ (22.7). Also we have

$$L(f, P_1) \leq \int_a^c f \leq U(f, P_1)$$

and

$$L(f, P_2) \leq \int_c^b \leq U(f, P_2)$$

Which means

$$L(f, P) \leq \int_a^c f + \int_c^b f \leq U(f, P).$$

But since this is true for any partition we must have

$$\sup\{L(f, P)\} \leq \int_a^c f + \int_c^b f \leq \inf\{U(f, P)\}$$

which gives

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

Conversely let f be integrable on $[a; c]$ and on $[c; b]$. Then for all $\varepsilon > 0$ there exists partitions P_1 of $[a; c]$ and P_2 of $[c; b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$$

and

$$U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then we have $L(f, P) = L(f, P_1) + L(f, P_2)$ and $U(f, P) = U(f, P_1) + U(f, P_2)$ so that

$$U(f, P) - L(f, P) = (U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) < \varepsilon$$

which means that f is integrable on $[a; b]$ (22.7). $\square$

Theorem 11 *If f and g are integrable functions on $[a;b]$, then $f+g$ is integrable on $[a;b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof. Suppose that f and g are integrable on $[a;b]$. Let $P = \{t_0, \ldots, t_n\}$ be some partition of $[a;b$ and define

$$m_i = \inf\{(f+g)(x) \mid t_{i-1} \leq x \leq t_i\},$$

$$m_i' = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

and

$$m_i'' = \inf\{g(x) \mid t_{i-1} \leq x \leq t_i\},$$

with M_i, M_i' and M_i'' defined in a similar fashion. We have $m_i \geq m_i' + m_i''$ and $M_i \leq M_i' + M_i''$ (18.4). Then $L(f,P) + L(g,P) \leq L(f+g,P)$ and $U(f,P) + U(g,P) \geq U(f+g,P)$ and so

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P).$$

Since f and g are integrable on $[a;b]$ there exists partitions P_1 and P_2 such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$$

and

$$U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}.$$

If $P = P_1 \cup P_2$ then we have

$$(U(f,P) + U(g,P)) - (L(f,P) + L(g,P) < \varepsilon$$

and so $U(f+g,P) - L(f+g,P) < \varepsilon$ which means $f+g$ is integrable on $[a;b]$ (22.7). Also we have

$$L(f,P) + L(g,P) \leq L(f+g)$$

□

Theorem 12 *If f is integrable on $[a;b]$, then for any number c, the function cf is integrable on $[a;b]$ and*

$$\int_a^b cf = c\int_a^b f.$$

Proof. Let f be integrable on $[a;b]$. Then for all $\varepsilon > 0$ there exists some partition $P = \{t_0, \ldots, t_n\}$ such that $U(f,P) - L(f,P) < \varepsilon/c$. Then note that for all i if $m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$ then $cm_i = \inf\{cf(x) \mid t_{i-1} \leq x \leq t_i\}$. A similar statement can be made for M_i and cM_i. Thus

$$U(cf,P) - L(cf,P) = \sum_{i=1}^{n}(cM_i - cm_i)(t_i - t_{i-1}) = c\sum_{i=1}^{n}(M_i - m_i)(t_i - t_{i-1}) = c(U(f,P) - L(f,P)) < \varepsilon$$

which shows that cf is integrable on $[a;b]$ (22.7). □

Exercise 13 *If f is integrable on $[a;b]$, then so is $|f|$.*
Exercise 14 *If f is integrable on $[a;b]$, then*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Lemma 15 *Suppose f is integrable on $[a;b]$ and that*

$$m \leq f(x) \leq M$$

for all $x \in [a;b]$. Then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. For some partition P of $[a;b]$ we have

$$m(b-a) \leq L(f,P) \leq \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\} \leq U(f,P) \leq M(b-a).$$

□

Theorem 16 *If f is integrable on $[a;b]$ and F is defined on $[a;b]$ by*

$$F(x) = \int_a^x f,$$

then F is continuous on $[a;b]$.

Proof. Let $c \in [a;b]$. Since f is integrable on $[a;b]$ it is bounded on $[a;b]$. Then there exists M such that $-M \leq f(x) \leq M$ for all $x \in [a;b]$. Let $h > 0$. Then we have

$$F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f$$

and because $-M \leq f(x) \leq M$ for all $x \in [a;b]$ we have

$$-Mh \leq \int_c^{c+h} f \leq Mh$$

from Lemma 15 (22.15). Thus $-Mh \leq F(c+h) - F(c) \leq Mh$ and a similar inequality will result if $h < 0$ so that $Mh \leq F(c+h) - F(c) \leq -Mh$. Combining these we have $|F(c+h) - F(c)| \leq M|h|$ and so if $|h| < \varepsilon/M$ we have $|F(c+h) - F(c)| < \varepsilon$. Thus

$$\lim_{h \to 0} F(c+h) = F(c)$$

and so F is continuous at c. □

Theorem 17 (The First Fundamental Theorem of Calculus) *Let f be integrable on $[a;b]$, and define F on $[a;b]$ by*

$$F(x) = \int_a^x f.$$

If f is continuous at $c \in [a;b]$, then F is differentiable at c, and

$$F'(c) = f(c).$$

(If $c = a$ or $c = b$, then $F'(c)$ is understood to mean the right- or left-hand derivative of F.)

Proof. Let $c \in (a;b)$ and suppose that $h > 0$. Define

$$m_h = \{f(x) \mid c \leq x \leq c+h\}$$

and

$$M_h = \{f(x) \mid c \leq x \leq c+h\}.$$

Then we have

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h}$$

and

$$m_h h \leq \int_c^{c+h} f \leq M_h h$$

from Lemma 15 (22.15). Then since $h > 0$

$$F(c+h) - F(c) = \int_c^{c+h} f$$

and

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h.$$

If $h < 0$ then the same result can be obtained using the fact that

$$\int_{c+h}^{c} f = -\int_c^{c+h} f.$$

Then since f is continuous at c we have

$$\lim_{h \to 0} m_h = \lim_{h \to 0} M_h = f(c)$$

which means that

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c).$$

□

Theorem 18 (The Second Fundamental Theorem of Calculus) *If f is integrable on $[a;b]$ and $f = g'$ for some function g, then*

$$\int_a^b f = g(b) - g(a).$$

Proof. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a;b]$. By the Mean Value Theorem there exists $x_i \in [t_{i-1}; t_i]$ such that

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1}).$$

Then we have

$$m_i(t_i - t_{i-1}) \leq f(x)(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1})$$

which means

$$m_i(t_i - t_{i-1}) \leq g(t_i) - g(t_{i-1}) \leq M_i(t_i - t_{i-1}).$$

If we then take the sum for the entire interval $[a;b]$ we obtain

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \leq g(b) - g(a) \leq \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = U(f,P).$$

Since this is true for every partition P we must have

$$g(b) - g(a) = \int_a^b f.$$

□