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Homework 8

Problem 1. *Let A and B be disjoint compact subspace of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.*

Proof. Note that since A and B are compact subspaces of a Hausdorff space they are necessarily closed. Then $X\backslash A$ and $X\backslash B$ are open. Since A and B are disjoint, $V = X\backslash A$ contains B and $U = X\backslash B$ contains A. $\square$

Problem 2. *Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the* graph *of f,*

$$G_f = \{x \times f(x) \mid x \in X\},$$

is closed in $X \times Y$.

Proof. Let $f(x_0) \in Y$ and consider a neighborhood V of $f(x_0)$. Note that $Y\backslash V$ is closed. If G_f is closed then $G_f \cap (X \times (Y\backslash V))$ is closed as well. But we also know that the projection $\pi_1 : X \times Y \to X$ is a closed map. If we apply π_1 to this set, we get all the $x \in X$ such that $f(x) \in Y\backslash V$. In particular, this set is closed and doesn't contain x. The complement of this set is a neighborhood U of x such that $U \times Y$ doesn't intersect $G_f \cap (X \times (Y\backslash V))$. This means that $f(U) \subseteq V$ and that f is continuous.

Conversely, suppose that f is continuous. Let $(x, y) \in (X \times Y)\backslash G_f$. Then $y \neq f(x)$ so we can find disjoint neighborhoods U and V of y and $f(x)$ respectively. Since f is continuous there exists a neighborhood W of x such that $f(W) \subseteq V \subseteq Y\backslash U$. Therefore $W \times U \subseteq (X \times Y)\backslash G_f$. Thus G_f is closed. $\square$

Problem 3. *Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that*

$$A \times B \subseteq U \times V \subseteq N.$$

Proof. Let $a \in A$. Cover the set $\{a\} \times B$ with basis elements $U_i \times V_i$. Since B is compact we can choose finitely many of these sets to cover this set. Furthermore, we can choose U_i and V_i, which are open in A and B, to be basis elements in the subspace topology so that $U_i = U_i' \cap A$ and $V_i = V_i' \cap B$ where U_i' and V_i' are open in X and Y respectively. Let $W_a = \bigcap_i U_i'$. This is an open set in X which contains a. Also, let $V_a = \bigcup_i V_i'$. This is an open set in Y which contains B. Now, the sets W_a form an open cover for A so some finite subcover covers A. Let U be the union of this finite collection and let V be the intersection of the corresponding V_a. Since there are only finitely many of these sets, V is open in Y. Furthermore, since each V_a contains B, $B \subseteq V$ as well. Thus, $A \subseteq U$ and $B \subseteq V$. Since for each $a \in A$ we have $W_a \times V_a \subseteq N$, we also have $U \times V \subseteq N$. $\square$

Problem 4. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subset of X that is simply ordered by proper inclusion. Then*

$$Y = \bigcap_{A \in \mathcal{A}} A$$

is connected.

Proof. Let $\{C, D\}$ be a separation of Y. Note that C and D are necessarily open in Y so they are of the form $\bigcup_i (U_i \cap Y) = Y \cap \bigcup_i U_i$ and $\bigcup_i (V_i \cap Y) = Y \cap \bigcup_i V_i$ where U_i and V_i are open in X. Letting $U = \bigcup_i U_i$ and $V = \bigcup_i V_i$ we have disjoint sets U and V containing C and D. Note that for $A \in \mathcal{A}$ the set $A\backslash(U \cup V)$ is closed. To see this, note that $X\backslash(A\backslash(U \cup V)) = X\backslash A$ which is open. Furthermore, since $\mathcal{A}$ was assumed to be ordered by inclusion, it follows that the sets of $A\backslash(U \cup V)$ is also ordered by inclusion. Finally, note that $A\backslash(U \cup V) \neq \emptyset$ since each A is connected and otherwise $\{U \cap A, V \cap A\}$ would form a separation of A. Therefore this collection of $A\backslash(U \cup V)$ with $A \in \mathcal{A}$ is a collection of nonempty, nested, closed sets in a compact space X. Thus the set

$$\bigcap_{A \in \mathcal{A}} (A\backslash(U \cup V)) = \left(\bigcap_{A \in \mathcal{A}} A \right) \backslash(U \cup V) \neq \emptyset.$$

But this is a contradiction since $U \cap Y$ and $V \cap Y$ were assumed to form a separation of Y. Thus Y must be connected. □

Problem 5. *Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X.*

Proof. Let U be an open set of X. We wish to find a point of U which is not in $\bigcup A_n$. Otherwise we would have $U \subseteq \bigcup A_n$ and the interior of the union would not be empty. Since the interior of A_1 is empty, we know $U \nsubseteq A_1$. Let $y \in U\backslash A_1$. Since X is compact and Hausdorff and A_1 is closed we can find a neighborhood U_1 of y such that $\overline{U_1} \cap A_1 = \emptyset$ and $\overline{U_1} \subseteq U$. For each n and each set U_{n-1} we choose a point $y \in U_{n-1}\backslash A_n$ and find a neighborhood U_n of y such that $\overline{U_n} \cap A_n = \emptyset$ and $\overline{U_n} \subseteq U_{n-1}$. Now note that $\{\overline{U_n}\}$ is a collection of nested, closed, nonempty sets in a compact space with the finite intersection property. Thus $\bigcap \overline{U_n}$ contains some point x. But $x \notin A_n$ for each n and $U_1 \subseteq U$ so $x \in U$. Thus $x \in U\backslash \bigcup A_n$ and $\bigcup A_n$ must have empty interior. □

Problem 6. *Let X be limit point compact.*
(a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?
(b) If A is a closed subset of X, does it follow that A is limit point compact?
(c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?

Proof. (a) No. Consider the set Y, the indiscrete topology on two points, and let $X = \mathbb{Z}_+ \times Y$. Then X is limit point compact. Let f be $\pi_1 : \mathbb{Z}_+ \times Y \to \mathbb{Z}_+$. This map is continuous since given some subset $A \subseteq \mathbb{Z}_+$ we have $\pi_1^{-1}(A) = A \times Y = \{a_1\} \times Y \cup \{a_2\} \times Y \cup \ldots$ which is open. But the image $\pi_1(X) = \mathbb{Z}_+$ is not limit point compact because $\mathbb{Z}_+$ has no limit points.

(b) Let A be a closed subset of X and let B be an infinite subset of A. Then B has a limit point $x \in X$ because X is limit point compact. But since A is closed, $x \in A$ and A is limit point compact as well.

(c) No. Consider $\overline{S_\Omega}$ which is Hausdorff and contains S_Ω. But S_Ω isn't closed in $\overline{S_\Omega}$ since it doesn't contain all its limit points. □