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Problem 1 (10.3.1)**.** *Prove that if A and B are sets of the same cardinality, then the free modules $F(A)$ and $F(B)$ are isomorphic.*

Proof. Since A and B have the same cardinality there exists a bijection $f : A \to B$. Let $\varphi : F(A) \to F(B)$ be given by $\varphi(r_1a_1 + \cdots + r_na_n) = r_1f(a_1) + \cdots + r_nf(a_n)$. Note that φ is surjective since given the element $r_1b_1 + \cdots + r_nb_n \in F(B)$ we know $r_1f^{-1}(b_1) + \cdots + r_nf^{-1}(b_n)$ is mapped to it by φ. It's also injective in that given two distinct elements $r_1a_1 + \cdots + r_na_n \neq r'_1a'_1 + \cdots + r'_ma'_m$ we see that there must exist some i such that $r_ia_i \neq r'_ia'_i$. Then in $F(A)$ we have $\varphi(r_1a_1 + \cdots + r_na_n) = r_1f(a_1) + \cdots + r_nf(a_n)$ and since f is injective, $r_if(a_i) \neq r'_if(a'_i)$. Therefore the images of the two elements are distinct and φ is injective.

If $r_1a_1 + \cdots + r_na_n$ and $r'_1a'_1 + \cdots + r'_ma'_m$ are two elements of $F(A)$ then

$$\begin{aligned}\varphi((r_1a_1 + \cdots + r_na_n) + (r'_1a'_1 + \cdots + r'_ma'_m)) &= r_1f(a_1) + \cdots + r_nf(a_n) + r'_1f(a'_1) + \cdots + r'_mf(a'_m)\\ &= \varphi(r_1a_1 + \cdots + r_na_n) + \varphi(r'_1a'_1 + \cdots + r'_ma'_m)\end{aligned}$$

so φ is additive. Finally, let $r \in R$ so we have

$$\begin{aligned}r\varphi(r_1a_1 + \cdots + r_na_n) &= r(r_1f(a_1) + \cdots + r_nf(a_n))\\ &= rr_1f(a_1) + \cdots + rr_nf(a_n)\\ &= \varphi(rr_1a_1 + \cdots + rr_na_n)\\ &= \varphi(r(r_1a_1 + \cdots + r_na_n))\end{aligned}$$

and φ is scalar multiplicative. Therefore φ is an R-module isomorphism between $F(A)$ and $F(B)$. □

Problem 2 (10.3.4)**.** *An R-module M is called a* torsion *module if for each $m \in M$ there is a nonzero element $r \in R$ such that $rm = 0$, where r may depend on m (i.e., $M = \mathrm{Tor}(M)$ in the notation of Exercise 8 of Section 1). Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.*

Proof. Let A be a finite abelian group of order n. Then for each $a \in A$ we have $na = 0$. Therefore A is a torsion $\mathbb{Z}$ module. As an example of an infinite abelian group, consider $\mathbb{Q}/\mathbb{Z}$. Every element of this group has finite order ($a/b + \mathbb{Z}$ has order at most b), so for each element we can find an element of $\mathbb{Z}$ which sends it to 0. Therefore $\mathbb{Q}/\mathbb{Z}$ is a torsion module. □

Problem 3 (10.3.6)**.** *Prove that if M is a finitely generated R-module that is generated by n elements then every quotient of M may be generated by n (or fewer) elements. Deduce that quotients of cyclic modules are cyclic.*

Proof. Suppose M is generated by the set A with $|A| = n$. Let N be a submodule of M and consider the projection map $\pi : M \to M/N$. Let $\overline{m} \in M/N$ and let $m' \in \pi^{-1}(\overline{m})$. Then $m' = r_1a_1 + \cdots + r_1a_n$ and $\pi(m') = \overline{m}$. But then $\pi(m') = \pi(r_1a_1 + \cdots + r_na_n) = r_1\pi(a_1) + \cdots + r_n\pi(a_n)$. Thus, every element $\overline{m} \in M/N$ can be written as a finite linear combination of elements of the set $\pi(A)$ and M/N is finitely generated. A cyclic module only has one generator and we've shown that a quotient of such a module will have one or fewer generators. Thus, it must also be cyclic. □

Problem 4 (10.3.9)**.** *An R-module M is called* irreducible *if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator. Determine all the irreducible $\mathbb{Z}$-modules.*

Proof. Suppose that M is irreducible and that M requires at least two generators, $a \neq b$. Then $Ra \neq Rb$ (since R has 1). But note that $R\{a, b\}$ contains both Ra and Rb since it contains a and b. Therefore Ra is a nonzero submodule of M which is properly contained in M, a contradiction. Therefore $M = Ra$ for some a. Conversely, suppose $M \neq 0$ and M is cyclic with generator a. Suppose N is a submodule of M. Note that $N \subseteq M$ so for each nonzero $n \in N$ we have $n = ra$ for some $r \in R$. Therefore N contains a and thus also contains Ra. But then $N = M$ and so M is irreducible.

The $\mathbb{Z}$ modules are the same as abelian groups, so the irreducible $\mathbb{Z}$-modules are all finitely generated abelian groups with 1 generator. $\square$

Problem 5 (10.4.11)**.** *Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.*

Proof. Given the basis elements e_1 and e_2 of V, we know $e_1 \otimes e_1$, $e_2 \otimes e_2$, $e_1 \otimes e_2$ and $e_2 \otimes e_1$ form a basis for $V \otimes_{\mathbb{R}} V$. Thus, $\{e_1 \otimes e_2, e_2 \otimes e_1\}$ is a linearly independent set which means $e_1 \otimes e_2 + e_2 \otimes e_1$ cannot equal a simple tensor $v \otimes w$. $\square$

Problem 6 (10.4.12)**.** *Let V be a vector space over the field F and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = av'$ for some $a \in F$.*

Proof. Suppose there exists $a \in F$ such that $v = av'$. Then $v \otimes v' = av' \otimes v' = v' \otimes av' = v' \otimes v$. Conversely, suppose $v \otimes v' = v' \otimes v$. Then $v \otimes v' - v' \otimes v = 0$. Since v and v' are nonzero, these two simple tensors are linearly dependent so there exists $a \in F$ such that $v \otimes v' = a(v' \otimes v) = av' \otimes v = v' \otimes av$. This is only possible if $v = av'$. $\square$

Problem 7 (10.5.14)**.** *Let* $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ *be a sequence of R-modules.*
(a) Prove that the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\varphi'} \mathrm{Hom}_R(D, N) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence.
(b) Prove that the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\psi'} \mathrm{Hom}_R(M, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(L, D) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence.

Proof. (a) Suppose the original sequence splits and let D be an R-module. Note then that we can write $M \cong L \oplus N$. But now we know $\mathrm{Hom}_R(D, M) \cong \mathrm{Hom}_R(D, L \oplus N) \cong \mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$. Then the associated sequence also splits and is an exact sequence. Conversely, suppose that the associated sequence is a short exact sequence. Let $D = N$ and let $f \in \mathrm{Hom}_R(N, N)$ be the identity. This lifts to some map $f' \in \mathrm{Hom}_R(N, M)$ such that $\varphi \circ f' = f$. But since f is the identity on N, we see that f' is a splitting homomorphism for φ. Thus the original sequence must be exact.

(b) If the original sequence is exact the associated sequence is a short exact sequence using the same proof as in part (a). Namely, using the fact that $\mathrm{Hom}_R(M, D) \cong \mathrm{Hom}_R(L \oplus N, D) \cong \mathrm{Hom}_R(L, D) \oplus \mathrm{Hom}_R(N, D)$. Conversely, if the associated sequence is exact, then let $D = L$ and let $f \in \mathrm{Hom}_R(L, L)$ be the identity. Then f lifts into an element $f' \in \mathrm{Hom}_R(M, L)$ such that $f' \circ \psi = f$. But since f is the identity on L f' is a splitting homomorphism for ψ and the original sequence is short exact. $\square$