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Homework 3

Problem 1. *Let $A = A_1 \times \cdots \times A_r$ be the direct product of rings A_i, $1 \leq i \leq r$. Show that any ideal I of A is of the form $I = I_1 \times \cdots \times I_r$. Deduce that A is Noetherian (respectively Artinian) if and only if each of the A_i are Noetherian (Artinian).*

Proof. Let I be an ideal of A. For each $1 \leq i \leq r$, consider the set

$$I_i = \{(0, 0, \ldots, a, \ldots, 0, 0) \cdot I \mid a \in A\}$$

where the (potentially) nonzero coordinate is the i^{th} coordinate. Since I is an ideal, $I_i \subseteq I$. Furthermore, each element of I_i has a zero in each coordinate except the i^{th} coordinate, thus multiplication by any element $(a_1, \ldots, a_r) \in A$ will remain in I_i. Finally, I_i is nonempty (since $0 \in I_i$) and closed under addition and multiplication. Thus I_i is isomorphic to some ideal of A_i. Since this is true for each i, we have $I = I_1 \times \cdots \times I_r$.

If each A_i is Noetherian, then every ideal I_i is finitely generated, which means every ideal $I = I_1 \times \cdots \times I_r$ is finitely generated. Thus A is Noetherian. Conversely, if A is Noetherian, then given an ideal $I_i \subseteq A_i$, we know $I_i = 0 \times \cdots \times I_i \times \cdots \times 0$ is finitely generated. Thus A_i is Noetherian. $\square$

Problem 2. *Let $A \subseteq B$ be a commutative rings. Suppose B is a finite A-module.*
(a) Let $I \subsetneq A$ be an ideal. Show that $IB \neq B$.
(b) Suppose further that B is an integral domain. Show that A is a field if and only if B is a field.
(c) Show that a prime ideal Q of B is maximal if and only if $A \cap Q$ is maximal.
(d) Let M be a maximal ideal of A. Show that there exists a maximal ideal Q of B such that $Q \cap A \subseteq M$.

Proof. (a) Suppose $IB = B$. Then since B is a finite A-module, there exists $a \in I$ such that $(1 - a)B = 0$. In particular, if we pick $x \in A$ then $0 = (1 - a)x = x - ax$ so $x = ax$. But $a \in I$ so $ax \in I$ as well which means $x \in I$. Thus $A \subseteq I$ and this contradicts the assumption $I \subsetneq A$. Thus $IB \neq B$.

(b) Suppose A is field. Then A is Artinian. Since B is a finitely generated A-module, it too is Artinian. But we know Artinian integral domains are fields. So B is a field.

Conversely, suppose B is a field. If A is not a field then there exists some nontrivial proper ideal $I \subsetneq A$. By part (a) we know $IB \neq B$. But IB is an ideal of B and B is field so $IB = 0$. Thus $I = 0$ and the only proper ideal of A is trivial. Therefore A is a field.

(c) Note that since B is a finite A-module, we know B/Q is a finite $A/(Q \cap A)$ module. Since Q is a prime ideal, B/Q is an integral domain. Thus, by part (b), B/Q is a field if and only if $A/(Q \cap A)$ is a field. Therefore Q is maximal if and only if $A \cap Q$ is maximal.

(d) Let M be a maximal ideal of A and let $Q = MB$. Then $Q \cap A = M$. By part (a) we know $Q \neq B$. Now suppose $x, y \in B \backslash Q$ and write $x = \sum_i a_i x_i$, $y = \sum_j b_j x_j$, $a_i \in A$, $x_i \in B$. Since $x, y \notin MB$, at least one a_i and b_j, say a_1 and b_1 are not in M. Then $xy = \sum_{ij} a_i b_j x_i$. Since M is maximal, it's prime and thus $a_1 b_1 \notin M$. Thus $xy \in B \backslash Q$, so Q is prime as well. Now by part (c) we know Q is maximal in B since $Q \cap A = M$ is maximal in A. $\square$

Problem 3. *Let $A \subseteq B$, with A and B being finitely generated algebras over an algebraically closed field K. Let $f : A \to K$ be a K-algebra homomorphism (i.e. a ring homomorphism such that $f(a) = a$ for all $a \in K$). Show that there exists a K-algebra homomorphism $\tilde{f} : B \to K$ such that $\tilde{f}|_A = f$.*

Proof. Let $M = \ker f$ so that $A/M \cong K$. Note $M \subseteq A \subseteq B$, so pick any maximal ideal $N \subseteq B$ with $M \subseteq N$. Note that since K is algebraically closed, by Nullstellensatz we have $B/N \cong K$, which shows that N is $\ker g$ for some homomorphism $g : B \to K$. Furthermore since $M \subseteq N$ then for $a \in A$ we have $g(a) = f(a)$ so g restricts to f on A. Take $\tilde{f} = g$. $\square$

Problem 4. *Let K be an algebraically closed field and $A = K[t]$, a polynomial ring in one variable t. Let $f_i(t) \in A$, $1 \leq i \leq n$. Suppose that $f_k(t) \notin K$ for some k. Let $R = K[f_1(t), \dots, f_n(t)]$ be the subring of A generated by $f_i(t)$, $1 \leq i \leq n$. Let $\varphi : R \to K$ be a K-algebra homomorphism and $\varphi(f_i(t)) = a_i$, $1 \leq i \leq n$. Show that there exists an $a \in K$ such that $a_i = f_i(a)$, $1 \leq i \leq n$.*

Proof. Since $R \subseteq A$ are both finitely generated K-algebras and φ is a K-algebra homomorphism, we can apply Problem 3 to get an extension $\tilde{\varphi} : A \to K$ which restricts to φ on R. Let $\tilde{\varphi}(t) = a$. Then since $\tilde{\varphi}$ fixes elements of K, for each $1 \leq i \leq n$ we have

$$a_i = \varphi(f_i(t)) = \tilde{\varphi}(f_i(t)) = f_i(\tilde{\varphi}(t)) = f_i(a)$$

as desired. □

Problem 5. *Let K be an algebraically closed field and $f_i \in K[x]$, $1 \leq i \leq n$. Let*

$$V = \{(f_1(a), \dots, f_n(a)) \mid a \in K\}.$$

Show that V is a K-affine algebraic set in K^n.

Proof. Let $\varphi : K[x_1, \dots, x_n] \to K[x]$ be the K-algebra homomorphism defined by $\varphi(x_i) = f_i(x)$, $1 \leq i \leq n$. Let $I = \ker \varphi$. Note that we must have $V \subseteq V(I)$ since if $x \in V$ we know $x = (f_1(a), \dots, f_n(a))$ and for all $f \in I$ we have $f(f_1(a), \dots, f_n(a)) = 0$.

Conversely, let $a = (a_1, \dots, a_n) \in V(I)$. Form the evaluation map $E_a : K[x_1, \dots, x_n] \to K$ which evaluates at a. Note that we have the projection map $K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]/I$. If we take $f + I$ in this quotient and evaluate at a then since $a \in V(I)$ we know I will map to 0 so f maps to $E_a(f)$. Thus the following diagram is well defined and commutes.

$$\begin{tikzcd}
K[x_1, \dots, x_n] \arrow[r] \arrow[d, "E_a"'] & K[x_1, \dots, x_n]/I \arrow[dl, "\tilde{E}_a"] \\
K &
\end{tikzcd}$$

Now note that by the first isomorphism theorem we have $K[x_1, \dots, x_n]/I \cong K[f_1(x), \dots, f_n(x)]$ and we have a K-algebra homomorphism $\tilde{E}_a : K[f_1(x), \dots, f_n(x)] \to K$ with $\tilde{E}_a(f_i(x)) = a_i$. Now we can apply Problem 4 to find an element $c \in K$ with $f_i(c) = a_i$, $1 \leq i \leq n$. Now $(a_1, \dots, a_n) = (f_1(c), \dots, f_n(c))$ so $a \in V$ as desired. □

Problem 6. *Let K be any field. Show that any maximal ideal M of $A = K[x_1, \dots, x_n]$ is generated by n irreducible polynomials $f_i \in K[x_1, \dots, x_n]$, $1 \leq i \leq n$.*

Proof. Suppose $M \cap K[x_1] = 0$. Then note that $K \subseteq K(x_1)$. Since M contains no polynomials with x_1, we have an injection from $K[x_1] \to A/M$. Since $K(x)$ is the field of fractions for $K[x_1]$ and A/M is a field, by the universal property of the field of fractions, we also have $K(x_1) \subseteq A/M$ (or at the very least, $K(x_1)$ injects into A/M).

Since M is maximal, A/M is a field and since A is a finitely generated K-algebra, we have A/M is a finite algebraic extension of K. Thus A/M is a finite K-module, and thus a finite $K(x)$-module. Since K is Noetherian, we can conclude that $K(x)$ is a finite K-algebra. But, since K is algebraically closed, we know $K(x)$ can't be generated as a K-algebra (since it's not a finite algebraic extension). Thus $M \cap K[x_1] \neq 0$.

Since K is a field, $K[x_1]$ is a principle ideal domain and all prime ideals are maximal. Pick $f, g \in K[x_1]$ with $fg \in M \cap K[x_1]$. Then $fg \in M$ and since M is maximal in $K[x_1, \dots, x_n]$, it's prime so $f \in M$ without loss of generality. Then $f \in M \cap K[x_1]$ so $M \cap K[x_1]$ is prime in $K[x_1]$ (note that it can't be the whole ring otherwise M would contain 1). Therefore $M \cap K[x_1]$ is a maximal ideal in the principle ideal domain $K[x_1]$ and thus $M \cap K[x_1] = (f_1)$ for some irreducible element $f_1 \in K[x_1]$.

Now suppose the statement is true for $n - 1$ and take a maximal ideal $M \subseteq K[x_1, \dots, x_n]$. Then by the above we can consider the field $K[x_1]/(M \cap K[x_1])$ and the ring $K[x_1]/(M \cap K[x_1])[x_2, \dots, x_n]$. Now note $M/(M \cap K[x_1])$ in this ring is a maximal ideal, so $M = (f_2, \dots, f_n)$. Now take the preimage under the quotient and we get $M = (f_1, \dots, f_n)$. □

Problem 7. *Let $A = \mathbb{Z}[x_1, \ldots, x_n]$. Show that for any maximal ideal M of A, A/M is a finite field.*

Proof. Suppose $M \cap \mathbb{Z} = 0$. Then each element of $\mathbb{Z}$ maps injectively into A/M. Since A/M is a field, by the universal property of the field of fractions we have $\mathbb{Q}$ injects into A/M as well. Thus we have $\mathbb{Z} \subseteq \mathbb{Q} \subseteq A/M$. Since $\mathbb{Z}$ is a principle ideal domain, it's Noetherian. Also A/M is a finite $\mathbb{Z}$-algebra and A/M is a finitely algebraic extension of $\mathbb{Q}$. Thus $\mathbb{Q}$ is a finite $\mathbb{Z}$-algebra, a contradiction.

Thus $M \cap \mathbb{Z} \neq 0$ and since M is a maximal in $\mathbb{Z}$, $M \cap \mathbb{Z} = p\mathbb{Z}$ for some prime $p \in \mathbb{Z}$. Consider the quotient $A/(M \cap \mathbb{Z})A$. Note that the quotient ring is $p\mathbb{Z}A$, which in particular contains p. Thus, this quotient is simply $\mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n]$, a finite $\mathbb{Z}/p\mathbb{Z}$-algebra. Now note that A/M also contains $\mathbb{Z}/p\mathbb{Z}$ and $A/p\mathbb{Z}A$ is a finite A/M-module. Thus we can conclude that A/M is a finite $\mathbb{Z}/p\mathbb{Z}$ algebra. By Nullstellensatz we know that since A/M is a field, it's a finite algebraic extension of $\mathbb{Z}/p\mathbb{Z}$ and therefore a finite field. □

Problem 8. *Let $A = \mathbb{Z}[x_1, \ldots, x_n]$ and M a maximal ideal of A. Then there exists $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$, $1 \leq i \leq n$ such that $M = Ap + \sum_{i=1}^n Af_i$ for some prime p.*

Proof. By Problem 7 we know $M \cap \mathbb{Z} = p\mathbb{Z}$ for some prime $p \in \mathbb{Z}$. Now consider $M/p\mathbb{Z} \subseteq A/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n]$. Since $\mathbb{Z}/p\mathbb{Z}$ is a field, we can apply Problem 6 to get $M = \sum_{i=1}^n A/p\mathbb{Z}f_i$ for some $f_i \in A/p\mathbb{Z}$, $1 \leq i \leq n$. Now take the preimage under the quotient map to get $M = Ap + \sum_{i=1}^n Af_i$ as desired. □

Here K, L denote fields with $K \subseteq L$.

Problem 9. *Let V, W be K-affine algebraic sets in L^m and L^n respectively. Identifying $L^m \times L^n$ with L^{m+n} (identify $((a_1, \ldots, a_m), (b_1, \ldots, b_n))$ with $(a_1, \ldots a_m, b_1, \ldots b_n)$), show that $V \times W$ is a K-affine algebraic set in L^{m+n}.*

Proof. Let $V = V(\{f_1, \ldots, f_r\})$ and $W = V(\{g_1, \ldots, g_s\})$. Then consider $U = V(\{f_1, \ldots, f_r, g_1, \ldots, g_s\})$ as a subset of L^{m+n} where each f_i and g_j are considered as elements of $L[x_1, \ldots, x_r, y_1, \ldots, y_s]$. An element $a \in U$ vanishes on each f_i and g_j so its first r coordinates form an element of V and its last s coordinates form an element of W. Thus $U \subseteq V \times W$. Similarly, if we take element $a \in V \times W$, then the first r coordinates of a must be 0 on each f_i and the last s must be 0 on each g_j. Thus $a \in U$ and $U = V \times W$. Thus $V \times W$ is an algebraic set in L^{m+n}. □

Problem 10. *(a) Let $f = a_0x^n + a_1x^{n-1} + \cdots + a_n \in K[x]$ be an irreducible polynomial of degree $n \geq 2$. Let $F(x, y) = y^n f(x/y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$. $F(x, y) \in K[x, y]$. Show that $F(t_1, t_2) = 0$, $t_1, t_2 \in K$ if and only if $t_1 = t_2 = 0$.*
(b) Let K be a non-algebraically closed field*. Show that for all $r \geq 1$ there exists a polynomial $F_r(x_1, \ldots x_r) \in K[x_1, \ldots, x_r]$ such that $F_r(t_1, \ldots, t_r) = 0$, $(t_1, \ldots, t_r) \in K^r$ if and only if $t_1 = t_2 = \cdots = t_r = 0$.*

Proof. (a) Suppose $F(t_1, t_2) = 0$ with $t_1 \neq 0$ or $t_2 \neq 0$. First suppose $t_2 = 0$. Then we have $a_0t_1^n = 0$ and $t_1 = 0$, contrary to assumption. Then $t_2 \neq 0$ and we have $t_2^n f(t_1/t_2) = 0$ so $f(t_1/t_2) = 0$. But we assumed f was irreducible. Thus $t_1 = t_2 = 0$. The other direction is trivial.

(b) For $r = 1$ we have $F_1(x_1) = x_1$. For $r = 2$ take $F_2(x_1, x_2) = F(x, y)$ from part (a). Assume we have such a polynomial from some positive integer $r - 1$. Then we construct $F_r(x_1, \ldots, x_r)$ as $F_2(F_{r-1}(x_1, \ldots, x_{r-1}), x_r)$. Now F_r will be 0 at $(t_1, \ldots, t_r)$ if and only if $t_r = 0$ and $F_{r-1}(t_1, \ldots, t_{r-1}) = 0$. But this this last condition is only true when $t_1 = t_2 = \cdots = t_{r-1} = 0$ by the inductive hypothesis. □

Problem 11. *Let K be a non-algebraically closed field. Let V be a K-affine algebraic set in K^n. Show that there exists a $\varphi \in K[x_1, \ldots, x_n]$ such that $V = V(\varphi)$.*

Proof. Let $V = V(\{f_1, \ldots, f_r\})$. Then form $F_r(x_1, \ldots, x_r)$ as in Problem 10 and set

$$\varphi(x_1, \ldots, x_n) = F_r(f_1(x_1, \ldots, x_n), \ldots f_r(x_1, \ldots, x_n)).$$

Then by Problem 10, φ will be 0 if and only each of f_i are 0. □

Problem 12. *Let K be any field. Let I be an ideal in $A = K[x_1, \ldots, x_n]$. Suppose for all $f \in I$, there exists a $k \in K^n$ such that $f(a) = 0$ (depending on f). Show that there exists $t = (t_1, \ldots, t_n) \in K^n$ such that $f(t) = 0$ for all $f \in I$.*

Proof. If K is not algebraically closed, use problem 11 to form the function φ for the set $V = V(I)$. Now let t be a 0 of φ. Then t must be a 0 of each of the $f \in I$.

If K is algebraically closed then we know $I \subseteq A$ is a proper ideal if and only if $V(I) \neq \emptyset$. But for each $f \in I$ we have $f(a) = 0$ for some $a \in K^n$. Thus all constant functions are not in I so I is proper and $V(I)$ is nonempty. Pick any $t \in V(I)$. $\square$

Problem 13. *Let $f : A \to B$ be a ring homomorphism of commutative rings. Show that for any prime ideal Q of B, $f^{-1}(Q)$ is a prime ideal of A. Give an example to show that in general $M \in \mathrm{Max}(B)$ does* not *imply $f^{-1}(M) \in \mathrm{Max}(A)$.*

Proof. First note that we can't have $f(A) \subseteq Q$ since $f(1) = 1$ and $Q \subsetneq B$. Thus $f^{-1}(Q) \subsetneq A$. Now let $ab \in f^{-1}(Q)$. Then $f(ab) = f(a)f(b)$ is in Q. Since Q is prime, $f(a), f(b) \in Q$. Thus $a \in f^{-1}(Q)$ and $b \in f^{-1}(Q)$, so $f^{-1}(Q)$ is prime in A.

Take the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$. Then since $\mathbb{Q}$ is a field, $\{0\}$ is a maximal ideal. But $\{0\} \subseteq \mathbb{Z}$ is not maximal since there are nontrivial ideals in $\mathbb{Z}$. $\square$

Problem 14. *Let B be a finitely generated K-algebra. Let A be a commutative ring with $K \subseteq A$. Let $f : A \to B$ be a K-algebra homomorphism (i.e. a ring homomorphism with $f(a) = a$ for all $a \in K$). Show that for all $M \in \mathrm{Max}(B)$, $f^{-1}(M) \in \mathrm{Max}(A)$.*

Proof. Let $M \in \mathrm{Max}(B)$. Then B/M is a field. Furthermore, since B is a finite K-algebra, B/M is also a finite K-algebra where we take generators $x_1 + M, \ldots, x_n + M$ if $B = K[x_1, \ldots, x_n]$. Now we have B/M is a finite algebraic extension of K by Nullstellensatz. Note that $f^{-1}(M)$ is an ideal in A and so $A/f^{-1}(M)$ is a ring containing K. We can view $A/f^{-1}(M) \subseteq B/M$ using $f(a + f^{-1}(M)) = f(a) + M$. Note that this gives an injection for if $a + f^{-1}(M) \neq 0$ then it's image in B/M is $f(a) + M$ which can't be 0 since $a \notin f^{-1}(M)$. Since $K \subseteq A/f^{-1}(M) \subseteq B/M$, we know $A/f^{-1}(M)$ is a field so $f^{-1}(M)$ is maximal. $\square$

Problem 15. *Let $f : A \to B$ be a ring homomorphism with B a finitely generated $\mathbb{Z}$-algebra. Show that for all $M \in \mathrm{Max}(B)$, $f^{-1}(M) \in \mathrm{Max}(A)$.*

Proof. Since B is a finite $\mathbb{Z}$-algebra, B is isomorphic quotient of $Z[x_1, \ldots, x_n]$. We can now apply Problem 7 to get B/M is a finite field, say with characteristic p. Thus B/M contains $\mathbb{Z}/p\mathbb{Z}$. Then we have a sequence $A \to B \to B/M$ where $p \cdot 1$ in A maps to 0 in B/M. Now, similar to Problem 14 we have an injection $A/f^{-1}(M) \to B/M$. Furthermore, since A contains an element of characteristic p, it must contain $\mathbb{Z}/p\mathbb{Z}$ as well and the same can be said about $A/f^{-1}(M)$. Now we can apply Problem 14. Pick a maximal ideal in B/M. This must be the 0 ideal. Then the preimage of 0 is 0 in $A/f^{-1}(M)$ and this is a maximal ideal. Thus $A/f^{-1}(M)$ is a field and $f^{-1}(M)$ is maximal. $\square$