Sheet 22: Integrals

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We want to define a semblance of area for functions on a closed interval. To do this we will create rectangles to approximate the area. Then we will make the approximation more precise. For the purposes of this sheet, a function f is a real function $f : [a; b] \to \mathbb{R}$.

Definition 1 *Let $a < b$. A* partition *of the interval $[a; b]$ is a finite collection of points in $[a, b]$, one of which is a and one of which is b.*

The points of a partition can be numbered $t_0, \ldots, t_n$ so that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

We will always assume that such a numbering has been assigned.

This partition defines the width of each rectangle. To define the height we use lower and upper sums.

Definition 2 *Suppose f is bounded on $[a; b]$ and $P = \{t_0, \ldots t_n\}$ is a partition of $[a; b]$. Let*

$$m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

$$M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}.$$

The lower sum *of f for P, denoted by $L(f, P)$, is defined as*

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum *of f for P, denoted by $U(f, P)$, is defined as*

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Theorem 3 *Let P_1 and P_2 be partitions of $[a; b]$, and let f be a function which is bounded on $[a; b]$. Then*

$$L(f, P_1) \leq U(f, P_2).$$

What does this imply about the set of lower sums and the set of upper sums for arbitrary partitions on $[a;b]$? We can define a specific property of functions on a closed interval using lower and upper sums.

Definition 4 *A function f which is bounded on $[a;b]$ is* integrable *on $[a;b]$ if*

$$\sup\{L(f,P) \mid P \text{ is a partition of } [a;b]\} = \inf\{U(f,P) \mid P \text{ is a partition of } [a;b]\}.$$

In this case, this common number is called the integral *of f on $[a;b]$ and is denoted by*

$$\int_a^b f = \int_a^b f(x)dx.$$

When $f(x) \geq 0$ for all $x \in [a;b]$, the integral is also called the area *of the region defined by f, $x = a$, $x = b$ and $f(x) = 0$.*

Exercise 5 *Show that for $c \in \mathbb{R}$, the function $f(x) = c$ is integrable on the interval $[a;b]$.*

Exercise 6 *Let f be defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

Show that f is not integrable on the closed interval $[a;b]$.

Notice that showing non-constant functions are integrable directly from the definition is difficult.

Theorem 7 *If f is bounded on $[a;b]$, then f is integrable on $[a;b]$ if and only if for every $\varepsilon > 0$ there exists a partition, P, of $[a;b]$ such that*

$$U(f,P) - L(f,P) < \varepsilon.$$

Exercise 8 *Show that $y = x$ is integrable on the closed interval $[a;b]$.*

Now we want to show some nice properties about integrable functions.

Theorem 9 *If f is continuous on $[a;b]$, then f is integrable on $[a;b]$.*

Hint: Remember that continuous functions on closed intervals are uniformly continuous. How does this help us pick a useful partition?

Theorem 10 *Let $a < c < b$ for $a, b, c \in \mathbb{R}$. Then f is integrable on $[a; b]$ if and only if f is integrable on $[a; c]$ and on $[c; b]$. Also, if f is integrable on $[a; b]$, then*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 11 *If f and g are integrable functions on $[a; b]$, then $f+g$ is integrable on $[a; b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Theorem 12 *If f is integrable on $[a; b]$, then for any number c, the function cf is integrable on $[a; b]$ and*

$$\int_a^b cf = c \int_a^b f.$$

Here is an interesting result.

Exercise 13 *If f is integrable on $[a; b]$, then so is $|f|$.*

Exercise 14 *If f is integrable on $[a; b]$, then*

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

The derivative does not display its full strength, nay display any strength at all, until amalgamated with the integral.

Lemma 15 *Suppose f is integrable on $[a; b]$ and that*

$$m \leq f(x) \leq M$$

for all $x \in [a; b]$. Then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Theorem 16 *If f is integrable on $[a; b]$ and F is defined on $[a; b]$ by*

$$F(x) = \int_a^x f,$$

then F is continuous on $[a;b]$.

Theorem 17 (The First Fundamental Theorem of Calculus) *Let f be integrable on $[a;b]$, and define F on $[a;b]$ by*

$$F(x) = \int_a^x f.$$

If f is continuous at $c \in [a;b]$, then F is differentiable at c, and

$$F'(c) = f(c).$$

(If $c = a$ or $c = b$, then $F'(c)$ is understood to mean the right- or left-hand derivative of F.)

Theorem 18 (The Second Fundamental Theorem of Calculus) *If f is integrable on $[a;b]$ and $f = g'$ for some function g, then*

$$\int_a^b f = g(b) - g(a).$$