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Homework 6

Problem 1. *(a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.*
(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ where X is a closed orientable surface of genus two and A and B are the circles shown.

Proof. (a) We know $H_2(S^2) \approx \mathbb{Z}$ and $H_n(S^2) = 0$ if $n \neq 2$ and $n > 0$. Additionally, for $n > 0$ the homology of a point is trivial and the homology of a space with multiple path components is the direct sum of the homology of each path component. Thus $H_n(A) = 0$ for $n > 0$. Now from the long exact sequence of the pair and the fact that ever third term is 0 when $n \neq 2$, we have $H_n(X, A) \approx H_{n-1}(A) = 0$ for $n > 3$. For $n = 3$ we have $0 = H_3(S^2) \to H_3(S^2, A) \to H_2(A) = 0$ so $H_3(S^2, A) = 0$. For $n = 2$ we have $0 = H_2(A) \to H_2(S^2) \approx \mathbb{Z} \to H_2(S^2, A) \to H_1(A) = 0$. So by exactness $\mathbb{Z}$ injects into $H_2(S^2, A)$ and its image is all of $H_2(S^2, A)$ so $H_2(S^2, A) \approx \mathbb{Z}$. When $n = 1$ we have the sequence

$$0 = H_1(A) \to H_1(S^2) = 0 \to H_1(S^2, A) \to H_0(A) \approx \mathbb{Z}^r \to H_0(S^2) \approx \mathbb{Z} \to H_0(S^2, A) \approx \mathbb{Z} \to 0.$$

Thus by exactness and since the last maps are surjective we get $H_1(S^2, A) \approx \mathbb{Z}^{r-1}$. We thus have $H_n(S^2, A) = 0$ for $n \geq 3$, $H_n(S^2, A) \approx \mathbb{Z}$ for $n = 2$, $H_n(S^2, A) \approx \mathbb{Z}^{r-1}$ for $n = 1$ and $H_n(S^2, A) \approx \mathbb{Z}$ for $n = 0$ where $r = |A|$.

The groups $H_n(S^1 \times S^1, A)$ are computed similarly. In this case $H_n(S^1 \times S^1) \approx \mathbb{Z}$ for $n = 0$ or $n = 2$, $H_n(S^1 \times S^1) \approx \mathbb{Z} \oplus \mathbb{Z}$ if $n = 1$ and $H_n(S^1 \times S^1) = 0$ otherwise. The exact argument as above gives $H_3(S^1 \times S^1, A) = 0$ and $H_2(S^1 \times S^1, A) \approx \mathbb{Z}$. Now we also have the sequence

$$0 = H_1(A) \to H_1(S^1 \times S^1) \approx \mathbb{Z} \oplus \mathbb{Z} \to H_1(S^1 \times S^1, A) \to H_0(A) \approx \mathbb{Z}^r \to 0.$$

Note that in this case, we can take generators $a_i \in \mathbb{Z}^r$ and through surjectivity, find preimages $b_i \in H_1(S^1 \times S^1, A)$. Given that the composition of this preimage map and the map $H_1(S^1 \times S^1, A) \to \mathbb{Z}^r$ is the identity, we must have that $H_1(S^1 \times S^1, A) \approx \mathbb{Z}^{r+2}$. Furthermore we also have the sequence

$$\mathbb{Z} \oplus \mathbb{Z} \approx H_1(S^1 \times S^1) \to H_1(S^1 \times S^1, A) \approx \mathbb{Z}^{r+2} \to H_0(A) \approx \mathbb{Z}^r \to H_0(S^1 \times S^1) \approx \mathbb{Z} \to H_0(S^1 \times S^1, A) \to 0.$$

and by exactness of this sequence we know $H_0(S^1 \times S^1, A) \approx \mathbb{Z}$. Thus $H_n(S^1 \times S^1, A)$ is $\mathbb{Z}$ if $n = 0$ or $n = 2$, is $\mathbb{Z}^{r+2}$ if $n = 1$ and trivial if $n > 2$.

(b) Since A is a closed set and is a deformation retract of an open band around it, we know (X, A) is a good pair. Thus $H_n(X, A) \approx \widetilde{H}_n(X/A)$. Note that $X/A \approx (S^1 \times S^1) \vee (S^1 \times S^1)$ and thus

$$H_n(X, A) \approx \widetilde{H}_n(X/A) \approx \widetilde{H}_n(S^1 \times S^1) \oplus \widetilde{H}_n(S^1 \times S^1) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0, n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now note that $H_n(X, B) \approx \widetilde{H}_n(X/B)$ since (X, B) is a good pair. But note that X/B is simply $S^1 \times S^1$ with one point identified. We can then take this point to be the set A so that $H_n(X/B) \approx H_n(X/A) \approx H_n(S^1 \times S^1, A)$ which we've computed in part (a). $\square$

Problem 2. *Show that $\widetilde{H}_n(X) \approx \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of n cones CX with their bases identified.*

Proof. Use the pair (CX, X) and note that $CX/X = SX$. Then we have the exact sequence

$$\widetilde{H}_{n+1}(CX) \to \widetilde{H}_{n+1}(CX, X) \approx \widetilde{H}_{n+1}(SX) \to \widetilde{H}_n X \to \widetilde{H}_n(CX).$$

But note that CX is contractible so it has trivial reduced homology group which means we get the exact sequence

$$0 \to \widetilde{H}_{n+1}(SX) \to \widetilde{H}_n(X) \to 0$$

giving the desired isomorphism. Let Y be the disjoint union of the n cones CX and note that the homology group of Y is the direct sum of the homology groups of each cone. In particular, since CX has trivial reduced homology group, Y has trivial homology group. Furthermore, Y/X is the space of n cones with their bases identified since X forms the base of each cone. Let Z be this space. Then we have a similar exact sequence

$$\widetilde{H}_{n+1}(Y) \to \widetilde{H}_{n+1}(Y, X) \approx \widetilde{H}_{n+1}(Z) \to \widetilde{H}_n X \to \widetilde{H}_n(Y).$$

And once again since $H_n(Y) = 0$ we have an isomorphism $\widetilde{H}_{n+1}(Z) \approx \widetilde{H}_n(X)$. $\square$

Problem 3. *Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \to C_{n+1}(SX)$ inducing isomorphisms $\widetilde{H}_n(X) \to \widetilde{H}_{n+1}(SX)$.*

Proof. Let $\sigma \in C_n(X)$ so we know $\sigma : \Delta^n \to X$. This gives a map $\sigma' : \Delta^{n+1} \to CX$ where we send each boundary face of Δ^{n+1} to $X \hookrightarrow CX$. That is, $\sigma' \mid_{\partial\Delta^{n+1}} = \sigma$. Thus we have a map $s : C_n(X) \to C_{n+1}(CX)$ where $s : \sigma \mapsto \sigma'$. If take another map t identical to s then $f = s + t$ is a map $C_n(X) \to C_{n+1}(SX)$ since s and t coincide on $\partial\Delta^{n+1}$. Note that $\partial f(\sigma) = \partial s(\sigma) + \partial t(\sigma) = s\partial(\sigma) + t\partial(\sigma) = f\partial(\sigma)$ because s and t are the identity on the boundary. Thus f is a chain map and so we get a homomorphism $f_* : \widetilde{H}_n(X) \to \widetilde{H}_{n+1}(SX)$. Note that from Problem 2 we already have a map $g : \widetilde{H}_{n+1}(SX) \to \widetilde{H}_n(X)$ where g is a bijection. Then we have $f_* g$ is a bijection as well since f_* is injective. $\square$

Problem 4. *Prove by induction on the dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:*
(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.
(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.
(c) If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof. (a) If $n = 0$ then X is a collection of points so the homology group $H_0(X) \approx \mathbb{Z}^r$ where r is the cardinality of the set of points. Furthermore, the homology groups of a point for $i > 0$ are trivial, so we have $H_i(0) = 0$ for $i > n = 0$.

Now suppose the statement is true for some $n > 0$ and consider a space X with dimension $n+1$. We have the exact sequence

$$H_i(X^n) \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) \to H_{i-1}(X^n).$$

If $i > n+1$ then $H_i(X^n) = 0$ by assumption and $H_i(X^{n+1}/X^n) \approx H_i(Y)$ where Y is a wedge sum of $(n+1)$-spheres which is homeomorphic to X^{n+1}/X^n. But the i^{th} homology of an $(n+1)$-sphere is trivial for $i > n+1$ so $H_i(X^{n+1}/X^n) \approx H_i(Y) = 0$. Thus we're left with the exact sequence

$$0 = H_i(X^n) \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) = 0$$

forcing $H_i(X^{n+1}) = 0$.

In the case $i = n+1$ the i^{th} homology of an $(n+1)$-sphere is $\mathbb{Z}$ so $H_i(X^{n+1}/X^n) \approx H_i(Y) \approx \mathbb{Z}^r$ where r is the number of $(n+1)$-spheres being wedged together. Then we're left with the exact sequence

$$0 = H_i(X^n) \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) \approx \mathbb{Z}^r \to H_{i-1}(X^n) = 0.$$

By exactness, $H_i(X^{n+1})$ injects into $\mathbb{Z}^r$ and its image is the entire space thus $H_i(X^{n+1})$ is free on r generators.

(b) If X has dimension $n = 0$ then $H_0(X) \approx \mathbb{Z}^r$ where r is the cardinality of the set of 0-cells and $H_i(X) = 0$ for $i > 0$ by the comments in part (a). Since there are no i-cells for $i > 0$, we see $H_i(X)$ is free with basis in bijective correspondence with the i-cells for $n = 0$.

Now suppose the statement is true for some $n > 0$ and note that we have the exact sequence

$$H_{i+1}(X^{n+1}/X^n) \to H_i(X^n) \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) \to H_{i-1}(X^n).$$

Suppose X has no $(i-1)$-cells or $(i+1)$-cells. Then this assumption is true for X^n as well so we have $H_i(X^n) \approx \mathbb{Z}^r$ where r is the number of i-cells. First suppose $i < n$ or $i > n+1$. Then by the same proof as in part (a) we know $H_{i+1}(X^{n+1}/X^n) \approx H_i(X^{n+1}/X^n) = 0$. This then gives the exact sequence

$$0 = H_{i+1}(X^{n+1}/X^n) \to H_i(X^n) \approx \mathbb{Z}^r \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) = 0$$

and by exactness, $H_i(X^{n+1}) \approx \mathbb{Z}^r$ where r is the number of i-cells. Now suppose $i = n$. Then since there are no $n+1$-cells we know $X^{n+1} = X^n$ which means $\mathbb{Z}^r \approx H_i(X^n) \approx H_i(X^{n+1})$. Finally suppose $i = n+1$. Then we once again have $H_{i+1}(X^{n+1}/X^n) = 0$ and $H_i(X^n) \approx \mathbb{Z}^r$ so we get the exact sequence

$$0 = H_{i+1}(X^{n+1}/X^n) \to H_i(X^n) \approx \mathbb{Z}^r \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) \to H_{i-1}(X^n).$$

But in this case there are no n-cells so $X^{n+1}/X^n = X^{n+1}$ and by exactness we're left with an isomorphism between $\mathbb{Z}^r$ and $H_i(X^{n+1})$.

(c) Suppose the dimension of X is n with $n = 0$. Then as we've seen in parts (a) and (b), $H_0(X) \approx \mathbb{Z}^k$ where k is the number of 0-cells in X. Furthermore, X has no i-cells for $i > 0$ and as we've also seen before, $H_i(X) = 0$ for $i > 0$. Thus the statement holds for $n = 0$.

Now suppose it's true for some n and let X be a space of dimension $n+1$ with k i-cells. We have the long exact sequence

$$H_{i+1}(X^{n+1}/X^n) \to H_i(X^n) \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) \to H_{i-1}(X^n).$$

If $i < n$ or $i > n+1$ then $H_{i+1}(X^{n+1}/X^n) = H_i(X^{n+1}/X^n) = 0$. Furthermore, $H_i(X^n) \approx \mathbb{Z}^j$ where $j < k$. Then we have the exact sequence

$$0 = H_{i+1}(X^{n+1}/X^n) \to H_i(X^n) \approx \mathbb{Z}^j \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) = 0$$

and by exactness $H_i(X^{n+1}) \approx \mathbb{Z}^j$ as in part (b). If $i = n$ then we have $H_{i+1}(X^{n+1}/X^n) \approx \mathbb{Z}^{j'}$ for some j' and we get an exact sequence

$$\mathbb{Z}^{j'} \approx H_{i+1}(X^{n+1}/X^n) \to H_i(X^n) \approx \mathbb{Z}^j \to H_i(X^{n+1}) \to H_i(X^{n+1}/X^n) = 0.$$

Thus $H_i(X^{n+1})$ is a quotient $\mathbb{Z}^j/\mathbb{Z}^m$ so it has fewer than k generators. If $i = n+1$ then we have $H_i(X^n) = 0$ by part (a) and $H_i(X^{n+1}/X^n) = \mathbb{Z}^{j''}$ with $j'' < k$ since there are k $(n+1)$-cells. Thus $H_i(X^{n+1})$ injects into $\mathbb{Z}^{j''}$ and the statement is true for all n. ☐

Problem 5. *Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.*
(a) Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.
(b) For the case of the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n \backslash \{0\})$, show that f is not a homotopy equivalence of pairs — there is no $g : (D^n, D^n\backslash\{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Proof. (a) We have the property of naturality in the long exact sequence of pairs which means the following diagram commutes.

$$\begin{tikzcd}
H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_{n-1}(X) \arrow[d, "f_*"] \\
H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r, "i_*"] & H_{n-1}(Y)
\end{tikzcd}$$

Since $f : X \to Y$ and $f : A \to B$ are homotopy equivalences we know f_* is an isomorphism in the first two and the last two vertical maps. By the Five-Lemma $f_* : (X, A) \to (Y, B)$ is also an isomorphism.

(b) Let $g : (D^n, D^n \backslash \{0\}) \to (D^n, S^{n-1})$ be a map of pairs. Since S^{n-1} is closed we know $g^{-1}(S^{n-1})$ is closed in D^n and also that $D^n \backslash \{0\} \subseteq g^{-1}(S^{n-1})$. Note also that $0 \in \overline{D^n \backslash \{0\}}$ so $g(0) \in S^{n-1}$. We then have a sequence $D^n \backslash \{0\} \hookrightarrow D^n \to S^{n-1}$ which splits g. This translates to induced maps on the homology groups $H_{n-1}(D^n \backslash \{0\}) \hookrightarrow H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$ with the composition being g_*. But since $H_{n-1}(D^n) = 0$ we see that $g_* = 0$ and since $H_{n-1}(S^{n-1}) \approx \mathbb{Z}$ we have that g_* is not an isomorphism on homology. Thus g cannot be a homotopy equivalence of pairs by part (a) and since g was arbitrary, neither is f. □

Problem 6. *Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.*

Proof. We know the homology groups for the torus and the homology groups for $n > 0$ for $S^1 \vee S^1 \vee S^2$ can be computed by taking direct sums of the corresponding homology groups. Since this space is path connected it has homology group $\mathbb{Z}$ for $n = 0$. Also $H_n(S^1) \approx \mathbb{Z}$ for $n = 1$ and is trivial for $n > 1$. The same is true for S^2 for $n = 2$. All this gives

$$H_n(S^1 \times S^1) \approx \begin{cases} \mathbb{Z} & n = 0, n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad H_n(S^1 \vee S^1 \vee S^2) \approx \begin{cases} \mathbb{Z} & n = 0, n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, the universal cover of $S^1 \times S^1$ is the plane $\mathbb{R}^2$ so the homology groups are trivial in all nonzero dimensions. The covering space of $S^1 \vee S^1 \vee S^2$ is an infinite graph in which every vertex has four edges and has a copy of S^2 attached to each edge. Let X be this covering space and view X as a CW complex with the obvious construction. Let X^1 and X^2 be the 1 and 2 skeleton structures of X. Then we have an exact sequence $H_2(X^1) \to H_2(X^2) \to H_2(X^2/X^1) \to H_1(X^1)$. Note that by Problem 4 we know $H_2(X^1) = 0$. In a similar fashion, $H_2(X^2, X^1) \approx H_2(X^2/X^1) \approx \mathbb{Z}^r$ where r is the number of 2-cells in X since X^2/X^1 is a wedge sum of 2-cells. In particular, $H_2(X^2, X^1)$ is not trivial. Furthermore, if we view X^1 as a Δ-complex with 1-simplexes as each edge and 0-simplexes at each vertex then every edge has two distinct vertices so $\ker \partial_1 = 0$. Thus $H_1(X^1) = 0$ and we have an isomorphism $H_2(X^2) \approx \mathbb{Z}^r$ where r is countably infinite. This means that the homology groups differ at $n = 2$. □

Problem 7. *Let M_g be the orientable surface of genus g, and let $A \subseteq M_g$ be a subspace homeomorphic to $M_1 \backslash e_2$, where e_2 is an open disk. (Draw a picture.) Consider the long exact sequence of (M_g, A) in homology. Compute $H_n(M_g)$ for all n by induction on g, starting with the case $M_1 = S^1 \times S^1$, as follows. Let $g \geq 2$ and assume that we already know $H_2(M_g, A) \cong H_2(M_g/A) = H_2(M_{g-1})$. First compute the boundary map*

$$\partial : H_2(M_g, A) \to H_1(A)$$

and then use this to compute $H_n(M_g)$ for all n using the long exact sequence of the pair.

Proof. We've already computed the homology groups for the case $g = 1$ and these have been listed in Problem 6. Assume we know the homology groups for some $g - 1$ and let $g \geq 2$. Now let $c \in C_2(M_g, A) = C_2(X)/C_2(A)$ be a 2-cycle. By surjectivity there is some element $b \in C_2(X)$ such that $j(b) = c$ where j is the quotient map. Note that $\partial_n(b) \in \ker j$ so $\partial_n b = i(a)$ for some $a \in C_1(A)$ where i is the inclusion map. This defines $\partial c = a$. But $a \in C_1(A)$ and since A is homeomorphic to $M_g \backslash e_2$ we must have $a = 0$. Thus ∂ is the 0-map from $H_2(M_g, A) \to H_1(A)$.

We have the long exact sequence

$$H_n(A) \to H_n(M_g) \to H_n(M_g, A) \to H_{n-1}(A).$$

For $n > 2$ we know $H_n(M_g) = 0$ by Problem 4 since M_g is a CW complex. For $n = 2$ using the induction hypothesis we're left with the sequence

$$H_2(A) \to H_2(M_g) \to H_2(M_g, A) \cong H_2(M_{g-1}) \cong \mathbb{Z} \to H_1(A).$$

We know $H_2(A) = 0$ since A is a 1-dimensional CW complex. Furthermore, the map $\mathbb{Z} \to H_1(A)$ is the 0-map by our above statements so $H_2(M_g) \cong \mathbb{Z}$. For $n = 1$ applying the induction hypothesis we have

$$H_2(M_g, A) \cong \mathbb{Z} \to H_1(A) \to H_1(M_g) \to H_1(M_g, A) \cong H_1(M_{g-1}) \cong \mathbb{Z}^{2(g-1)} \to H_0(A) \cong \mathbb{Z} \to H_0(M_g) \cong \mathbb{Z}.$$

Given that the first map is injective and the last map is surjective by exactness of the sequence we must have $H_1(M_g) \cong \mathbb{Z}^{2g}$. Finally, since M_g is path connected we have $H_0(M_g) \cong \mathbb{Z}$. $\square$