Kris Harper
MATH 16300
Miklós Abért
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Sheet 17: More About Metric Spaces

Theorem 1 *Let (X, d) be a metric space and let (a_n) be a sequence in X. Then $\lim_{n\to\infty} a_n = a$ if and only if $\lim_{n\to\infty} d(a_n, a) = 0$.*

Proof. Let $\lim_{n\to\infty} a_n = a$. Then for every open set $A \subseteq X$ with $a \in A$ there are finitely many n with $a_n \notin A$. But then for $r \in \mathbb{R}$, there are finitely many n with $a_n \notin B(a, r)$. Then there are finitely many n such that $d(a_n, a) < r$ which means there are finitely many n such that $d(a_n, a) \notin (-r, r)$. Thus, $\lim_{n\to\infty} d(a_n, a) = 0$.

Conversely, let $\lim_{n\to\infty} d(a_n, a) = 0$. Then for all $r \in \mathbb{R}$ there are finitely many n such that $d(a_n, a) \notin (-r, r)$ which means there are finitely many n such that $d(a_n, a) > r$. But then there are finitely many n such that $a_n \notin B(a, r)$. If we consider some open set $A \subseteq X$ such that $a \in A$, there there exists some ball $B(a, r) \subseteq A$. But since there are finitely many n with $a_n \notin B(a, r)$, there are only finitely n with $a_n \notin A$. Thus, $\lim_{n\to\infty} a_n = a$. $\square$

Definition 2 *Let $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$ denote the set of real n-tuples.*

Definition 3 *For $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, b_2, \ldots b_n) \in \mathbb{R}^n$ let*

$$d_0(\mathbf{a}, \mathbf{b}) = \max_{1\le i\le n} |a_i - b_i|,$$

$$d_1(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} |a_i - b_i|$$

and

$$d_2(\mathbf{a}, \mathbf{b}) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}.$$

Theorem 4 *The functions d_0, d_1 and d_2 are all metrics on $\mathbb{R}^n$.*

Proof. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. It's clear that $d_0(\mathbf{a}, \mathbf{b})$, $d_1(\mathbf{a}, \mathbf{b})$ and $d_2(\mathbf{a}, \mathbf{b})$ are all greater than or equal to 0. Let $d_0(\mathbf{a}, \mathbf{b}) = 0$. Then $\max_{1\le i\le n} |a_i - b_i| = 0$ and so $a_i = b_i$. Since the maximum positive difference between two coordinates is 0, all the distances must be 0 as well. Now let $\mathbf{a} = \mathbf{b}$. Then $a_i = b_i$ for $1 \le i \le n$. Thus $\max_{1\le i\le n} |a_i - b_i| = 0$ and $d_0(\mathbf{a}, \mathbf{b}) = 0$.

Let $d_1(\mathbf{a}, \mathbf{b}) = 0$. Then $\sum_{i=1}^{n} |a_i - b_i| = 0$. But since $|a_i - b_i| \ge 0$ for $1 \le i \le n$ we have $|a_i - b_i| = 0$ for $1 \le i \le n$. Thus $a_i = b_i$ and $\mathbf{a} = \mathbf{b}$. Now suppose that $\mathbf{a} = \mathbf{b}$. Then we have $a_i = b_i$ for $1 \le i \le n$ and so $|a_i - b_i| = 0$. But then $\sum_{i=1}^{n} |a_i - b_i| = 0$ and so $d_1(\mathbf{a}, \mathbf{b}) = 0$.

Let $d_2(\mathbf{a}, \mathbf{b}) = 0$. Then $\sqrt{\sum_{i=1}^{n} (a_i - b_i)^2} = 0$ which means $\sum_{i=1}^{n} (a_i - b_i)^2 = 0$. From here the proof follows similarly to that of $d_1(\mathbf{a}, \mathbf{b})$.

Since $|a - b| = |b - a|$ and $(a - b)^2 = (b - a)^2$ for all $a, b \in \mathbb{R}$, we have $d_i(\mathbf{a}, \mathbf{b}) = d_i(\mathbf{b}, \mathbf{a})$ for $0 \le i \le 2$. Finally, note that using the triangle inequality we have

$\max_{1\le i\le n}|a_i-b_i|+\max_{1\le i\le n}|b_i-c_i|\ge|a_i-b_i|+|b_i-c_i|$ for arbitrary $1\le i\le n$ which is in turn greater than $\max_{1\le i\le n}|a_i-c_i|$. Note also that by the triangle inequality we have $|a_i-b_i|+|b_i-c_i|\ge|a_i-c_i|$ for $1\le i\le n$. But then if we sum this inequality n times we have $\sum_{i=1}^n|a_i-b_i|+\sum_{i=1}^n|b_i-c_i|\ge\sum_{i=1}^n|a_i-c_i|$. Lastly note that

$$\sqrt{\sum_{i=1}^n(a_i-b_i)^2}+\sqrt{\sum_{i=1}^n(b_i-c_i)^2}\ge\sqrt{\sum_{i=1}^n((a_i-b_i)^2+(b_i-c_i)^2)}\ge\sqrt{\sum_{i=1}^n(a_i-c_i)^2}.$$

Thus all three distance functions satisfy the triangle inequality. Therefore all three are metrics. $\square$

Theorem 5 *For all $0\le i\le 2$, $0\le j\le 2$ and for all $\mathbf{x}\in\mathbb{R}^n$ and $r>0$ there exists $r'>0$ such that*

$$B_{d_i}(x,r')\subseteq B_{d_j}(x,r).$$

Proof. Let $\mathbf{x}\in\mathbb{R}^n$ and let $r>0$. Consider $B_{d_0}(\mathbf{x},r)$, let $r=r'$ and let $\mathbf{y}\in B_{d_1}(\mathbf{x},r')$. Then $\sum_{i=1}^n|x_i-y_i|<r'$ and so $d_0(\mathbf{x},\mathbf{y})=\max_{1\le i\le n}|x_i-y_i|<r'=r$. Thus $\mathbf{y}\in B_{d_0}(\mathbf{x},r)$ and $B_{d_1}(\mathbf{x},r')\subseteq B_{d_0}(\mathbf{x},r)$. Now let $r=r'$ again and let $\mathbf{y}\in B_{d_2}(\mathbf{x},r')$. Then

$$\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<r'$$

so $\max_{1\le i\le n}(x_i-y_i)^2<\sum_{i=1}^n(x_i-y_i)^2<r'^2$ and $d_0(\mathbf{x},\mathbf{y})=\max_{1\le i\le n}|x_i-y_i|<r'=r$. Thus $\mathbf{y}\in B_{d_0}(\mathbf{x},r)$ and $B_{d_2}(\mathbf{x},r')\subseteq B_{d_0}(\mathbf{x},r)$.

Next consider $B_{d_1}(\mathbf{x},r)$, let $r'=r/n$ and let $\mathbf{y}\in B_{d_0}(\mathbf{x},r')$. Then $\max_{1\le i\le n}|x_i-y_i|<r/n$ which means $d_1(\mathbf{x},\mathbf{y})=\sum_{i=1}^n|x_i-y_i|<r$ and $\mathbf{y}\in B_{d_1}(\mathbf{x},r)$. Thus $B_{d_0}(\mathbf{x},r')\subseteq B_{d_1}(\mathbf{x},r)$. Now let $r'=r/\sqrt{n}$ and let $\mathbf{y}\in B_{d_2}(\mathbf{x},r')$. Then

$$\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<\frac{r}{\sqrt{n}}$$

so $\sum_{i=1}^n(x_i-y_i)^2<r^2/n$ and $(x_i-y_i)^2<r^2/n^2$ for $1\le i\le n$. Thus $|x_i-y_i|<r/n$ for $1\le i\le n$ and so $d_1(\mathbf{x},\mathbf{y})=\sum_{i=1}^n|x_i-y_i|<r$. Thus $B_{d_2}(\mathbf{x},r')\subseteq B_{d_1}(\mathbf{x},r)$.

Finally, consider $B_{d_2}(\mathbf{x},r)$, let $r'=r/\sqrt{n}$ and let $\mathbf{y}\in B_{d_0}(\mathbf{x},r')$. Then $\max_{1\le i\le n}|x_i-y_i|<r/\sqrt{n}$ which means $\max_{1\le i\le n}(x_i-y_i)^2<r^2/n$ and

$$d_2(\mathbf{x},\mathbf{y})=\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<r.$$

Thus $\mathbf{y}\in B_{d_2}(\mathbf{x},r)$ and $B_{d_0}(\mathbf{x},r')\subseteq B_{d_2}(\mathbf{x},r)$. Now let $r'=r/n\sqrt{n}$ and let $\mathbf{y}\in B_{d_1}(\mathbf{x},r')$. Then $\sum_{i=1}^n|x_i-y_i|<r/n\sqrt{n}$ so $|x_i-y_i|<r/\sqrt{n}$ and $(x_i-y_i)^2<r^2/n$ for $1\le i\le n$. Then $\sum_{i=1}^n(x_i-y_i)^2<r^2$ and

$$d_2(\mathbf{x},\mathbf{y})=\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<r.$$

Thus $\mathbf{y}\in B_{d_2}(\mathbf{x},r)$ and $B_{d_1}(\mathbf{x},r')\subseteq B_{d_2}(\mathbf{x},r)$. $\square$

Corollary 6 *The metrics d_0, d_1 and d_2 generate the same topology on $\mathbb{R}^n$, namely, a subset $A\subseteq\mathbb{R}^n$ is open in $(\mathbb{R}^n,d_i)$ if it is open in $(\mathbb{R}^n,d_j)$ ($0\le i\le 2$, $0\le j\le 2$).*

Proof. Let $A \subseteq \mathbb{R}^n$ be an open set in $(\mathbb{R}^n, d_j)$. Then for all $a \in A$ there exists $r \in \mathbb{R}$ such that $B_{d_j}(a, r) \subseteq A$. But from Theorem 5 we know that there exists $r' \in \mathbb{R}$ such that $B_{d_i}(a, r') \subseteq B_{d_j}(a, r) \subseteq A$ (17.5). Thus A is open for $(\mathbb{R}, d_i)$. This is true for arbitrary $0 \leq i \leq 2$, $0 \leq j \leq 2$. $\square$

Definition 7 *Let (X, d) be a metric space. A sequence (a_n) on X has the Cauchy property if for all $\varepsilon > 0$ there exists N such that for all $n, m > N$ we have $d(a_n, a_m) < \varepsilon$.*

Definition 8 *A metric space (X, d) is complete if every Cauchy sequence on X is convergent.*

Theorem 9 *$\mathbb{R}^n$ is complete.*

Proof. Let $(\mathbf{a}_n)$ be a Cauchy sequence on $\mathbb{R}^d$ and let $\varepsilon' > 0$. Then there exists N such that for all $n, m > N$ we have

$$d_2(\mathbf{a}_n, \mathbf{a}_m) = \sqrt{\sum_{i=1}^{d} (a_{ni} - a_{mi})^2} < \varepsilon'$$

so $(a_{ni} - a_{mi})^2 \leq \sum_{i=1}^{d} (a_{ni} - a_{mi})^2 < \varepsilon'^2$ and $|a_{ni} - a_{mi}| < \varepsilon'$. Thus the ith coordinate of the terms of $(\mathbf{a}_n)$ forms a Cauchy sequence which converges to some b_i (14.5). Then let $\mathbf{b} = (b_1, b_2, \ldots, b_d)$, let $\varepsilon > 0$ and consider $\varepsilon/\sqrt{d}$. For all $i \leq d$ there exists some N_i such that for $n > N_i$ we have $|a_{ni} - b_i| < \varepsilon/\sqrt{d}$ by convergence (13.3). Let N be the largest of all such N_i so that for all $n > N$ we have $|a_{ni} - b_i| < \varepsilon/\sqrt{d}$. Then $(a_{ni} - b_i)^2 < \varepsilon^2/d$ and $\sum_{i=1}^{d} (a_{ni} - b_i)^2 < \varepsilon^2$. Then $d_2(\mathbf{a}_n, \mathbf{b}) < \varepsilon$ for all $n > N$ and $|d(\mathbf{a}_n, \mathbf{b})| < \varepsilon$ for all $n > N$. Thus $\lim_{n \to \infty} \mathbf{a}_n = \mathbf{b}$ because $\lim_{n \to \infty} d(\mathbf{a}_n, \mathbf{b}) = 0$ (13.3, 17.1). $\square$

Theorem 10 *Every compact metric space is complete.*

Proof. Let (X, d) be a compact metric space and suppose that (X, d) is not complete. Then there exists some Cauchy sequence $(a_n) \in X$ such that (a_n) does not converge. Therefore for all $x \in X$ there exists some ball $B(x, \varepsilon)$ such that there are infinitely many n with $a_n \notin B(x, \varepsilon)$. Let $\mathcal{A}$ be the set of all such balls and let $\mathcal{A}' = \{B(x, \varepsilon/2) \mid B(x, \varepsilon) \in \mathcal{A}\}$. Then $\mathcal{A}'$ is an open cover for X and X is compact so let $\mathcal{B}$ be a finite subcover for $\mathcal{A}'$. Let $B(x, \varepsilon/2) \in \mathcal{B}$. Note that there are infinitely many n such that $a_n \notin B(x, \varepsilon)$ so there are infinitely many n such that $a_n \notin B(x, \varepsilon/2)$. We have (a_n) is Cauchy so there exists N such that for all $n, m > N$ we have $d(a_n, a_m) < \varepsilon/2$. Suppose that there are infinitely many n with $a_n \in B(x, \varepsilon/2)$. Since there are infinitely many n with $a_n \in B(x, \varepsilon/2)$ and $a_n \notin B(x, \varepsilon/2)$, choose $n, m > N$ with $a_n \in B(x, \varepsilon/2)$ and $a_m \notin B(x, \varepsilon/2)$. But then $d(x, a_m) \leq d(x, a_n) + d(a_n, a_m) < \varepsilon$. Thus there are infinitely many n with $a_n \notin B(x, \varepsilon)$ which is a contradiction. Therefore there are finitely many n with $a_n \in B(x, \varepsilon/2)$. But this is true for all $B(x, \varepsilon/2) \in \mathcal{B}$ and there are finitely many elements of $\mathcal{B}$ which is an open cover for X. So we have finitely many n with $a_n \in X$ which is a contradiction. Therefore (X, d) is complete. $\square$

Theorem 11 *Let $(\mathbf{a}_n)$ be a bounded sequence in $\mathbb{R}^d$. Show that $(\mathbf{a}_n)$ has a convergent subsequence.*

Proof. Consider the sequence (a_{1n}) where a_{1n} is the 1st coordinate in the nth term of $(\mathbf{a}_n)$. Then we have (a_{1n}) is a bounded sequence so there exists some convergent subsequence (b_{1k}). Use induction on n. We have shown the base case for $n = 1$. Now assume that a bounded sequence $(\mathbf{a}_n) \in \mathbb{R}^d$ has a convergent subsequence for $d \in \mathbb{N}$. Consider a bounded sequence $(\mathbf{a}_n) \in \mathbb{R}^{d+1}$. By our Inductive Hypothesis there exists a convergent subsequence in $\mathbb{R}^d$ formed by the first d coordinates of terms in $(\mathbf{a}_n)$. Let the corresponding terms in $(\mathbf{a}_n)$ be the sequence $(\mathbf{b}_k = \mathbf{a}_{n_k})$ Form a subsequence $(\mathbf{c}_k = \mathbf{a}_{n_k})$ of $(\mathbf{a}_n)$ where the kth term has the coordinates of $\mathbf{b}_k$ as the first d coordinates and the $d + 1$th coordinate of $\mathbf{a}_{n_k}$ as the $d + 1$th coordinate. Now take the sequence in $\mathbb{R}$ where the kth term is the $d + 1$th coordinate of $\mathbf{c}_k$. Then this sequence is bounded so there exists a convergent subsequence $(e_i = c_{k_i(d+1)})$. Finally form a subsequence of $(\mathbf{a}_n)$ where the ith term is c_{k_i}. Now every coordinate in $(\mathbf{c}_{k_i})$ forms a convergent sequence in $\mathbb{R}$ so $(\mathbf{c}_{k+i})$ converges to some $\mathbf{f} \in \mathbb{R}^{d+1}$ using a similar proof as in Theorem 9. $\square$

Theorem 12 *Show that a set $A \subseteq \mathbb{R}^d$ is open if and only if for all $\mathbf{x} \in A$ there is a rational ball O such that $\mathbf{x} \in O$ and $O \subseteq A$.*

Proof. Suppose that for all $\mathbf{x} \in A$ there is a rational ball $B(\mathbf{a}, r) \subseteq A$ such that $\mathbf{x} \in B(\mathbf{a}, r)$. Then consider the ball $B(\mathbf{x}, r - d(\mathbf{a}, \mathbf{x}))$. For $\mathbf{y} \in B(\mathbf{x}, r - d(\mathbf{a}, \mathbf{x}))$ we have $d(\mathbf{x}, \mathbf{y}) < r - d(\mathbf{a}, \mathbf{x})$ which means $d(\mathbf{a}, \mathbf{y}) \leq d(\mathbf{a}, \mathbf{x}) + d(\mathbf{x}, \mathbf{y}) < r$ and so $\mathbf{y} \in B(\mathbf{a}, r)$. Thus $B(\mathbf{x}, r - d(\mathbf{a}, \mathbf{x})) \subseteq B(\mathbf{a}, r) \subseteq A$. Then for all $\mathbf{x} \in A$ there exists a ball $B(\mathbf{x}, r') \subseteq A$ so A is open.

Conversely let $A \subseteq \mathbb{R}^d$ be open. Let $\mathbf{x} \in A$. There exists a ball $B(\mathbf{x}, r) \subseteq A$ where r may be rational or not. If $r \notin \mathbb{Q}$ then consider some $r' \in \mathbb{Q}$ such that $0 < r' < r$ and then $B(\mathbf{x}, r') \subseteq B(\mathbf{x}, r) \subseteq A$ (9.12). We have $B(\mathbf{x}, r'/2) \subseteq B(\mathbf{x}, r') \subseteq A$. Let $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ where $y_i \in \mathbb{Q}$ and $0 < y_i < r'/(2\sqrt{d}) + x_i$ (9.12). Then $y_i - x_i < r'/(2\sqrt{d})$ and $|x_i - y_i| < r'/(2\sqrt{d})$. Also $(x_i - y_i)^2 < r'^2/(4d)$ so $\sum_{i=1}^{d}(x_i - y_i)^2 < r'^2/4$ and $d(\mathbf{x}, \mathbf{y}) < r'/2$. Finally consider $\mathbf{z} \in B(\mathbf{y}, r'/2)$. Then $d(\mathbf{y}, \mathbf{z}) < r'/2$. But also $d(\mathbf{x}, \mathbf{y}) < r'/2$ so we have $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) < r'/2 + r'/2 = r'$. Thus $B(\mathbf{y}, r'/2) \subseteq B(\mathbf{x}, r') \subseteq A$. Also $d(\mathbf{y}, \mathbf{x}) < r'/2 < r'$ so $\mathbf{x} \in B(\mathbf{y}, r'/2)$. Note that $r'/2 \in \mathbb{Q}$ and $\mathbf{y} \in \mathbb{Q}^d$. $\square$

Theorem 13 *Let C be a closed, bounded subset of $\mathbb{R}^d$ and let $\mathcal{A}$ be an open cover for C. Then $\mathcal{A}$ has a countable subcover.*

Proof. Let $\mathbf{x} \in C$. Then there exists $A \in \mathcal{A}$ such that $\mathbf{x} \in A$. We have A is open, so there exists some rational ball $O \subseteq A$ such that $\mathbf{x} \in O$. Let $\mathcal{B}$ be the set of all such rational balls for all $\mathbf{x} \in C$. Each of these balls has a center in $\mathbb{Q}^d$ which is countable, so there are countably many of them. Then let $\mathcal{C} \subseteq \mathcal{A}$ be set set of elements of $\mathcal{A}$ which have subsets in $\mathcal{B}$. Since every element of $\mathcal{B}$ is a subset of some element of $\mathcal{A}$, there are countable many elements of $\mathcal{C}$. But $\mathcal{C}$ covers C. $\square$

Theorem 14 *Closed bounded subsets of $\mathbb{R}^d$ are compact.*

Proof. Assume C is a closed bounded subset of $\mathbb{R}^d$ which is not compact. Let $\mathcal{A}$ be an open cover for C which does not have a finite subcover. Let $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\} \subseteq \mathcal{A}$ be a countably infinite subcover for C (17.13). Create a sequence $(\mathbf{a}_n) \in C$ such that $\mathbf{a}_1 \in B_1$ and for $n > 1$

$$\mathbf{a}_n \in C \backslash (B_1 \cup B_2 \cup \ldots B_{n-1}).$$

Then for all $j > i$, $\mathbf{a}_j \notin B_i$. Thus for all $B_i \in \mathcal{B}$, there are infinitely many n with $\mathbf{a}_n \notin B_i$. Note that $(\mathbf{a}_n)$ is bounded since C is bounded, so there exists a subsequence $(\mathbf{b}_n) \in C$ such that $\lim_{n\to\infty} \mathbf{b}_n = \mathbf{b}$. Note that C is closed, so if $\mathbf{b} \notin C$ then there exists some ball $B(\mathbf{b}, r) \subseteq \mathbb{R}^d \backslash C$ because $\mathbb{R}^d \backslash C$ is open. But $(\mathbf{b}_n) \in C$ so there are infinitely many n such that $\mathbf{b}_n \notin \mathbb{R}^d \backslash C$. Thus $\mathbf{b} \in C$. Then $\mathbf{b} \in B_i$ for some $B_i \in \mathcal{B}$. But there are infinitely many n such that $\mathbf{a}_n \notin B_i$ and so $\lim_{n\to\infty} \mathbf{b}_n \notin B_i$. Thus, $(\mathbf{b}_n)$ does not converge which is a contradiction. Therefore C is compact. $\square$