Kris Harper
MATH 26700
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Homework 1

arcsec x
sin x

Problem 1. *Let $V = \mathbb{C}^4$. Suppose that $\sigma : V \to V$ is the function defined by*

$$\sigma(z_1, z_2, z_3, z_4) = (z_3, z_4, z_1, z_2).$$

Show that σ is a $\mathbb{C}$-linear transformation. Choose a basis for V and determine the matrix of σ relative to it. Determine the characteristic and minimal polynomials of σ and conclude that there is a basis for V consisting of eigenvectors of σ.

Proof. Let $z \in \mathbb{C}$ and let $(u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in V$. Then

$$\sigma(z(u_1, u_2, u_3, u_4)) = \sigma(zu_1, zu_2, zu_3, zu_4) = (zu_3, zu_4, zu_1, zu_2) = z(u_3, u_4, u_1, u_2) = z\sigma(u_1, u_2, u_3, u_4)$$

and

$$\begin{aligned}\sigma((u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4)) &= \sigma(u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4)\\ &= (u_3 + v_3, u_4 + v_4, u_1 + v_1, u_2 + v_2)\\ &= (u_3, u_4, u_1, u_2) + (v_3, v_4, v_1, v_2)\\ &= \sigma(u_1, u_2, u_3, u_4) + \sigma(v_1, v_2, v_3, v_4).\end{aligned}$$

This shows that σ is $\mathbb{C}$-linear.

We pick the standard basis for V, $\{e_1, e_2, e_3, e_4\}$ where e_i has a 1 in the i^{th} place and 0s elsewhere. Then $\sigma(e_1) = (0, 0, 1, 0)$, $\sigma(e_2) = (0, 0, 0, 1)$, $\sigma(e_3) = (1, 0, 0, 0)$ and $\sigma(e_4) = (0, 1, 0, 0)$. We know $\sigma = (a_{ij})$ where $\sigma(e_j) = \sum_{i=1}^{4} a_{ij} e_i$. Using this definition with the previous calculations gives

$$\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of σ is given by

$$\det(\lambda I - \sigma) = \det \begin{pmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{pmatrix} = \lambda^4 - 2\lambda^2 + 1 = (\lambda - 1)^2(\lambda + 1)^2.$$

From this, we can easily find the minimal polynomial for σ as the irreducible polynomial of least degree which divides the characteristic polynomial, namely $(\lambda - 1)(\lambda + 1)$.

We now know that the eigenvalues for σ are ± 1. To find the eigenvectors we solve the equations $\sigma(v) = \pm v$. Taking the positive value first, $(v_3, v_4, v_1, v_2) = (v_1, v_2, v_3, v_4)$ so $v_3 = v_1$ and $v_4 = v_2$. This gives the two vectors $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ which span the eigenspace corresponding to 1. A similar calculation shows that $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$ span the second eigenspace. Since the sum of the dimensions of the eigenspaces is equal to $\dim(V)$, we know there exists a basis of V consisting of these eigenvectors. $\square$

Problem 2. *For the matrix*
$$A = \begin{pmatrix} 18 & 5 & 15 \\ -6 & 5 & -9 \\ -2 & -1 & 5 \end{pmatrix}$$
show that the characteristic polynomial is $\mathrm{char}_A(x) = (x-12)(x-8)$. *Find a basis for* $\mathbb{C}^3$ *consisting of* A. *Find the minimal polynomial for* A.

Proof. The characteristic polynomial of A is given by
$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 18 & -5 & -15 \\ 6 & \lambda - 5 & 9 \\ 2 & 1 & \lambda - 5 \end{pmatrix} = \lambda^3 - 28\lambda^2 + 256\lambda - 768 = (\lambda - 12)(\lambda - 8)^2.$$
This immediately gives that the minimal polynomial for A is $(\lambda - 12)(\lambda - 8)$.

We now have the two distinct eigenvalues 12 and 8. For the former, we compute
$$\begin{pmatrix} 18 & 5 & 15 \\ -6 & 5 & -9 \\ -2 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18x + 5y + 15z \\ -6x + 5y - 9z \\ -2x - y + 5z \end{pmatrix}.$$
So we're left with the equations $18x+5y+15z = 12x$, $-6x+5y-9z = 12y$ and $-2x-y+5z = 12z$. Solving, we get $y = -3x/5$ and $z = -x/5$. This gives the eigenvector $(5,-3,-1)$. For the eigenvalue 8, we have similar equations which reduce to $2x+y+3z = 0$. This gives the two eigenvectors $(-3,0,2)$ and $(-1,2,0)$. Thus our basis is $\{(5,-3,-1),(-3,0,2),(-1,2,0)\}$. $\square$

Problem 3. *Let* V *be an* S_n*-representation. Write out a proof that the obvious action of* S_n *on* $V \otimes V$ *is indeed a* G*-representation.*

Proof. Let $\rho : S_n \to GL(V)$ be the representation in question. Then we also have a function $\sigma : S_n \to GL(V \otimes V)$ defined as $\sigma(g)(v \otimes w) = \rho(g)(v) \otimes \rho(g)(w)$. If σ is to be a representation we need to show it's a homomorphism. Let $g, h \in S_n$. Then
$$\sigma(gh)(v \otimes w) = \rho(gh)(v) \otimes \rho(gh)(w) = \rho(g)\rho(h)(v) \otimes \rho(g)\rho(h)(w) = \sigma(g)(\rho(h)(v) \otimes \rho(h)(w)) = \sigma(g)\sigma(h)(v \otimes w).$$
Thus $\sigma : S_n \to V \otimes V$ is a homomorphism and thus a representation of S_n. $\square$

Problem 4. *(a) Let* V *and* W *be finite-dimensional vector spaces. Prove that there is an isomorphism of vector spaces* $W \otimes V^* \to \mathrm{Hom}(V, W)$.
(b) Now suppose that V *and* W *are* G*-representations of some group* G. *Prove that the isomorphism above is an isomorphism of* G*-representations.*

Proof. (a) We have a homomorphism $W \times V^* \to \mathrm{Hom}(V, W)$ given by $(w, T) \mapsto (v \mapsto T(v)w)$. This map is bilinear because the linearity of T. By the universal property, this gives a unique homomorphism $\varphi : W \otimes V^* \to \mathrm{Hom}(V, W)$ given by $\varphi : w \otimes T \mapsto (v \mapsto T(v)w)$. Suppose $\varphi(w \otimes T) = 0$, i.e., $\varphi(w \otimes T)$ is the map which takes v to 0 for all vectors $v \in V$. This will certainly happen if $w = 0$, so suppose otherwise. Then $T(v)w = 0$ for nonzero w and all v, thus, T is the 0 map. Therefore our original element is either $0 \otimes T = 0$ or $w \otimes 0 = 0$, showing that φ is injective. Since $\dim(W \otimes V^*) = \dim(\mathrm{Hom}(V, W))$ we see that φ must be an isomorphism.

(b) Suppose that $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ are the representations in question. Then $\rho^* : g \mapsto^t \rho(g^{-1})$ is a representation of V^*. From Problem 3 we now know $\tau : G \to GL(W \otimes V^*)$ given by $\tau(g)(w \otimes T) = \sigma(g)(w) \otimes \rho^*(g)(T)$ is a representation. We wish to show given $g \in G$ and $w \otimes T \in W \otimes V^*$, we have $\varphi(\tau(g)(w \otimes T)) = g(\varphi(w \otimes T))$. Note

$$\begin{aligned}
\varphi(\tau(g)(w \otimes T)) &= \varphi(\sigma(g)w \otimes \rho^*(g)(T)) \\
&= v \mapsto \rho^*(g)(T)(v)\sigma(g)(w) \\
&= v \mapsto {}^t\rho(g^{-1})(T)(v)(\sigma(g)(w)) \\
&= v \mapsto \sigma(g)(T(\rho(g^{-1})(v))w) \\
&= g(v \mapsto T(v)w) \\
&= g(\varphi(w \otimes T)).
\end{aligned}$$

□

Problem 5. *Verify that with this definition of ρ^*, the above relation is satisfied.*

Proof. Note that given a map between vector spaces such as $\rho(g)$ we can form its transpose $\rho*(g)(\varphi)$ as $\varphi \circ \rho$ for each φ in the dual space. In matrix notation, this is literally the transpose of the matrix representation of ρ. We then have

$$\begin{aligned}
\langle \rho^*(g)(v^*), \rho(g)(v)\rangle &= (\rho^*(g)(v^*))(\rho(g)(v)) \\
&= ({}^t\rho(g^{-1})(v^*))(\rho(g)(v)) \\
&= v^*(\rho(g^{-1}))(\rho(g)(v)) \\
&= v^*(v) \\
&= \langle v^*, v\rangle.
\end{aligned}$$

□

Problem 6. *Verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace of $\mathrm{Hom}(V,W)^G$ of elements of $\mathrm{Hom}(V,W)$ fixed under the action of G. This subspace is often denoted $\mathrm{Hom}_G(V,W)$.*

Proof. Let φ be a G-linear map from V to W. Then for $v \in V$ and $g \in G$ we have

$$\varphi(v) = gg^{-1}\varphi(v) = g\varphi(g^{-1}v) = (g\varphi)(v).$$

Thus φ is fixed under the action of G. Now suppose $\varphi \in \mathrm{Hom}(V,W)^G$. Then for $g \in G$ and $v \in V$ we have

$$\varphi(v) = (g^{-1}\varphi)(v) = g^{-1}\varphi(gv).$$

Acting with g on both sides shows φ is G-linear. □

Problem 7. *Use this approach to find the decomposition of the representations $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^3 V$.*

Proof. Let $\alpha = (\omega, 1, \omega^2)$ and $\beta = (1, \omega, \omega^2)$ where $\omega = e^{2\pi i/3}$. Note that α and β form a basis for V. Then a basis for $\mathrm{Sym}^2 V$ is $\{\alpha^2, \beta^2, \alpha\beta\}$. Let $\tau = (1\ 2\ 3)$ and $\sigma = (1\ 2)$ so that $S_3 = \langle \tau, \sigma \rangle$. Note that $\tau\alpha = \omega\alpha$ and $\tau\beta = \omega^2\beta$. Thus $\tau\alpha^2 = \omega^2\alpha^2$, $\tau\beta^2 = \omega\beta^2$ and $\tau\alpha\beta = \alpha\beta$ and these basis elements are eigenvectors for τ with eigenvalues ω^2, ω and 1 respectively. Note also that $\sigma\alpha^2 = \beta^2$, $\sigma\beta^2 = \alpha^2$ and $\sigma\alpha\beta = \alpha\beta$. Thus $\alpha\beta$ spans a subrepresentation isomorphic to the trivial representation, and α^2 and β^2 form a 2-dimensional invariant subspace which then must be isomorphic to V. Therefore $\mathrm{Sym}^2 V \cong U \oplus V$ where U is the trivial representation.

Now consider $\mathrm{Sym}^3 V$. This has basis $\{\alpha^3, \beta^3, \alpha^2\beta, \alpha\beta^2\}$. Note that $\tau\alpha^3 = \alpha^3$, $\tau\beta^3 = \beta^3$, $\tau\alpha^2\beta = \omega\alpha^2\beta$ and $\tau\alpha\beta^2 = \omega^2\alpha\beta^2$. These vectors are thus eigenvectors of τ with eigenvalues 1, 1, ω and ω^2 respectively. Note also that $\sigma\alpha^3 = \beta^3$, $\sigma\beta^3 = \alpha^3$, $\sigma\alpha^2\beta = \alpha\beta^2$ and $\sigma\alpha\beta^2 = \alpha^2\beta$. Thus the two sets of vectors $\{\alpha^3, \beta^3\}$ and $\{\alpha^2\beta, \alpha\beta^2\}$ each span a two dimensional subspace which is S_3-invariant. This subspace must be isomorphic to V, so we have $\mathrm{Sym}^3 V \cong V \oplus V$. □

Problem 8. *Consider the representation of S_n on $\mathbb{R}^n$ given by permuting coordinates. Prove that the subspace $W := \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\}$ is S_n-invariant, thus giving an $(n-1)$-dimensional representation of S_n on W. Prove that this representation is irreducible.*

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an element of W and $\sigma \in S_n$. Then $\sigma(\mathbf{x}) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. But note that $x_1 + \cdots + x_n = 0 = x_{\sigma(1)} + \cdots + x_{\sigma(n)}$, since we've just permuted the terms in the sum. Therefore $\sigma(\mathbf{x}) \in W$ and W is S_n-invariant.

Suppose now that W has some nontrivial subrepresentation U. Note that each nonzero vector in U must have a positive and a negative coordinate, since the sum of all the coordinates is 0. Let $\mathbf{x} = (x_1, \ldots x_n)$ be such a vector. Since U is G-invariant we can permute the coordinates of $\mathbf{x}$ and be assured the resulting vector is still in U. Choose an element of G which permutes the x_i so that the first and second coordinates of $\mathbf{x}$ are positive and negative respectively.

Call this new vector $\mathbf{a}$ and let $\mathbf{b}$ be the resulting vector after transposing the first two coordinates of $\mathbf{a}$. Since $\mathbf{a}$ and $\mathbf{b}$ are both in U, so is their difference $\mathbf{c} = \mathbf{a} - \mathbf{b}$. Note that $\mathbf{c} = (c_1, -c_1, 0, \ldots, 0) = c_1(1, -1, 0, \ldots, 0)$. Let $\mathbf{e_1} = (1, -1, 0, \ldots, 0)$. Then we can permute the coordinates of $\mathbf{e_1}$ to get the $(n-1)$ vectors $\mathbf{e_i} = (1, 0, \ldots, 0, -1, 0, \ldots, 0)$ which have a -1 in the $(i+1)^{\text{st}}$ coordinate. But it's clear that these $\mathbf{e_i}$ are linearly independent so they form a basis for the $(n-1)$-dimensional space W. Hence any nontrivial subspace of W is equal to W and W is thus irreducible. ☐

Problem 9. *Every irreducible complex representation of a finite abelian group is one-dimensional. Give an example to show that this is false for real representations.*

Proof. Consider the map $\mathbb{Z}/4\mathbb{Z} \to GL(2, \mathbb{R})$ which takes a generator g to the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It's easily verified that this matrix has order 4 and so this is indeed a representation. Suppose a one dimensional subspace is fixed by the action of g. Then for some vector (x, y) we would have $(x, y) = g(x, y) = \lambda(y, -x)$ for some nonzero λ. Then $x = \lambda y$, $y = -\lambda x$ and $x = -\lambda^2 x$. Since $\lambda \neq 0$, we must have $x = y = 0$. Hence, no one dimensional subspace of $\mathbb{R}$ is fixed under G and this representation is irreducible.

Alternatively, we could note that this matrix rotates the plane and so clearly only the zero vector is fixed under this action. ☐

Problem 10. *(a) Prove that S_n has no irreducible (say real) representations of dimension m where $2 \leq m \leq n-2$.*
(b) Classify all 1-dimensional and $(n-1)$-dimensional representations of S_n.