Sheet 10: Continuous Functions

Definition 1 *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for all open subsets $O \subseteq \mathbb{R}$ the preimage $f^{-1}(O)$ is open.*

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist $a, b \in \mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in \mathbb{R}$ such that $f(c) = 0$.*

Proof. Assume to the contrary that there exists no such point c. Then consider the sets $(-\infty; 0)$ and $(0; +\infty)$. We know these sets are open. If we take the preimages of each of these and name them we have $A = f^{-1}((-\infty; 0)) = \{x \in \mathbb{R} \mid f(x) < 0\}$ and $B = f^{-1}((0; +\infty)) = \{x \in \mathbb{R} \mid f(x) > 0\}$. Note that by definition A and B are disjoint. Additionally $\mathbb{R}\backslash(A \cup B) = \{x \in \mathbb{R} \mid f(x) = 0\}$, but we assumed that this set was empty. Thus $A \cup B = \mathbb{R}$. We have $f(a) < 0$ and so $a \in A$ and $f(b) > 0$ and so $b \in B$ so neither A or B is empty. But then since A and B are disjoint and union to $\mathbb{R}$ they are complements of each other. So then B is open but A is open and so $\mathbb{R}\backslash A = B$ is closed. Since $B \neq \mathbb{R}$ and $B \neq \emptyset$ this is contradiction of Axiom 4. $\square$

Theorem 3 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $a, b \in \mathbb{R}$ such that $a < b$. Let us define $g : \mathbb{R} \to \mathbb{R}$ as follows*

$$g(x) = \begin{cases} f(a) & \text{if } x \leq a \\ f(x) & \text{if } a < x < b \\ f(b) & \text{if } x \geq b \end{cases}$$

Then g is continuous.

Proof. Let $O \subseteq \mathbb{R}$ be an open set and consider $g^{-1}(O)$. If $g^{-1}(O)$ is empty, then it is open so assume that there exists some $x \in g^{-1}(O)$. If $x < a$ then we have $f(a) \in O$ and so $(-\infty; a) \subseteq g^{-1}(O)$. Thus there exists some $y \in \mathbb{R}$ such that $y < x$ and so $x \in (y; a)$ and $(y; a) \subseteq g^{-1}(O)$. A similar argument holds if $x > b$. If $x \in (a; b)$ then for $f(x) \in O$ there exists some region $R \subseteq O$ containing $f(x)$ by the open condition. But then $f^{-1}(R)$ is open since R is open, f is continuous and because of how g is defined, $g^{-1}(R)$ is open as well. If $x = a$ then $f(a) \in O$ and so $(-\infty; a) \subseteq g^{-1}(O)$. We know that $f^{-1}(O)$ is open so there exists some region $(p; q)$ containing a such that $(p; q) \subseteq f^{-1}(O)$. But then consider $(a; q) \subseteq f^{-1}(O)$. For all $y \in (a; q)$ we have $f(y) \in O$. But $y > a$ so $g(y) \in O$ as well. Thus $(a; q) \subseteq (p; q) \subseteq g^{-1}(O)$. A similar argument holds for when $x = b$. In all cases there exists a region R with $x \in R$ such that $R \subseteq g^{-1}(O)$ so $g^{-1}(O)$ is open by the open condition. $\square$

Theorem 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist $a, b \in \mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in (a; b)$ such that $f(c) = 0$.*

Proof. Define a new function $g : \mathbb{R} \to \mathbb{R}$ as in Theorem 3. Then g is continuous and so we know from Theorem 2 that there exists $c \in \mathbb{R}$ such that $g(c) = 0$. But we see that $c > a$ and $c < b$ because otherwise $g(c) \neq 0$. Thus there exists $c \in (a; b)$ such that $g(c) = 0$. But then $f(c) = 0$ as well. $\square$

Theorem 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $C \subseteq \mathbb{R}$ be a compact set. Show that the image $f(C)$ is compact.*

Proof. Let $\mathcal{A}$ be an open cover for $f(C)$. Then for all $x \in C$ we have $f(x) \in f(C)$ and so for all $x \in C$ there exists an open set $O \in \mathcal{A}$ such that $f(x) \in O$. But then for all $x \in C$, $x \in f^{-1}(O)$ for some $O \in \mathcal{A}$. Then $C \subseteq \bigcup_{O \in \mathcal{A}} f^{-1}(O)$ and since f is continuous $\{f^{-1}(O) \mid O \in \mathcal{A}\}$ covers C. But C is compact and so there exists a finite subcover, $\{f^{-1}(O_1), f^{-1}(O_2), \ldots, f^{-1}(O_k)\}$, which covers C. So for all $x \in C$ there exists some $O_i \in \mathcal{A}$ such that $x \in f^{-1}(O_i)$. But then $f(x) \in O_i$ and since $f(C) = \{y \in \mathbb{R} \mid x \in C, f(x) = y\}$, we have for all $y \in f(C)$, $y \in O_i$. Since every $O_i \in \mathcal{A}$ we have found a finite subcover for $\mathcal{A}$ which covers $f(C)$. Thus $f(C)$ is compact. $\square$

Theorem 6 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then for all $a < b$ the set $f([a;b])$ is bounded.*

Proof. For all $a < b$ we have $[a;b]$ is compact. By Theorem 5 we know that $f([a;b])$ is compact as well and we know that compact sets are bounded. $\square$

Lemma 7 *Let $C \subseteq \mathbb{R}$ be a nonempty compact set. Then $\sup C \in C$.*

Proof. Suppose that $\sup C \notin C$. Then by Theorem 6.8 we know that if $\sup C \notin C$ then it's a limit point of C. But C is compact and so it's closed. Thus C contains all its limit points so this is a contradiction. $\square$

Theorem 8 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $a < b$. Then there exists $c \in [a;b]$ such that for all $x \in [a;b]$ we have $f(a) \leq f(c)$.*

Proof. From Theorem 6 we know $f([a;b])$ is bounded and we know it's nonempty because $f(a) \in f([a;b])$ and so $\sup f([a;b])$ exists. Let $f(c) = \sup f([a;b])$. Lemma 7 tells us that $f(c) \in f([a;b])$ and so there exists some $d \in [a;b]$ such that $f(d) = f(c)$. $\square$

Theorem 9 *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all regions $A \subseteq \mathbb{R}$, the preimage $f^{-1}(A)$ is open.*

Proof. Let f be continuous. Then we have $A \subseteq \mathbb{R}$ is a region which is open by definition. But then $f^{-1}(A)$ is open by definition. Conversely, suppose that for all regions $A \subseteq \mathbb{R}$ we have $f^{-1}(A)$ is open. Consider some open set $O \subseteq \mathbb{R}$. By the open condition O is a union of regions and the preimage of each of these regions is open. But the preimage of a union of sets is the union of the preimages of each of those sets. To show this let $x \in f^{-1}(O)$. Then $f(x) \in O$ and so $f(x)$ is in some region which is a subset of O. But then x must be in the preimage of that region and so x is in the union of the preimages of all the regions which union to O. Since the preimage of each region is open, and the union of open sets is open, we have $f^{-1}(O)$ is open. Thus f is continuous. $\square$

Theorem 10 *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all $a \in \mathbb{R}$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $(a - \delta; a + \delta) \subseteq f^{-1}((f(a) - \varepsilon; f(a) + \varepsilon))$.*

Proof. Suppose that f is continuous and let $a \in \mathbb{R}$. Consider the region $(f(a) - \varepsilon; f(a) + \varepsilon)$ for some $\varepsilon > 0$. We know that $a \in f^{-1}((f(a) - \varepsilon; f(a) + \varepsilon))$ and we know that this preimage is open. Thus there exists some region $(a - m; a + n) \subseteq (f(a) - \varepsilon; f(a) + \varepsilon)$. Now let $\delta = \min(m, n)$ so that we have $(a - \delta; a + \delta) \subseteq (a - m; a + n) \subseteq (f(a) - \varepsilon; f(a) + \varepsilon)$. To prove the converse consider an open set $O \subseteq \mathbb{R}$. We have $f^{-1}(O)$ may be empty, but $\emptyset$ is open and so let $a \in f^{-1}(O)$. Then $f(x) \in O$ and so by the open condition there exists some region $(f(a) - \varepsilon; f(a) + \varepsilon) \subseteq O$ for $\varepsilon > 0$. Then there exists $\delta > 0$ such that $(a - \delta; a + \delta) \subseteq (f(a) - \varepsilon; f(a) + \varepsilon) \subseteq f^{-1}(O)$. Thus we have for all $a \in f^{-1}(O)$ there exists some region containing a which is a subset of $f^{-1}(O)$ and so $f^{-1}(O)$ is open. $\square$

Theorem 11 *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all $a \in \mathbb{R}$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|a - x| < \delta$ we have $|f(a) - f(x)| < \varepsilon$.*

Proof. Assume that f is continuous. From Theorem 10 we know that for all $a \in \mathbb{R}$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $(a - \delta; a + \delta) \subseteq (f(a) - \varepsilon; f(a) + \varepsilon)$. Consider $x \in (a - \delta; a + \delta)$. Then $-\delta < x - a < \delta$ and so $|a - x| < \delta$. But then $x \in f^{-1}((f(a) - \varepsilon; f(a) + \varepsilon))$ and so $f(x) \in (f(a) - \varepsilon; f(a) + \varepsilon)$. But then $|f(a) - f(x)| < \varepsilon$. To show the converse consider $x \in \mathbb{R}$ and $\epsilon > 0$ such that $|a - x| < \delta$ for $\delta > 0$. Then $x \in (a - \delta; a + \delta)$. But we also know that $|f(a) - f(x)| < \epsilon$ and so $f(x) \in (f(a) - \varepsilon; f(a) + \varepsilon)$. But then $x \in f^{-1}((f(a) - \varepsilon; f(a) + \varepsilon))$. Thus by Theorem 10, f must be continuous. $\square$

Definition 12 (f is Continuous at a *Let $a \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|a - x| < \delta$ we have $|f(a) - f(x)| < \varepsilon$.*