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MATH 27700
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Homework 6

Problem 1. *(a) Prove part (a) of Enderton's Homomorphism Theorem, page* 96.
(b) Give an example of two $\mathcal{L}$-structures M, N, a homomorphism $f : M \to N$ and a formula $\varphi(x_1, \ldots, x_k)$ such that for some $a_1, \ldots, a_k \in |M|$, $M \models \varphi(a_1, \ldots, a_k)$ but $N \models \neg\varphi(a_1 \ldots, a_k)$.

Proof. (a) Let $h : M \to N$ be a homomorphism and let s map the set of variables into $|M|$. If t is a constant symbol, then $h(\overline{s}(t)) = h(t^M) = t^N = \overline{h \circ s}(t)$. Suppose that the statement is true for a term built by applying n or fewer function symbols and let t be a term built by applying $n + 1$ function symbols. Then $t = f(t_1, \ldots, t_k)$ and

$$\begin{aligned} h(\overline{s}(t)) &= h(\overline{s}(f^M(t_1, \ldots, t_k))) \\ &= h(f^M(\overline{s}(t_1), \ldots, \overline{s}(t_k))) \\ &= f^N(h(\overline{s}(t_1)), \ldots, h(\overline{s}((t_k))) \\ &= f^N(\overline{h \circ s}(t_1), \ldots, \overline{h \circ s}(t_k)) \\ &= \overline{h \circ s}(t). \end{aligned}$$

(b) Let $M = (\mathbb{N}, <)$ and $N = (\mathbb{Q}, <)$. Let $h : M \to N$ be the identity map. Let $\varphi = \exists x \forall y((x < y) \vee (x = y))$. Then $M \models \varphi$ witnessed by $x = 0$ but $N \models \neg\varphi$ since there is no least element of $\mathbb{Q}$. □

Problem 2. *Let M, N be $\mathcal{L}$ structures. Say that M is an elementary submodel of N (written $M \preceq N$) if $|M| \subseteq |N|$ and 1(b) is not an issue, i.e. for every $k < \omega$,* every *$\mathcal{L}$-formula φ in k free variables, and $a_1, \ldots, a_k \in |M|$, $M \models \varphi(a_1, \ldots, a_k)$ iff $N \models \varphi(a_1, \ldots a_k)$.*
(a) Show that $M \preceq N$ iff for every $\mathcal{L}$-formula $\varphi(x, y_1, \ldots, y_k)$ and every $a_1, \ldots, a_k \in |M|$, if $N \models \exists x \varphi(x, a_1, \ldots, a_k)$ then $N \models \varphi(c, a_1, \ldots a_k)$ for some $c \in |M|$.
(b) Let $\langle M_i \mid i < \omega \rangle$ be an elementary chain, i.e. $i < j \implies M_i \preceq M_j$. Let $M = \bigcup_{i<\omega} M_i$. Show that for each $i < \omega$, $M_i \preceq M$.
(c) Give an example to show that ($M \subseteq N$ and $M \equiv N$) doesn't imply $M \preceq N$.

Proof. (a) Suppose that $M \preceq N$. Then $N \models \exists x \varphi(x, a_1, \ldots, a_k)$ if and only if $M \models \exists x \varphi(x, a_1, \ldots, a_k)$. But if that's true then there must be some $c \in |M|$ which witnesses it, so that $M \models \varphi(c, a_1, \ldots, a_k)$. But then $N \models \varphi(c, a_1, \ldots, a_k)$. Now suppose the hypothesis for the converse. If φ is a formula in k free variables and $M \models \varphi$, then certainly $N \models \varphi$ since M is a submodel of N. Now if $N \models \varphi$ then $N \models \exists x \varphi(x, a_1, \ldots, a_k)$ since we can always add a tautology using x to φ. But then by hypothesis, $N \models \varphi(c, a_1, \ldots, a_k)$ for some $c \in |M|$ and therefore $M \models \varphi(c, a_1, \ldots, a_k)$ which means $M \models \varphi(a_1, \ldots, a_k)$.

(b) Clearly $|M_i| \subseteq |M|$. If φ is a formula in k free variables and $M_i \models \varphi(a_1, \ldots, a_k)$ but $M \models \neg\varphi(a_1, \ldots, a_k)$, then there must exist j such that $M_j \models \neg\varphi(a_1, \ldots, a_k)$. But then $M_i \not\preceq M_j$ which is a contradiction. Conversely, suppose that $M \models \varphi(a_1, \ldots, a_k)$ for $a_1, \ldots, a_k \in M_i$. Then it must be the case that $M_i \models \varphi(a_1, \ldots, a_k)$ since M_i is a submodel of M.

(c) Let $M = (\mathbb{Q}, 1, +, \cdot)$ and $N = (\mathbb{R}, 1+, \cdot)$. Then let $\varphi = \neg(x \cdot x) = (1 + 1)$. □

Problem 3. *Suppose Γ is a set of $\mathcal{L}$-sentences, c is a constant symbol which does not occur in $\mathcal{L}$, and $\varphi = \varphi(x)$ is an $\mathcal{L}$-formula in one free variable. Then*

$$\Gamma \cup \{\exists x \varphi(x) \to \varphi(c)\}$$

is consistent.

Proof. Suppose $\Gamma' = \Gamma \cup \{\exists x\varphi(x) \to \varphi(c)\}$ is not consistent. Then we can derive any formula from Γ'. In particular $\Gamma' \vdash \varphi(c)$. From the generalization of constants, we know also that $\Gamma' \vdash \forall x\varphi(x)$. But then we have $\Gamma' \vdash \forall x\varphi(x) \to \varphi(c)$. This is a contradiction. $\square$

Problem 4. *A binary relation $\leq$ on a set P is a* partial ordering *if it is irreflexive and transitive. If $Y \subseteq P$ is any subset, $c \in P$ is an* upper bound *for Y if $y \leq c$ for every $y \in Y$. Zorn's lemma states that if $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element. A filter $\mathcal{F}$ is* principle *if there is $X \in \mathcal{F}$ such that for all $Z \in F$, $X \subseteq Z$.*
Show using Zorn's lemma that every filter on an infinite set I which does not contain any finite subsets of I can be extended to a nonprincipal ultrafilter.

Proof. Partially order the filters on I by containment. For a filter $\mathcal{F}$, let $\overline{\mathcal{F}}$ be the set of filters bigger than or equal to $\mathcal{F}$. Let C be a chain of filters in $\overline{\mathcal{F}}$ and let $U_C = \bigcup_{\mathcal{G} \in C} \mathcal{G}$. We know that $\mathcal{F} \subseteq U$, so $U \neq \emptyset$, If $G \in U$, then G is in some filter in C, which means that every superset of G is in U. If $A, B \in U$, then $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for filters $\mathcal{A}$ and $\mathcal{B}$. We can assume without loss of generality that $\mathcal{A} < \mathcal{B}$ so that $A \in \mathcal{B}$. But then $A \cap B \in \mathcal{B}$ and thus $A \cap B \in U$. Now U is an upper bound for C and so by Zorn's lemma $\mathcal{F}$ is in some maximal filter $\mathcal{M}$ of I. Since $\mathcal{F}$ doesn't contain any finite sets, we see that $\mathcal{M}$ cannot be principal. Since $\mathcal{M}$ is maximal, it must be an ultrafilter for I. $\square$

Problem 5. *Say that a class R of $\mathcal{L}$-structures is an* elementary class *if there is a first-order set of sentences T such that $M \in R$ iff $M \models T$.*
(a) Show that R is an elementary class iff it is closed under elementary equivalence and ultraproducts. (This means that any ultraproduct of elements of R is again in R, and if $M \equiv N$ with $N \in R$ then $M \in R$.)
(b) Using (a), give an example of a nonempty R which is not an elementary class.

Proof. Let R be an elementary class. It's clear that R is closed under elementary equivalence since if $M \models T$ and $M \equiv N$ then $N \models T$ as well. We also know that if M_i are elements of R and $M_i \models \varphi$, then $N = \prod_{i \in I} M_i / \mathcal{D}$ for some ultrafilter $\mathcal{D}$ is also a model such that $N \models \varphi$. Thus R is closed under ultraproducts as well.

Now let R be a set of $\mathcal{L}$-structures which is closed under ultraproducts and elementary equivalence. Let T be the set of $\mathcal{L}$-sentences such that if $\varphi \in T$ then $M \models \varphi$. Thus $M \models T$ for all $M \in R$. Now let M be a model of T and let S be the set of $\mathcal{L}$-sentences φ for which $M \models \varphi$. Let S' be the set of all finite subsets of S. We know for each $s \in S'$ with $s = \{\varphi_1, \ldots, \varphi_n\}$, there exists a model $M_s \in R$ such that $M_s \models s$ because otherwise, $\neg(\varphi_1 \wedge \cdots \wedge \varphi_n) \in T$, but would be false in M. But now we know there exists an ultraproduct $N = \prod_{i \in S'} M_i$ for which $N \models S$. Since R is closed under ultraproducts, $N \in R$. But since every model of R is elementary equivalent to M, we have $M \equiv N$. Thus $M \in R$ and R is the class of all models of T. $\square$