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Homework 8

**** Problem 1.** *Let*

$$f(x) = \begin{cases} 2x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Is f increasing or decreasing on any neighborhood of the origin?
No.

Proof. The derivative of f is $Df(x) = 4x\sin(1/x) - 2\cos(1/x)$ for all $x \neq 0$. But there are infinitely many zeros of this function in any neighborhood of the origin. Thus, Df is never strictly greater or less than 0 for any neighborhood of the origin and therefore f is neither increasing or decreasing for any neighborhood of the origin. ☐

**** Problem 2.** *Let*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show f is differentiable infinitely many times at 0 and the kth derivative at 0 is 0 for all k.

Proof. Note that

$$\lim_{x \to 0} e^{-\frac{1}{x^2}} = 0$$

and so this function is continuous and thus differentiable at $x = 0$. For $x \neq 0$, using the chain rule we have

$$Df(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^3}$$

where a_1 is some integer constant. Now suppose that the kth derivative for $x \neq 0$ is

$$D^k f(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^{(k+2)}} + \frac{a_2 e^{-\frac{1}{x^2}}}{x^{(k+4)}} + \cdots + \frac{a_k e^{-\frac{1}{x^2}}}{x^{(k+2k)}}$$

where $a_1, \ldots, a_k$ are integer constants. Using the chain rule and the product rule, we can differentiate again to obtain

$$D^{k+1} f(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^{(k+3)}} + \frac{a_2 e^{-\frac{1}{x^2}}}{x^{(k+5)}} + \cdots + \frac{a_k e^{-\frac{1}{x^2}}}{x^{(k+1+2(k))}} + \frac{a_{k+1} e^{-\frac{1}{x^2}}}{x^{(k+1+2(k+1))}}$$

for all $x \neq 0$ where $a_1, \ldots, a_k$ are different integer constants. Thus, by induction, the is the kth nonzero derivative. To show that each derivative is continuous at 0, note that the first derivative for $x \neq 0$ is

$$Df(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^3}.$$

Taking $\lim_{x \to 0} Df(x)$ we see that l'Hopital's Rule applies, and we end up with $\lim_{x \to 0} Df(x) = 0$. We can assume inductively that the kth derivative is continuous at 0, and then use that fact and l'Hopital's Rule to show the $k + 1$st derivative is continuous at 0. Thus $D^k f(0)$ exists for all k and $D^k f(0) = 0$ for all k. ☐

**** Problem 3.** *If $\alpha \notin \mathbb{Q}$ and $r \leq 2$, show f_r is not differentiable at α.*

Proof. Let $\alpha \notin \mathbb{Q}$ and let $r \leq 2$. Suppose that $Df_r(\alpha)$ exists. Then we have

$$Df_r(\alpha) = \lim_{x\to\alpha}\left|\frac{f_r(x) - f_r(\alpha)}{x-\alpha}\right| = \lim_{x\to\alpha}\left|\frac{f_r(x)}{x-\alpha}\right|.$$

For $x \notin \mathbb{Q}$ we have $f_r(x) = 0$, which means that $Df_r(\alpha) = 0$ on the irrationals. But also note that there are infinitely many $x \in \mathbb{Q}$ with $x = p/q$ such that $|\alpha - x| < 1/q^2 \leq |f_r(x)|$. Thus there are infinitely many $x \in \mathbb{Q}$ such that

$$1 < \left|\frac{f_r(x)}{\alpha - x}\right|.$$

But then $Df_r(\alpha)$ has different values on the rationals and irrationals. This is a contradiction and so $Df_r(\alpha)$ does not exist. $\square$

**** Problem 4.** *Let*

$$f_1 = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1-x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Extend $f_1(x) = f_1(x+1)$. *Let* $f_n(x) = \frac{1}{2}f_{n-1}(2x)$. *Let*

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Show that f is continuous on $\mathbb{R}$, but is differentiable nowhere.

Proof. Each f_n is piecewise linear and therefore continuous. Additionally, (f_n) uniformly converges and so f must be continuous. To see that f is not differentiable, let $a \in \mathbb{R}$. Then consider

$$\lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}\right| = \lim_{x\to a}\left|\frac{\sum_{n=1}^{\infty} f_n(x) - f_n(a)}{x-a}\right|.$$

But between every two points in $\mathbb{R}$ there exists a "spike" of f which separates the points. Thus, as x approaches a, $f(x)$ and $f(a)$ will have opposite slopes, forcing the limit to not exist. $\square$

**** Problem 5.** *What is the differentiability of f_r at 0?*

Proof. By definition we have $f_r(0) = 0$. Taking the difference quotient we have

$$\lim_{h\to 0}\left|\frac{f_r(0+h) - f_r(0) - Df_r(0)}{h}\right| = \lim_{h\to 0}\left|\frac{f_r(h) - Df_r(0)}{h}\right|.$$

For $h \notin \mathbb{Q}$ we have $f_r(h) = 0$. For $h \in \mathbb{Q}$ we consider $\lim_{h\to 0} f_r(h)$. But this limit is 0 since the denominator of h will tend toward infinity as h tends toward 0. Thus, it must be the case that $Df_r(0) = 0$. $\square$

**** Problem 6.** *Let α be a real, algebraic number of degree $n \geq 2$. Then there exists a constant $c(\alpha)$ depending only on α such that for all $p/q \in \mathbb{Q}$ we have*

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha)}{q^n}.$$

Proof. Let $\alpha \in \mathbb{A}_{\mathbb{R}}$ such that $f(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$ where each $a_i \in \mathbb{Z}$. Then choose

$$M > \max_{\alpha-1\leq x\leq\alpha+1} |f'(x)|.$$

Note that M is entirely determined by α. Suppose $p/q \in \mathbb{Q}$ such that $p/q \in (\alpha - 1; \alpha + 1)$ and $f(p/q) \neq 0$. We can do this because $n \geq 2$. Then we have

$$\left| f\left(\frac{p}{q}\right)\right| = \frac{|a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q|}{q^n} \geq \frac{1}{q^n}$$

because the numerator in a nonzero integer. Then by the mean value theorem there exists x such that

$$\frac{1}{q^n} \leq \left| f\left(\frac{p}{q}\right) - f(\alpha)\right| = \left| \left(\frac{p}{q} - \alpha\right) f'(x)\right| < M \left|\frac{p}{q} - \alpha\right|.$$

Taking $c(\alpha) = 1/M$ completes the proof. □

** **Problem 7.** *Let $A \subseteq \mathbb{R}$ and suppose there exists a countable collection of intervals $\{I_i\}_{i\in\mathbb{N}}$ such that 1)*

$$A \subseteq \bigcup_i I_i$$

2)

$$\sum_i Vol(I_i)$$

is finite.
3) If $a \in A$ then $a \in I_i$ for infinitely many $i \in \mathbb{N}$.
Show that this is equivalent to A having measure 0.

Proof. Suppose the above hypotheses are true for a set $A \subseteq \mathbb{R}$. Let $\varepsilon > 0$. If $\sum_i Vol(I_i) < \varepsilon$, we're finished. If $\sum_i Vol(I_i) \geq \varepsilon$ then we remove one interval, I_{j_1} from the collection. Since $a \in A$ implies $a \in I_i$ for infinitely many i, it follows that $A \subseteq \bigcup_i I_i \backslash I_{j_1}$. Continue in this way until $\sum_i Vol(I_i) - \sum_k Vol(I_{j_k}) < \varepsilon$.

Now let A be a set with measure 0 and let $\varepsilon > 0$. Suppose that there exists $a \in A$ such that $a \in I_i$ for only finitely many i, say $I_{j_1}, \ldots, I_{j_n}$. But then this is a contradiction because ε is finite and so we can simply choose a smaller ε to exclude a. Thus a must be in infinitely many intervals. □

** **Problem 8.** *Show that, if $r \geq 2$, then f_r is differentiable except on a set of measure 0.*

Proof. Use the method of ** Problem 9 to show that f is not differentiable on a thick set, then $r < 2$. □

Lemma 1. *Every thick subset of a complete metric space, X, is dense in X. If X is the union of countably many closed sets, at least one has nonempty interior.*

Proof. A thick subset of X is a countable intersection $G = \bigcap G_n$ of open dense subsets of X. A thin subset of X is a countable union $H = \bigcup H_n$ of closed nowhere dense subsets of X. If $X = \emptyset$ then we're finished, so assume $X \neq \emptyset$. Let G be a thick subset of X and let $p_0 \in X$ and $\varepsilon > 0$. Choose $r_n < 1/n$ and create a sequence $(p_n) \in X$ by

$$B_{2r_1}(p_1) \subseteq B_\varepsilon(p_0)$$
$$B_{2r_2}(p_2) \subseteq B_{r_1}(p_1) \cap G_1$$
$$\cdots$$
$$B_{2r_{n+1}}(p_{n+1}) \subseteq B_{r_n}(p_n) \cap G_1 \cap \cdots \cap G_n.$$

Since the $r_n \to 0$ as $n \to \infty$, (p_n) is a Cauchy sequence in X, which converges to $p \in X$ since X is complete. Since $p \in \overline{B_{r_n}(p_n)}$ we also have $p \in G_n$ and thus $p \in G \cap B_\varepsilon(p_0)$. Thus G is dense in X.

Now suppose that $X = H = \bigcup H_n$ with each H a nowhere dense closed set. Then each $G_n =^c H_n$ is open dense and

$$G = \bigcap G_n =^c \left(\bigcup K_n\right) = \emptyset$$

which contradicts the density of G. □

Lemma 2. *The set PL of piecewise linear functions is dense in the set C^0 of all continuous linear functions from $\mathbb{R}$ to $\mathbb{R}$.*

Proof. Let $f \in C^0$ and $\varepsilon > 0$. Since $[a,b]$ is compact, f is uniformly continuous on this interval and so there exists $\delta > 0$ such that $|t-s| < \delta$ implies $|f(t) - f(s)| < \varepsilon$ for all $s,t \in [a,b]$. Choose $n > (b-a)/\delta$ and partition $[a,b]$ into n subintervals $I_i = [x_i - 1, x_i]$, each of length less than δ. Now create a piecewise linear function $\phi : [a,b] \to \mathbb{R}$ which connects each of the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ for all $0 \leq i \leq n$. Then $\phi(t)$ for $t \in I_i$ is between $f(x_{i-1})$ and $f(x_i)$, each of which differ from $f(t)$ by less than ε. Thus for all $t \in [a,b]$ we have $|f(t) - \phi(t)| < \varepsilon$. ☐

Lemma 3. *If $\phi \in PL$ and $\varepsilon > 0$, then there exists a sawtooth function σ such that $||\sigma|| \leq \varepsilon$, σ has period less than or equal to ε, and*

$$\min(|\textit{slope } \sigma|) > \max(|\textit{slope } \phi|) + \frac{1}{\varepsilon}.$$

Proof. Choose $c \in \mathbb{R}$. The compressed sawtooth function $\sigma(x) = \varepsilon\sigma_0(cx)$ has $||\sigma|| = \varepsilon$, period $1/c$, and slope $\pm\varepsilon c$. Choosing c arbitrarily large will give the desired results. ☐

**** Problem 9.** *Let $f : [a,b] \to \mathbb{R}$. If f is monotonically increasing on $[a,b]$, then f is differentiable almost everywhere on $[a,b]$.*

Proof. We use the contrapositive. For $n \in \mathbb{N}$ define

$$R_n = \{f \in C^0 \mid \forall x \in [a, b - \frac{1}{n}] \exists h > 0 \text{ such that } \left|\frac{f(x+h) - f(x)}{h}\right| > n\}$$

$$L_n = \{f \in C^0 \mid \forall x \in [a + \frac{1}{n}], b \exists h < 0 \text{ such that } \left|\frac{f(x+h) - f(x)}{h}\right| > n\}$$

$$G_n = \{f \in C^0 \mid f \text{ restricted to any interval of length } \frac{1}{n} \text{ is non-monotone}\}.$$

We must show that each of these sets is open dense in C^0. Lemma 3 shows that the closure of PL is C^0, so we must only show that the closure of each set contains PL. Let $\phi \in PL$ and $\varepsilon > 0$. By Lemma 3 there exists a sawtooth function σ such that $||\sigma|| \leq \varepsilon$, σ has period less than $1/n$ and

$$\min(|\text{slope } \sigma|) > \max(|\text{slope } \phi|) + n.$$

Consider $f = \phi + \sigma$ and note that $f \in PL$. It has slopes that are dominated by σ and so they alternate in sign with period $1/2n$. At any $x \in [a, b + 1/n]$ there is a rightward slope either greater than n or less than $-n$. Thus $f \in R_n$ and a similar argument shows $f \in L_n$. Since any interval of length $1/n$ contains a minimum or maximum of σ, it must contain a subinterval where f is strictly increasing, and one where f is strictly decreasing. Thus $f \in G_n$. Since $||f - \phi|| = \varepsilon$ we have R_n, L_n and G_n are dense in C^0.

Now suppose that $f \in R_n$. The for all $x \in [a, b - 1/n]$ there exists $h > 0$ such that

$$\left|\frac{f(x+h) - f(x)}{h}\right| > n.$$

Since f is continuous, there exists some neighborhood $A_x \subseteq [a,b]$ of x and a constant $y > 0$ such that

$$\left|\frac{f(t+h) - f(t)}{h}\right| > n + y$$

for all $t \in A_x$. Since $[a, b - 1/n]$ is compact, we only need finitely many of these intervals, $A_{x_1}, \ldots, A_{x_m}$, to cover it. Again, since f is continuous for all $t \in A_{x_i}$ we have

$$\left|\frac{f(t+h_i) - f(t)}{h_i}\right| > n + y_i.$$

Since there are only finitely many inequalities we can replace f by a function g with the distance between f and g small enough so that we have

$$\left| \frac{g(t+h_i)-g(t)}{h_i} \right| > n.$$

Thus $g \in R_n$ and R_n is open in C^0. A similar proof holds for L_n open in C^0.

Let (f_k) be a sequence of functions in ${}^cG_n$ and such that f_k uniformly converges to f. We must show that $f \in^c G_n$. Each f_k is monotone on some interval of $1/n$. Taking these intervals as a sequence, there is a subsequence which converges to some interval I. This interval must have length $1/n$ and by uniform convergence, f is monotone on I. Thus ${}^cG_n$ is closed and therefore G_n is open. Hence all of R_n, L_n and G_n are open dense in C^0.

Finally, suppose that f belongs to the thick set $\bigcup_n R_n \cap L_n \cap G_n$. Then for each $x \in [a, b]$ there exists a sequence of nonzero h_n such that

$$\left| \frac{f(x+h_n)-f(x)}{h_n} \right| < n.$$

The numerator is at most $2||f||$ and so $h_n \to 0$ as $n \to \infty$. This shows that f is not differentiable at x. Moreover, since $f \in G_n$, f is not monotone on any interval of length $1/n$ and since any interval I contains an interval of length $1/n$ for n large enough, f is not monotone on I. $\square$