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Homework 3

Problem 1. *For the equation $u'' + u' = x$, find a particular integral by inspection. What is the most general solution? What is the solution with initial data $(u, u') = (0, 0)$ at $x = 0$?*

A particular solution is $U(x) = x(x-2)/2$. It's easy to see two solutions for $u'' + u' = 0$ are $u_1(x) = 1$ and $u_2(x) = e^{-x}$. Thus the most general solution is

$$u(x) = U(x) + c_1 u_1(x) + c_2 u_2(x) = \frac{x(x-2)}{2} + c_1 + c_2 e^{-x}.$$

Putting in $u(0) = c_1 + c_2 = 0$ and $u'(0) = -1 - c_2 = 0$ gives $c_2 = -1$ and $c_1 = 1$. Thus $u(x) = x(x-2)/2 - e^{-x} + 1$.

Problem 2. *Carry out the construction of the influence function for the equation $u'' + u = r$.*

The solutions to the homogeneous equation $u'' + u = 0$ are $u_1(x) = \cos(x)$ and $u_2(x) = \sin(x)$. Then $W(u_1, u_2; x) = u_1(x)u_2'(x) - u_2(x)u_1'(x) = \cos^2(x) + \sin^2(x) = 1$. Thus

$$G(x, s) = \begin{cases} \cos(s)\sin(x) - \sin(s)\cos(x) & s < x \\ 0 & s \geq x \end{cases}.$$

Problem 3. *Verify directly that if $G(x, \xi)$ satisfies the initial value problem (2.27), then the expression (2.26) provides a particular integral.*

Proof. We need to show that $U''(x) + p(x)U'(x) + q(x)U(x) = r(x)$. Note using differentiation under the integral we have

$$\begin{aligned} U''(x) &= \frac{d^2}{dx^2} \int_{x_0}^{x} \frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(u_1, u_2; s)} r(s)ds \\ &= \frac{d}{dx} \left(\frac{u_1(x)u_2(x) - u_2(x)u_1(x)}{W(u_1, u_2; x)} r(x) + \int_{x_0}^{x} \frac{\partial}{\partial x} \frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(u_1, u_2; s)} r(s)ds \right) \\ &= \frac{d}{dx} \int_{x_0}^{x} \frac{u_1(s)u_2'(x) - u_2(s)u_1'(x)}{W(u_1, u_2; s)} r(s)ds \\ &= \frac{u_1(x)u_2'(x) - u_2(x)u_1'(x)}{W(u_1, u_2; x)} r(x) + \int_{x_0}^{x} \frac{\partial}{\partial x} \frac{u_1(s)u_2'(x) - u_2(s)u_1'(x)}{W(u_1, u_2; s)} r(s)ds \\ &= r(x) + \int_{x_0}^{x} \frac{u_1(s)u_2''(x) - u_2(s)u_1''(x)}{W(u_1, u_2; s)} r(s)ds. \end{aligned}$$

and

$$\begin{aligned} U'(x) &= \frac{d}{dx} \int_{x_0}^{x} \frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(u_1, u_2; s)} r(s)ds \\ &= \frac{u_1(x)u_2(x) - u_2(x)u_1(x)}{W(u_1, u_2; x)} r(x) + \int_{x_0}^{x} \frac{\partial}{\partial x} \frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(u_1, u_2; s)} r(s)ds \\ &= \int_{x_0}^{x} \frac{u_1(s)u_2'(x) - u_2(s)u_1'(x)}{W(u_1, u_2; s)} r(s)ds \end{aligned}$$

Now, putting this together with $U(x)$ and given that u_1 and u_2 are solutions to the homogeneous equation $Lu_1 = 0$ and $Lu_2 = 0$, we have

$$\begin{aligned}
&U''(x) + p(x)U'(x) + q(x)U(x) \\
&= r(x) + \int_{x_0}^{x} \frac{u_1(s)u_2''(x) - u_2(s)u_1''(x)}{W(u_1,u_2;s)} r(s)ds \\
&\quad + p(x)\int_{x_0}^{x} \frac{u_1(s)u_2'(x) - u_2(s)u_1'(x)}{W(u_1,u_2;s)} r(s)ds + q(x)\int_{x_0}^{x} \frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(u_1,u_2;s)} r(s)ds \\
&= r(x) + \int_{x_0}^{x} \left(\frac{u_1(s)u_2''(x) - u_2(s)u_1''(x)}{W(u_1,u_2;s)} + p(x)\frac{u_1(s)u_2'(x) - u_2(s)u_1'(x)}{W(u_1,u_2;s)} \right. \\
&\quad \left. + q(x)\frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(u_1,u_2;s)} \right) r(s)ds \\
&= r(x) + \int_{x_0}^{x} \frac{u_1(s)(u_2''(x) + p(x)u_2'(x) + q(x)u_2(x)) - u_2(s)(u_1''(x) + p(x)u_1'(x) + q(x)u_1(x))}{W(u_1,u_2;s)} r(s)ds. \\
&= r(x).
\end{aligned}$$

□

Problem 4. *Consider the homogeneous equation $u'' + q(x)u = 0$ where*

$$q(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$

Require of solutions that they be C^1 for all real values of x (i.e., continuous with continuous derivatives: the second derivative will in general fail to exist at $x = 0$). Construct a basis of solutions. Check its Wronskian.

Proof. Define the functions

$$u_1(x) = \begin{cases} 1 & x < 0 \\ \cos(x) & x \geq 0 \end{cases}$$

and

$$u_2(x) = \begin{cases} x & x < 0 \\ \sin(x) & x \geq 0 \end{cases}.$$

Since both parts of both functions are continuous and they agree at 0, both u_1 and u_2 are continuous. Furthermore

$$u_1'(x) = \begin{cases} 0 & x < 0 \\ -\sin(x) & x \geq 0 \end{cases}$$

and

$$u_2'(x) = \begin{cases} 1 & x < 0 \\ \cos(x) & x \geq 0 \end{cases}$$

both of which have continuous parts which agree at 0, so are continuous. Furthermore, we've seen already that 1 and x are linearly independent and $\sin(x)$ and $\cos(x)$ are linearly independent. Thus u_1 and u_2 form a basis of solutions. The Wronskian is

$$W(u_1,u_2;x) = \det\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} = \begin{cases} 1 - 0 & x < 0 \\ \cos^2(x) + \sin^2(x) & x \geq 0 \end{cases}$$

so $W(u_1,u_2;x) = 1$ for all x. □

Problem 5. *Find a basis of solutions for the system $u''' + u'' = 0$. Calculate the Wronskian and check the result against Theorem 2.2.2 (equation 2.33).*

Clearly $u_1(x) = 1$ and $u_2(x) = x$ satisfy the equation. Note also that $d^3/dx^3(e^{-x}) = -e^{-x} = -d^2/dx^2(e^{-x})$ so e^{-x} is a solution as well. Thus $u_1(x)$, $u_2(x)$ and $u_3(x) = e^{-x}$ constitute a basis of solutions. The Wronskian is

$$\det \begin{pmatrix} 1 & x & e^{-x} \\ 0 & 1 & -e^{-x} \\ 0 & 0 & e^{-x} \end{pmatrix} = e^{-x}.$$

Theorem 2.2.2 tells us that $W(u_1, u_2, u_3; x) = W_0 \exp\left(-\int_{x_0}^{x} 1 ds\right) = W_0 e^{-x}$. This is consistent with our calculated Wronskian.

Problem 6. *For the operator $Lu = u''' + u'$, find the most general solution of the equation $Lu = 1$ (cf. Example 2.2.3).*

From the example we know 1, $\cos(x)$ and $\sin(x)$ form a basis of solutions to the homogeneous problem. Furthermore $U(x) = x$ gives a particular solution to the problem. Thus, a general solution is

$$u(x) = x + c_1 + c_2 \cos(x) + c_3 \sin(x).$$

Problem 7. *For the operator of the proceeding problem, obtain the influence function for solving the inhomogeneous problem.*

Let $U(x)$ be a particular solution and write

$$U(x) = \sum_{i=1}^{3} c_i(x) u_i(x) = c_1(x) + c_2(x) \cos(x) + c_3(x) \sin(x).$$

We now impose the two conditions

$$0 = \sum_{i=1}^{3} c_i'(x) u_i(x) = c_1'(x) + c_2'(x) \cos(x) + c_3'(x) \sin(x)$$

and

$$0 = \sum_{i=1}^{3} c_i'(x) u_i'(x) = -c_2'(x) \sin(x) + c_3'(x) \cos(x).$$

In order for $U(x)$ to satisfy $LU = r$ we must have

$$r = \sum_{i=1}^{3} c_i'(x) u_i''(x) = -c_2'(x) \cos(x) - c_3'(x) \sin(x).$$

Now we have three equations in the three functions $c_i'(x)$, $1 \leq i \leq 3$. We can solve for them as

$$c_1'(x) = r(x) = u_1(x) r(x)$$

$$c_2'(x) = -r(x) \cos(x) = -u_2(x) r(x)$$

and

$$c_3'(x) = -r(x) \sin(x) = -u_3(x) r(x).$$

Thus

$$c_1 = \int_{x_0}^{x} r(s) ds$$

$$c_2 = -\int_{x_0}^{x} \cos(s) r(s) ds$$

and

$$c_3 = -\int_{x_0}^{x} \sin(s) r(s) ds.$$

Putting these back into $U(x)$ we find

$$U(x) = \int_{x_0}^{x} (1 - \cos(s)\cos(x) - \sin(s)\sin(x)) r(s) ds.$$

Therefore an influence function must be

$$G(x,s) = \begin{cases} 1 - \cos(s)\cos(x) - \sin(s)\sin(x) & s < x \\ 0 & s \geq s \end{cases}.$$

Problem 8. *Find the equivalent first order system (that is, find the matrix A and the vector R of equation (2.40)) for the second order equation*

$$u'' + x^2 u' + x^4 u = \frac{1}{1+x^2}.$$

Define $a_1(x) = x^2$, $a_2(x) = x^4$ and $r(x) = 1/(1+x^2)$. Let $v_1(x) = u(x)$ and $v_2(x) = u'(x)$. Then $v_1'(x) = v_2(x)$ and $v_2'(x) = u''(x) = r(x) - a_1(x)v_2(x) - a_2(x)v_1(x)$. Now let

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}.$$

and

$$A = \begin{pmatrix} 0 & 1 \\ -a_2(x) & -a_1(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -x^4 & -x^2 \end{pmatrix}.$$

Then to fit the equation $v' = Av + R$ we must have

$$R = \begin{pmatrix} 0 \\ r(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{1+x^2} \end{pmatrix}.$$

Problem 9. *Deduce equation (3.11) and hence infer that for polynomials with real coefficients, if λ is a root, so is $\overline{\lambda}$.*

Proof. Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with real coefficients. Note then that $\overline{a_i} = a_i$. Also we know the basics of conjugation like $\overline{zw} = \overline{z}\overline{w}$ and $\overline{z+w} = \overline{z} + \overline{w}$. Then

$$p(\overline{\lambda}) = \sum_{i=0}^{n} a_i \overline{\lambda}^i = \sum_{i=0}^{n} a_i \overline{\lambda^i} = \sum_{i=0}^{n} \overline{a_i \lambda^i} = \overline{\sum_{i=0}^{n} a_i \lambda^i} = \overline{p(\lambda)}.$$

Thus, if $p(\lambda) = 0$ then $0 = \overline{p(\lambda)} = p(\overline{\lambda})$ as well. ∎

Problem 10. *Find the solution of $u'' + 2au' + bu = 0$ if (i) $a^2 > b$, (ii) $a^2 < b$, (iii) $a^2 = b$.*

The characteristic polynomial is $p(x) = x^2 + 2ax + b$ and using the quadratic formula we see that the roots are

$$x = \frac{-2a \pm \sqrt{4a^2 - 4b}}{2} = -a \pm \sqrt{a^2 - b}.$$

(i) Since $a^2 > b$, the root is positive so we have two distinct real roots, $\lambda_1 = -a + \sqrt{a^2 - b}$ and $\lambda_2 = -a - \sqrt{a^2 - b}$. Thus the solution is $u(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

(ii) If $a^2 < b$ then the root is complex so we have two complex solutions $\lambda = -a + \sqrt{a^2 - b} = -a + i\sqrt{b - a^2}$ and $\overline{\lambda} = -a - i\sqrt{b - a^2}$. Then $u = c_1 e^{-ax}\cos(x\sqrt{b - a^2}) + c_2 e^{-ax}\sin(x\sqrt{b - a^2})$.

(iii) If $a^2 = b$ then there is only one real root $\lambda = -a$ so $u(x) = c_1 e^{-ax} + c_2 x e^{-ax}$ is the solution.

Problem 11. *Find a basis of solutions for the equation $u''' + 4u'' + 4u' = 0$.*

Clearly 1 is a solution. Let $v(x) = u'(x)$ so we have the equation $v'' + 4v' + 4v = 0$. The characteristic polynomial of this equation is $p(x) = x^2 + 4x + 4 = (x+2)^2$ and has one repeated real root $x = -2$. Thus, a solution to v is $v(x) = c_1 e^{-2x} + c_2 x e^{-2x}$ so

$$u(x) = v'(x) = -2c_1 e^{-2x} + c_2(1-2x)e^{-2x}.$$

Adding in the first solution and renaming constants we see that

$$u(x) = c_1 + c_2 e^{-2x} + c_3(1-2x)e^{-2x}.$$

Problem 12. *The same for the equation $u^{iv} + 2u'' + 3u = 0$.*

The characteristic polynomial is

$$p(x) = x^4 + 2x^2 + 3 = \left(x - \sqrt{-1+i\sqrt{2}}\right)\left(x + \sqrt{-1+i\sqrt{2}}\right)\left(x - \sqrt{-1-i\sqrt{2}}\right)\left(x + \sqrt{-1-i\sqrt{2}}\right).$$

Let $a + bi = \sqrt{-1+i\sqrt{2}}$. Then $a^2 - b^2 + 2abi = (a+bi)^2 = -1 + i\sqrt{2}$ so $a^2 - b^2 = -1$ and $ab = \sqrt{2}/2$. Solving these we have

$$b = \pm\sqrt{\frac{1+\sqrt{3}}{2}}$$

and

$$a = \pm\sqrt{\frac{\sqrt{3}-1}{2}}.$$

A basis of solutions is then

$$\left\{\exp\left(\sqrt{\frac{\sqrt{3}+1}{2}}\right)\cos\left(\sqrt{\frac{\sqrt{3}-1}{2}}\right), \exp\left(\sqrt{\frac{\sqrt{3}+1}{2}}\right)\sin\left(\sqrt{\frac{\sqrt{3}-1}{2}}\right),\right.$$
$$\left.\exp\left(-\sqrt{\frac{\sqrt{3}+1}{2}}\right)\cos\left(-\sqrt{\frac{\sqrt{3}-1}{2}}\right), \exp\left(-\sqrt{\frac{\sqrt{3}+1}{2}}\right)\sin\left(-\sqrt{\frac{\sqrt{3}-1}{2}}\right)\right\}.$$