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Homework 4

Problem 1. *Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subseteq [a,b]$. Under what conditions does equality hold?*

Proof. We're asked to show that if $x \in \overline{(a,b)}$ then $x \in [a,b]$, and we will show the contrapositive, that if $x \notin [a,b]$ then $x \notin \overline{(a,b)}$. Suppose that $x \notin [a,b]$. Without loss of generality, we can take $x < a$. Then consider the open set (c,a) where $c < x$ (if x happens to be the least element of X, then take the set $[x,a)$). This open set contains x and doesn't intersect (a,b), so $x \notin \overline{(a,b)}$.

Now suppose $x \in [a,b]$. Since $(a,b) \subseteq \overline{(a,b)}$, if $x \in (a,b)$ we automatically have $x \in \overline{(a,b)}$. Thus we need only worry about the cases when $x = a$ or $x = b$. Suppose $x = a$ and let (c,d) be a basis element containing x. In order for $(c,d) \cap (a,b) \neq \emptyset$ we need some element $y \in X$ such that $a < y < d$. Since d is an arbitrary element greater than a, the condition we need is "there exists a point of X between any two given points of X". If we can find such a y, then $(c,d) \cap (a,b) \neq \emptyset$ and $x \in \overline{(a,b)}$. The case $x = b$ follows similarly. □

Problem 2. *Let A, B and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supseteq$ or $\subseteq$ holds.*
(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
(b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$.
(c) $\overline{A \backslash B} = \overline{A} \backslash \overline{B}$.

Proof. (a) Let $x \in \overline{A \cap B}$ and let U be an open set containing x. Then there exists $y \in U$ such that $y \in A \cap B$ which means $y \in A$ and $y \in B$. Since U is arbitrary, $x \in \overline{A}$ and $x \in \overline{B}$ or $x \in \overline{A} \cap \overline{B}$. Therefore $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Now consider $\mathbb{R}$ with the usual topology. Let $x = 0$ so that $x \in \overline{(-1,0)} \cap \overline{(0,1)}$. But note that $\overline{(-1,0) \cap (0,1)} = \overline{\emptyset} = \emptyset$ so it can't be that $x \in \overline{(-1,0) \cap (0,1)}$. Therefore in general we don't have the second inclusion.

(b) Let $x \in \overline{\bigcap A_\alpha}$ where $\alpha \in J$ and let U be a neighborhood of x. Then there exists $y \in U$ such that $y \in \bigcap A_\alpha$. Thus $y \in A_\alpha$ for each $\alpha \in J$. Since U is arbitrary, we have that $x \in \overline{A_\alpha}$ for each $\alpha \in J$ and so $x \in \bigcap \overline{A_\alpha}$. Therefore $\overline{\bigcap A_\alpha} \subseteq \bigcap \overline{A_\alpha}$.

We can produce a counter example similar to the one in part (a) even if J is infinite by letting $x = 0$ and taking intervals of the form $(0,r)$ and $(0,-r)$. Then x will be in the closure of all of these intervals and thus in the intersection of their closures, but the intersection of the sets themselves is empty so x cannot be in the closure of the intersection. Thus, as in part (a), the second inclusion doesn't hold in general.

(c) Suppose $x \in \overline{A} \backslash \overline{B}$ and let U be a neighborhood of x such that $U \cap B = \emptyset$. Now let V be any neighborhood of x and consider $U \cap V$. Note that $(U \cap V) \cap B = \emptyset$ as well since $(U \cap V) \subseteq U$. But also $(U \cap V) \cap A \neq \emptyset$ since $x \in \overline{A}$ and the intersection of two open sets is open. So there's some point $y \in V$ such that $y \notin B$ and $y \in A$, that is $V \cap (A \backslash B) \neq \emptyset$. Since V is arbitrary, we see that $x \in \overline{A \backslash B}$ and so $\overline{A} \cap \overline{B} \subseteq \overline{A \backslash B}$.

Consider $\mathbb{R}$ with the usual topology. Let $x = 0$ and note that $\overline{(-1,1) \backslash (-1,0)} = \overline{[0,1)} = [0,1]$ so $x \in \overline{(-1,1) \backslash (-1,0)}$. But then note that $\overline{(-1,1)} = [-1,1]$ and $\overline{(-1,0)} = [-1,0]$ so $x \notin \overline{(-1,1)} \backslash \overline{(-1,0)}$. □

Problem 3. *If $A \subseteq X$, we define the* boundary *of A by the equation*

$$\mathrm{Bd}A = \overline{A} \cap \overline{(X \backslash A)}.$$

(a) Show that $\mathrm{Int}A$ *and* $\mathrm{Bd}A$ *are disjoint, and* $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$.
(b) Show that $\mathrm{Bd}A = \emptyset \iff A$ *is both open and closed.*
(c) Show that U is open $\iff \mathrm{Bd}U = \overline{U} \backslash U$.
(d) If U is open, is it true that $U = \mathrm{Int}(\overline{U})$*? Justify your answer.*

Proof. (a) Let $x \in \text{Int}A$. Since $\text{Int}A$ is open, there exist some open set $U \subseteq A$ such that $x \in U$. But then $U \cap (X \backslash A) = \emptyset$ and so $x \notin \overline{(X \backslash A)}$. Therefore $x \notin \text{Bd}A$. Conversely, suppose that $x \in \text{Bd}A$. Then $x \in \overline{(X \backslash A)}$ so every open set containing x intersects $X \backslash A$. In particular, there is no open set U such that $x \in U$ and $U \subseteq \text{Int}A$. Therefore $x \notin \text{Int}A$ since $\text{Int}A$ is open.

Now suppose that $x \in \overline{A}$. If $x \in \text{Int}A$ we're done, so suppose otherwise. Let U be a neighborhood of x. Since $x \notin \text{Int}A$, we know that $U \nsubseteq \text{Int}A$, so $U \cap (X \backslash A) \neq \emptyset$. Since U is arbitrary, we must have that $x \in \overline{(X \backslash A)}$. Therefore, because $x \in \overline{A}$ we have $x \in \text{Bd}A$ and $\overline{A} \subseteq \text{Int}A \cup \text{Bd}A$.

Conversely, suppose that $x \in \text{Int}A \cup \text{Bd}A$. If $x \in \text{Int}A$ then $x \in \overline{A}$ since $\text{Int}A \subseteq A \subseteq \overline{A}$, so assume $x \in \text{Bd}A$. But then $x \in \overline{A}$ by definition, so $\text{Int}A \cup \text{Bd}A \subseteq \overline{A}$.

(b) Suppose that $\text{Bd}A = \emptyset$. Then from part (a) we have $\overline{A} = \text{Int}A \cup \emptyset = \text{Int}A$. But then $\text{Int}A \subseteq A \subseteq \overline{A} \subseteq \text{Int}A$ so we must have $A = \text{Int}A = \overline{A}$ and thus A is both open and closed.

Conversely, suppose that A is both open and closed. Then $A = \overline{A} = \text{Int}A = \text{Int}A \cup \emptyset$. By part (a), $\text{Int}A$ and $\text{Bd}A$ are disjoint, so we must have $\text{Bd}A = \emptyset$.

(c) Suppose that U is open. Then $U = \text{Int}U$. From part (a) we know $\overline{U} = \text{Int}U \cup \text{Bd}U$ and basic set operations gives $\overline{U} \backslash U = \overline{U} \backslash \text{Int}U = (\text{Int}U \cup \text{Bd}U) \backslash \text{Int}U = \text{Bd}U$. Conversely, suppose that $\overline{U} \backslash U = \text{Bd}U$. Then union U to both sides to obtain $\overline{U} = \text{Bd}U \cup U$. From the equation in part (a) we're forced to conclude that $U = \text{Int}U$ and U is open.

(d) No. Consider the set $(-1,0) \cup (0,1)$. This set is open in the standard topology on $\mathbb{R}$. Note that $\overline{(-1,0) \cup (0,1)} = [-1,1]$ and $\text{Int}(\overline{(-1,0) \cup (0,1)}) = (-1,1) \neq (-1,0) \cup (0,1)$. □

Problem 4. *Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?*

Proof. No. Consider the constant function, $f(x) = a$ for $a \in Y$. Then $f(A) = \{a\}$ and $f(x) = \{a\}$ so it's impossible for an open set containing $f(x)$ to contain any point of $f(A)$ other than $f(x)$. The statement holds, however, if f is injective as the following proof shows.

Let U be a neighborhood of $f(x)$. Then $f^{-1}(U)$ is an open set in X containing x. Therefore there exists some $y \in f^{-1}(U)$ such that $y \in A$ and $y \neq x$. Then $f(y) \in f(A)$ and since $y \in f^{-1}(U)$, $f(y) \in U$. But also, f is injective so $f(x) \neq f(y)$. Since U is arbitrary, we see that $f(x)$ must be a limit point of $f(A)$. □

Problem 5. *Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.*
(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.
(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous.

Proof. (a) Let $h : X \to Y \times Y$ be defined as $h(x) = (f(x), g(x))$. Since f and g are continuous, we also have that h is continuous. Let $A = \{(x,y) \mid x \leq y\}$ and note that $h^{-1}(A) = \{x \in X \mid h(x) \in A\} = \{x \in X \mid f(x) \leq g(x)\}$. Therefore, it suffices to show that A is closed, or equivalently, that it's complement B is open. Let $(x,y) \in B$ so we have $y < x$. Note that since Y is ordered in the order topology, it is a Hausdorff space. Namely, we can find open intervals (a,b) and (c,d) containing x and y so that $c < y < d \leq a < x < b$. Now consider the basis element $(a,b) \times (c,d)$. A point $(x_0, y_0) \in (a,b) \times (c,d)$ has the property that $c < y_0 < d \leq a < x_0 < b$. In particular $y_0 < x_0$ so we have $(a,b) \times (c,d) \subseteq B$. Thus, any point of B has a basis element containing it which is contained in B so B must be open and A is closed. Then since h is continuous, we also have $h^{-1}(A) = \{x \mid f(x) \leq g(x)\}$ is closed.

(b) From part (a) we know the set $A = \{x \mid f(x) \leq g(x)\}$ is closed and a similar proof shows that $B = \{x \mid g(x) \leq f(x)\}$ is closed. It's clear that $A \cup B = X$ and $f(x) = g(x)$ for $x \in A \cap B$. Then it immediately follows that $h : X \to Y$ where $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is continuous. But this is precisely the function $h(x) = \min\{f(x), g(x)\}$. □

Problem 6. *Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f \mid A_\alpha$, is continuous for each α.*
(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.
(c) An indexed family of sets $\{A_\alpha\}$ is said to be locally finite *if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.*

Proof. (a) We use induction on n, the number of sets in $\{A_\alpha\}$. The case when $n = 2$ is a special case of the pasting lemma, since both functions in this case are $f \mid A_1$ and $f \mid A_2$, so they clearly agree on the intersection. Now suppose that for some positive integer n we have that f is continuous. Since a finite union of closed sets is closed, we have $\bigcup_{\alpha=1}^{n} A_\alpha$ is closed. By assumption, f restricted to this set is continuous, as is $f \mid A_{n+1}$. Therefore, by the same reasoning as in the $n = 2$ case, we have f is continuous on the union $\bigcup_{\alpha=1}^{n} A_\alpha$. Thus by induction, f is continuous if the collection $\{A_\alpha\}$ is finite.

(b) Let $f : [0,1] \to \mathbb{R}$ where both sets have the order topology. For $p \in \mathbb{Q}$ with $0 < p < 1$ define $f \mid [0,p]$ as $f(x) = x$ and $f(1) = 2$. Note that since $[0,1]$ has the order topology, it's a Hausdorff space and so $\{1\}$ is closed. Thus f is continuous on each of the closed sets $[0,p]$ and the closed set $\{1\}$ and $[0,1] = \{1\} \cup \bigcup_{p \in \mathbb{Q}} [0,p]$. But f is not continuous. Namely, $f^{-1}(3/2, 5/2) = \{1\}$ which is not open.

(c) Let $f : X \to Y$ and let U be a closed subset of Y. Then $f^{-1}(U) \cap A_\alpha$ is closed for each α since A_α is closed and f is continuous when restricted to A_α. Note that since $\{A_\alpha\}$ is locally finite, it follows that $\{f^{-1}(U) \cap A_\alpha\}$ is locally finite as well. Thus, it suffices to show that the union of $\{f^{-1}(U) \cap A_\alpha\}$ is closed.

Let x be a point not in this union and let V be a neighborhood of x which intersects some n sets in this collection. Call them $V_1, \dots, V_n$. Now for each V_i choose an open set V_i' which doesn't intersect V_i. Then $\bigcap_i V_i'$ will contain x and have no intersection with any element of $\{f^{-1}(U) \cap A_\alpha\}$. Furthermore, since there are only finitely many V_i', this is an open set which shows the intersection of $\{f^{-1}(U) \cap A_\alpha\}$ is closed. But note that this intersection is precisely the set $f^{-1}(U)$, so f must be continuous since the preimage of a closed set is closed. ☐

Problem 7. *Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \dots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies. Justify your answer.*

Proof. First consider $\mathbb{R}^\omega$ in the box topology. Let $\mathbf{x}$ be an element of $\mathbb{R}^\omega \backslash \mathbb{R}^\infty$. Then $\mathbf{x}$ contains infinitely many nonzero coordinates. For each coordinate x_i of $\mathbf{x}$ different from 0 we can find an interval containing x_i which doesn't contain 0. For every coordinate where $x_i = 0$ simply take an interval around 0. Then the product of all of these intervals is an open set in the box topology. Since infinitely many coordinates of this product don't contain 0, this set cannot contain any elements from $\mathbb{R}^\infty$. Therefore the closure of $\mathbb{R}^\infty$ is itself in the box topology.

Now consider $\mathbb{R}^\omega$ in the product topology. Let $\mathbf{x} \in \mathbb{R}^\omega$ and let U be an open set in $\mathbb{R}^\omega$ containing $\mathbf{x}$. Note that all but finitely many coordinates of U are $\mathbb{R}$ and so U contains some element of $\mathbb{R}^\infty$. Thus, for an arbitrary element of $\mathbb{R}^\omega$ and any open set containing it, that open set intersects $\mathbb{R}^\infty$. Thus $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ in the product topology. ☐

Problem 8. *Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.*
(a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.
(b) Let

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$

and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Show that $\mathcal{T}$ is a subbasis for $\mathcal{T}$.
(c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and on;y if each map $f_\alpha \circ g$ is continuous.
(d) Let $f : A \to \prod X_\alpha$ be defined by the equation

$$f(x) = (f_\alpha(a))_{\alpha \in J};$$

let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

Proof. (a) Let $\{\mathcal{T}_i\}$ denote the set of all topologies on A for which f_α is continuous. Now let $\mathcal{T} = \bigcap_i \mathcal{T}_i$. The empty set and A are in $\mathcal{T}$ since they are both in each of the $\mathcal{T}_i$. An arbitrary union and a finite intersection of elements of $\mathcal{T}$ are in $\mathcal{T}$ since each of the elements involved in the union and intersection are in each of the $\mathcal{T}_i$ and each of these is closed under arbitrary union and finite intersection. Now consider any topology $\mathcal{U}$ of A such that each f_α is continuous. By assumption this is some $\mathcal{T}_i$ and so $\mathcal{T} \subseteq \mathcal{U}$.

(b) It's clear that the union of all the elements in $\mathcal{S}$ is A since $f_\alpha^{-1}(X_\alpha) = A$ and X_α is open in itself. Let $\mathcal{U}$ be any topology on A such that f_α is continuous for each α. Then $\mathcal{U}$ necessarily contains at the very least every preimage of open sets in X_α. Furthermore, $\mathcal{U}$ is a topology so it contains finite intersections and arbitrary unions of these sets. But these are precisely the elements of the topology generated by $\mathcal{S}$. Thus, this topology is coarser than $\mathcal{U}$ and since $\mathcal{U}$ is arbitrary, we must have that the topology generated by $\mathcal{S}$ is $\mathcal{T}$.

(c) Suppose $f_\alpha \circ g$ is continuous and let $U \in \mathcal{T}$ be a subbasis element from part (b). Note that this implies $U = f_\alpha^{-1}(V)$ where V is open in X_α for some α because of how we constructed the subbasis for $\mathcal{T}$ in part (b). Then $(f_\alpha \circ g)^{-1}(V) = g^{-1}(f_\alpha^{-1}(V)) = g^{-1}(U)$ is open since $f_\alpha \circ g$ is continuous. Therefore g^{-1} takes subbasis elements to open sets and is continuous. For the converse, suppose that g is continuous. Since we're using the topology $\mathcal{T}$ on A, we know f_α is continuous as well, so this is merely a composition of two continuous functions and $f_\alpha \circ g$ is continuous.

(d) Note that f is continuous because each f_α is continuous. Furthermore, note that $f^{-1}(Z) = A$ since everything in A is mapped to Z. Since A is open in A, Z is open in $\prod X_\alpha$. Note also that $f' : A \to Z$ is surjective where f' is just f with a restricted domain. Now consider an open set $U \subseteq Z$. Since f' is continuous, $f'^{-1}(U)$ is open in A and must be some element of $\mathcal{T}$. But since f' is surjective we also have $f'(f'^{-1}(U)) = U$. Therefore this element of $\mathcal{T}$ is actually an open set in Z under f' and therefore also under f since Z is open.

Note that any element of $\mathcal{T}$ can be found this way because $\mathcal{T}$ has preimages of open sets in X_α as a subbasis. Namely, if $U \in \mathcal{T}$ is an arbitrary union of finite intersections of the sets $f^{-1}(U_\beta$ for $U_\beta \subseteq X_\beta$, then we can identify U_β with the subset of $\prod X_\alpha$ which has U_β in the β^{th} coordinate and X_α in all other coordinates. Then taking the intersection and union of these sets is preserved under f^{-1} and is precisely the same as $U \in \mathcal{T}$. $\square$