Kris Harper
MATH 20900
May 11, 2009

Homework 6

** **Problem 1.** *Let (f_n) be an increasing sequence of nonnegative, measurable functions on X and f a function on X such that*

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Then f is measurable and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Proof. Because (f_n) is monotonically increasing, we know that $f(x) = \sup_n f_n(x)$ and thus f is measurable since f_n is measurable for all n. Also, since $f \geq f_n$ for all n by monotonicity of the Lebesgue integral we immediately have

$$\sup_n \int_X f_n d\mu \leq \int_X f d\mu.$$

Let S be the set of all simple functions on X such that $0 \leq s \leq f$. For $\alpha < 1$ and $s \in S$ define

$$E_n = \{x \in X \mid f_n(x) \geq \alpha s(x)\}.$$

Note that E_n is measurable and $E_n \subseteq E_{n+1}$. Additionally, $\bigcup_n E_n = X$ since $\lim_{n\to\infty} f_n(x) = f(x) \geq s(x) > \alpha s(x)$. Furthermore

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq \alpha \int_{E_n} s d\mu.$$

We now use the measure $\nu(E) = \int_X s d\mu$ to obtain

$$\sup_n \int_X f_n d\mu \geq \alpha \int_X s d\mu.$$

Note that this inequality is true for every $\alpha < 1$ and every $s \leq f$. Then we have

$$\sup_n \int_X f_n d\mu \leq \sup_{s,\alpha} \int_X s d\mu = \int_X f d\mu.$$

Since both inequalities are satisfied, the proof is complete. □

** **Problem 2.** *Suppose f and g are measurable. Then $f+g$ is measurable and if f and g are nonnegative then*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Proof. We know that $f+g$ is measurable because

$\{x \in X \mid (f+g)(x) \leq a, a \in \mathbb{R}\} = (\{f = -\infty, g \neq \infty\} \cup \{f \neq \infty, g = -\infty\}) \cup (\{x \in X \mid f(x) < a - g(x), a \in \mathbb{R}\} \cap \{f \neq \pm\infty, g \neq$

and we can pick a rational number between f and $a-g$. Now we have

$$\int_X (f+g) d\mu = \sup_{s \leq f+g} \int_X s d\mu = \sup_{s \leq f, t \leq g} \int_X (s+t) d\mu.$$

Let α_i and β_i be constants and A_i and B_i sets such that $s(x) = \sum_{i=1}^k \alpha_i \chi_{A_i}$ and similarly for $t(x)$. Then if $E_i j = A_i \cap B_j$ we have

$$\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$$

and

$$\int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu = \alpha_i \mu(E_{ij}) + \beta_i \mu(E_{ij}).$$

Thus, the statement holds for the sets E_{ij}. But note that E_{ij} is a disjoint union of X. Finally note that we can use the $\nu(E_{ij})$ metric so that the final statement holds for all simple functions. Then since f and g are simply supremums of the integrals of simple functions, we have the desired result. $\square$