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Homework 3

Exercise 1 *Let*

$$A_1 \supseteq A_2 \supseteq A_3 \supset \dots$$

be a sequence of closed nonempty subsets of $\mathbb{R}$. *Assume that* A_1 *is bounded. Then*

$$\bigcap_n A_n \neq \emptyset.$$

Proof. Since A_1 is bounded and any A_i is a subset of A_1 we have A_i is bounded as well. Thus there exists a greatest lower bound for every set in the sequence. Let $I = \{x \in \mathbb{R} \mid x = \inf A_i \text{ for some } A_i\}$. I is nonempty and it is bounded above by $\sup A_1$ since every element of I is less than every element of some A_n, and every element of A_n is less than $\sup A_1$. So $\sup I$ exists. Suppose to the contrary that $\sup I \notin \bigcap_n A_n$. Then there exists some A_i such that $\sup I \notin A_i$. We know that $\inf A_i \in A_i$ by Theorem 6.8 and so $\sup I$ cannot be a lower bound of A_i because it must be greater than or equal to $\inf A_i$ and not in A_i. Consider the case where $\sup I$ is between two elements of A_i. We have $\inf A_i$ is greater than or equal to $\inf A_j$ for $j \leq i$ and we have $\inf A_j \in A_i$ for $j \geq i$. But since $\sup I$ is not in A_i and A_i is closed, $\sup I$ is not a limit point of A_i and so there exists a disjoint region from A_i which contains $\sup I$. But then there exists some other point in this region which is less than $\sup I$ and still greater than every point in I since all of these points are in A_i or are less than $\inf A_i$. This is a contradiction and so $\sup I$ is not between two elements of A_i. So we have $\sup I$ is an upper bound for A_i. But $\sup A_i \in A_i$ and so $\sup A_i < \sup I$. But we must have $\sup A_i$ is greater than every greatest upper bound of all the sets otherwise two sets would be disjoint. So then we have an upper bound for I which is less than $\sup I$. This is a contradiction and so $\sup I \in \bigcap_n A_n$. □

Exercise 2 *Show that Exercise 1 does not hold for open intervals.*

Proof. Define a series of sets where $A_1 = (0; 1)$ and $A_n = (0; \frac{1}{n})$. Suppose that $\bigcap_n A_n \neq \emptyset$. Then suppose $x \in \bigcap_n A_n$. We have $x \in \mathbb{R}$ and so there exists some $q \in (0; x)$ such that q is rational. But then since $0 < q < 1$ and $q \in \mathbb{Q}$, by the Archimedean property there exists an integer k such that $\frac{1}{q} < k$. But then $\frac{1}{k} < q < x$ and so $x \notin (0; \frac{1}{k})$. This means that $x \notin \bigcap_n A_n$ and so the intersection must be empty. □

Exercise 3 *Show that Exercise 1 does not hold if we omit boundedness.*

Proof. Consider $\mathbb{N}$ and make a series of subsets of $\mathbb{N}$ where each succeeding subset removes the least element from the previous one. That is $A_{n+1} = A_n \backslash \{\text{the least element of } A_n\}$. Each succeeding subset in this sequence is a subset of the previous one, but the intersection of all of them will be empty because every natural number is eventually excluded from some set. □

Exercise 4 *Let* $A_1, A_2, \dots$ *be a sequence of closed intervals. Assume that for all* $i, j > 0$, $A_i \cap A_j \neq \emptyset$. *Show that*

$$\bigcap_n A_n \neq \emptyset.$$

Proof. Let $\mathcal{A}$ be the set of all A_i in the sequence. Let $I = \{x \in \mathbb{R} \mid x = \inf A_i \text{ for some } A_i\}$ and let $S = \{x \in \mathbb{R} \mid x = \sup A_i \text{ for some } A_i\}$. We know that I is nonempty and bounded above because every A_i is bounded above. So we have $\sup I$ exists and by a similar argument $\inf S$ exists. Note that for all $A_i, A_j \in \mathcal{A}$ we have $\inf A_i < \sup A_j$ because $A_i \cap A_j \neq \emptyset$. Also note that $\sup I$ is either in I or is a limit point of I and the same is true for $\inf S$ and S by Theorem 6.8. Assume that $\sup I > \inf S$. There are four cases:

Case 1: Let $\sup I \in I$ and $\inf S \in S$. This is a contradiction because every element of I and S is either $\inf A_i$ or $\sup A_i$ for some $A_i \in \mathcal{A}$ and we never have $\inf A_i > \sup A_j$.

Case 2: Let $\sup I \in I$ and let $\inf S$ be a limit point of S. Then we let $(a; b)$ be a region containing $\inf S$ such that there exists some $x \in S$ where $x \in (a; b)$. Since this is true for any region containing $\inf S$, suppose that $b < \sup I$. But then we have some $x \in S$ and $\sup I \in I$ such that $x < \sup I$ which is a contradiction.

Case 3: Let $\inf S \in S$ and let $\sup I$ be a limit point of I. This is a contradiction by a similar argument to *Case 2*.

Case 4: Let $\inf S$ be a limit point of S and let $\sup I$ be a limit point of I. So there exist two disjoint regions, $(a; b)$ and $(b; c)$ such that $\inf S \in (a; b)$ and $\sup I \in (b; c)$ and there exist elements $s \in S$ and $i \in I$ such that $s \in (a; b)$ and $i \in (b; c)$. But then we have $s < i$ which is a contradiction.

So we have $\sup I \leq \inf S$. In the case where $\inf S = \sup I$ we have $\inf S \leq \sup A_i$ and $\sup I \geq \inf A_i$ for all $A_i \in \mathcal{A}$. But then $\inf S \leq \sup A_i$ and $\inf S \geq \inf A_i$ for all $A_i \in \mathcal{A}$. But by definition $A_i = [\inf A_i; \sup A_i]$ and so we have $\inf S \in \bigcap_n A_n$. In the case where $\sup I < \inf S$ we consider $x \in [\sup I; \inf S]$. Then $x \leq \sup A_i$ and $x \geq \inf A_i$ for all $A_i \in \mathcal{A}$. Thus we have $x \in \bigcap_n A_n$. □

Exercise 5 *Show that if A is an uncountable set of positive real numbers, then there exists $\varepsilon > 0$ such that there are uncountably many elements of A that are bigger than ε.*

Proof. Suppose that for an uncountable set of positive reals and for all $\varepsilon > 0$ there are countably many elements of this set greater than ε. Then consider some $\varepsilon > 0$ and an uncountable set of positive reals A. We have countably many elements of A greater than ε and since two countable sets will union to a countable set, there must be uncountably many elements of A less than ε. But then consider the reciprocals of every element in A. We now have an uncountable set with countably many elements less than $1/\varepsilon$ and uncountably many elements of greater than $1/\varepsilon$. But $1/\varepsilon > 0$ and so this is a contradiction. □

Exercise 6 *Is there an uncountable set of pairwise disjoint real regions?*

No.

Proof. Let S be a set of pairwise disjoint real regions. Every element of S contains some rational number because between every two real numbers there exists a rational number. But since every two elements are disjoint, we have every element containing a unique rational number. But then we can make a function from S to a subset of $\mathbb{Q}$ by mapping each element of S to a rational representative. This function is clearly surjective and it must be injective because every element maps to a distinct and unique rational number. This subset of $\mathbb{Q}$ is countable since $\mathbb{Q}$ is countable. Thus S is countable. □

Exercise 7 *Let A be an uncountable subset of the reals. Show that there exists $a \in A$ which is a limit point of A.*

Proof. Assume that there exists no limit point $a \in A$ for an uncountable set A. For all $x \in A$ there exists some region $(a; b)$ such that $x \in (a; b)$. Choose these regions to be disjoint as in Exercise 6 from Homework 2. But now we have an uncountable set of pairwise disjoint real regions which is a contradiction from Exercise 6. Thus there must exist some limit point of A in A. □