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MATH 16300
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Homework 2

Problem 13.4 *Let P be a partition $P = \{t_0, \dots, t_n\}$ such that the ratio $r = t_i/t_{i-1}$ is equal for $1 \le i \le n$. Then we have*

$$t_i = a\,(c)^{\frac{i}{n}}.$$

for $c = b/a$.

Proof. Note that

$$\frac{b}{a} = \frac{t_n}{t_0} = \frac{t_n}{t_{n-1}} \cdot \frac{t_{n-1}}{t_{n-2}} \cdot \dots \cdot \frac{t_1}{t_0} = r^n$$

so $r = (b/a)^{1/n} = c^{1/n}$. In a similar fashion,

$$\frac{t_i}{a} = r^i$$

so

$$t_i = ar^i = a\,(c)^{\frac{i}{n}}.$$

□

If $f(x) = x^p$ then show

$$U(f, P) = (b^{p+1} - a^{p+1})c^{p/n} \cdot \frac{1}{1 + c^{1/n} + \dots + c^{p/n}}$$

and find a similar statement about $L(f, P)$.

Proof. We have

$$\begin{aligned}
U(f, P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left(ac^{i/n}\right)^p \left(ac^{i/n} - ac^{(i-1)/n}\right) \\
&= a^{p+1}(1 - c^{-1/n}) \sum_{i=1}^{n} \left(c^{(p+1)/n}\right)^i \\
&= a^{p+1}(1 - c^{-1/n})c^{(p+1)/n} \sum_{i=0}^{n-1} \left(c^{(p+1)/n}\right)^i \\
&= a^{p+1}(1 - c^{-1/n})c^{(p+1)/n} \frac{1 - c^{p+1}}{1 - c^{(p+1)/n}} \\
&= a^{p+1}(1 - c^{(p+1)})c^{(p+1)/n} \frac{1 - c^{-1/n}}{1 - c^{(p+1)/n}} \\
&= (a^{p+1} - b^{(p+1)})c^{(p+1)/n} \frac{1 - c^{-1/n}}{1 - c^{(p+1)/n}} \\
&= (a^{p+1} - b^{(p+1)})c^{p/n} \frac{c^{1/n} - 1}{1 - c^{(p+1)/n}} \\
&= (b^{p+1} - a^{p+1})c^{p/n} \frac{1}{1 + c^{1/n} + \dots + c^{p/n}}.
\end{aligned}$$

A similar proofs shows that

$$L(f,P) = (b^{p+1} - a^{p+1})\frac{1}{1 + c^{1/n} + \cdots + c^{p/n}}.$$

$\square$

Show that

$$\int_a^b x^p dx = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

Proof. We take

$$\lim_{n\to\infty}(b^{p+1} - a^{p+1})c^{p/n}\frac{1}{1 + c^{1/n} + \cdots + c^{p/n}} = \frac{b^{p+1} - a^{p+1}}{p+1}$$

because $\lim_{n\to\infty} c^{i/n} = 1$. $\square$

Problem 13.11 *Which functions have the property that every lower sum equals every upper sum?*

Proof. We have

$$\sum_{i=1}^n m_i(t_i - t_{i-1}) = \sum_{i=1}^n M_i(t_i - t_{i-1})$$

and so $m_i = M_i$ for all $1 \le i \le n$ regardless of our partition. But then f must be constant on $[a;b]$. $\square$

Which functions have the property that some upper some equals some other lower sum?

Proof. Let P_1 and P_2 be partitions on $[a;b]$. Then if $L(f,P_1) = U(f,P_2)$ and P contains both P_1 and P_2 then we have $L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2) = L(f,P_1)$ so $L(f,P) = U(f,P)$ which means f is constant again. $\square$

Which continuous functions have the property that all lowers sums are equal?

Proof. Only constant functions again. If not, then we can choose a minimum value, m, on $[a;b]$ and take a partition such that f is greater than m on some interval. Then the lower sum will be greater than $m(b-a)$ but if we just use one interval then $L(f,[a;b]) = m(b-a)$. $\square$

Which integrable functions have the property that all lower sums are equal?

Proof. Problem 13.30 shows that f is continuous at infinitely many points on $[a;b]$ which means that we can use the above proof to show that f must be constant everywhere. $\square$

Problem 13.15 *Show*

$$\int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{t}dt = \int_1^{ab} \frac{1}{t}dt.$$

Proof. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[1,a]$. We have
$b\inf\{1/t \mid t_{i-1} \le x \le t_i\} = \inf\{1/t \mid bt_{i-1} \le x \le bt_i\}$. Let P' and m_i' correspond to the second inf. Then

$$L(f,P') = \sum_{i=1}^n m_i'(bt_i - bt_{i-1}) = \sum_{i=1}^n bm_i'(t_i - t_{i-1}) = \sum_{i=1}^n m_i(t_i - t_{i-1}) = L(f,P).$$

Thus the interval $[1;a]$ has been mapped to the interval $[b;ab]$ but since $f(t) = 1/t$ we still have

$$\int_1^a \frac{1}{t}dt = \int_b^{ab} \frac{1}{t}dt.$$

But then
$$\int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{t}dt = \int_1^b \frac{1}{t}dt + \int_b^{ab} \frac{1}{t}dt = \int_1^{ab} \frac{1}{t}dt.$$
□

Problem 13.27 *Let f be integrable on $[a;b]$. Then for all $\varepsilon > 0$ there exists continuous functions $g \leq f \leq h$ with*
$$\int_a^b h - \int_a^b g < \varepsilon.$$

Proof. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a;b]$ and let $\varepsilon > 0$. First create step functions on $[a;b]$ where the value of each function on the ith interval equals m_i or M_i respectively. Then the integral for each step function is just the lower and upper sum for f, the difference of which we know is less than ε. Now connect the step functions by making a line from $f(t_{i-1})$ to m_i at some value in $[t_{i-1};t_i]$ so that a triangle is formed. Do this for the upper step function as well. The area of one of these triangles is $1/2(m_i - m_{i-1})(b_i)$ where b_i is the necessary value on $[t_{i-1};t_i]$. But since there are a finite number of intervals we can take b_i small enough such that
$$frac12 \sum_{i=1}^n (M_i - M_{i-1})B_i - frac12 \sum_{i=1}^n (m_i - m_{i-1})b_i < \varepsilon - U(f,P) + L(f,P).$$
□

Problem 13.30 *Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a;b]$ with $U(f,P) - L(f,P) < b - a$. Show that for some i we have $M_i - m_i < 1$.*

Proof. Note that
$$1 > \frac{U(f,P) - L(f,P)}{b-a} = \frac{\sum_{i=1}^n M_i(t_i - t_{i-1}) - \sum_{i=1}^n m_i(t_i - t_{i-1})}{b-a} = \frac{(b-a)\left(\sum_{i=1}^n M_i - \sum_{i=1}^n m_i\right)}{b-a} = \sum_{i=1}^n M_i - \sum_{i=1}^n m_i$$
and so there must exists i such that $M_i - m_i < 1$. □

Show that there are numbers a_1 and b_1 such that $a < a_1 < b_1 < b$ and $\sup\{f(x) \mid a_1 \leq x \leq b_1\} - \inf\{f(x) \mid a_1 \leq x \leq b_1\} < 1$.

Proof. From before we know there exists i such that $M_i - m_i < 1$. But then if we let $[a_1;b_1] = [t_{i-1};t_i]$ we're done so long as $i \neq 1$ and $i \neq n$. In the case where $i = 1$ we have $a_1 \in [a;b_1]$ and we already know that since $[a;a_1] \subseteq [a;b_1]$ we have $\sup\{f(x) \mid a_1 \leq x \leq b_1\} \leq \sup\{f(x) \mid a \leq x \leq b_1\}$ and a similar statement holds for inf and in the case where $i = n$. □

Show that there are numbers a_2 and b_2 with $a_1 < a_2 < b_2 < b_1$ and $\sup\{f(x) \mid a_2 \leq x \leq b_2\} - \inf\{f(x) \mid a_2 \leq x \leq b_2\} < 1/2$.

Proof. Choose a partition P of $[a_1;b_1]$ such that $U(f;P) - L(f,P) < (b_1 - a_1)/2$. Then $M_i - m_i < 1/2$ for some i. Choose $[a_2;b_2] = [t_{i-1};t_i]$ unless $i = 1$ or $i = n$ in which case we use a similar method as above. □

Find a sequence of intervals $I_n = [a_n;b_n]$ such that $\sup\{f(x) \mid x \in I_n\} - \inf\{f(x) \mid x \in I_n\} < 1/n$.

Proof. Let $x \in I_n$ for all n. We know x exists from the Nested Interval Theorem. Then $x \neq a_n$ and $x \neq b_n$ for all n because $x \in [a_{n+1};b_{n+1}]$ and $a_n < a_{n+1} < b_{n+1} < b_n$. For $\varepsilon > 0$ there exists some n such that $1/n < \varepsilon$ and so there exists n such that
$$\sup\{f(x) \mid x \in I_n\} - \inf\{f(x) \mid x \in I_n\} < \varepsilon/2.$$
Thus if $\delta = \min(x - a_n, x - b_n)$ then for all $y \in [a;b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. □

Show that f is continuous at infinitely many points in $[a;b]$.

Proof. We have f is continuous at some point for every interval contained in $[a;b]$ since f is integrable on each interval. There are infinitely many of these. $\square$

Problem 13.39 *Let f and g be integrable on $[a;b]$. Show*

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right)\left(\int_a^b g^2\right).$$

Proof. Note that for all $c \in \mathbb{R}$ we have

$$0 \leq \int_a^b (f-cg)^2 = c^2 \int_a^b g^2 - 2c \int_a^b fg + \int_a^b f^2$$

and from the quadratic formula we have

$$4\left(\int_a^b f^2\right)\left(\int_a^b g^2\right) \geq 4\left(\int_a^b fg\right)^2.$$

$\square$

Problem 14.7 *Find all continuous functions f such that*

$$\int_0^x f = (f(x))^2 + C$$

for some constant C.

Proof. If we differentiate f^2 we have $f(x) = 2f(x)f'(x)$ which means that for all $x \neq 0$ we have $f'(x) = 0$ for the equality to hold. Then f is constant on intervals where f is nonzero and since f is continuous it must be constant everywhere. Thus for all x we have

$$\int_0^x c = c^2 + C$$

so $cx = c^2 + C$ which can only be true if $c = 0$. $\square$