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Homework 1

**** Problem 1.** *For two partitions P and P' such that $P \subseteq P'$, we have $L(f,P) \leq L(f,P')$ and $U(f,P') \leq U(f,P)$.*

Proof. Suppose first that P' contains just one more point that P and write $P = \{a_0, a_2, \ldots, a_n\}$ and $P' = \{a_0, a_2, \ldots a_{k-1}, b, a_k, \ldots a_n\}$. Let $m_1 = \inf\{f(x) \mid a_{k-1} \leq x \leq b\}$ and $m_2 = \inf\{f(x) \mid b \leq x \leq a_k\}$. We have

$$L(f,P) = \sum_{i=1}^{n} m_i(a_i - a_{i-1})$$

and

$$L(f,P') = \sum_{i=1}^{k-1} m_i(a_i - a_{i-1}) + m_1(b - a_{k-1}) + m_2(a_k - b) + \sum_{i=k+1}^{n} m_i(a_i - a_{i-1}).$$

Note that

$$\{f(x) \mid a_{k-1} \leq x \leq b\} \subseteq \{f(x) \mid a_{k-1} \leq x \leq a_k\}$$

and

$$\{f(x) \mid b \leq x \leq a_k\} \subseteq \{f(x) \mid a_{k-1} \leq x \leq a_k\}.$$

Thus $m_k \leq m_1$ and $m_k \leq m_2$. Therefore

$$m_k(a_k - a_{k-1}) = m_k(b - a_{k-1}) + m_k(a_k - b) \leq m_1(b - a_{k-1}) + m_2(a_k - b)$$

and so $L(f,P) \leq L(f,P')$. Now suppose that P' contains n more points that P. Then we can create a sequence of partitions, each with one more point than the one before it, $P, P_1, \ldots, P_{n-1}, P'$. Then

$$L(f,P) \leq L(f,P_1) \leq \cdots \leq L(f,P_{n-1}) \leq L(f,P').$$

A similar proof holds using the least upper bound to show that $U(f,P') \leq U(f,P)$. □

**** Problem 2.** *If P and P' are partitions then $L(f,P) \leq U(f,P')$.*

Proof. Consider $P'' = P \cup P'$. Then by **Problem 1 we have

$$L(f,P) \leq L(f,P'') \leq U(f,P'') \leq U(f,P').$$

□

**** Problem 3.** *Let f and g be integrable on $[a,b]$ and $\alpha \in \mathbb{R}$. Show the following:*
1) A function f is integrable on $[a,b]$ if and only if for all $\varepsilon > 0$ there exists $P \in \mathcal{P}$ such that $U(f,P) - L(f,P) < \varepsilon$.
2) The function $\alpha f + g$ is integrable on $[a,b]$ and

$$\int_a^b \alpha f + g = \alpha \int_a^b f + \int_a^b g.$$

3) If $f(x) \leq g(x)$ for $x \in [a,b]$ then

$$\int_a^b f \leq \int_a^b g.$$

4) The function $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

5) A function f is integrable on $[a,b]$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|P| < \delta$ implies $U(f,P) - L(f,P) < \varepsilon$.

Proof. 1) Suppose that for all $\varepsilon > 0$ there exists a partition, P, such that $U(f,P) - L(f,P) < \varepsilon$ and let $\varepsilon > 0$. Note that $\inf \mathcal{U}(f,P) \leq U(f,P)$ and $\sup \mathcal{L}(f,P) \geq L(f,P)$ so we have

$$\inf \mathcal{U}(f,P) - \sup \mathcal{L}(f,P) < \varepsilon.$$

Note that it's never the case that $\inf \mathcal{U}(f,P) < \sup \mathcal{L}(f,P)$ and if $\inf \mathcal{U}(f,P) > \sup \mathcal{L}(f,P)$ then we have $\inf \mathcal{U}(f,P) - \sup \mathcal{L}(f,P) > 0$. Then there exists $c \in \mathbb{R}$ such that

$$\inf \mathcal{U}(f,P) - \sup \mathcal{L}(f,P) > c > 0$$

and letting $c = \varepsilon$ we have a contradiction. Thus $\inf \mathcal{U}(f,P) = \sup \mathcal{L}(f,P)$ which shows that f is integrable on $[a,b]$. Conversely, assume that f is integrable on $[a,b]$. Then $\inf \mathcal{U}(f,P) = \sup \mathcal{L}(f,P)$. Thus for all $\varepsilon > 0$ there exist partitions P_1 and P_1 of $[a,b]$ such that $U(f,P_1) - L(f,P_2) < \varepsilon$. Letting P be a partition such that $P_1 \subseteq P$ and $P_2 \subseteq P$ we have

$$U(f,P) - L(f,P) \leq U(f,P_1) - L(f,P_1) \leq \varepsilon.$$

2) Let $P = \{a_0, \dots a_n\}$ be a partition of $[a,b]$. Define

$$m_i = \inf\{(f+g)(x) \mid a_{i-1} \leq x \leq a_i\},$$

$$m_i' = \inf\{f(x) \mid a_{i-1} \leq x \leq a_i\}$$

and

$$m_i'' = \inf\{g(x) \mid a_{i-1} \leq x \leq a_i\},$$

with M_i, M_i', and M_i'' defined in a similar fashion. Since f and g are bounded, we have $m_i \geq m_i' + m_i''$ and $M_i \leq M_i' + M_i''$. It then follows that $L(f,P) + L(g,P) \leq L(f+g,P)$ and $U(f,P) + U(g,P) \geq U(f+g,P)$ and so

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P).$$

Since f and g are integrable, for $\varepsilon > 0$, there exist partitions P_1 and P_2 such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$$

and

$$U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}.$$

If $P = P_1 \cup P_2$ then we have

$$(U(f,P) + U(f,P)) - (L(f,P) + L(f,P)) < \varepsilon$$

and so $U(f+g,P) - L(f+g,P) < \varepsilon$ which means $f+g$ is integrable on $[a,b]$. Also we have

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P).$$

for all partitions, P, of $[a,b]$. Thus

$$\sup \mathcal{L}(f,P) + \sup \mathcal{L}(g,P) \leq \sup \mathcal{L}(f+g,P) \leq \inf \mathcal{U}(f+g,P) \leq \inf \mathcal{U}(f,P) + \inf \mathcal{U}(f,P)$$

which means

$$\int_a^b f + \int_a^b g = \int_a^b f + g.$$

Now suppose that $\alpha \geq 0$. Then for all $\varepsilon > 0$ there exists some partition $P = \{a_0, \dots, a_n\}$ such that $U(f,P) - L(f,P) < \varepsilon/\alpha$. Then note that for all $1 \leq i \leq n$ if $m_i = \inf\{f(x) \mid a_{i-1} \leq x \leq a_i\}$ then $\alpha m_i = \inf\{\alpha f(x) \mid a_{i-1} \leq x \leq a_i\}$. A similar statement follows for M_i and αM_i. Thus

$$U(\alpha f,P) - L(\alpha f,P) = \sum_{i=1}^{n} (\alpha M_i - \alpha m_i)(t_i - t_{i-1}) = \alpha \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) = \alpha(U(f,P) - L(f,P)) < \varepsilon$$

which shows αf is integrable on $[a,b]$. Since $L(\alpha f,P) = \alpha L(f,P)$ for all partitions, P, we have

$$\int_a^b \alpha f = \sup \mathcal{L}(\alpha f,P) = \alpha \mathcal{L}(f,P) = \alpha \int_a^b f.$$

3) Suppose that $f(x) \leq g(x)$ for all $x \in [a,b]$. Then for some partition, $P = \{a_0, \dots, a_n\}$, we have

$$m_i = \inf\{f(x) \mid a_{i-1} \leq x \leq a_i\} \leq \inf\{g(x) \mid a_{i-1} \leq x \leq a_i\} = m_i'$$

and similarly for M_i and M_i'. Then

$$L(f,P) = \sum_{i=1}^{n} m_i(a_i - a_{i-1}) \leq \sum_{i=1}^{n} m_i'(a_i - a_{i-1}) = L(g,P).$$

Since this is true for all $P \in \mathcal{P}$ we must have

$$\int_a^b f \leq \int_a^b g.$$

4) Let $P = \{a_0, \dots, a_n\}$ be a partition. Define

$$m_i = \inf\{f(x) \mid a_{i-1} \leq x \leq a_i\}$$

and

$$m_i' = \inf\{|f(x)| \mid a_{i-1} \leq x \leq a_i\}.$$

Define M_i and M_i' similarly. If $f \geq 0$ on $[a_{i-1}, a_i]$ we have $m_i = m_i'$ and $M_i = M_i'$. Thus $M_i' - m_i' \leq M_i - m_i$. If $f(x) \leq 0$ on $[a_{i-1}, a_i]$ then $m_i = -M_i'$ and $m_i' = -M_i$. Thus $M_i' - m_i' \leq M_i - m_i$. Now suppose that f takes on negative and positive values on $[a_{i-1}, a_i]$. Then we have $m_i \leq 0 \leq M_i$. First suppose that $-m_i \leq M_i$. Then $M_i = M_i'$ and since $m_i < 0$ we have $M_i' - m_i' \leq M_i' = M_i \leq M_i - m_i$. We can consider $-f$ for the case where $-m_i \geq M_i$ and obtain the same result. Now let $\varepsilon > 0$ so that $U(f,P) - L(f,P) < \varepsilon$. Then since $M_i' - m_i' \leq M_i - m_i$ we have

$$U(|f|,P) - L(|f|,P) = \sum_{i=1}^{n} (M_i' - m_i')(a_{i-1} - a_i) \leq \sum_{i=1}^{n} (M_i - m_i)(a_{i-1} - a_i) = U(f,P) - L(f,P) < \varepsilon.$$

Thus $|f|$ is integrable on $[a,b]$. Moreover, we know that

$$\left|\sum_{i=1}^{n} m_i\right| \leq \sum_{i=1}^{n} |m_i|$$

from the triangle inequality. Then since $(a_i - a_{i-1}) \geq 0$ for $1 \leq i \leq n$ we have

$$L(f,P) = \left|\sum_{i=1}^{n} m_i(a_i - a_{i-1})\right| \leq \sum_{i=1}^{n} |m_i|(a_i - a_{i-1}) \leq \sum i = 1^n m_i'(a_i - a_{i-1}) = L(|f|,P).$$

Thus

$$|\sup \mathcal{L}(f,P)| = \left|\int_a^b f\right| \leq \int_a^b |f| = \sup \mathcal{L}(|f|,P).$$

5) Let $|f(x)| \leq M$ for some constant M. Suppose first that f is integrable on $[a,b]$. Let $\varepsilon > 0$ and choose a partition P' such that $U(f,P') - L(f,P') < \varepsilon/2$. Let N be the number of partition points in P' and let $\delta = \varepsilon/(8MN(b-a))$. Suppose that $|P| < \delta$. Using the common refinement of P and P', it follows that $U(f,P) - L(f,P) < \varepsilon$.

Conversely, suppose that for $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ we have $U(f,P) - L(f,P) < \varepsilon$. Let $\varepsilon > 0$ and consider a partition, P, such that $|P| < \delta$. This partition clearly exists. But then for all $\varepsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$. Therefore, f is integrable on $[a,b]$. □

** **Problem 4.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Show that f is Riemann-integrable on $[a,b]$.*

Proof. Note that since f is continuous on $[a,b]$ and $[a,b]$ is compact, f is uniformly continuous. Consider $\varepsilon/(b-a)$. Then there exists $\delta > 0$ such that for all $x, y \in [a,b]$ with $|x-y| < \delta$ we have $|f(x) - f(y)| < \varepsilon/(b-a)$. Now choose a partition $P = \{a_0, \ldots, a_n\}$ such that $|a_i - a_{i-1}| < \delta$ for all $1 \leq i \leq n$. Then if $x, y \in [a_i - a_{i-1}|$ we have $|f(x) - f(y)| < \varepsilon/(b-a)$. Since f is continuous we know that it takes on minimum and maximum values m_i and M_i on this interval so for all i we have $M_i - m_i < \varepsilon/(b-a)$. Thus

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(M_i - m_i)(a_i - a_{i-1}) < \frac{\varepsilon}{b-a}\sum_{i=1}^{n}(a_i - a_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Therefore f is integrable on $[a,b]$. □

** **Problem 5.** *Let $f : [a,b] \to \mathbb{R}$ be bounded. Show that f is Riemann-integrable if and only if f is continuous almost everywhere.*

Proof. Let A be the set of measure 0 on which f is not continuous. Let $\varepsilon > 0$ and let B_j be a series of intervals which cover A, such that $\sum_{j=1}^{\infty} Vol(B_j) < \varepsilon$. We want to find the intervals on which f has a large change in values. For $B \subseteq [a,b]$ define $d(B)$ to be $\sup_{x \in B} f(x) - \inf_{x \in B} f(x)$. Now let $J = \{j \in \mathbb{N} \mid d(B_j) > \varepsilon\}$ and let $V = \bigcup_{j \in J} B_j$. Note that the total length of V is still less than ε. We want to find a partition in which every interval has a small change in value or is in V. We consider equidistant partitions with each interval having length $(b-a)/N$.

Now suppose that for every $N \in \mathbb{N}$ we can can find i with $1 \leq i \leq N$ such that $d([a_i - a_{i-1}]) > \varepsilon$, but $[a_i - a_{i-1}] \cap V \neq \emptyset$. Then for every N we have $s_N, t_N, z_N \in [a_i - a_{i-1}]$ such that $d([a_i - a_{i-1}]) \geq f(s_N) - f(t_N) < \varepsilon$ and $z_N \in^c V$. The sequence (s_N) lies in $[a,b]$ and so it's bounded. Thus it has a convergent subsequence such that $\lim_{k\to\infty} s_{N_k} = y$. Since t_N and z_N have at most distance $(b-a)/N$ from s_N, they both converge to under the same subsequence to y as well. Note that f is discontinuous at y since $f(s_N) - f(t_N) > \varepsilon$ doesn't converge to 0. So $y \in A$ and $y \in B_j$ for some $j \in \mathbb{N}$. But also (z_N) is a sequence in ${}^cV$. Since V is open ${}^cV$ is closed and must contain the limit $\lim_{k\to\infty} z_{N_k} = y$. Thus $y \notin V$ and so $y \in B_j$ such that $d(B_j) \leq \varepsilon$. But B_j is open and thus must contain s_{N_k} and t_{N_k} for large enough k. Thus $\varepsilon < f(s_{N_k}) - f(t_{N_k}) \leq d(B_j) \leq \varepsilon$. Therefore, there exists $N \in \mathbb{N}$ such that for all i with $1 \leq i \leq N$, if $d([a_i - a_{i-1}]) > \varepsilon$ we have $[a_i - a_{i-1}] \subseteq V$.

Now let $\varepsilon' > 0$. Let $\varepsilon = \varepsilon'((b-a) + d([a,b]))^{-1}$. Note that $d([a,b])$ exists because f is bounded. Now we have

$$U(f,P) - L(f,P) = \sum_{i=1}^{N} \frac{b-a}{N} d([a_i - a_{i-1}]) \leq \sum_{i=1}^{N} \frac{b-a}{N}\varepsilon + C\frac{b-a}{N} d([a_i - a_{i-1}])$$

where C represents the number of intervals $[a_i - a_{i-1}]$ which are subsets of V. The total length of these intervals is $K(b-a)/N$, but they are all contained in V and overlap at most at endpoints. Thus the total length is bounded by ε since V is bounded by this. Then we have

$$U(f,P) - L(f,P) < (b-a)\varepsilon + \varepsilon d([a_i - a_{i-1}]) = \varepsilon'.$$

This shows that f is integrable on $[a,b]$.

Conversely, suppose that f is integrable on $[a,b]$. We can write $A = B_1 \cup B_{1/2} \cup B_{1/3} \cup \ldots$ where $B_{1/n} = \{x \in [a,b] \mid d([a,b]) \geq \varepsilon\}$. Let $\varepsilon > 0$ and choose a partition P such that $U(f,P) - L(f,P) < \varepsilon/n$. Let S be the set of subintervals of P which contain points in $B_{1/n}$. Then S is a cover of $B_{1/n}$. Now for $I \in S$ we have $\sup_{x\in I} f(x) - \inf_{x \in I} f(x) \geq 1/n$. Thus

$$\frac{1}{n}\sum_{I\in S} Vol(I) \leq \sum_{I \in S}\left(\sup_{x\in I} f(x) - \inf_{x\in I} f(x)\right) Vol(I) \leq \sum_{I}\left(\sup_{x\in I} f(x) - \inf_{x\in I} f(x)\right) Vol(I) < \frac{\varepsilon}{n}.$$

Therefore $\sum_{I\in S} Vol(S) < \varepsilon$. This shows that $B_{1/n}$ has measure 0 which shows that A has measure 0. ☐

**** Problem 6.** *What about the Fundamental Theorem of Calculus when f is not everywhere continuous?*

Proof. The second part of the fundamental theorem of calculus only requires that f have a primitive. If f is everywhere continuous then the result from the second part can be obtained from the first part. The second part strengthens this result by removing continuity from f and only assuming a primitive exists. ☐

**** Problem 7.** *Find*

$$\int_0^\infty \frac{\sin x}{x} dx$$

Proof. Let

$$f(a,b) = \int_0^\infty e^{-ax}\frac{\sin bx}{x} dx.$$

Differentiate with respect to a as

$$\frac{df}{da} = \frac{d}{da}\int_0^\infty e^{-ax}\frac{\sin bx}{x} dx.$$

Since the integrand and it's derivative are both continuous we can write

$$\frac{d}{da}\int_0^\infty e^{-ax}\frac{\sin bx}{x} dx = \int_0^\infty e^{-ax}\frac{\partial}{\partial a}\frac{\sin bx}{x} dx = \int_0^\infty e^{-ax}\sin(bx) dx.$$

Note that $e^{ibx} = \cos(bx) + i\sin(bx)$. Then if Im represents the imaginary part, we have

$$-\mathrm{Im}\int_0^\infty e^{-ax}e^{ibx} dx = \mathrm{Im}\frac{1}{-a+ib} = \mathrm{Im}\frac{-a-ib}{a^2+b^2} = \frac{-b}{a^2+b^2}.$$

Now we have
$$\int_0^\infty \frac{df}{da}da = \int_0^\infty \frac{-b}{a^2+b^2}da = -\lim_{a\to\infty} \arctan\frac{a}{b} + \arctan(0) = \frac{\pi}{2}.$$

□

**** Problem 8.** *1)* $\Gamma(1) = 1$.
2) $\Gamma(s+1) = s\Gamma(s)$.
3) If $n \in \mathbb{N}$ *then* $\Gamma(n+1) = n!$.

Proof. 1) We have
$$\Gamma(1) = \int_0^\infty e^{-t}dt = \lim_{a\to\infty}\int_0^a e^{-t}dt = \lim_{a\to\infty} -e^{-t} - (-1) = 1.$$

2) We have
$$\Gamma(s+1) = \int_0^\infty e^{-t}t^s dt.$$
Letting $u = t^s$ and $dv = e^{-t}dt$ we have $du = st^{s-1}dt$ and $v = -e^{-t}$. Then
$$\Gamma(s+1) = uv|_0^\infty - \int_0^\infty vdu = -t^s e^{-t}\big|_0^\infty + \int_0^\infty e^{-t}st^{s-1}dt = s\Gamma(s).$$

3) Use induction on n. We already know from Part 1) that $\Gamma(1) = 1$. Supposing that $\Gamma(n) = (n-1)!$, we consider $\Gamma(n+1)$. But then by Part 2) we have
$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!.$$

□

**** Problem 9.** *What happens if* $n < 0$ *for* $-n \in \mathbb{N}$ *and we take* $\lim_{s\to n^+}\Gamma(s)$ *and* $\lim_{s\to n^-}\Gamma(s)$*?*

Proof. Note that at 0 we have $\Gamma(0) = \int_0^\infty e^{-t}t^{-1}dt$ which approaches $+\infty$ from the right and $-\infty$ from the left. Now note that $\Gamma(-1) = \Gamma(0)/(-1)$. This means that the signs of the function are reversed so that $\Gamma(-1)$ approaches $-\infty$ from the right and $+\infty$ from the left. Inductively, we have $\lim_{s\to n^+}\Gamma(s) = +\infty$ and $\lim_{s\to n^-}\Gamma(s) = -\infty$ for n even. The signs are switched for n odd. □

**** Problem 10.** *Find* $\min_{s>0}\Gamma(s)$.

Proof. We have $\min_{s>0}\Gamma(s) = 1.46163\ldots$. □

**** Problem 11.** *Show* $\Gamma(1/2) = \sqrt{(\pi)}$.

Proof. We have
$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{-1/2}dt.$$
Letting $u = t^2$ we have
$$\Gamma\left(\frac{1}{2}\right) = 2\int_{u(0)}^\infty e^{-x^2} = \sqrt{\pi}.$$

□

**** Problem 12.** *Suppose* $\phi : [a,b] \to \mathbb{R}$ *is* C^1 *and* $\phi' > 0$ *on* $[a,b]$*. Then if* f *is integrable on* $[a,b]$ *then*
$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx.$$

Proof. Since ϕ and ϕ' are continuous, they are integrable and thus the above integrals exist. Let F be the function which has derivative f. This must exist by the fundamental theorem of calculus. Now consider $F \circ \phi : [a, b] \to \mathbb{R}$. Using the chain rule we have

$$(F \circ \phi)'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t).$$

Now using the fundamental theorem of calculus we have

$$\int_a^b f(\phi(t))\phi'(t)dt = (F \circ \phi)(b) - (F \circ \phi)(a) = F(\phi(b)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(x)dx.$$

$\square$