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Homework 3

Problem 1. *Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.*

Proof. Let $\mathcal{T}$ be the topology A inherits as a subspace of Y and let $\mathcal{T}'$ be the topology A inherits as a subspace of X. Let $B \in \mathcal{T}$. Then $B = U \cap A$ where U is open in Y. But then $U = V \cap Y$ where V is open in X. Note now that since $A \subseteq Y$, we have $B = V \cap Y \cap A = V \cap A$. Thus $C \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$.

Conversely, suppose that $C \in \mathcal{T}'$. Then $C = U \cap A$ where U is open in X. Since U is open in X, we know $V = U \cap Y$ is open in Y. But because $A \subseteq Y$, we have $C = U \cap A = U \cap A \cap Y = V \cap A$ where C is open in Y. Thus $C \in \mathcal{T}$ and $\mathcal{T}' = \mathcal{T}$. □

Problem 2. *If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what an you say about the corresponding subspace topologies on the subset Y of X?*

Proof. Let $\mathcal{U}$ and $\mathcal{U}'$ be the respective subspace topologies Y inherits from $\mathcal{T}$ and $\mathcal{T}'$. It's clear that $\mathcal{U} \subseteq \mathcal{U}'$. To see this, let $U \in \mathcal{U}$ and write $U = V \cap X$ where $V \in \mathcal{T}$. Then $V \in \mathcal{T}'$ as well, and so $U \in \mathcal{U}'$.

Now, if $Y = X$, then $\mathcal{U} = \mathcal{T}$ and $\mathcal{U}' = \mathcal{T}'$. In this case, we have that $\mathcal{U}'$ is strictly finer than $\mathcal{U}$. On the other hand, if $Y = \{x\}$ a single point, then Y inherits the indiscrete topology as a subspace. That is, any set from $\mathcal{T}$ or from $\mathcal{T}'$ intersected with Y will either be Y or $\emptyset$. In this case $\mathcal{U} = \mathcal{U}'$ and so we no longer have strict containment. Thus, while $\mathcal{U}'$ is necessarily finer than $\mathcal{U}$, it may or may not be strictly finer depending on Y. □

Problem 3. *Let X and X' prime denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$ respectively. Assume these sets are nonempty.*
(a) Show that if $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
(b) Does the converse of (a) hold? Justify your answer.

Proof. (a) Let $U \times V$ be a basis element for the product topology on $X \times Y$ and let $(u, v) \in U \times V$. Then $u \in U$ and $v \in V$ where U and V are open in X and Y respectively. Thus $U \in \mathcal{T}$ and $V \in \mathcal{U}$. By assumption then, $U \in \mathcal{T}'$ and $V \in \mathcal{U}'$ so $U \times V$ is a basis element of the product topology on $X' \times Y'$ which contains (u, v). Therefore the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

(b) Assume that the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$. Let $\mathcal{B}$ be a basis for $\mathcal{T}$, $\mathcal{C}$ be a basis for $\mathcal{U}$, $\mathcal{B}'$ be a basis for $\mathcal{T}'$ and $\mathcal{C}'$ be a basis for $\mathcal{U}'$. Let $x \in X$ and $y \in Y$ and let $B \in \mathcal{B}$ and $C \in \mathcal{C}$ be basis elements containing x and y respectively. Then $(x, y) \in B \times C$ and there exists a basis element $B' \times C'$ such that $(x, y) \in B' \times C'$ and $B' \times C' \subseteq B \times C$. But then $B' \subseteq B$, $C' \subseteq C$, $x \in B'$ and $y \in C'$. Thus $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$. □

Problem 4. *If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.*

Proof. Note that open intervals (a, b) form a basis for $\mathbb{R}$ and half-open intervals $[a, b)$ form a basis for $\mathbb{R}_\ell$. Thus, sets of the form $[a, b) \times (c, d)$ form a basis for $\mathbb{R}_\ell \times \mathbb{R}$. These are rectangles in the plane, where the "left side" is closed and the other three sides are open. Since these sets are a basis for $\mathbb{R}_\ell \times \mathbb{R}$, their intersection with L gives a basis for the subspace topology on L.

If L is a vertical line in the plane, then its intersection with any of these open sets is an open interval in L, and so the subspace topology is just the standard topology on $\mathbb{R}$. Now suppose L is not vertical. Then given a basis element of $\mathbb{R}_\ell \times \mathbb{R}$, L will either intersect the "left side" of this element or it won't. In the former case, the intersection forms a half-open interval $[a, b)$ in L and in the later case the intersection is an open interval (a, b) in L. But note that an open interval of this form can be expressed as an infinite union of half open intervals, so a basis for the subspace topology on L is given by half-open intervals $[a, b)$, which is the lower limit topology $\mathbb{R}_\ell$.

Now consider the basis elements for the product topology on $\mathbb{R}_\ell \times \mathbb{R}_\ell$. These are sets of the form $[a, b) \times [c, d)$ which are rectangles with the "left" and "bottom" sides closed and other sides open. Now if L is vertical or horizontal, its intersection with these basis elements gives half open intervals $[a, b)$ on L and so the subspace topology is $\mathbb{R}_\ell$ as above. On the other hand, if L has some positive slope, then L intersects one or two of the "left" and "bottom" sides of a given basis element and one of the "top" or "right" sides. In both cases, the intersection is a half open interval of the form $[a, b)$ and so the subspace topology is once again $\mathbb{R}_\ell$. Finally, suppose that L has negative slope. Then for each point on L there exists some basis element which intersects the corner where the "left" and "bottom" sides meet. Note that this is a single point, which means that every point in L is open. Thus, the subspace topology on L is the discrete topology. □

Problem 5. *Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.*

Proof. Let $\mathcal{T}$ be the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ and let $\mathcal{T}'$ be the product topology on $\mathbb{R}_d \times \mathbb{R}$. Let $x \times y \in \mathbb{R} \times \mathbb{R}$ and let $(a \times b, c \times d)$ be a basis element of $\mathcal{T}$ containing $x \times y$. First suppose that $a = c$. Then $(a \times b, c \times d) = \{a \times i \mid b < i < d\}$. But this is precisely the set $\{a\} \times (b, d)$ in $\mathcal{T}'$. Since $\{a\}$ is open in $\mathbb{R}_d$ and (b, d) is open in $\mathbb{R}$, this is a basis element of $\mathbb{R}_d \times \mathbb{R}$. This basis element contains $x \times y$ and is clearly contained in $(a \times b, c \times d)$. If it's the case that $a < c$, then the same argument follows since (a, c) is open in $\mathbb{R}_d$ as well. Thus $\mathcal{T} \subseteq \mathcal{T}'$.

Now let $x \times y \in \mathbb{R}_d \times \mathbb{R}$ and let $U \times (a, b)$ be a basis element of $\mathcal{T}'$ containing $x \times y$. This means that $x \in U$ and $a < y < b$. Note then that $(x \times a, x \times b)$ must contain $x \times y$ and is contained in $U \times (a, b)$. Since $(x \times a, x \times b)$ is a basis element from $\mathcal{T}$, we have $\mathcal{T}' \subseteq \mathcal{T}$ and since both inclusions hold, we must have $\mathcal{T} = \mathcal{T}'$.

Let $\mathcal{T}''$ be the standard topology on $\mathbb{R} \times \mathbb{R}$. Let $(a, b) \times (c, d)$ be a basis element of $\mathcal{T}''$ and let $x \times y \in (a, b) \times (c, d)$. But note that this set is also a basis element of $\mathcal{T}'$ since (a, b) is open in $\mathbb{R}_d$. Thus $\mathcal{T}'' \subseteq \mathcal{T}' = \mathcal{T}$. On the other hand, the element $\{a\} \times (b, c)$ is a basis element of $\mathcal{T}'$, but no basis element of $\mathcal{T}''$ is contained in this set since $\{a\}$ is not open in the standard topology on $\mathbb{R}$. Thus, the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as the product topology on $\mathbb{R}_d \times \mathbb{R}$ which is strictly finer than the standard topology on $\mathbb{R} \times \mathbb{R}$. □

Problem 6. *Prove Theorem 19.2.*

Proof. We first consider the box topology on $\prod_{\alpha \in J} X_\alpha$. Let $(x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$. Then $x_\alpha \in X_\alpha$ for each $\alpha \in J$. But since each X_α has a basis $\mathcal{B}_\alpha$, for each $\alpha \in J$ there exists some B_α which contains x_α. Then $(x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} B_\alpha$ so the first condition of bases is satisfied. For the second condition note that $\prod_{\alpha \in J} B_\alpha \cap \prod_{\alpha \in J} C_\alpha = \prod_{\alpha \in J} (B_\alpha \cap C_\alpha)$ where $B_\alpha, C_\alpha \in \mathcal{B}_\alpha$. Then since each $\mathcal{B}_\alpha$ is a basis, there exists a $D_\alpha \in \mathcal{B}_\alpha$ such that $D_\alpha \subseteq B_\alpha \cap C_\alpha$. But then $\prod_{\alpha \in J} D_\alpha \subseteq \prod_{\alpha \in J} B_\alpha \cap \prod_{\alpha \in J} C_\alpha$ so the second condition is also satisfied.

Now consider the product topology. Note that $\prod_{\alpha \in J} X_\alpha$ is one of the sets which we are considering since $B_\alpha \in \mathcal{B}_\alpha$ for finitely (namely zero) indices α and is equal to X_α for the remaining indices. Thus, the first condition of being a basis is trivially satisfied. Now suppose we have two such sets $\prod_{\alpha \in J} B_\alpha$ and $\prod_{\alpha \in J} C_\alpha$ where $B_\alpha, C_\alpha \in \mathcal{B}_\alpha$ for finitely many indices (not necessarily the same ones). Then $\prod_{\alpha \in J} B_\alpha \cap \prod_{\alpha \in J} C_\alpha = \prod_{\alpha \in J} (B_\alpha \cap C_\alpha)$. There are four possibilities for the sets involved in this product—$B_\alpha \cap X_\alpha$, $X_\alpha \cap C_\alpha$, $X_\alpha \cap X_\alpha$ or $B_\alpha \cap C_\alpha$. The first three cases evaluate to B_α, C_α and X_α respectively, and in the last case we know there exists some $D_\alpha \in \mathcal{B}_\alpha$ such that $D_\alpha \subseteq B_\alpha \cap C_\alpha$. Since all but finitely many of these terms are in $\mathcal{B}_\alpha$ we see that there exists some product containing finitely many basis elements from the sets $\mathcal{B}_\alpha$ which is a subset of the intersection $\prod_{\alpha \in J} B_\alpha \cap \prod_{\alpha \in J} C_\alpha$. This completes the second criterion for a basis and so we're done. □

Problem 7. *Prove Theorem 19.3.*

Proof. Let U be a basis element in $\prod_{\alpha \in J} A_\alpha$ when given the box topology. Then $U = \prod_{\alpha \in J} U_\alpha$ where each U_α is open in A_α. But since each A_α is a subspace of X_α, we can write $U_\alpha = V_\alpha \cap A_\alpha$ where V_α is open in X_α. Then $U = \prod_{\alpha \in J} U_\alpha = \prod_{\alpha \in J}(V_\alpha \cap A_\alpha) = \prod_{\alpha \in J} V_\alpha \cap \prod_{\alpha \in J} A_\alpha$. Since each V_α is open in X_α, this is the intersection of $\prod_{\alpha \in J} A_\alpha$ with an open set in $\prod_{\alpha \in J} X_\alpha$. Thus each basis element of $\prod_{\alpha \in J} A_\alpha$ can be written this way which shows that any open set can be written this way since open sets are unions of basis elements. Therefore $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod_{\alpha \in J} X_\alpha$.

Now let U be a subbasis element in $\prod_{\alpha \in J} A_\alpha$ when given the product topology. Then $U = \prod_{\alpha \in J} U_\alpha$ where all but finitely many U_α are A_α and the rest are open in A_α. Note that the finitely many U_α which are open in A_α can be written as $V_\alpha \cap A_\alpha$ where V_α is open in X_α. So now $U = \prod_{\alpha \in J}(V_\alpha \cap A_\alpha) = \prod_{\alpha \in J} V_\alpha \cap \prod_{\alpha \in J} A_\alpha$ where all but finitely many V_α are A_α. Note that all but finitely many of these intersections are $A_\alpha \cap A_\alpha = X_\alpha \cap A_\alpha$. Thus $U = \prod_{\alpha \in J} V_\alpha \cap \prod_{\alpha \in J} A_\alpha$ where finitely many of the V_α are X_α and the rest are open in $\prod_{\alpha \in J} X_\alpha$. This shows that any subbasis element of $\prod_{\alpha \in J} A_\alpha$ can be written as the intersection of an open set in $\prod_{\alpha \in J} X_\alpha$ with $\prod_{\alpha \in J} A_\alpha$ and is thus open in the subspace topology on $\prod_{\alpha \in J} A_\alpha$. Since open sets are just finite intersections of subbasis elements, we see that the result holds for any open set in $\prod_{\alpha \in J} A_\alpha$. $\square$