Sheet 12: Uniform Continuity

Definition 1 *Let f be a real function and let $a \in \mathbb{R}$. We say that f approaches a at l from the left, or*

$$\lim_{x \to a^-} f(x) = l$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $0 < a - x < \delta$ we have $|l - f(x)| < \varepsilon$. We say that f approaches a at l from the right, or

$$\lim_{x \to a^+} f(x) = l$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $0 < x - a < \delta$ we have $|l - f(x)| < \varepsilon$.

Definition 2 *A real function $f : [a; b] \to \mathbb{R}$ is continuous on $[a; b]$ if it is continuous for every $x \in (a; b)$, $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.*

Theorem 3 *Let f be a real function and let $a \in \mathbb{R}$. Then $\lim_{x \to a} f(x) = l$ if and only if $\lim_{x \to a^+} f(x) = l$ and $\lim_{x \to a^-} f(x) = l$.*

Proof. Suppose that $\lim_{x \to a^+} f(x) = l$ and $\lim_{x \to a^-} f(x) = l$. Then for all $\varepsilon > 0$ there exist $\delta_1 > 0$ and δ_2 such that for all $x \in \mathbb{R}$ when $0 < a - x < \delta_1$ and $0 < x - a < \delta_2$ we have $|l - f(x)| < \varepsilon$. Let $\delta = \min(\delta_1, \delta_2)$. Then for all $x \in \mathbb{R}$ when $0 < |a - x| < \delta$ we have $|l - f(x)| < \varepsilon$. Thus $\lim_{x \to a} f(x) = l$.

Conversely, assume $\lim_{x \to a} f(x) = l$. Then for all $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x \in \mathbb{R}$ with $0 < |a - x| < \delta$ we have $|l - f(x)| < \varepsilon$. But then for all $x \in \mathbb{R}$ with $0 < x - a < \delta$ we have $|l - f(x)| < \varepsilon$ and so $\lim_{x \to a^+} f(x) = l$ and likewise for all $x \in \mathbb{R}$ with $0 < a - x < \delta$ we have $|l - f(x)| < \varepsilon$ and so $\lim_{x \to a^-} f(x) = l$. □

Definition 4 *A function $f : \mathbb{R} \to \mathbb{R}$ is increasing if for all $x \leq y$ we have $f(x) \leq f(y)$.*

Theorem 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing real function. Then for all $a \in \mathbb{R}$ the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist.*

Proof. Let $L = \{f(x) \mid a < x\}$. Since f is defined for all $x > a$, $L \neq \emptyset$ and since L is bounded below by $f(a)$, $\inf L$ exists. For all $\varepsilon > 0$ we have $\varepsilon + \inf L > \inf L$. So there exists some $y \in L$ such that $y \leq \inf L + \varepsilon$. Since $y \in L$, there exists some $x' > a$ such that $y = f(x')$. For $\varepsilon > 0$ let $\delta = x' - a > 0$. Now consider all $x \in \mathbb{R}$ such that $0 < x - a < x' - a$. Then we have $x < x'$ so $f(x) < f(x') \leq \inf L + \varepsilon$. So we have $|f(x) - \inf L| < \varepsilon$ when $0 < x - a < x' - a = \delta$. Thus $\inf L$ is the right hand limit of f. A similar proof holds for the left hand limit. □

Theorem 6 (Intermediate Value Theorem) *Let $f : [a; b] \to \mathbb{R}$ be continuous. Then f takes on every value between $f(a)$ and $f(b)$ on $[a; b]$.*

Proof. Let $f : [a; b] \to \mathbb{R}$ be continuous. Without loss of generality suppose that $f(a) < f(b)$. For all $y \in (f(a); f(b))$ let $g(x) = f(x) - y$. We have $f(a) < y < f(b)$ for all $y \in (f(a); f(b))$ and so $g(a) < 0$ and $0 < g(b)$. But then for all $y \in (f(a); f(b))$ there exists $c \in [a; b]$. Such that $g(c) = f(c) - y = 0$. Then $f(c) = y$ and so for all $y \in (f(a); f(b))$ there exists $x \in [a; b]$ such that $f(x) = y$. □

Theorem 7 (Positive Continuous Functions are Bounded Away From Zero) *Let $f : [a; b] \to \mathbb{R}$ be continuous. If $f(x) > 0$ for all $x \in [a; b]$ then there exists $C > 0$ such that $f(x) > C$ for all $x \in [a; b]$.*

Proof. From Theorem 10.8 we know that there exists some $c \in [a; b]$ such that $f(c) \leq f(x)$ for all $x \in [a; b]$. Let $C = f(c)/2$. Then we have $C < f(x)$ for all $x \in [a; b]$. □

Theorem 8 *A real function $f : [a; b] \to \mathbb{R}$ is continuous on $[a; b]$ if and only if for all $x \in [a; b]$ and for all $\varepsilon > 0$ there exists $\delta(x, \varepsilon) > 0$ such that for all $y \in [a; b]$ with $|x - y| < \delta(x, \varepsilon)$ we have $|f(x) - f(y)| < \varepsilon$.*

Proof. Let f be continuous on $[a; b]$. Then for all $y \in (a; b)$ and all $\varepsilon > 0$ there exists $\delta > 0$ so that for all $x \in \mathbb{R}$ when $|y - x| < \delta$ we have $|f(y) - f(x)| < \varepsilon$. We can then confine our delta so that our definition holds only for $x \in [a; b]$. Let $\delta' = \min(\delta, |y - b|, |y - a|)$. But also $\lim_{x \to a^+} f(x) = f(a)$ so for all $\varepsilon > 0$ there exists $\delta > 0$ so that for all $x \in \mathbb{R}$, if $0 < x - a < \delta$ we have $|f(a) - f(x)| < \varepsilon$. But if $0 < x - a < \delta$ then $|a - x| < \delta$. Again truncate the δ so that $\delta' = \min(\delta, b)$. A similar statement can be said for the left hand limit and $f(b)$. Thus we have for all $y \in [a; b]$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in [a; b]$ with $|y - x| < \delta$ we have $|f(y) - f(x)| < \varepsilon$.

Conversely suppose that for all $x \in [a; b]$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in [a; b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Then the statement is true for all $x \in (a; b)$ as well. Note that for continuity we need to be able to choose y's from $\mathbb{R}$, not just $[a; b]$, but as we've shown we can make equivalent statements about continuity for closed intervals if we restrict δ to be within the confines of $[a; b]$. We also have for $x = a$, there exists $\delta > 0$ such that for all $y \in [a; b]$ with $|a - y| < \delta$ we have $|f(a) - f(y)| < \varepsilon$. But if $|a - y| < \delta$ then $x - a < \delta$. So $\lim_{x \to a^+} f(x) = f(a)$. A similar statement can be made about $f(b)$. So we have these conditions implying continuity. □

Exercise 9 *Calculate some good $\delta(x, \varepsilon)$ for the following real functions: 1) $f(x) = 17$ $(x \in \mathbb{R})$ 2) $f(x) = x$ $(x \in \mathbb{R})$ 3) $f(x) = x^2$ $(x \in \mathbb{R})$ 4) $f(x) = 1/x$ $(x \in \mathbb{R}\backslash\{0\})$.*

1) δ can be any value because for all $x \in \mathbb{R}$ we have $f(x) = 17$. Then for all $a \in \mathbb{R}$ when $|a - x| < \delta$ we have $|f(a) - f(x)| = 0 < \varepsilon$.

2) Let $\delta = \varepsilon$. Then for all $a \in \mathbb{R}$ if $|a - x| < \delta = \varepsilon$ we have $|f(a) - f(x)| = |a - x| < \varepsilon = \delta$.

3) Let $\delta = \sqrt{\varepsilon}$. Then for all $a \in \mathbb{R}$ if $|a - x| < \delta$ we have $|f(a) - f(x)| = |a^2 - x^2| < \varepsilon$.

4) Let $\delta = 1/\varepsilon$. Then for all $a \in \mathbb{R}$ if $|a - x| < \delta = 1/\varepsilon$ we have $|f(a) - f(x)| = |1/a - 1/x| < \varepsilon$.

Definition 10 *Let f be a real function and let A be a subset of the domain of f. Then f is uniformly continuous on A if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $x, y \in A$ with $|x - y| < \delta(\varepsilon)$ we have $|f(x) - f(y)| < \varepsilon$.*

Theorem 11 (Continuous Functions on Closed Intervals are Uniformly Continuous) *Let $f : [a; b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a; b]$.*

Proof. Let $\varepsilon > 0$. Then for all $x \in [a; b]$ there exists $\delta_x > 0$ such that for all $y \in [a; b] \cap (x - \delta; x + \delta)$ we have $f(y) \in (f(x) - \varepsilon; f(x) + \varepsilon)$. Create an open cover for $[a; b]$ using $(x - \delta_x; x + \delta_x)$ for all $x \in [a; b]$. Then $[a; b]$ is compact so there exist finitely many of these regions which will cover $[a; b]$. Choose the region with the smallest δ_x and call it δ, note that δ will work for all the other regions in our cover since it is smaller than all of them. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [a; b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. □

Theorem 12 *Let $f : [a; b] \to \mathbb{R}$ be continuous and let $\varepsilon > 0$. For $x \in [a; b]$ let*

$$\Delta(x) = \sup\{\delta \mid \textit{for all } y \in [a; b] \textit{ with } |x - y| < \delta,\ |f(x) - f(y)| < \varepsilon\}.$$

Then Δ is a continuous function of x.