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Problem 1. *Let G be a finite group with Sylow p-subgroup P. Prove that any subgroup of G that contains $N_G(P)$ (the normalizer in G of P) is equal to its own normalizer.*

Proof. Let $H \leq G$ be a subgroup such that $N_G(P) \leq H$. It's clear that $H \leq N_G(H)$ so we must show the other inclusion. Let $x \in G$ such that $xHx^{-1} = H$. Then since $P \leq N_G(P) \leq H$, we have $xPx^{-1} \in Syl_p(H)$. But since any two Sylow p-subgroups of H are conjugates of each other in H, $xPx^{-1} = yPy^{-1}$ for some $y \in H$. Then $y^{-1}xPx^{-1}y = y^{-1}xP(y^{-1}x)^{-1} = P$ and $y^{-1}x \in N_G(P)$. Thus $y^{-1}x \in H$ and since $y^{-1} \in H$, we must also have $x \in H$. Therefore $N_G(H) \leq H$ and we're done. $\square$

Problem 2. *Prove that the only group of order* 255 *is cyclic.*

Proof. Let $|G| = 255$. Note that $255 = 3 \cdot 5 \cdot 17$ and by the Sylow divisibility rules, $n_{17} = 1$. Thus G has some normal Sylow 17-subgroup P. Now, recall that $N_G(P)/C_G(P) \cong \text{Aut}(P)$. Since P is normal $N_G(P) = G$ and since $|P| = 17$ $|\text{Aut}(P)| = \varphi(17) = 16$. Thus $|G/C_G(P)| \mid |16$. But also, $C_G(P) \leq G$ and so $|G/C_G(P)| \mid 255$. This forces $|G/C_G(P)| = 1$ and hence $C_G(P) = G$. Therefore $P \leq Z(G)$ implying $|Z(G)|$ is a multiple of 17 and thus $|G/Z(G)|$ is either 1, 3, 5 or 15. We've shown that all groups of these orders are cyclic, and $G/Z(G)$ being cyclic implies G is abelian. Since G is abelian, every subgroup of G is normal, so $n_3 = 1$ and $n_5 = 1$. Let $Q \in Syl_3(G)$ and $R \in Syl_5(G)$. Note that P, Q and R are all cyclic and are generated by the elements x, y and z respectively. But since G is abelian, $|xyz| = |P||Q||R| = 255$. Thus G is cyclic. $\square$