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Homework 4

Problem 1. *Let* $H = A_5 \subseteq G = S_5$. *Show that* $\mathrm{Ind}\ U = U \oplus U'$, $\mathrm{Ind}\ V = V \oplus V'$, *and* $\mathrm{Ind}\ W = W \oplus W'$, *whereas* $\mathrm{Ind}\ Y = \mathrm{Ind}\ Z = \wedge^2 V$.

Proof. Using Frobenius reciprocity and examining the character tables we have

$$(\chi_{\mathrm{Ind}\ U}, \chi_U) = (\chi_U, \chi_{\mathrm{Res}\ U}) = 1,$$
$$(\chi_{\mathrm{Ind}\ U}, \chi_{U'}) = (\chi_U, \chi_{\mathrm{Res}\ U'}) = 1,$$
$$(\chi_{\mathrm{Ind}\ U}, \chi_V) = (\chi_U, \chi_{\mathrm{Res}\ V}) = (\chi_U, \chi_V) = 0,$$
$$(\chi_{\mathrm{Ind}\ U}, \chi_{V'}) = (\chi_U, \chi_{\mathrm{Res}\ V'}) = (\chi_U, \chi_V) = 0,$$
$$(\chi_{\mathrm{Ind}\ U}, \chi_{\wedge^2 V}) = (\chi_U, \chi_{\mathrm{Res}\ \wedge^2 V}) = (\chi_U, \chi_{Y \oplus Z}) = 0,$$
$$(\chi_{\mathrm{Ind}\ U}, \chi_W) = (\chi_U, \chi_{\mathrm{Res}\ W}) = (\chi_U, \chi_W) = 0,$$

and

$$(\chi_{\mathrm{Ind}\ U}, \chi_{W'}) = (\chi_U, \chi_{\mathrm{Res}\ W'}) = (\chi_U, \chi_W) = 0$$

so $\mathrm{Ind}\ U \cong U \oplus U'$. Also,

$$(\chi_{\mathrm{Ind}\ V}, \chi_U) = (\chi_V, \chi_{\mathrm{Res}\ U}) = 0,$$
$$(\chi_{\mathrm{Ind}\ V}, \chi_{U'}) = (\chi_V, \chi_{\mathrm{Res}\ U'}) = 0,$$
$$(\chi_{\mathrm{Ind}\ V}, \chi_V) = (\chi_V, \chi_{\mathrm{Res}\ V}) = (\chi_V, \chi_V) = 1,$$
$$(\chi_{\mathrm{Ind}\ V}, \chi_{V'}) = (\chi_V, \chi_{\mathrm{Res}\ V'}) = (\chi_V, \chi_V) = 1,$$
$$(\chi_{\mathrm{Ind}\ V}, \chi_{\wedge^2 V}) = (\chi_V, \chi_{\mathrm{Res}\ \wedge^2 V}) = (\chi_V, \chi_{Y \oplus Z}) = 0,$$
$$(\chi_{\mathrm{Ind}\ V}, \chi_W) = (\chi_V, \chi_{\mathrm{Res}\ W}) = (\chi_V, \chi_W) = 0,$$

and

$$(\chi_{\mathrm{Ind}\ V}, \chi_{W'}) = (\chi_V, \chi_{\mathrm{Res}\ W'}) = (\chi_V, \chi_W) = 0$$

so $\mathrm{Ind}\ V \cong V \oplus V'$. Also,

$$(\chi_{\mathrm{Ind}\ W}, \chi_U) = (\chi_W, \chi_{\mathrm{Res}\ U}) = 0,$$
$$(\chi_{\mathrm{Ind}\ W}, \chi_{U'}) = (\chi_W, \chi_{\mathrm{Res}\ U'}) = 0,$$
$$(\chi_{\mathrm{Ind}\ W}, \chi_V) = (\chi_W, \chi_{\mathrm{Res}\ V}) = (\chi_W, \chi_V) = 0,$$
$$(\chi_{\mathrm{Ind}\ W}, \chi_{V'}) = (\chi_W, \chi_{\mathrm{Res}\ V'}) = (\chi_W, \chi_V) = 0,$$
$$(\chi_{\mathrm{Ind}\ W}, \chi_{\wedge^2 V}) = (\chi_W, \chi_{\mathrm{Res}\ \wedge^2 V}) = (\chi_W, \chi_{Y \oplus Z}) = 0,$$
$$(\chi_{\mathrm{Ind}\ W}, \chi_W) = (\chi_W, \chi_{\mathrm{Res}\ W}) = (\chi_W, \chi_W) = 1,$$

and

$$(\chi_{\mathrm{Ind}\ W}, \chi_{W'}) = (\chi_W, \chi_{\mathrm{Res}\ W'}) = (\chi_W, \chi_W) = 1$$

so $\mathrm{Ind}\ W \cong V \oplus V'$. Also,

$$(\chi_{\mathrm{Ind}\ Y}, \chi_U) = (\chi_Y, \chi_{\mathrm{Res}\ U}) = 0,$$
$$(\chi_{\mathrm{Ind}\ Y}, \chi_{U'}) = (\chi_Y, \chi_{\mathrm{Res}\ U'}) = 0,$$
$$(\chi_{\mathrm{Ind}\ Y}, \chi_V) = (\chi_Y, \chi_{\mathrm{Res}\ V}) = (\chi_Y, \chi_V) = 0,$$

$$(\chi_{\mathrm{Ind}\ Y}, \chi_{V'}) = (\chi_Y, \chi_{\mathrm{Res}\ V'}) = (\chi_Y, \chi_V) = 0,$$

$$(\chi_{\mathrm{Ind}\ Y}, \chi_{\wedge^2 V}) = (\chi_Y, \chi_{\mathrm{Res}\ \wedge^2 V}) = (\chi_Y, \chi_{Y \oplus Z}) = 1,$$

$$(\chi_{\mathrm{Ind}\ Y}, \chi_W) = (\chi_Y, \chi_{\mathrm{Res}\ W}) = (\chi_Y, \chi_W) = 0,$$

and

$$(\chi_{\mathrm{Ind}\ Y}, \chi_{W'}) = (\chi_Y, \chi_{\mathrm{Res}\ W'}) = (\chi_Y, \chi_W) = 0$$

so $\mathrm{Ind}\ Y \cong \wedge^2 V$. Also,

$$(\chi_{\mathrm{Ind}\ Z}, \chi_U) = (\chi_Z, \chi_{\mathrm{Res}\ U}) = 0,$$

$$(\chi_{\mathrm{Ind}\ Z}, \chi_{U'}) = (\chi_Z, \chi_{\mathrm{Res}\ U'}) = 0,$$

$$(\chi_{\mathrm{Ind}\ Z}, \chi_V) = (\chi_Z, \chi_{\mathrm{Res}\ V}) = (\chi_Z, \chi_V) = 0,$$

$$(\chi_{\mathrm{Ind}\ Z}, \chi_{V'}) = (\chi_Z, \chi_{\mathrm{Res}\ V'}) = (\chi_Z, \chi_V) = 0,$$

$$(\chi_{\mathrm{Ind}\ Z}, \chi_{\wedge^2 V}) = (\chi_Z, \chi_{\mathrm{Res}\ \wedge^2 V}) = (\chi_Z, \chi_{Z \oplus Z}) = 1,$$

$$(\chi_{\mathrm{Ind}\ Z}, \chi_W) = (\chi_Z, \chi_{\mathrm{Res}\ W}) = (\chi_Z, \chi_W) = 0,$$

and

$$(\chi_{\mathrm{Ind}\ Z}, \chi_{W'}) = (\chi_Z, \chi_{\mathrm{Res}\ W'}) = (\chi_Z, \chi_W) = 0$$

so $\mathrm{Ind}\ Z \cong \wedge^2 V$. □

Problem 2. *If $\mathbb{C}G$ is identified with the space of functions on G, the function φ corresponding to $\sum_{g \in G} \varphi(g) e_g$, show that the product in $\mathbb{C}G$ corresponds to the convolution $*$ of induced functions:*

$$(\varphi * \psi)(g) = \sum_{h \in G} \varphi(h) \psi(h^{-1} g).$$

Proof. Note that $(\varphi * \psi)(g)$ corresponds to

$$\begin{aligned}
\sum_{g \in G} (\varphi * \psi)(g) e_g &= \sum_{g \in G} \left(\sum_{h \in G} \varphi(h) \psi(h^{-1} g) \right) e_g \\
&= \sum_{g \in G} \sum_{h \in G} \varphi(h) \psi(h^{-1} g) e_g \\
&= \sum_{g \in G} \sum_{hk = g \in G} \varphi(h) \psi(k) e_g \\
&= \left(\sum_{h \in G} \varphi(h) e_h \right) \left(\sum_{k \in G} \psi(k) e_k \right)
\end{aligned}$$

□

Problem 3. *If $\rho : G \to GL(V_\rho)$ is a representation, and φ is a function on G, define the* Fourier transform *$\widehat{\varphi}$ in $\mathrm{End}(V_\rho)$ by the formula*

$$\widehat{\varphi}(\rho) = \sum_{g \in G} \varphi(g) \cdot \rho(g).$$

(a) Show that $\widehat{\varphi * \psi}(\rho) = \widehat{\varphi}(\rho) \cdot \widehat{\psi}(\rho)$.
(b) Prove the Fourier inversion formula

$$\varphi(g) = \frac{1}{|G|} \sum \dim(V_\rho) \cdot \mathrm{Tr}(\rho(g^{-1}) \cdot \widehat{\varphi}(\rho)),$$

the sum over the irreducible representations ρ of G. This formula is equivalent to formula (2.19) and (2.20).
(c) Prove the Plancherel formula *for functions φ and ψ on G:*

$$\sum_{g\in G}\varphi(g^{-1})\psi(g) = \frac{1}{|G|}\sum_{\rho}\dim(V_\rho)\cdot\mathrm{Tr}(\widehat{\varphi}(\rho)\widehat{\psi}(\rho)).$$

Proof. (a) We have

$$\begin{aligned}
\widehat{\varphi*\psi}(\rho) &= \sum_{g\in G}\widehat{\varphi*\psi}(g)\rho(g)\\
&= \sum_{g\in G}\left(\sum_{h\in G}\varphi(h)\psi(h^{-1}g)\right)\rho(g)\\
&= \sum_{g\in G}\sum_{hk=g\in G}\varphi(h)\psi(k)\rho(g)\\
&= \sum_{g\in G}\sum_{hk=g\in G}\varphi(h)\psi(k)\rho(h)\rho(k)\\
&= \left(\sum_{h\in G}\varphi(h)\rho(h)\right)\left(\sum_{k\in G}\psi(k)\rho(k)\right)\\
&= \widehat{\varphi}(\rho)\widehat{\psi}(\rho).
\end{aligned}$$

(b) We have

$$\begin{aligned}
\frac{1}{|G|}\sum_{i=1}^{r}\dim(V_i)\mathrm{Tr}(\rho_i(g^{-1})\widehat{\varphi}(\rho_i)) &= \frac{1}{|G|}\sum_{i=1}^{r}\dim(V_i)\mathrm{Tr}\left(\rho_i(g^{-1})\sum_{h\in G}\varphi(h)\rho_i(h)\right)\\
&= \frac{1}{|G|}\sum_{i=1}^{r}\dim(V_i)\mathrm{Tr}\left(\sum_{h\in G}\varphi(h)\rho_i(g^{-1}h)\right)\\
&= \frac{1}{|G|}\sum_{i=1}^{r}\dim(V_i)\left(\sum_{h\in G}\varphi(h)\mathrm{Tr}(\rho_i(g^{-1}h))\right)\\
&= \frac{1}{|G|}\sum_{h\in G}\varphi(h)\left(\sum_{i=1}^{r}\dim(V_i)\chi_i(g^{-1}h)\right).
\end{aligned}$$

Note that the inner sum is 0 for $h\neq g$ and $\dim(V_i)$ for $h=g$ by column orthogonality of group characters. Putting in $h=g$ simplifies the equation to

$$\frac{1}{|G|}\varphi(g)\sum_{i=1}^{r}(\dim(V_i))^2 = \frac{1}{G}\varphi(g)|G| = \varphi(g).$$

(c) Let $\varphi : G\to\mathbb{C}$ be the identifier function for $g^{-1}\in G$ so that

$$\varphi(h) = \begin{cases}0 & h\neq g^{-1}\\ 1 & h=g^{-1}.\end{cases}$$

Then $\widehat{\varphi}(\rho) = \sum_{g\in G}\varphi(g)\rho(g) = \varphi(g^{-1})\rho(g^{-1}) = \rho(g^{-1})$. Now using part (b) we have

$$\frac{1}{|G|}\sum_{i=1}^{r}\dim(V_i)\mathrm{Tr}(\widehat{\varphi}(\rho)\widehat{\psi}(\rho)) = \frac{1}{|G|}\sum_{i=1}^{r}\dim(V_i)\mathrm{Tr}(\rho(g^{-1})\widehat{\psi}(\rho)) = \psi(g) = \varphi(g^{-1})\psi(g) = \sum_{g\in G}\varphi(g^{-1})\psi(g).$$

Now since trace is additive and multiplicative, we can extend this formula linearly for all possible functions φ. $\square$

Problem 4. *Let $A \leq S_n$ be an abelian subgroup that acts transitively on $\mathcal{N} = \{1, \ldots, n\}$.*
(a) Show that for each $k \in \mathcal{N}$ the stabilizer of k in A is trivial. Deduce that A has n elements.
(b) Show that the permutation representation V of A on $\mathcal{N}$ decomposes as

$$V = V_1 \oplus \cdots \oplus V_n$$

where the V_i are distinct irreducible representations of A.

Proof. (a) Let $\sigma \in A$ fix $k \in \mathcal{N}$. Then $\tau\sigma(k) = \tau(k)$ for all $\tau \in A$. But A is abelian, so $\sigma\tau(k) = \tau(k)$ for all $\tau \in A$. Since A acts transitively, there is some $\tau \in A$ such that $\tau(k) = k'$ for each $k' \in \mathcal{N}$. Then $\sigma(k') = k'$ for all $k' \in \mathcal{N}$, so σ is the identity.

Let $S_A(k)$ be the stabilizer of k and Ak be the orbit of k. By the orbit-stabilizer theorem we know $S_A(k) = |A|/|Ak|$ for each k. Since A acts transitively we also know $|Ak| = |\mathcal{N}| = n$. Then we know

$$\sum_{k \in \mathcal{N}} |S_A(k)| = \sum_{k \in \mathcal{N}} \frac{|A|}{Ak} = \sum_{k \in \mathcal{N}} \frac{|A|}{n} = |A|.$$

Since $|S_A(k)| = 1$ for each k, this immediately gives $|A| = n$.

(b) Since V is a permutation representation $\chi_V(a)$ is determined by how many elements a fixes. But since the stabilizer for any non-identity element is trivial, we know that

$$\chi_V(a) = \begin{cases} 0 & a \neq 1 \\ n & a = 1. \end{cases}$$

Now note that since A is abelian, $\chi_{V_i}(1) = 1$ for each irreducible representation V_i of A. Then we have

$$(\chi_V, \chi_{V_i}) = \frac{1}{|A|} \sum_{a \in A} \chi_V(a)\overline{\chi_{V_i}(a)} = \frac{1}{n}\chi_V(1)\overline{\chi_{V_i}(1)} = 1$$

and this is the multiplicity of V_i in the decomposition for A. Thus $V = V_1 \oplus \cdots \oplus V_n$. □

Problem 5. *Verify the statement given in class that for an H-representation W there is an isomorphism of G-representations:*

$$\operatorname{Ind}{}_H^G W \cong \mathbb{C}G \otimes_{\mathbb{C}H} W.$$

Proof. Note that W is a $\mathbb{C}H$ under the extension of the action of H on W. Furthermore, we clearly have $\mathbb{C}H \subseteq \mathbb{C}G$ so we are in a position to use the universal property of extension of scalars. We have the diagram

$$\begin{tikzcd} W \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathbb{C}G \otimes_{\mathbb{C}H} W \arrow[d, "\Phi"] \\ & \operatorname{Ind} W \end{tikzcd}$$

where $\iota : w \mapsto 1 \otimes_{\mathbb{C}H} w$ and $\varphi : w \mapsto w$ is clearly a $\mathbb{C}H$-module homomorphism. By the universal property we know Φ is a unique $\mathbb{C}G$-module map, so it only remains to show it's an isomorphism. Recall that we've shown $\dim(\operatorname{Ind} W) = |G : H| \dim(W)$ and the dimension of $\mathbb{C}G \otimes_{\mathbb{C}H} W$ is $(\dim(\mathbb{C}G)/\dim(\mathbb{C}H))\dim(W) = |G : H| \dim(W)$. So it suffices to show Φ takes basis elements to basis elements.

A basis for $\mathbb{C}G \otimes_{\mathbb{C}H} W$ is

$$\{\sigma_i \otimes w_j \mid 1 \leq i \leq |G : H|, 1 \leq j \leq \dim(W)\}$$

where each $\sigma_i \in G$ is a coset representative. A basis for $\operatorname{Ind} W$ is

$$\{w_j^{\sigma_i} \mid 1 \leq j \leq \dim(W), \sigma_i \in G/H\}.$$

Now from the definition of φ we know $\Phi : 1 \otimes w \mapsto w$. Furthermore, we know the action of g on $\mathbb{C}G \otimes W$, namely $g(g' \otimes w) = gg' \otimes w$. Also the action of g on $w_j^{\sigma_i}$ is $g(w_j^{\sigma_i}) = g(g_{\sigma_i} w_j) = g_\tau(hw_j)$ where $gg_{\sigma_i} = g_\tau h$. Since Φ is a $\mathbb{C}G$-module map, it respects the action of G so we have

$$\Phi(\sigma_i \otimes w_j) = \Phi(\sigma_i(1 \otimes w_j) = \sigma_i \Phi(1 \otimes w_j) = \sigma_i w_j = g_\tau h w_j = gg_{\sigma_i} w_j = w_j^{\sigma_i}$$

where $g_\tau h$ is the way to write σ_i as an element of a coset of H and gg_{σ_i} is the appropriate element to move w_j back into W_i^σ. Thus Φ takes basis elements to basis elements so it must be an isomorphism. □

Problem 6. *Let S_n act by permutations on the set $X = \{1, \ldots, n\}$. Let X_r be the set of all r-element subsets of X. Then the S_n action on X_r gives a permutation representation on a $|X_r|$-dimensional vector space. Let χ_r denote the character of this representation.*
(a) Suppose $r \leq s \leq n/2$. Prove that S_n has $r+1$ orbits for its action on $X_r \times X_s$.
(b) Deduce that $\langle \chi_r, \chi_s \rangle = r + 1$. It follows that the "generalized character" $\chi_r - \chi_s$ is irreducible (i.e. has norm $s - r$) for $1 \leq r \leq n/2$.

Proof. (a) Let $(A, B) \in X_r \times X_s$ so that A is an r-element subset and B is an s-element subset. Define $m = |A \backslash B|$ to be the number of elements of A which are not in B. Note that for a fixed m, the possible choices for A and B determine an orbit. To see this, fix m and let $\sigma \in S_n$. Note that the action of σ on B determines exactly $r - m$ elements of A (since they are also in B). On the other hand, σA cannot have more than $r - m$ elements in common with B because this would mean that two distinct elements were mapped to a single element. Thus the orbit of (A, B) is precisely the set of (A', B') such that A has exactly $r - m$ elements in common with B. Now note that there are only $r + 1$ choices for m (namely, $0 \leq m \leq r$), so S_n has $r + 1$ orbits under this action.

(b) Note that $(\chi_r, \chi_s) = \frac{1}{|S_n|} \sum_g \chi_r(g)\chi_s(g)$, where $\chi_r(g)$ is the number of r-element subsets that g fixes and likewise for $\chi_s(g)$. Then $\chi_r(g)\chi_s(g)$ is the number of elements in $X_r \times X_s$ fixed by g. In other words $\chi_r(g)\chi_s(g)$ is the size of $(X_r \times X_s)^g$, the fixed set under the action of g. This now becomes a straightforward application of Burnside's Lemma. In particular, if we denote $(X_r \times X_s)/S_n$ as the set of orbits under the action of S_n, S_n^x as the stabilizer of x and $S_n x$ as the orbit of x, then we have

$$\begin{aligned}
\sum_g \chi_r(g)\chi_s(g) &= |\{(g, x) \in S_n \times (X_r \times X_s) \mid gx = x\}| \\
&= \sum_{x \in X_r \times X_s} |S_n^x| \\
&= \sum_{x \in X_r \times X_s} \frac{|S_n|}{|S_n x|} \\
&= |S_n| \sum_{x \in X_r \times X_s} \frac{1}{|S_n x|} \\
&= |S_n| \sum_{A \in (X_r \times X_s)/S_n} \sum_{x \in A} \frac{1}{|A|} \\
&= |S_n| \sum_{A \in (X_r \times X_s)/S_n} 1 \\
&= |S_n||(X_r \times X_s)/S_n|.
\end{aligned}$$

Dividing by $|S_n|$ and noting that $|(X_r \times X_r)/S_n| = r + 1$ by part (a) gives the desired formula.

For the second statement we can use bilinearity and the above calculation to get

$$(\chi_r - \chi_s, \chi_r - \chi_s) = (\chi_r, \chi_r) + (\chi_s, \chi_s) - 2(\chi_r, \chi_s) = (r + 1) + (s + r) - 2(r - 1) = s - r.$$

□