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# Sheet 14: Cauchy Sequences

**Definition 1 (Cauchy Sequence)** *We say that a sequence $(a_n)$ is a Cauchy sequence if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then $|a_n - a_m| < \varepsilon$.*

**Lemma 2** *Every convergent sequence has the Cauchy property.*

*Proof.* Let $(a_n)$ converge to $a$ and let $\varepsilon > 0$. Consider $\varepsilon/2$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n \in (a - \varepsilon/2; a + \varepsilon/2)$. But then also for all $m, n > N$ we have $a_m, a_n \in (a - \varepsilon/2; a + \varepsilon/2)$. Then the distance between $a_m$ and $a_n$ is no more than $\varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus, there exists $N \in \mathbb{N}$ such that for all $m, n > N$ we have $|a_m - a_n| < \varepsilon$. $\square$

**Lemma 3** *Let $(a_n)$ be a Cauchy sequence and let $(b_k = a_{n_k})$ be a subsequence. If $(b_k)$ converges then so does $(a_n)$.*

*Proof.* Let $(b_k = a_{n_k})$ be a subsequence of $(a_n)$ which converges to $a$ and let $\varepsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$ we have $|a - b_k| < \varepsilon/2$. But also $(a_n)$ is a Cauchy sequence and so there exists some $N_2 \in \mathbb{N}$ such that for all $n, m > N_2$ we have $|a_m - a_n| < \varepsilon/2$. Let $N = \max(N_1, N_2)$. Then for all $n, m > N$ we have $|a - b_n| < \varepsilon/2$ and $|a_m - a_n| < \varepsilon/2$. Thus by the triangle inequality for all $n > N$ we have $|a - a_n| < \varepsilon$ and so $(a_n)$ converges to $a$. $\square$

**Lemma 4** *Every Cauchy sequence is bounded.*

*Proof.* Let $(a_n)$ be a Cauchy sequence and let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_N - a_n| < \varepsilon$. Then there are finitely many $n \in \mathbb{N}$ such that $a_n \notin (-\varepsilon + a_N; \varepsilon + a_N)$. Then the largest of these $a_n$ is greater than or equal to every other term of $(a_n)$. Note that if there are no terms of $(a_n)$ greater than $a_N + \varepsilon$, then we can choose a smaller epsilon so that such a term exists. A similar argument shows that there is a lower bound of $(a_n)$. $\square$

**Theorem 5** *A sequence is convergent if and only if it is Cauchy.*

*Proof.* Let $(a_n)$ be a Cauchy sequence. Then by Lemma 4 we know $(a_n)$ is bounded and therefore there exists a convergent subsequence of $(a_n)$ (13.16, 14.4). But then by Lemma 3 we know $(a_n)$ converges (14.3). Conversely if a sequence is convergent then it Cauchy by Lemma 2 (14.2). $\square$

**Definition 6** *Let $(a_n)$ be a bounded sequence and $A$ be the set of its accumulation points. We define its limes inferior, $\liminf_{n\to\infty} a_n$, to be the first point of $A$ and the limes superior, $\limsup_{n\to\infty} a_n$, to be the last point of $A$.*

**Corollary 7** *Let $(a_n)$ be a bounded sequence. Then $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ and equality holds if and only if the sequence is convergent.*

*Proof.* Let $A$ be the set of accumulation points for $(a_n)$. Since $\liminf_{n\to\infty} a_n$ is the first point of $A$, we have $\liminf_{n\to\infty} a_n \leq a$ for all $a \in A$. But since $\limsup_{n\to\infty} a_n \in A$ we have $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$. Suppose now that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. Then the first and last points of $A$ are equal and so $A$ only has one accumulation point. But then since $(a_n)$ is bounded we have $(a_n)$ is convergent (13.17). Conversely assume that $(a_n)$ is convergent. Then $(a_n)$ only has one accumulation point and so $A$ contains one point (13.17). But then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. $\square$

**Theorem 8** *Let $(a_n)$ be a bounded sequence. Then*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty}(\inf\{a_k \mid k > n\})$$

*and*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_k \mid k > n\}).$$

*Proof.* Consider the sequence $(b_n)$ where $b_n = \inf\{a_k \mid k > n\}$. Then $(b_n)$ is bounded because $(a_n)$ is bounded and it's increasing because each infimum will either be less than or equal to the previous one. Thus $\lim_{n\to\infty} b_n = \sup\{b_n \mid n \in \mathbb{N}\} = s$ (13.18). Now consider some region $(p;q)$ with $s \in (p;q)$. Note that $p < \inf\{a_k \mid k > n\} = r$ for some $n$, otherwise there would exist some point in $(p;s)$ which would be an upper bound for $\{b_n \mid n \in \mathbb{N}\}$. Note that there are finitely many $n$ such that $a_n < r$ because of how $r$ is defined. Thus there are finitely many $n$ with $a_n < p$. But also there must be finitely many $n$ with $a_n > q$ because if there were infinitely many then there would exist $a_k > q$ such that $k$ is greater than every index of $a_n \le q$. But this contradicts how $s$ is defined. Thus there are infinitely many $n$ with $a_n \in (p;q)$ and so $s$ is an accumulation point of $(a_n)$. But there can't be an accumulation point of $(a_n)$ less than $s$ because for each term or $(b_n)$ there are finitely many $n$ with $a_n$ less that it and an accumulation point would imply infinitely many such $n$. Thus $s = \liminf_{n\to\infty} a_n$. A similar proof holds to show $\limsup_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_k \mid k > n\})$. □

**Theorem 9** *Let $(a_n)$ be a bounded sequence. Then*

$$\liminf_{n\to\infty} a_n = \sup\{x \mid \text{ there are finitely many } n \text{ with } a_n \in (-\infty;x)\}$$

*and*

$$\limsup_{n\to\infty} a_n = \inf\{x \mid \text{ there are finitely many } n \text{ with } a_n \in (x;\infty)\}$$

*Proof.* Let $S = \{x \mid \text{ there are finitely many } n \text{ with } a_n \in (-\infty;x)\}$. Note that $S$ is nonempty because $(a_n)$ is bounded. Thus a lower bound for $(a_n)$ shows that $S$ is nonempty and an upper bound for $(a_n)$ shows that $S$ is bounded. Thus $\sup S = t$ exists. Let $(b_n)$ be defined such that $b_n = \inf\{a_k \mid k > n\}$ and let $s = \lim_{n\to\infty} b_n = \sup\{b_n \mid n \in \mathbb{N}\}$ (13.18, 14,8). First suppose that $t > s$. Then there exists $x \in (s;t)$ such that there are finitely many $n$ with $a_n < x$. But then if we take the largest index, $i$, of all such $a_n$ we have $\inf\{a_k \mid k > i\} > s$ which is a contradiction. So $t \le s$. Suppose that $t < s$. Then for all $x \in (t;s)$ there are infinitely many $n$ with $a_n < x$. But this implies that there are infinitely many $n$ with $a_n \in (t;s)$ because there exists $x < t$ such that there are finitely many $n$ with $a_n < x$. But then there exists some element of $b_n$ which is less than $s$, but greater than infinitely many terms of $(a_n)$. This cannot happen and so $s = t$. But then using Theorem 8 we have $t = \liminf_{n\to\infty} a_n$ (14.8). □

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# Sheet 15: Series

**Definition 1** *A series of real numbers is an expression $\sum_{n=1}^{\infty} a_n$, where $(a_n)$ is a real sequence.*

**Definition 2 (Convergent Series)** *Let $\sum_{n=1}^{\infty} a_n$ be a series. The sequence of partial sums is defined as*

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

*We say that the series $\sum_{n=1}^{\infty} a_n$ converges to $s$ (or $\sum_{n=1}^{\infty} a_n = s$) if $\lim_{n\to\infty} s_n = s$. If such an $s$ exists, we say that $\sum_{n=1}^{\infty} a_n$ is convergent, otherwise it is divergent.*

**Exercise 3** *Reformulate convergence using the Cauchy property.*

We say a series $\sum_{n=1}^{\infty} a_n$ is convergent if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have $|s_n - s_m| < \varepsilon$.

**Lemma 4** *If $\sum_{n=1}^{\infty} a_n$ is a convergent series, the the sequence $(a_n)$ converges to 0.*

*Proof.* Let $\sum_{n=1}^{\infty} a_n = s$. Then the sequence of partial sums $(s_n)$ converges to $s$ and $(s_n)$ is a Cauchy sequence. Thus for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have $|s_n - s_m| < \varepsilon$. But note that $s_{n+1} - s_n = a_n$ so for $n > N + 1$ we have $|a_n| < \varepsilon$ which means $\lim_{n\to\infty} a_n = 0$. □

**Lemma 5** *Let $\sum_{n=1}^{\infty} a_n$ be convergent with a partial sum sequence $(s_n)$. Let $n_0 = 0$ and $n_1 < n_2 < \ldots$ be a sequence of natural numbers. For $k \in \mathbb{N}$ let*

$$b_k = a_{n_{k-1}+1} + \cdots + a_{n_k} = \sum_{i=n_{k-1}+1}^{n_k} a_i.$$

*Then*

$$\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n.$$

*Proof.* Let $s_{b_k} = \sum_{i=1}^{k} b_i$ and $s_{a_n} = \sum_{i=1}^{n} a_i$. Then note that

$$s_{b_k} = \sum_{i=1}^{k} b_i = \sum_{i=1}^{n_1} a_i + \sum_{i=n_1+1}^{n_2} a_i + \cdots + \sum_{i=n_{k-1}+1}^{n_k} a_i = s_{a_{n_k}}.$$

We know $\sum_{n=1}^{\infty} a_n$ is convergent so $(s_{a_n})$ converges. Also $(s_{a_{n_k}})$ is a subsequence of $(s_{a_n})$ so it converges as well (13.12). But $(s_{b_k}) = (s_{a_{n_k}})$ so $\lim_{n\to\infty} s_{b_k} = \lim_{n\to\infty} s_{a_{n_k}}$ which implies

$$\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n.$$

□

**Theorem 6 (Geometric Series)** *For all $t < |1|$, we have*

$$\sum_{n=0}^{\infty} t^n \frac{1}{1-t}.$$

*Proof.* Consider a partial sum of $\sum_{n=0}^{\infty} t^n$,

$$s_k = \sum_{n=0}^{\infty} t^n = 1 + t + \cdots + t^k = \frac{1-t^{k+1}}{1-t} = \frac{1}{1-t} - \frac{t^k}{1-t}.$$

But since $t < |1|$ we have $\lim_{k\to\infty} t^k/(1-t) = 0$. So then $\lim_{k\to\infty} s_k = 1/(1-t) + 0$ which means

$$\sum_{n=0}^{\infty} t^n \frac{1}{1-t}.$$

□

**Theorem 7** *The series $\sum_{n=1}^{\infty} 1/n$ is not convergent.*

*Proof.* Suppose that $\sum_{n=1}^{\infty} 1/n$ is convergent. Create a sequence $(b_k)$ as in Lemma 5 such that

$$b_k = \sum_{i=n_{k-1}+1}^{n_k} \frac{1}{n}$$

where $n_k = 2^{k-1}$ for $k \in \mathbb{N}$ and $n_0 = 0$. Note that for $k \geq 2$, $b_k$ has $2^{k-1} - 2^{k-2} = 2^{k-2}$ terms, the smallest of which is $1/2^{k-1}$. Thus, for all $k \geq 2$, $b_k \geq 2^{k-2}/2^{k-1} = 1/2$. Also $b_1 = \sum_{n=1}^{1} 1/n = 1$. So for all $k \in \mathbb{N}$ we have $b_k \geq 1/2$. But then there are infinitely many $k \in \mathbb{N}$ such that $b_k \notin (-1/2; 1/2)$ so $\lim_{k\to\infty} b_k \neq 0$. Thus, $\sum_{k=1}^{\infty} k_n$ is not convergent (15.4). But we know that $\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n$ which is a contradiction (15.5). Thus $\sum_{n=1}^{\infty} 1/n$ is not convergent. □

**Theorem 8 (Alternating Sign Series)** *Let $\sum_{n=1}^{\infty} a_n$ be a series with the following properties: 1) $a_n$ is positive if $n$ is odd and negative if $n$ is even; 2) $|a_{n+1}| < |a_n|$ for all $n$; 3) $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty} a_n$ is convergent.*

*Proof.* Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n| < \varepsilon$. Let $n \in \mathbb{N}$ such that $n > N$ and $n$ is even. Then $a_{n+1} > 0$. We have $s_{n+1} = s_n + a_{n+1} > s_n$. Also $a_{n+2} < 0$ and $|a_{n+2}| < |a_{n+1}|$ so $a_{n+1} + a_{n+2} > 0$. Then $s_{n+1} > s_{n+1} + a_{n+2} = s_{n+2} = s_n + a_{n+1} + a_{n+2} > s_n$. So for $n > N$ even we have $s_n \leq s_{n+2} \leq s_{n+1}$ and a similar proof shows that for $n > N$ odd we have $s_n \geq s_{n+2} \geq s_{n+1}$. Use strong induction on $n$ to show that for $k + N$ even $s_N \leq s_{k+N} \leq s_{N+1}$. We see that for $k = 1$ we have $s_N \leq s_{N+1} \leq s_{N+1}$ which is true since $a_{N+1}$ is positive. We've also shown the case for $k = 2$. Assume that for $n + N$ even we have $s_N \leq s_{N+n} \leq s_{N+1}$. Consider $s_{N+n+2}$. We know $s_{N+n} \leq s_{N+n+2} \leq s_{N+n+1}$ and $s_{N+n-1} \leq s_{N+n+1} \leq s_{N+n}$. Combining these three inequalities we have $s_N \leq s_{N+n+2} \leq s_{N+1}$. Thus for all even $N + n$ we have $s_N \leq s_{N+n} \leq s_{N+1}$. A similar proof holds to show that for odd $N + n$ we have $s_N \leq s_{N+n} \leq s_{N+1}$. Since this is true for any $N$ given $\varepsilon$, for any region $(s_N; s_{N+1})$ there are finitely many $n$ with $s_n$ not in the region. Thus $\sum_{n=1}^{\infty} a_n$ is convergent. □

**Exercise 9** *The series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

*is convergent.*

*Proof.* Note that for $n$ odd we have $a_n = (-1)^{n+1}/n$ and since $n+1$ is even and $n > 0$ we have $a_n = 1/n > 0$. For $n$ even $n+1$ is odd so $a_n = (-1)^{n+1}/n = -1/n < 0$. Also $|a_{n+1}| = 1/(n+1) < 1/n = |a_n|$. Finally we know that $\lim_{n\to\infty} a_n = 0$ (13.4). Since this series fulfills the requirements of Theorem 8, it must be convergent. $\square$

**Definition 10** *A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

**Lemma 11** *$\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if there exists $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$, $\sum_{n=1}^{N} |a_n| \leq C$.*

*Proof.* Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Let $s_k = \sum_{n=1}^{k} |a_n|$. Then $(s_n)$ is convergent and therefore bounded (13.15). Thus there exists $C \in \mathbb{R}$ such that for all $N$ we have $s_N = \sum n = 1^N |a_n| \leq C$.

Now suppose there exists $C \in \mathbb{R}$ such that $s_N \leq C$ for all $N$. Thus $(s_n)$ is bounded. Note that $s_n = s_{n-1} + |a_n|$ and since $|a_n| \geq 0$ for all $n$ we have $(s_n)$ is an increasing sequence. Since $(s_n)$ is bounded and increasing we know it is convergent (13.18). Thus $\sum_{n=1}^{\infty} |a_n|$ is convergent and so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. $\square$

**Theorem 12 (Comparison Criterion)** *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series. Suppose there is some $N$ such that for all $n \geq N$ we have $|a_n| \leq |b_n|$. Then if $\sum_{n=1}^{\infty} b_n$ is absolutely convergent so is $\sum_{n=1}^{\infty} a_n$.*

*Proof.* For all $M \geq N$ note that

$$\sum_{n=N}^{M} |a_n| \leq \sum_{n=N}^{M} |b_n| \leq \sum_{n=1}^{M} \leq C$$

for some $C \in \mathbb{R}$ because every term in $(|b_n|)$ is greater than or equal to zero (15.11). Also note that

$$\sum_{n=1}^{M} |a_n| \leq C + \sum_{n=1}^{N-1} |a_n| \leq C'$$

for some $C' \in \mathbb{R}$ because every term of $(|a_n|)$ is greater than or equal to zero. Also note that for $M' < N \leq M$ we have

$$\sum_{n=1}^{M'} |a_n| \leq \sum_{n=1}^{M} \leq C'$$

so that for all $M$ we have $\sum_{n=1}^{M} |a_n| \leq C'$. By Lemma 11 $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (15.11). $\square$

**Corollary 13 (Quotient Criterion)** *Let $\sum_{n=1}^{\infty} a_n$ be a series. Suppose that there is an $N \in \mathbb{N}$ and $0 < r < 1$, such that $|a_{n+1}/a_n| \leq r$ for all $n \geq N$. Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

*Proof.* Use induction on $n$ to show that $|a_{N+n}| \leq |a_N| r^n$. For the base case, $n = 1$ we have $|a_{N+1}| \leq |a_N| r$ by assumption. Assume that for all $n \in \mathbb{N}$ we have $|a_{N+n}| \leq |a_N| r^n$ so $|a_{N+n}| r \leq |a_N| r^{n+1}$. Then note that $|a_{N+n+1}| \leq |a_{N+n}| r \leq |a_N| r^{n+1}$ as desired. Thus for $n \geq N$ we have $|a_n| \leq |a_N| r^{n-N}$. Let $b_n = |a_N| r^{n-N}$. Then for $n > N$ we have $|a_n| \leq |a_N| r^{n-N} = |a_N r^{n-N}| = |b_n|$ since $r > 0$. But also

$$\sum_{n=1}^{\infty} |a_N r^{n-N}| = \sum_{n=0}^{\infty} |a_N| r^{n-N+1} = |a_N| r^{-N+1} \sum_{n=0}^{\infty} r^n$$

and so $\sum_{n=1}^{\infty} b_n$ is absolutely convergent by Theorem 6, because $r > 0$ and because $|a_N| r^{-N+1}$ is a constant value (15.6). Thus, by Theorem 12 we have $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. $\square$

**Definition 14** *Let $\sum_{n=1}^{\infty} a_n$ be a series. A reordering of $\sum_{n=1}^{\infty} a_n$ is a series of the form $\sum_{n=1}^{\infty} b_n$, where $b_n = a_{f(n)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$.*

**Lemma 15** *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, and let $\sum_{n=1}^{\infty} b_n$ be a reordering of it. Then for every $k \in \mathbb{N}$ there exists $L \in \mathbb{N}$ such that for all $l \geq L$,*

$$\left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{l} b_n \right| \leq \sum_{n=k+1}^{\infty} |a_n|.$$

*Proof.* Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(x) = |x|$. We know that since $g$ is continuous, for a sequence $(a_n)$, if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} |a_n| = |a|$ (13.7). We have $\sum_{n=1}^{\infty} a_n$ is absolutely convergent so $|\sum_{n=1}^{\infty} a_n| = \lim_{n\to\infty} |s_n|$. Then use induction on $n$ to show that $|s_n| \leq \sum_{k=1}^{n} |a_k|$. For $n = 1$ we have $|s_1| = |a_1| = \sum_{k=1}^{1} |a_1|$. Assume that for $n \in \mathbb{N}$, $\sum_{k=1}^{n} |a_k| \geq |s_n|$. Then

$$\sum_{k=1}^{n+1} |a_k| = \sum_{k=1}^{n} |a_k| + |a_{n+1}| \geq |s_n| + |a_{n+1}| \geq |s_n + a_{n+1}| = |s_{n+1}|$$

by the triangle inequality and our inductive hypothesis (9.36). Therefore we have

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Let $k \in \mathbb{N}$ and consider the sets $A = \{a_n \mid n \leq k\}$ and $S = \{f(n) \mid n \leq k\}$ Let $L = \sup S$. Consider $l \geq L$ and let $B = \{b_n \mid n \leq l\}$ and $T = \{n \mid b_n \in B\}$. Note that every element of $B$ is in $A$ because $L \geq k$. Finally let $C = \{a_n \mid n \notin T\}$. Make a new sequence $c_n$ where $n$ is the $n$th element of $C$. Note that by definition, $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{l} b_n$. Then

$$\left| \sum_{n=1}^{\infty} c_n \right| = \left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{l} b_n \right| \leq \sum_{n=1}^{\infty} |c_n| \leq \sum_{k+1}^{\infty} |a_n|.$$

The last inequality holds because $(c_n)$ is the sequence $(a_n)$, but with at least $k$ terms missing. □

**Theorem 16 (Abel Resummation Theorem)** *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, and let $\sum_{n=1}^{\infty} b_n$ be a reordering of it. Then $\sum_{n=1}^{\infty} b_n$ absolutely convergent and*

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

*Proof.* Let $k \in \mathbb{N}$ and consider the sets $A = \{a_n \mid n \leq k\}$ and $S = \{f(n) \mid n \leq k\}$ Let $L = \sup S$ and let $B = \{b_n \mid n \leq L\}$. Note that every element of $B$ is in $A$ because $L \geq k$. But then

$$\sum_{n=1}^{L} |b_n| = \sum_{n=1}^{L} |a_n| \leq C$$

for some $C \in \mathbb{R}$ (15.11). Since $f$ is a bijection, $L$ can be any value of $\mathbb{N}$, so every partial sum of $\sum_{n=1}^{\infty} |b_n|$ is bounded and thus $\sum_{n=1}^{\infty} b_n$ is absolutely convergent (15.11). Now consider $\sum_{n=k+1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |a_n| - \sum_{n=1}^{k} |a_n|$ (15.5). Take the limit as $k$ goes to infinity. We have

$$\lim_{k\to\infty} \sum_{n=k+1}^{\infty} |a_n| = \lim_{k\to\infty} \left( \sum_{n=1}^{\infty} |a_n| - \sum_{n=1}^{k} |a_n| \right) = \sum_{n=1}^{\infty} |a_n| - \lim_{k\to\infty} s_k = 0.$$

But then we have

$$\lim_{l\to\infty}\left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{l} b_n\right| = \left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n\right| \leq \lim_{n\to\infty}\sum_{n=k+1}^{\infty}|a_n| = 0 \text{ (15.15)}.$$

Thus,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

□

**Theorem 17** *Let $\sum_{n=1}^{\infty} a_n$ be a convergent, but not absolutely convergent series. Then for all $c \in \mathbb{R}$ there exists a reordering $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that*

$$\sum_{n=1}^{\infty} b_n = c.$$

*Proof.* Let $A = \{a_n \mid n \in \mathbb{N}\}$. Then $A$ is nonempty and bounded, so $\sup A$ exists (6. 11, 13.15). Suppose that for any positive term of $(a_n)$ there are infinitely many terms greater than or equal to it. Consider some term $a_k > 0$ and the region $(-a_k; a_k)$. Then there are infinitely many terms of $(a_n)$ which are not in $(-a_k; a_k)$. But then $(a_n)$ does not converge to zero which means $\sum_{n=1}^{\infty} a_n$ is not convergent (13.4). This is a contradiction and so for all positive terms of $(a_n)$ there are finitely many terms greater than or equal to it. A similar proof holds to show that for a negative term of $(a_n)$, there are finitely many terms less than or equal to it.

We have $\sum_{n=1}^{\infty} a_n = a$ for some $a \in \mathbb{R}$. Assume that $a_n = 0$ for finitely many $n$. We can order the positive elements of $(a_n)$ in decreasing order and the negative elements of $(a_n)$ in increasing order because there are finitely many positive or negative terms of $(a_n)$ greater than or less than any given term respectively. Define $(x_k)$ where $x_k$ is the $k$th positive element of $(a_n)$ and $(y_k)$ where $y_k$ is the $k$th negative element of $(a_n)$. Then for all $k \in \mathbb{N}$ we have $y_k < 0 \leq x_k$. Suppose there are finitely many negative terms of $(a_n)$. Then there exists a largest element, $j$, of $N$ so that

$$\sum_{k=1}^{j} y_k = q \text{ and } \sum_{k=1}^{j}|y_k| = -q$$

for some $q \in \mathbb{R}$ because $y_k < 0$ for all $k$. Then we have

$$\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} x_k + \sum_{k=1}^{j} y_k \text{ and so } \sum_{k=1}^{\infty} x_n = \sum_{k=1}^{\infty}|x_k| = a - q.$$

This follows from Lemma 5. But then

$$(a-q) + q = \sum_{k=p_1}^{\infty}|x_k| + \sum_{k=n_1}^{n_j}|y_k| = \sum_{n=1}^{\infty}|a_n|$$

which means $\sum_{n=1}^{\infty} a_n$ is absolutely convergent which is a contradiction. Thus there are infinitely many terms of $(y_k)$ and a similar proof shows there are infinitely many terms of $(x_k)$.

Let $c \in \mathbb{R}$. Now suppose that for all $j \in \mathbb{N}$ we have $\sum_{k=1}^{j} x_k \leq c$. Since $x_k > 0$ for all $k$, we have the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded and increasing so it must converge to $x$ for some $x \in \mathbb{R}$ (13.18). Suppose that $\sum_{k=1}^{\infty}|y_k| = y$ for some $y \in \mathbb{R}$. Then $\sum_{n=1}^{\infty}|a_n| = \sum_{k=1}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = x + y$ which is a contradiction (15.16). Thus $\sum_{k=1}^{\infty} y_k$ is not absolutely convergent so there exists $l \in \mathbb{N}$ such that $\sum_{k=1}^{l} |y_k| > c$ (15.11). But since $y_k < 0$ for all $k$ we have $-c < \sum_{k=1}^{l} y_k$.

Now consider the sequence $(a'_n)$ where $a'_n = a_n$ if $a_n < 0$ and 0 if $a_n \geq 0$. Then a partial sum of

$$\sum_{n=1}^{\infty} a'_n \text{ is } s_{a'_n} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n'} x_k$$

supposing there are $n'$ positive terms in the first $n$ terms of $(a_n)$. Then if we consider $\lim_{n\to\infty} s_{a'_n}$ we simply have $a - x$ since $n'$ will go to $\infty$ as $n$ does. Hence

$$\sum_{n=1}^{\infty} a'_n = \sum_{k=1}^{\infty} y_k + 0 = a - x.$$

Thus $\sum_{k=1}^{\infty} y_k$ is convergent, but we just showed that the partial sums of this series are unbounded which is a contradiction (13.15). Thus, for $c \in \mathbb{R}$ there exists $j \in \mathbb{N}$ such that $\sum_{k=1}^{j} x_k > c$. A similar proof shows that for $c \in \mathbb{R}$ there exists $j \in \mathbb{N}$ such that $\sum_{k=1}^{j} y_k < c$

Define a reordering of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ where the first $n_1$ terms of $b_n$ are the least number of terms of $(x_k)$ such that $\sum_{k=1}^{n_1} x_k > c$. Then let the next $n_2$ terms be the least number of terms of $(y_k)$ such that $\sum_{k=1}^{n_1} x_k + \sum_{k=1}^{n_2} y_k < c$. Note that we can always do this because the partial sums of

$$\sum_{k=1}^{\infty} x_k \text{ and } \sum_{k=1}^{\infty} y_k$$

are unbounded. Then for odd $i \in \mathbb{N}$, $n_i$ is the least number of terms of $(x_k)$ such that

$$\sum_{n=1}^{n_i} b_n = \sum_{k=1}^{n_i} x_k + \sum_{k=1}^{n_{i-1}} y_k > c$$

and for even $i$, $n_i$ is the least number of terms of $(y_k)$ such that

$$\sum_{n=1}^{n_i} b_n = \sum_{k=1}^{n_{i-1}} + \sum_{k=1}^{n_i} y_k < c.$$

Let $s_k = \sum_{n=1}^{k} b_n$. Note that $s_k$ for $k$ between $n_i$ and $n_{i+1}$ for $i \in \mathbb{N}$ is between $s_{n_i}$ and $s_{n_{i+1}}$ because the terms of $b_n$ change sign at $n_i$. Consider some region $(p;q)$ such that $c \in (p;q)$. Since the least number of elements of $(y_k)$ are added to $s_{n_{i-1}}$ so that $s_{n_i} < c$, we have $|c - s_{n_i}|$ is always less than or equal to the absolute value of some element of $(y_k)$. Suppose that $p > s_{n_i}$ for an infinite number of odd $i$. Then $|c - p|$ is less than or equal to an infinite number of absolute values of terms of $(y_k)$. But then if we consider some $|y_k| > |c - p|$ there are an infinite number of $n$ such that $|y_n| > |y_k|$. This is a contradiction and so $p > s_{n_i}$ for finitely many odd $i$. But also for all $s_{n_i}$ with odd $i$ there are finitely many $s_k$ such that $i < k < i + 1$ because the positive and negative partial sums are unbounded. Thus there are finitely many $n$ such that $s_n < p$. A similar proof shows that there are finitely many $n$ with $s_n > q$ so there are finitely many $n$ with $s_n \notin (p;q)$. Therefore $\lim_{n\to\infty} s_n = c$ and so

$$\sum_{n=1}^{\infty} b_n = c.$$

If there are infinitely many $n$ such that $a_n = 0$ the change $b_n$ so that a zero term is added to each $n_i$th partial sum. This will not change the resulting series convergence. $\square$

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# Sheet 16: Metric Spaces

**Definition 1** *Let $X$ be a set. A topology on $X$ is a set $\mathcal{A}$ of subsets of $X$, that we call open sets, satisfying the following:*
*1) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$;*
*2) if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$;*
*3) if $\mathcal{B} \subset \mathcal{A}$ then*

$$\bigcup_{B \in \mathcal{B}} B \in \mathcal{A}.$$

**Definition 2** *A topological space is a pair $(X, \mathcal{A})$ such that $\mathcal{A}$ is a topology on $X$.*

**Definition 3** *Let $X$ be a set and let $d : X \times X \to \mathbb{R}$ be a function. We say that $(X, d)$ is a metric space if the following hold:*
*1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;*
*2) $d(x, y) = d(y, x)$;*
*3) $d(x, y) + d(y, z) \geq d(x, z)$.*

**Definition 4** *For $c \in X$ and $r \in \mathbb{R}$ with $r > 0$ let*

$$B(c, r) = \{x \in X \mid d(c, x) < r\}$$

*be the ball of radius $r$ centered at $c$.*

**Definition 5** *A subset $A \subseteq X$ is open if for every $a \in A$ there exists $r > 0$ such that $B(a, r) \subseteq A$. This topology is the topology generated by $d$.*

**Theorem 6** *For all $c \in X$ and $r > 0$ the ball $B(c, r)$ is open.*

*Proof.* Let $a \in B(c, r)$. Then $d(c, a) < r$. Consider the ball $B(a, r - d(c, a))$. For $x \in B(a, r - d(c, a))$ we have $d(a, x) < r - d(c, a)$ so $d(c, a) + d(a, x) < r$. By the triangle inequality we have $d(c, x) < r$ so $x \in B(c, r)$. Thus, $B(a, r - d(c, a)) \subseteq B(c, r)$ and $B(c, r)$ is open. $\square$

**Proposition 7** *There is a topology on $\{0, 1\}$ that cannot be generated by any metric on $\{0, 1\}$.*

*Proof.* Consider the topology $\mathcal{A} = \{\emptyset, \{0, 1\}\}$ and consider some arbitrary metric on $\{0, 1\}$, $d(0, 1) = a$ for $a \in \mathbb{R}$. Then the ball $B(0, a)$ will be in the topology generated by this metric, but $B(0, a) = \{0\}$ which is not in $\mathcal{A}$. $\square$

**Theorem 8 (Metric Spaces are Hausdorff)** *Let $(X, d)$ be a metric space and let $a, b \in X$ with $a \neq b$. Then there exist $A, B \subseteq X$ open such that $a \in A$, $b \in B$ and $A \cap B = \emptyset$.*

*Proof.* Consider the two balls $B(a, d(a, b)/2)$ and $B(b, d(a, b)/2)$. Suppose there exists $x \in X$ such that $x \in B(a, d(a, b)/2)$ and $x \in B(b, d(a, b)/2)$. Then $d(a, x) < d(a, b)/2$ and $d(b, x) < d(a, b)/2$ so $d(a, x) + d(x, b) < d(a, b)$ which contradicts the triangle inequality. Thus $B(a, d(a, b)/2) \cap B(b, d(a, b)/2) = \emptyset$. We also have $B(a, d(a, b)/2)$ and $B(a, d(a, b)/2)$ are open (16.6). $\square$

**Definition 9** *Let $A \subseteq X$ be a subset. We say that $x \in X$ is a limit point of $A$ if for all open sets $B \subseteq X$ with $x \in B$ the intersection $A \cap B$ is infinite.*

**Lemma 10** *Let $A \subseteq X$ be a subset. Then $x \in X$ is a limit point of $A$ if for all $r > 0$ the intersection $A \cap B(x, r)$ is infinite.*

*Proof.* Suppose that for $x \in X$ and all $r > 0$ we have $A \cap B(x, r)$ is infinite. Consider some open set $B \subseteq X$ with $x \in B$. Then there exists $B(x, r) \subseteq B$ because $B$ is open. But then $B \cap A$ is infinite since $B(x, r) \cap A$ is infinite. □

**Theorem 11** *A subset of $X$ is closed if and only if it contains all its limit points.*

*Proof.* Let $A \subseteq X$ be closed and consider some point $p \in X \backslash A$. Since $X \backslash A$ is open, there exists some ball $B(p, r) \subseteq X \backslash A$. But since this ball is open and disjoint from $X$ we have $p$ is not a limit point of $A$ (16.6). Thus there are no limit points of $A$ in $X \backslash A$ so $A$ must contain all its limit points. Conversely let $A \subseteq X$ be a subset which contains all its limit points and let $p \in X \backslash A$. Since $p$ is not a limit point of $A$, there exists some ball $B(p, r)$ such that $B(p, r) \cap A$ is finite. Then consider the point $x \in B(p, r) \cap A$ such that $d(p, x) = \min\{d(p, y) \mid y \in B(p, r) \cap A\}$. The ball $B(p, x)$ will then contain no points of $A$ which means $B(p, x) \subseteq X \backslash A$ and thus $X \backslash A$ is open. Then $A$ is closed. □

**Theorem 12 (Metric Spaces are T3)** *Let $C \subseteq X$ be closed and let $b \in X$ such that $b \notin C$. Then there exist $A, B \subseteq X$ open such that $C \subseteq A$, $b \in B$ and $A \cap B = \emptyset$.*

*Proof.* Since $C$ is closed, $X \backslash C$ is open and so there exists a ball $B = B(b, r) \subseteq X \backslash C$. Consider the set $S = \{B(a, (d(a, b) - r)/2) \mid a \in C\}$. Then let

$$A = \bigcup_{B(a,r) \in S} B(a, r)$$

so that $C \subseteq A$. Now let $x \in A$. Then there exists some ball $B(a, (d(a, b) - r)/2) \subseteq A$ such that $a \in C$ and $x \in B(a, (d(a, b) - r)/2)$. Then $d(x, a) < d(a, b) - r$ so $r < d(a, b) - d(a, x) \leq d(x, b)$. Thus $x \notin B(b, r)$ and so $A \cap B = \emptyset$. □

**Definition 13** *A subset $C \subseteq X$ is compact if every open cover of $C$ has a finite subcover.*

**Definition 14** *A sequence on $X$ is a function from $\mathbb{N}$ to $X$. The sequence $(a_n)$ converges to $a$ (or $\lim_{n \to \infty} a_n = a$) if for every open set $A \subseteq X$ with $a \in A$ there are only finitely many $n$ with $a_n \notin A$.*

**Proposition 15** *There is a topological space on every set where every sequence converges to every element.*

*Proof.* Consider the trivial topology, $\{\emptyset, X\}$. Consider some sequence $(a_n) \in X$ and let $a \in X$. The only open set which contains $a$ is $X$, but there are no terms of $(a_n)$ not in $X$ so we have for all open sets $A$ with $a \in A$, there are finitely many terms of $(a_n)$ not in $A$. Thus $(a_n)$ converges to $a$. This is true of all sequences and points in $X$. □

**Proposition 16** *There is a topological space on every set where the only convergent sequences are the ones that are constant up to finitely many elements.*

*Proof.* Consider the full topology where every subset is open. Then for all $x \in X$, the set $\{x\}$ is open. Thus for a sequence $(a_n)$, there are finitely many $n$ such that $a_n \notin \{x\}$ which means there are finitely many $n$ such that $a_n \neq x$. □

**Definition 17** *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces. A function $f : X \to Y$ is continuous if for all $B \in \mathcal{B}$ the preimage $f^{-1}(B) \in \mathcal{A}$*

**Theorem 18** *Let $(X, \mathcal{A})$ be a Hausdorff topological space and let $(a_n)$ be a sequence on $X$. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$ then $a = b$.*

*Proof.* Suppose that $a \neq b$. Then there exist two open sets $A$ and $B$ such that $a \in A$ and $b \in B$ and $A \cap B = \emptyset$ by the Hausdorff property. There are finitely many $n$ with $a_n \notin A$ so there are infinitely many $n$ with $a_n \in A$. But then there are finitely many $n$ with $a_n \notin B$ which is a contradiction because $\lim_{n\to\infty} a_n = b$. Thus $a = b$. □

**Theorem 19** *Let $(X, d)$ and $(X', d')$ be metric spaces and let $f : X \to X'$ be a function. Then the following are equivalent:*
*1) $f$ is continuous;*
*2) for all $x \in X$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in X$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$;*
*3) for all convergent sequences $a_n \in X$ we have*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

*Proof.* Let $f$ be continuous and let $x \in X$ and consider the ball $B(f(x), \varepsilon)$ for $\varepsilon > 0$. Then since $f$ is continuous, $f^{-1}(B(f(x), \varepsilon))$ is open. And since $x \in f^{-1}(B(f(x), \varepsilon))$ there exists some ball $B(x, \delta) \subseteq B(f(x), \varepsilon)$. But then for all $y \in B(x, \delta)$, $f(y) \in B(f(x), \varepsilon)$. Thus for all $y \in X$ such that $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$.

Now suppose that for all $x \in X$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in X$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$. Let $a_n \in X$ be a sequence which converges to $a$ and let $\varepsilon > 0$. Consider $B(a, \delta)$. Since $\lim_{n\to\infty} a_n = a$, there are finitely many $n$ with $a_n \notin B(a, \delta)$. But then there are finitely many $n$ such that $d(a, a_n) \geq \delta$ which means there are finitely many $n$ with $d'(f(a), f(a_n)) \geq \varepsilon$. Therefore there are finitely many $n$ with $f(a_n) \notin B(f(a), \varepsilon)$ and since this is true for all $\varepsilon > 0$, we have $\lim_{n\to\infty} f(a_n) = f(a)$.

Finally use the contrapositive and assume that $f$ is not continuous. Then there exists some set $A \subseteq X'$ such that $f^{-1}(A)$ is not open. Then there exists $a \in f^{-1}(A)$ such that for all $r > 0$ there exists $x \in B(a, r)$ such that $x \notin A$. Create a sequence $a_n \in X$ where $a_n \in B(a, 1/n)$, but $a_n \notin A$. We know that $a_n$ exists for all $n$ because $f^{-1}(A)$ is not open. Note that for the ball $B(a, r)$ with $r > 1$ there are no terms of $(a_n)$ not in $B(a, r)$ and for $r \leq 1$ we can use the Archimedean Property to show that there are finitely many terms not in $B(a, r)$. Thus $(a_n)$ converges to $a$. Note that for all $n$, $a_n \notin f^{-1}(A)$ and thus $f(a_n) \notin A$, while $a \in f^{-1}(A)$ and so $f(a) \in A$. But $A$ is open so there exists some ball $B(a, r) \subseteq A$ for which $a_n \notin B(a, r)$ for all $n$. But then $\lim_{n\to\infty} f(a_n) \neq f(a)$. □

**Theorem 20** *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and let $f : X \to Y$ be continuous. Then for every compact subset $C \subseteq X$ the image $f(C)$ is also compact.*

*Proof.* Let $\mathcal{E} \subseteq \mathcal{B}$ be an open cover of $f(C)$. For all $x \in C$ we have $x \in f(C)$ and so for all $x \in C$ there exist an open set $B \in \mathcal{E}$ such that $f(x) \in B$. But then for all $x \in C$, $x \in f^{-1}(B)$ for some $B \in \mathcal{E}$. So we have $C \subseteq \bigcup_{B\in\mathcal{E}} f^{-1}(B)$ and since $f$ is continuous $\{f^{-1}(B) \mid B \in \mathcal{E}\} \subseteq \mathcal{A}$ is an open cover for $C$. But $C$ is compact so there exists a finite subcover, $\{f^{-1}(B_1), f^{-1}(B_2), \ldots, f^{-1}(B_n)\}$ which covers $C$. So for all $x \in C$ there exists some $B_i \in \mathcal{E}$ such that $x \in f^{-1}(B_i)$. But then $f(x) \in B_i$ and since $f(C) = \{y \in Y \mid x \in C, y = f(x)\}$, we have for all $y \in f(C)$, $y \in B_i$. Since every $B_i \in \mathcal{E}$ we have found a finite subcover of $\mathcal{E}$ which covers $f(C)$. Thus $f(C)$ is compact. □

**Theorem 21** *Let $(X, d)$ be a metric space. Then every compact subset of $X$ is bounded and closed.*

*Proof.* Let $C$ be a compact subset of $X$ and suppose that $C$ is not bounded below. Let $\mathcal{A}$ be the set of all balls centered at $c \in C$. Then $\mathcal{A}$ covers $C$ and since $C$ is compact there exists a finite subcover $\mathcal{B} \subseteq \mathcal{A}$ which covers $C$. Then $\mathcal{B} = \{B(c, r_1), B(c, r_2), \ldots, B(c, r_n)\}$. Take the largest $r_i$ such that $B(c, r_i) \in \mathcal{B}$. But we have $C$ is not bounded below so there exists $x \in C$ such that $d(x, c) > r_i$. Thus $C \nsubseteq \bigcup_{B \in \mathcal{B}} B$ and so $\mathcal{B}$ doesn't cover $C$. This is a contradiction and so compact sets are bounded below. A similar proof holds to show compact sets must be bounded above.

Now suppose that $C \subseteq X$ is compact and $C$ is not closed. Let $p \notin C$ be a limit point of $C$. Let $\mathcal{A} = \{X \backslash B(p, r) \mid r \in \mathbb{R}\}$. Since $p \notin C$ we see that $\mathcal{A}$ covers $C$. Since $C$ is compact, let $\mathcal{B}$ be a finite subset of $\mathcal{A}$ which covers $C$. We have $X$ is open and $X \backslash \emptyset$ is closed so $X \neq \emptyset$. Thus if $\mathcal{B} = \emptyset$, $\mathcal{B}$ does not cover $X$. Then $\mathcal{B} = \{X \backslash B_1(p, r_1), X \backslash B_2(p, r_2), \ldots, X \backslash B_n(p, r_n)\}$. Take the smallest $r_i$ such that $B_i(p, r_i) \in \mathcal{B}$ and consider $B(p, r_i/2)$. This ball contains $p$, which is a limit point of $C$, and since balls are open, $B(p, r_i/2) \cap C \neq \emptyset$. But $B(p, r_i/2)$ is defined such that $B(p, r_i/2) \nsubseteq \bigcup_{B \in \mathcal{B}} B$ and so $C \nsubseteq \bigcup_{B \in \mathcal{B}} B$. But then $\mathcal{B}$ doesn't cover $C$ which is a contradiction. Therefore compact sets are closed. $\square$

**Proposition 22** *Let $X$ be an infinite set. Then there is a metric on $X$ such that there exists a bounded and closed set that is not compact.*

*Proof.* Consider the metric $d(x, y) = a$ for some $a \in \mathbb{R}$. Let $Y \subseteq X$ be a bounded closed infinite set and let $\mathcal{A} = \{B(y, a) \mid y \in Y\}$. This set covers $Y$, but each element contains only one element of $Y$ so a finite subset of $\mathcal{A}$ will only contain finitely many elements of $Y$. $\square$

**Definition 23** *Let $(X, d)$ and $(X', d')$ be metric spaces and let $f : X \to X'$ be a function. We say that $f$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$.*

**Theorem 24** *Let $(X, d)$ and $(X', d')$ be metric spaces and let $f : X \to X'$ be a continuous function. If $X$ is compact then $f$ is uniformly continuous.*

*Proof.* Let $\varepsilon > 0$ and consider $\varepsilon/2 > 0$. We have $f$ is continuous so for all $x \in X$ there exists $\delta(x) > 0$ such that for all $y \in X$ with $d(x, y) < \delta(x)$ we have $d'(f(x), f(y)) < \varepsilon/2$ (16.19). Consider the set of balls $\mathcal{A} = \{B(x, \delta(x)) \mid x \in X\}$ and let $\mathcal{A}' = \{B(x, \delta(x)/2) \mid B(x, \delta(x)) \in \mathcal{A}\}$. $\mathcal{A}'$ is an open cover for $X$ and since $X$ is compact there exists a finite subcover, $\mathcal{B} \subseteq \mathcal{A}'$. Let $\delta = \min\{\delta(x)/2 \mid B(x, \delta(x)/2) \in \mathcal{B}\}$. Then consider two points $x, y \in X$ such that $d(x, y) < \delta$. $\mathcal{B}$ is an open cover for $X$ so there exists some ball $B(z, \delta(z)/2) \in \mathcal{B}$ such that $x \in B(z, \delta(z)/2)$. Then $d(x, z) < \delta(z)/2 < \delta(z)$ and $d(x, y) < \delta \leq \delta(z)/2$ so $d(z, y) \leq d(z, x) + d(x, y) < \delta(z)$. But then $d'(f(z), f(x)) < \varepsilon/2$ and $d'(f(z), f(y)) < \varepsilon/2$ so $d'(f(x), f(y)) \leq d'(f(x), f(z)) + d'(f(z), f(y)) < \varepsilon$. Therefore for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$. $\square$

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# Sheet 17: More About Metric Spaces

**Theorem 1** *Let $(X, d)$ be a metric space and let $(a_n)$ be a sequence in $X$. Then $\lim_{n\to\infty} a_n = a$ if and only if $\lim_{n\to\infty} d(a_n, a) = 0$.*

*Proof.* Let $\lim_{n\to\infty} a_n = a$. Then for every open set $A \subseteq X$ with $a \in A$ there are finitely many $n$ with $a_n \notin A$. But then for $r \in \mathbb{R}$, there are finitely many $n$ with $a_n \notin B(a, r)$. Then there are finitely many $n$ such that $d(a_n, a) < r$ which means there are finitely many $n$ such that $d(a_n, a) \notin (-r, r)$. Thus, $\lim_{n\to\infty} d(a_n, a) = 0$.

Conversely, let $\lim_{n\to\infty} d(a_n, a) = 0$. Then for all $r \in \mathbb{R}$ there are finitely many $n$ such that $d(a_n, a) \notin (-r, r)$ which means there are finitely many $n$ such that $d(a_n, a) > r$. But then there are finitely many $n$ such that $a_n \notin B(a, r)$. If we consider some open set $A \subseteq X$ such that $a \in A$, there there exists some ball $B(a, r) \subseteq A$. But since there are finitely many $n$ with $a_n \notin B(a, r)$, there are only finitely $n$ with $a_n \notin A$. Thus, $\lim_{n\to\infty} a_n = a$. □

**Definition 2** *Let $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}$ denote the set of real $n$-tuples.*

**Definition 3** *For $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, b_2, \dots b_n) \in \mathbb{R}^n$ let*

$$d_0(\mathbf{a}, \mathbf{b}) = \max_{1\le i\le n} |a_i - b_i|,$$

$$d_1(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} |a_i - b_i|$$

*and*

$$d_2(\mathbf{a}, \mathbf{b}) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}.$$

**Theorem 4** *The functions $d_0$, $d_1$ and $d_2$ are all metrics on $\mathbb{R}^n$.*

*Proof.* Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. It's clear that $d_0(\mathbf{a}, \mathbf{b})$, $d_1(\mathbf{a}, \mathbf{b})$ and $d_2(\mathbf{a}, \mathbf{b})$ are all greater than or equal to 0. Let $d_0(\mathbf{a}, \mathbf{b}) = 0$. Then $\max_{1\le i\le n} |a_i - b_i| = 0$ and so $a_i = b_i$. Since the maximum positive difference between two coordinates is 0, all the distances must be 0 as well. Now let $\mathbf{a} = \mathbf{b}$. Then $a_i = b_i$ for $1 \le i \le n$. Thus $\max_{1\le i\le n} |a_i - b_i| = 0$ and $d_0(\mathbf{a}, \mathbf{b}) = 0$.

Let $d_1(\mathbf{a}, \mathbf{b}) = 0$. Then $\sum_{i=1}^{n} |a_i - b_i| = 0$. But since $|a_i - b_i| \ge 0$ for $1 \le i \le n$ we have $|a_i - b_i| = 0$ for $1 \le i \le n$. Thus $a_i = b_i$ and $\mathbf{a} = \mathbf{b}$. Now suppose that $\mathbf{a} = \mathbf{b}$. Then we have $a_i = b_i$ for $1 \le i \le n$ and so $|a_i - b_i| = 0$. But then $\sum_{i=1}^{n} |a_i - b_i| = 0$ and so $d_1(\mathbf{a}, \mathbf{b}) = 0$.

Let $d_2(\mathbf{a}, \mathbf{b}) = 0$. Then $\sqrt{\sum_{i=1}^{n} (a_i - b_i)^2} = 0$ which means $\sum_{i=1}^{n} (a_i - b_i)^2 = 0$. From here the proof follows similarly to that of $d_1(\mathbf{a}, \mathbf{b})$.

Since $|a - b| = |b - a|$ and $(a - b)^2 = (b - a)^2$ for all $a, b \in \mathbb{R}$, we have $d_i(\mathbf{a}, \mathbf{b}) = d_i(\mathbf{b}, \mathbf{a})$ for $0 \le i \le 2$. Finally, note that using the triangle inequality we have

$\max_{1\le i\le n}|a_i-b_i|+\max_{1\le i\le n}|b_i-c_i|\ge|a_i-b_i|+|b_i-c_i|$ for arbitrary $1\le i\le n$ which is in turn greater than $\max_{1\le i\le n}|a_i-c_i|$. Note also that by the triangle inequality we have $|a_i-b_i|+|b_i-c_i|\ge|a_i-c_i|$ for $1\le i\le n$. But then if we sum this inequality $n$ times we have
$\sum_{i=1}^n|a_i-b_i|+\sum_{i=1}^n|b_i-c_i|\ge\sum_{i=1}^n|a_i-c_i|$. Lastly note that

$$\sqrt{\sum_{i=1}^n(a_i-b_i)^2}+\sqrt{\sum_{i=1}^n(b_i-c_i)^2}\ge\sqrt{\sum_{i=1}^n((a_i-b_i)^2+(b_i-c_i)^2)}\ge\sqrt{\sum_{i=1}^n(a_i-c_i)^2}.$$

Thus all three distance functions satisfy the triangle inequality. Therefore all three are metrics. □

**Theorem 5** *For all $0\le i\le 2$, $0\le j\le 2$ and for all $\mathbf{x}\in\mathbb{R}^n$ and $r>0$ there exists $r'>0$ such that*

$$B_{d_i}(x,r')\subseteq B_{d_j}(x,r).$$

*Proof.* Let $\mathbf{x}\in\mathbb{R}^n$ and let $r>0$. Consider $B_{d_0}(\mathbf{x},r)$, let $r=r'$ and let $\mathbf{y}\in B_{d_1}(\mathbf{x},r')$. Then $\sum_{i=1}^n|x_i-y_i|<r'$ and so $d_0(\mathbf{x},\mathbf{y})=\max_{1\le i\le n}|x_i-y_i|<r'=r$. Thus $\mathbf{y}\in B_{d_0}(\mathbf{x},r)$ and $B_{d_1}(\mathbf{x},r')\subseteq B_{d_0}(\mathbf{x},r)$. Now let $r=r'$ again and let $\mathbf{y}\in B_{d_2}(\mathbf{x},r')$. Then

$$\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<r'$$

so $\max_{1\le i\le n}(x_i-y_i)^2<\sum_{i=1}^n(x_i-y_i)^2<r'^2$ and $d_0(\mathbf{x},\mathbf{y})=\max_{1\le i\le n}|x_i-y_i|<r'=r$. Thus $\mathbf{y}\in B_{d_0}(\mathbf{x},r)$ and $B_{d_2}(\mathbf{x},r')\subseteq B_{d_0}(\mathbf{x},r)$.

Next consider $B_{d_1}(\mathbf{x},r)$, let $r'=r/n$ and let $\mathbf{y}\in B_{d_0}(\mathbf{x},r')$. Then $\max_{1\le i\le n}|x_i-y_i|<r/n$ which means $d_1(\mathbf{x},\mathbf{y})=\sum_{i=1}^n|x_i-y_i|<r$ and $\mathbf{y}\in B_{d_1}(\mathbf{x},r)$. Thus $B_{d_0}(\mathbf{x},r')\subseteq B_{d_1}(\mathbf{x},r)$. Now let $r'=r/\sqrt{n}$ and let $\mathbf{y}\in B_{d_2}(\mathbf{x},r')$. Then

$$\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<\frac{r}{\sqrt{n}}$$

so $\sum_{i=1}^n(x_i-y_i)^2<r^2/n$ and $(x_i-y_i)^2<r^2/n^2$ for $1\le i\le n$. Thus $|x_i-y_i|<r/n$ for $1\le i\le n$ and so $d_1(\mathbf{x},\mathbf{y})=\sum_{i=1}^n|x_i-y_i|<r$. Thus $B_{d_2}(\mathbf{x},r')\subseteq B_{d_1}(\mathbf{x},r)$.

Finally, consider $B_{d_2}(\mathbf{x},r)$, let $r'=r/\sqrt{n}$ and let $\mathbf{y}\in B_{d_0}(\mathbf{x},r')$. Then $\max_{1\le i\le n}|x_i-y_i|<r/\sqrt{n}$ which means $\max_{1\le i\le n}(x_i-y_i)^2<r^2/n$ and

$$d_2(\mathbf{x},\mathbf{y})=\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<r.$$

Thus $\mathbf{y}\in B_{d_2}(\mathbf{x},r)$ and $B_{d_0}(\mathbf{x},r')\subseteq B_{d_2}(\mathbf{x},r)$. Now let $r'=r/n\sqrt{n}$ and let $\mathbf{y}\in B_{d_1}(\mathbf{x},r')$. Then $\sum_{i=1}^n|x_i-y_i|<r/n\sqrt{n}$ so $|x_i-y_i|<r/\sqrt{n}$ and $(x_i-y_i)^2<r^2/n$ for $1\le i\le n$. Then $\sum_{i=1}^n(x_i-y_i)^2<r^2$ and

$$d_2(\mathbf{x},\mathbf{y})=\sqrt{\sum_{i=1}^n(x_i-y_i)^2}<r.$$

Thus $\mathbf{y}\in B_{d_2}(\mathbf{x},r)$ and $B_{d_1}(\mathbf{x},r')\subseteq B_{d_2}(\mathbf{x},r)$. □

**Corollary 6** *The metrics $d_0$, $d_1$ and $d_2$ generate the same topology on $\mathbb{R}^n$, namely, a subset $A\subseteq\mathbb{R}^n$ is open in $(\mathbb{R}^n,d_i)$ if it is open in $(\mathbb{R}^n,d_j)$ $(0\le i\le 2,\ 0\le j\le 2)$.*

*Proof.* Let $A \subseteq \mathbb{R}^n$ be an open set in $(\mathbb{R}^n, d_j)$. Then for all $a \in A$ there exists $r \in \mathbb{R}$ such that $B_{d_j}(a, r) \subseteq A$. But from Theorem 5 we know that there exists $r' \in \mathbb{R}$ such that $B_{d_i}(a, r') \subseteq B_{d_j}(a, r) \subseteq A$ (17.5). Thus $A$ is open for $(\mathbb{R}, d_i)$. This is true for arbitrary $0 \leq i \leq 2$, $0 \leq j \leq 2$. $\square$

**Definition 7** *Let $(X, d)$ be a metric space. A sequence $(a_n)$ on $X$ has the Cauchy property if for all $\varepsilon > 0$ there exists $N$ such that for all $n, m > N$ we have $d(a_n, a_m) < \varepsilon$.*

**Definition 8** *A metric space $(X, d)$ is complete if every Cauchy sequence on $X$ is convergent.*

**Theorem 9** *$\mathbb{R}^n$ is complete.*

*Proof.* Let $(\mathbf{a}_n)$ be a Cauchy sequence on $\mathbb{R}^d$ and let $\varepsilon' > 0$. Then there exists $N$ such that for all $n, m > N$ we have

$$d_2(\mathbf{a}_n, \mathbf{a}_m) = \sqrt{\sum_{i=1}^{d} (a_{ni} - a_{mi})^2} < \varepsilon'$$

so $(a_{ni} - a_{mi})^2 \leq \sum_{i=1}^{d} (a_{ni} - a_{mi})^2 < \varepsilon'^2$ and $|a_{ni} - a_{mi}| < \varepsilon'$. Thus the $i$th coordinate of the terms of $(\mathbf{a}_n)$ forms a Cauchy sequence which converges to some $b_i$ (14.5). Then let $\mathbf{b} = (b_1, b_2, \ldots, b_d)$, let $\varepsilon > 0$ and consider $\varepsilon/\sqrt{d}$. For all $i \leq d$ there exists some $N_i$ such that for $n > N_i$ we have $|a_{ni} - b_i| < \varepsilon/\sqrt{d}$ by convergence (13.3). Let $N$ be the largest of all such $N_i$ so that for all $n > N$ we have $|a_{ni} - b_i| < \varepsilon/\sqrt{d}$. Then $(a_{ni} - b_i)^2 < \varepsilon^2/d$ and $\sum_{i=1}^{d} (a_{ni} - b_i)^2 < \varepsilon^2$. Then $d_2(\mathbf{a}_n, \mathbf{b}) < \varepsilon$ for all $n > N$ and $|d(\mathbf{a}_n, \mathbf{b})| < \varepsilon$ for all $n > N$. Thus $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{b}$ because $\lim_{n\to\infty} d(\mathbf{a}_n, \mathbf{b}) = 0$ (13.3, 17.1). $\square$

**Theorem 10** *Every compact metric space is complete.*

*Proof.* Let $(X, d)$ be a compact metric space and suppose that $(X, d)$ is not complete. Then there exists some Cauchy sequence $(a_n) \in X$ such that $(a_n)$ does not converge. Therefore for all $x \in X$ there exists some ball $B(x, \varepsilon)$ such that there are infinitely many $n$ with $a_n \notin B(x, \varepsilon)$. Let $\mathcal{A}$ be the set of all such balls and let $\mathcal{A}' = \{B(x, \varepsilon/2) \mid B(x, \varepsilon) \in \mathcal{A}\}$. Then $\mathcal{A}'$ is an open cover for $X$ and $X$ is compact so let $\mathcal{B}$ be a finite subcover for $\mathcal{A}'$. Let $B(x, \varepsilon/2) \in \mathcal{B}$. Note that there are infinitely many $n$ such that $a_n \notin B(x, \varepsilon)$ so there are infinitely many $n$ such that $a_n \notin B(x, \varepsilon/2)$. We have $(a_n)$ is Cauchy so there exists $N$ such that for all $n, m > N$ we have $d(a_n, a_m) < \varepsilon/2$. Suppose that there are infinitely many $n$ with $a_n \in B(x, \varepsilon/2)$. Since there are infinitely many $n$ with $a_n \in B(x, \varepsilon/2)$ and $a_n \notin B(x, \varepsilon/2)$, choose $n, m > N$ with $a_n \in B(x, \varepsilon/2)$ and $a_m \notin B(x, \varepsilon/2)$. But then $d(x, a_m) \leq d(x, a_n) + d(a_n, a_m) < \varepsilon$. Thus there are infinitely many $n$ with $a_n \notin B(x, \varepsilon)$ which is a contradiction. Therefore there are finitely many $n$ with $a_n \in B(x, \varepsilon/2)$. But this is true for all $B(x, \varepsilon/2) \in \mathcal{B}$ and there are finitely many elements of $\mathcal{B}$ which is an open cover for $X$. So we have finitely many $n$ with $a_n \in X$ which is a contradiction. Therefore $(X, d)$ is complete. $\square$

**Theorem 11** *Let $(\mathbf{a}_n)$ be a bounded sequence in $\mathbb{R}^d$. Show that $(\mathbf{a}_n)$ has a convergent subsequence.*

*Proof.* Consider the sequence $(a_{1n})$ where $a_{1n}$ is the 1st coordinate in the $n$th term of $(\mathbf{a}_n)$. Then we have $(a_{1n})$ is a bounded sequence so there exists some convergent subsequence $(b_{1k})$. Use induction on $n$. We have shown the base case for $n = 1$. Now assume that a bounded sequence $(\mathbf{a}_n) \in \mathbb{R}^d$ has a convergent subsequence for $d \in \mathbb{N}$. Consider a bounded sequence $(\mathbf{a}_n) \in \mathbb{R}^{d+1}$. By our Inductive Hypothesis there exists a convergent subsequence in $\mathbb{R}^d$ formed by the first $d$ coordinates of terms in $(\mathbf{a}_n)$. Let the corresponding terms in $(\mathbf{a}_n)$ be the sequence $(\mathbf{b}_k = \mathbf{a}_{n_k})$ Form a subsequence $(\mathbf{c}_k = \mathbf{a}_{n_k})$ of $(\mathbf{a}_n)$ where the $k$th term has the coordinates of $\mathbf{b}_k$ as the first $d$ coordinates and the $d + 1$th coordinate of $\mathbf{a}_{n_k}$ as the $d + 1$th coordinate. Now take the sequence in $\mathbb{R}$ where the $k$th term is the $d + 1$th coordinate of $\mathbf{c}_k$. Then this sequence is bounded so there exists a convergent subsequence $(e_i = c_{k_i(d+1)})$. Finally form a subsequence of $(\mathbf{a}_n)$ where the $i$th term is $c_{k_i}$. Now every coordinate in $(\mathbf{c}_{k_i})$ forms a convergent sequence in $\mathbb{R}$ so $(\mathbf{c}_{k+i})$ converges to some $\mathbf{f} \in \mathbb{R}^{d+1}$ using a similar proof as in Theorem 9. $\square$

**Theorem 12** *Show that a set $A \subseteq \mathbb{R}^d$ is open if and only if for all $\mathbf{x} \in A$ there is a rational ball $O$ such that $\mathbf{x} \in O$ and $O \subseteq A$.*

*Proof.* Suppose that for all $\mathbf{x} \in A$ there is a rational ball $B(\mathbf{a}, r) \subseteq A$ such that $\mathbf{x} \in B(\mathbf{a}, r)$. Then consider the ball $B(\mathbf{x}, r - d(\mathbf{a}, \mathbf{x}))$. For $\mathbf{y} \in B(\mathbf{x}, r - d(\mathbf{a}, \mathbf{x}))$ we have $d(\mathbf{x}, \mathbf{y}) < r - d(\mathbf{a}, \mathbf{x})$ which means $d(\mathbf{a}, \mathbf{y}) \leq d(\mathbf{a}, \mathbf{x}) + d(\mathbf{x}, \mathbf{y}) < r$ and so $\mathbf{y} \in B(\mathbf{a}, r)$. Thus $B(\mathbf{x}, r - d(\mathbf{a}, \mathbf{x})) \subseteq B(\mathbf{a}, r) \subseteq A$. Then for all $\mathbf{x} \in A$ there exists a ball $B(\mathbf{x}, r') \subseteq A$ so $A$ is open.

Conversely let $A \subseteq \mathbb{R}^d$ be open. Let $\mathbf{x} \in A$. There exists a ball $B(\mathbf{x}, r) \subseteq A$ where $r$ may be rational or not. If $r \notin \mathbb{Q}$ then consider some $r' \in \mathbb{Q}$ such that $0 < r' < r$ and then $B(\mathbf{x}, r') \subseteq B(\mathbf{x}, r) \subseteq A$ (9.12). We have $B(\mathbf{x}, r'/2) \subseteq B(\mathbf{x}, r') \subseteq A$. Let $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ where $y_i \in \mathbb{Q}$ and $0 < y_i < r'/(2\sqrt{d}) + x_i$ (9.12). Then $y_i - x_i < r'/(2\sqrt{d})$ and $|x_i - y_i| < r'/(2\sqrt{d})$. Also $(x_i - y_i)^2 < r'^2/(4d)$ so $\sum_{i=1}^{d}(x_i - y_i)^2 < r'^2/4$ and $d(\mathbf{x}, \mathbf{y}) < r'/2$. Finally consider $\mathbf{z} \in B(\mathbf{y}, r'/2)$. Then $d(\mathbf{y}, \mathbf{z}) < r'/2$. But also $d(\mathbf{x}, \mathbf{y}) < r'/2$ so we have $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) < r'/2 + r'/2 = r'$. Thus $B(\mathbf{y}, r'/2) \subseteq B(\mathbf{x}, r') \subseteq A$. Also $d(\mathbf{y}, \mathbf{x}) < r'/2 < r'$ so $\mathbf{x} \in B(\mathbf{y}, r'/2)$. Note that $r'/2 \in \mathbb{Q}$ and $\mathbf{y} \in \mathbb{Q}^d$. □

**Theorem 13** *Let $C$ be a closed, bounded subset of $\mathbb{R}^d$ and let $\mathcal{A}$ be an open cover for $C$. Then $\mathcal{A}$ has a countable subcover.*

*Proof.* Let $\mathbf{x} \in C$. Then there exists $A \in \mathcal{A}$ such that $\mathbf{x} \in A$. We have $A$ is open, so there exists some rational ball $O \subseteq A$ such that $\mathbf{x} \in O$. Let $\mathcal{B}$ be the set of all such rational balls for all $\mathbf{x} \in C$. Each of these balls has a center in $\mathbb{Q}^d$ which is countable, so there are countably many of them. Then let $\mathcal{C} \subseteq \mathcal{A}$ be set set of elements of $\mathcal{A}$ which have subsets in $\mathcal{B}$. Since every element of $\mathcal{B}$ is a subset of some element of $\mathcal{A}$, there are countable many elements of $\mathcal{C}$. But $\mathcal{C}$ covers $C$. □

**Theorem 14** *Closed bounded subsets of $\mathbb{R}^d$ are compact.*

*Proof.* Assume $C$ is a closed bounded subset of $\mathbb{R}^d$ which is not compact. Let $\mathcal{A}$ be an open cover for $C$ which does not have a finite subcover. Let $\mathcal{B} = \{B_i \mid i \in \mathbb{N}\} \subseteq \mathcal{A}$ be a countably infinite subcover for $C$ (17.13). Create a sequence $(\mathbf{a}_n) \in C$ such that $\mathbf{a}_1 \in B_1$ and for $n > 1$

$$\mathbf{a}_n \in C \backslash (B_1 \cup B_2 \cup \ldots B_{n-1}).$$

Then for all $j > i$, $\mathbf{a}_j \notin B_i$. Thus for all $B_i \in \mathcal{B}$, there are infinitely many $n$ with $\mathbf{a}_n \notin B_i$. Note that $(\mathbf{a}_n)$ is bounded since $C$ is bounded, so there exists a subsequence $(\mathbf{b}_n) \in C$ such that $\lim_{n\to\infty} \mathbf{b}_n = \mathbf{b}$. Note that $C$ is closed, so if $\mathbf{b} \notin C$ then there exists some ball $B(\mathbf{b}, r) \subseteq \mathbb{R}^d \backslash C$ because $\mathbb{R}^d \backslash C$ is open. But $(\mathbf{b}_n) \in C$ so there are infinitely many $n$ such that $\mathbf{b}_n \notin \mathbb{R}^d \backslash C$. Thus $\mathbf{b} \in C$. Then $\mathbf{b} \in B_i$ for some $B_i \in \mathcal{B}$. But there are infinitely many $n$ such that $\mathbf{a}_n \notin B_i$ and so $\lim_{n\to\infty} \mathbf{b}_n \notin B_i$. Thus, $(\mathbf{b}_n)$ does not converge which is a contradiction. Therefore $C$ is compact. □

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# Sheet 18: Convergence of Functions

**Definition 1** *For $a < b$ with $a, b \in \mathbb{R}$ let*

$$B[a;b] = \{f : [a;b] \to \mathbb{R} \mid f \text{ is bounded on } [a;b]\}$$

*be the set of bounded real functions on $[a;b]$.*

**Definition 2** *We say that $f$ is the pointwise limit of $(f_n)$, or*

$$\overset{\bullet}{\lim_{n\to\infty}} f_n = f$$

*if for all $x \in [a;b]$ we have*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

**Definition 3** *For $f, g \in B$ let*

$$d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)|.$$

**Theorem 4** *$d$ is a metric on $B$.*

*Proof.* Let $f, g, h \in B$. We have $|f(x) - g(x)| \geq 0$ for all $x \in [a;b]$ so then $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| \geq 0$. Also if $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = 0$ then $|f(x) - g(x)| = 0$ for all $x \in [a;b]$ because $d(f,g)$ is an upper bound. But then $f(x) = g(x)$ for $x \in [a;b]$. Conversely suppose that $f(x) = g(x)$ for all $x \in [a;b]$. Then $|f(x) - g(x)| = 0$ for all $x \in [a;b]$ and so $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = 0$. Also $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = \sup_{x\in[a;b]} |g(x) - f(x)| = d(g,f)$. Finally from the triangle inequality we have $|f(x) - g(x)| + |g(x) - h(x)| \geq |f(x) - h(x)|$ for all $x \in [a;b]$ so $|f(x) - g(x)| + |g(x) - h(x)| \geq \sup_{x\in[a;b]} |f(x) - h(x)|$ for all $x \in [a;b]$. But then $d(f,g) + d(g,h) = \sup_{x\in[a;b]} |f(x) - g(x)| + \sup_{x\in[a;b]} |g(x) - h(x)| \geq |f(x) - g(x)| + |g(x) - h(x)| \geq \sup_{x\in[a;b]} |f(x) - h(x)| = d(f,h)$ for all $x \in [a;b]$. □

**Definition 5** *We say that $f$ is the uniform limit of $(f_n)$, or*

$$\lim_{n\to\infty} f_n = f$$

*if $\lim_{n\to\infty} f_n = f$ in the metric $d$.*

**Theorem 6** *W have $\lim_{n\to\infty} f_n = f$ if and only if for all $\varepsilon > 0$ there exists $N$ such that for all $n > N$ and for all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$.*

*Proof.* Suppose that $\lim_{n\to\infty} f_n = f$. Then $\lim_{n\to\infty} f_n = f$ in the metric $d$. Thus $\lim_{n\to\infty} d(f, f_n) = 0$ which means $\lim_{n\to\infty} \sup_{x\in[a;b]} |f(x) - f_n(x)| = 0$ (17.1). Then for all $\varepsilon > 0$ there exists $N$ such that for all $n > N$ we have $|\sup_{x\in[a;b]} |f(x) - f_n(x)|| < \varepsilon$. But then for all $\varepsilon > 0$ there exists $N$ such that for all $n > N$ and for all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$.

Conversely suppose that for all $\varepsilon > 0$ there exists $N$ such that for all $n > N$ and for all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$. Since this is true for all $x \in [a;b]$ then for all $\varepsilon > 0$ there exists $N$ such that for all $n > N$ we have $\sup_{x\in[a;b]} |f(x) - f_n(x)| = |\sup_{x\in[a;b]} |f(x) - f_n(x)| - 0| = |d(f, f_n) - 0| < \varepsilon$. But then $\lim_{n\to\infty} d(f, f_n) = 0$ and so $\lim_{n\to\infty} f_n = f$ (17.1). ☐

**Theorem 7** *If* $\lim_{n\to\infty} f_n = f$ *then* $\lim^{\bullet}_{n\to\infty} f_n = f$.

*Proof.* We have $\lim_{n\to\infty} f_n = f$ and so for all $\varepsilon > 0$ there exists $N$ such that for all $n > N$ and all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$. But then for all $x \in [a;b]$ and all $\varepsilon > 0$ there exists $N$ such that for all $n > N$ we have $|f(x) - f_n(x)| < \varepsilon$. Thus $\lim^{\bullet}_{n\to\infty} f_n = f$. ☐

**Theorem 8** *The sequence* $f_n(x) = x^n$ *on the interval* $[0;1]$ *converges pointwise but not uniformly.*

*Proof.* Let

$$f = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

and let $x \in [0;1)$. Since $0 \le x < 1$ we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0 = f(x)$. If $x = 1$ then $x^n = 1$ for all $n$ and so $\lim_{n\to\infty} x^n = 1 = f(x)$. Thus, $(f_n)$ converges pointwise. Suppose that $(f_n)$ converges uniformly and let $1 > \varepsilon > 0$. Then there exists an $N$ such that for all $n > N$ and for all $x \in [0;1]$ we have $|f(x) - f_n(x)| < \varepsilon$. But since $f(x) = 0$ for $x \in [0;1)$ we can choose $x$ large enough such that $x^n \ge \varepsilon < 1$. Thus there exists $\varepsilon > 0$ such that for all $N$ there exists $n > N$ and $x \in [0;1]$ such that $|f(x) - f_n(x)| \ge \varepsilon$ and so $(f_n)$ doesn't converge uniformly. ☐

**Theorem 9** *Let* $(f_n)$ *be a sequence of continuous functions on* $[a;b]$ *that uniformly converges to* $f$ *on* $[a;b]$. *Then* $f$ *is continuous on* $[a;b]$.

*Proof.* Let $\varepsilon > 0$ and consider $\varepsilon/3$. We know $(f_n)$ uniformly converges to $f$ so there exists $N$ such that for all $n > N$ and for all $x, y \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon/3$ and $|f(y) - f_n(y)| < \varepsilon/3$. Also $f_n$ is continuous for all $n$ so for all $n > N$ and for all $x \in [a;b]$ there exists $\delta_n > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta_n$ we have $|f_n(x) - f_n(y)| < \varepsilon/3$. Consider $\delta_{N+1}$. Then for all $x \in [a;b]$ there exists $\delta_{N+1} > 0$, which may depend on $x$, such that for all $y \in [a;b]$ with $|x - y| < \delta_{N+1}$ we have $|f_{N+1}(x) + f_{N+1}(y)| < \varepsilon/3$. By the triangle inequality we have $|f(x) - f_{N+1}(y)| \le |f_{N+1}(x) - f_{N+1}(y)| + |f(x) - f_{N+1}(x)| < 2\varepsilon/3$ and then $|f(x) - f(y)| < |f(x) - f_{N+1}(y)| + |f(y) - f_{N+1}(y)| < \varepsilon$. Thus for all $x \in [a;b]$ there exists some $\delta > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Therefore $f$ is continuous on $[a;b]$. ☐

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# Sheet 19: Polynomials

**Definition 1** *A real polynomial of degree $n$ is a function of the form*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

*where $a_i \in \mathbb{R}$ $(0 \le i \le n)$ and $a_n \neq 0$. If $p(x) = 0$ then we define the degree $\deg p = -\infty$. The set of real polynomials is denoted by $\mathbb{R}[x]$.*

**Theorem 2** *For all $p, q \in \mathbb{R}[x]$ we have*

$$\deg(p+q) \le \max(\deg(p), \deg(q))$$

*and*

$$\deg(pq) = \deg(p) + \deg(q)$$

*Proof.* Let $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{i=0}^{m} b_i x^i$. Then

$$p + q(x) = p(x) + q(x) = \left(\sum_{i=0}^{n} a_i x^i\right) + \left(\sum_{i=0}^{m} b_i x^i\right)$$

and so $\deg(p+q) = \max(n, m) = \max(\deg(p), \deg(q))$. Also

$$pq(x) = p(x)q(x) = \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{i=0}^{m} b_i x^i\right)$$

and so using the product of powers $\deg pq = n + m = \deg(p) + \deg(q)$. $\square$

**Theorem 3 (Division Remainder)** *Let $a, b \in \mathbb{R}[x]$ be polynomials with $b \neq 0$. Then there exists unique $q, r \in \mathbb{R}[x]$ such that*

$$a = bq + r$$

*and*

$$\deg r < \deg b.$$

*Proof.* To show existence consider the set $S = \{a - bc \mid c \in \mathbb{R}[x]\}$. Suppose that for all $r \in S$, $\deg(r) \ge \deg(b)$. Choose $p \in S$ such that $\deg(p)$ is the minimum degree of all elements of $S$ using the Well Ordering Principle. Note that $p = a - bc$ for some $c \in \mathbb{R}[x]$. Now let $q = p - bd$ for some $d \in \mathbb{R}[x]$. Then $q = a - bc - bd = a - b(c + d)$ and so $q \in S$. Thus $\deg(q) \ge \deg(p)$. But then if $p(x) = \sum_{i=0}^{n} a_i x^i$ and $b(x) = \sum_{i=0}^{m} b_i x^i$ then consider $d = (a_n/b_m)x^{(n-m)}$. Then $\deg(bd) = n$ and so $\deg(q) < \deg(p)$ since $q = p - bd$. This is a contradiction and so there exists $r \in S$ such that $\deg(r) < \deg(b)$.

For uniqueness suppose that there exists $q, q', r, r'$ with $q \neq q'$ and $r \neq r'$ such that $a = bq + r$, $a = bq' + r'$, $\deg(r) < b$ and $\deg(r') < b$. Then $bq + r = bq' + r'$ and $b(q - q') = r' - r$. Note that since $q \neq q'$ and $r \neq r'$, $\deg(q - q') \ge 0$ and $\deg(r - r') \ge 0$. But then using Theorem 2 we have $\deg(r - r') < b$ and $\deg(b(q - q')) = \deg(b) + \deg(q - q') \ge \deg(b)$ (19.2). This is a contradiction and so $q = q'$ and $r = r'$ which means $q$ and $r$ are unique. $\square$

**Definition 4** *We call $r$ the remainder of $a$ divided by $b$.*

**Exercise 5** *Divide $x^3 + 4$ by $2x^2 - 1$ with remainder. Also $x^4 - 1$ by $x^2 - 1$.*

$x^3 + 4$ divided by $2x^2 - 1$ is $x/2$ with $x/2 + 4$ as a remainder because $x^3 + 4 = x^3 + x/2 - x/2 + 4 = (2x^2 - 1)(x/2) + x/2 + 4$. Also $(x^2 - 1)(x^2 + 1) = x^4 - 1$ so $x^4 - 1$ divided by $x^2 - 1$ is $x^2 + 1$ with no remainder.

**Definition 6** *A real number $\alpha$ is a root of $p(x) \in \mathbb{R}[x]$ if $p(\alpha) = 0$.*

**Theorem 7** *Let $p, q \in \mathbb{R}[x]$. Then $\alpha$ is a root of $pq$ if and only if $\alpha$ is a root of $p$ or $q$.*

*Proof.* Let $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{i=0}^{m} b_i x^i$ and suppose that $\alpha$ is a root of $p$ or $q$. Without loss of generality suppose that $\alpha$ is a root of $p$. Then $\sum_{i=0}^{n} a_i \alpha^i = 0$ and so

$$pq(\alpha) = p(\alpha)q(\alpha) = \left(\sum_{i=0}^{n} a_i \alpha^i\right)\left(\sum_{i=1}^{m} b_i \alpha^i\right) = 0 \cdot \left(\sum_{i=1}^{m} b_i \alpha^i\right) = 0$$

which means $\alpha$ is a root of $pq$. For the converse we use the contrapositive. Suppose that $\alpha$ is not a root of $p$ and $q$. Then $p(\alpha) \neq 0$ and $q(\alpha) \neq 0$. But then $pq(\alpha) = p(\alpha)q(\alpha) \neq 0$. $\square$

**Theorem 8** *Let $p \in \mathbb{R}[x]$. Then $\alpha$ is a root of $p$ if and only if $p = (x - \alpha)q$ for some $q \in \mathbb{R}[x]$.*

*Proof.* Suppose that $p = (x - \alpha)q$ for some $q \in \mathbb{R}[x]$. Then $p(\alpha) = (\alpha - \alpha)q = 0$ and so $\alpha$ is a root of $p$. Conversely suppose that $\alpha$ is a root of $p$. From Theorem 3 we know that $p = (x - \alpha)q + r$ for $q, r \in \mathbb{R}[x]$ and $\deg(r) = 0$ (19.3). Thus $r$ is a constant and since $\alpha$ is a root of $p$ we have $p(\alpha) = (\alpha - \alpha)q + r = r = 0$. Thus $p = (x - \alpha)q$ for some $q \in \mathbb{R}[x]$. $\square$

**Theorem 9** *Let $p \in \mathbb{R}[x]$ be a nonzero polynomial of degree $n$. Then $p$ has at most $n$ roots.*

*Proof.* Suppose that $\deg(p) = n$ and $p$ has $m$ distinct roots with $m > n$. Let the $m$ roots be $\alpha_1, \alpha_2, \ldots, \alpha_m$. From Theorem 8 we know that $p = (x - \alpha_1)q_1$ for some $q \in \mathbb{R}[x]$ (19.8). From Theorem 7 we know that since $\alpha_2$ is a root of $p$ it is a root of $(x - \alpha_1)$ or $q$ (19.7). Since $\alpha_2 - \alpha_1 \neq 0$, $\alpha_2$ is a root of $q_1$. This $q_1 = (x - \alpha_2)q_2$ and $p = (x - \alpha_1)(x - \alpha_2)q_2$ (19.8). We can continue in this process $m$ times until we have

$$p = \prod_{i=1}^{m}(x - \alpha_i)q_m.$$

But then $\deg(p) = m \neq n$ which is a contradiction. $\square$

**Theorem 10** *For every even $n$ there exists a real polynomial of degree $n$ with no roots. Every real polynomial of odd degree has a root.*

*Proof.* Let $n$ be even. Consider the polynomial $p(x) = x^n + 1$. Since $n$ is even, $n = 2k$ for some $k \in \mathbb{N}$. Then $p(x) = x^{2k} + 1 = (x^k)^2 + 1$. But then $p(x) > 0$ for all $x \in \mathbb{R}$ and so $p(x)$ has no roots.

Now let $p$ be a polynomial of degree $n$ with $n$ odd such that $p(x) = \sum_{i=0}^{n} a_i x^i$. Suppose that $a_n > 0$. We know $\lim_{x\to\infty} p(x)/(a_n x^n) = 1$. Let $\varepsilon = 1/2$. Then there exists $m \in \mathbb{R}$ such that for all $x > m$ we have $|p(x)/(a_n x^n) - 1| < 1/2$. Thus there exists $x_1 > 0$ such that $1/2 < p(x_1)/(a_n x_1^n)$. Since $x_1, a_n > 0$ and $n$ is odd we have $0 < (a_n x_1^n)/2 < p(x_1)$. Thus $p(x_1)$ is positive. Similarly take $\lim_{x\to-\infty} p(x)/(a_n x^n) = 1$ and let $\varepsilon = 1/2$. Then there exists $m \in \mathbb{R}$ such that for all $x < m$ we have $|p(x)/(a_n x^n) - 1| < 1/2$. Then there exists $x_2 < 0$ such that $1/2 < p(x)/(a_n x^n)$. But since $x_2 < 0$ and $a_n > 0$ we have $a_n x^n < 0$ so then $p(x) < (a_n x^n)/2 < 0$. Thus $p(x_2) < 0$. Therefore there exist $x_1, x_2 \in \mathbb{R}$ with $p(x_2) < 0$ and $p(x_1) > 0$ so there must exist $c \in (x_2; x_1)$ with $p(c) = 0$ by the Intermediate Value Theorem. A very similar proof holds if $a_n < 0$ where the limits give values of opposite signs as in this proof. $\square$

**Theorem 11 (Lagrange Interpolation)** *Let $a_1 < a_2 < \cdots < a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then there exists a polynomial $p(x)$ of degree at most $n-1$ such that*

$$p(a_i) = b_i \ (1 \leq i \leq n).$$

*Proof.* Consider the polynomial

$$p(x) = \sum_{i=1}^{n} b_i \prod_{j=1, j\neq i}^{n} \frac{(x-a_j)}{(a_i - a_j)}.$$

Note that

$$p(a_k) = \sum_{i=1}^{n} b_i \prod_{j=1, j\neq i}^{n} \frac{(a_k-a_j)}{(a_i - a_j)} = b_k \prod_{j=1, j\neq k}^{n} \frac{(a_k-a_j)}{(a_k - a_j)} = b_k.$$

□

**Exercise 12** *Is this polynomial unique?*

Yes.

*Proof.* Let $a_1 < a_2 < \cdots < a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Consider two polynomials $f(x)$ and $g(x)$ such that $f(a_i) = b_i$ and $g(a_i) = b_i$ $(1 \leq i \leq n)$. Then consider $h(x) = f(x) - g(x)$. We see $h(x) = 0$ for each $a_i$ and so $h$ has $n$ roots. But then $n \leq \deg(h) \leq \max(\deg(p), \deg(q))$ (19.2, 19.9). Thus $\deg(p)$ or $\deg(q)$ is greater than or equal to $n$ which means there exists only one such polynomial with degree less than $n$. □

**Theorem 13** *Let $p$ be a real polynomial which maps rationals to rationals. Then all the coefficients of $p$ are rational.*

*Proof.* Take $n+1$ rational points $a_1 < a_2 < \cdots < a_{n+1}$ and their images $p(a_1) = b_1, p_(a_2) = b_2, \ldots, p(a_{n+1}) = b_{n+1}$. From Theorem 11 we know that there exists a polynomial of degree $n$

$$p'(x) = \sum_{i=1}^{n} b_i \prod_{j=1, j\neq i}^{n} \frac{(x-a_j)}{(a_i - a_j)}$$

such that $p'(a_i) = b_i$ $(1 \leq i \leq n+1)$ (19.11). Note that the coefficients of $p'$ are all rational because $a_i, b_i \in \mathbb{Q}$ $(1 \leq i \leq n)$. From Exercise 12 we know that this polynomial is unique and so $p = p'$ (19.12). Thus $p$ has all rational coefficients. □