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Homework 7

**** Problem 1.** *Determine whether for a closed set A and a single point x the distance $d(x, A)$ is assumed for the following:*
1) For $\ell_n^2(\mathbb{R})$.
2) For an arbitrary metric space, (X, d).

Proof. 1) Suppose that $a = d(x, A) = \inf\{d(x, y) \mid y \in A\}$ is not assumed. Then we can choose points of A which have a distance from x which is arbitrarily close to a. Consider the set $S = \{y \in \mathbb{R}^n \mid d(x, y) = a\}$. Since $d(x, A)$ is not assumed, none of the points in S are in A. Also, since A is closed, none of these points are accumulation points of A. Thus for all $y \in A$ there exists $r_y \in \mathbb{R}$ such that $B_{r_y}(y) \cap A = \emptyset$. Let $s = \inf\{r_y \mid y \in S\}$. Then note that the set

$$T = \bigcup_{y \in S} B_s(y)$$

contains no points of A. Since $d(x, A)$ is not assumed, there exists a point of $y \in A$ such that $d(x, y) = a + s/2$. But then $y \in T$ as well. This is a contradiction and so $d(x, A)$ must be assumed.

2) Consider the $\mathbb{R}\backslash\{0\}$ with the usual metric. Then the set $(0, 1]$ is closed since it contains all it's accumulation points, but $d(-1, (0, 1])$ is not assumed since 0 is not in the metric space. ☐

**** Problem 2.** *Given $p(x)/q(x) \in \mathbb{R}(x)$ with $p(x)/q(x) > 0$ show that there exists $N \in \mathbb{N}$ such that $1/x^N < p(x)/q(x)$.*

Proof. Choose $N > \deg(q(x))$. Since $p(x)/q(x) \neq 0$ we have $\deg(p(x)) \geq 0$. Then $\deg(p(x)x^N) \geq N > \deg(q(x))$ which implies that $q(x) < p(x)x^N$ and so $1/x^N < p(x)/q(x)$. ☐

**** Problem 3.** *For $p(x)/q(x) \in \mathbb{R}(x)$ define $|p(x)/q(x)| = 2^{\deg(p(x))-\deg(q(x))}$. Show that for $u, v \in \mathbb{R}(x)$ we have $|u + v| \leq \max(|u|, |v|)$ and equality holds if $|u| \neq |v|$.*

Proof. Note that for polynomials p, q we have $\deg(p + q) \leq \max(\deg(p) + \deg(q))$. Let $u, v \in \mathbb{R}(x)$ such that $u = p/q$ and $v = r/s$ with $p, q, r, s \in \mathbb{R}[x]$. Then $u + v = (ps - qr)/qs$ and so

$$\begin{aligned}
|u + v| &= \left|\frac{ps - qr}{qs}\right| \\
&= 2^{\deg(ps-qr)-\deg(qs)} \\
&\leq 2^{\max(\deg(ps),\deg(qr))-\deg(q)-\deg(s)} \\
&= 2^{\max(\deg(p)+\deg(s),\deg(q)+\deg(r))-\deg(q)-\deg(s)} \\
&= \max(2^{\deg(p)+\deg(s)-\deg(q)-\deg(s)}, 2^{\deg(q)+\deg(r)-\deg(q)-\deg(s)}) \\
&= \max(|u|, |v|).
\end{aligned}$$

Suppose that $|u| \neq |v|$ and without loss of generality suppose that $|u| < |v|$. Then

$$2^{\deg(p)-\deg(q)} < 2^{\deg(r)-\deg(s)}$$

and

$$\deg(p) + \deg(s) < \deg(r) + \deg(q).$$

Then in the above calculation note that

$$\max(\deg(p) + \deg(s), \deg(r), \deg(q)) = \deg(r) + \deg(q)$$

and so we have

$$|u + v| = 2^{\deg(r)+\deg(q)-\deg(q)-\deg(s)} = 2^{\deg(r)-\deg(s)} = |v| = \max(|u|, |v|).$$

□

**** Problem 4.** *Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Show that if we have a norm defined on V then for $u, v \in V$ $d(u,v) = ||u - v||$ is a metric on V.*

Proof. By definition of a norm $||v|| \geq 0$ and $||v|| = 0$ if and only if $v = 0$. Because of closure under addition, this directly implies that $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$. Next, note that in a vector space we have commutativity of addition and so $u - v = -v + u$ and from the definition of a norm for some $a \in \mathbb{R}$ we have $||av|| = |a| \cdot ||v||$. Then note that

$$d(u, v) = ||u - v|| = |1| \cdot ||u - v|| = | - 1| \cdot ||u - v|| = || - 1(u - v)|| = || - u + v|| = ||v - u|| = d(v, u).$$

Finally, let $w \in V$. From the definition of a norm we have $||u + v|| \leq ||u|| + ||v||$ and so

$$d(u, w) = ||u - w|| = ||(u - v) + (v - w)|| \leq ||u - v|| + ||v - w||.$$

Thus d is a metric on V. □

**** Problem 5.** $\mathbb{R}(x)$ *is not complete.*

Proof. Let

$$a_n = \sum_{i=0}^{n} \frac{1}{x^i}.$$

Then let $N \in \mathbb{N}$ so that we have $1/x^N > 0$. Choose $M \in \mathbb{N}$ such that $M > N$. Let $m, n > M$ and without loss of generality suppose that $m < n$. Then we have

$$|a_n - a_m| = \left| \sum_{i=m+1}^{n} \frac{1}{x^i} \right| = \left| \sum_{i=m+1}^{n} \frac{x^i}{x^n} \right| < \left| \frac{1}{x^M} \right| < \left| \frac{1}{x^N} \right|$$

where the final sum is a ratio of a polynomial of degree $m + 1$ over a polynomial of degree n with $m < n$ and $m, n > M$. Thus, the sequence is a Cauchy sequence. Suppose that it converges to $p(x)/q(x) \in \mathbb{R}(x)$. Then consider

$$\left| a_n - \frac{p(x)}{q(x)} \right| = \left| \sum_{i=0}^{n} \frac{1}{x^i} - \frac{p(x)}{q(x)} \right| = \left| \sum_{i=0}^{n} \frac{x^i}{x^n} - \frac{p(x)}{q(x)} \right| = \left| \frac{q(x) \sum_{i=0}^{n} x^i - x^n p(x)}{x^n q(x)} \right| \geq 2^{n+\min(\deg(p(x),q(x))-n-\deg(q(x))}.$$

Since we can bound the degree of the difference between the nth term and $p(x)/q(x)$ below, we see that the sequence cannot converge. For it to converge, the difference in degrees of the numerator and the denominator would have to tend towards $-\infty$. □

**** Problem 6.** *A set $A \subseteq \mathbb{R}(x)$ is open in the order topology if and only if it is open in the metric topology.*

Proof. Let $A \subseteq \mathbb{R}(x)$ be open in the order topology. Let $u = p/q$. Then there exists $N \in \mathbb{N}$ such that $(u - 1/x^N, u + 1/x^N) \subseteq A$. Note that this implies that $-1/x^N < u < 1/x^N$. Define

$$B_{2^{-N}}(u) = \{a \in \mathbb{R}(x) \mid d(u, a) < 2^{-N}\}$$

and let $v \in B_{2^-N}(a)$ such that $v = r/s$. Then $|u - v| < 2^{-N}$ and so

$$\deg(ps - qr) - \deg(qs) < -N.$$

This implies

$$\deg(ps - qr) + N < \deg(qs)$$

which means $(ps - qr)x^N < qs$. We then have $u - v < 1/x^N$. Thus $v \in (u - 1/x^N, u + 1/x^N)$ and so $B_{2^{-N}}(u) \subseteq (u - 1/x^N, u + 1/x^N) \subseteq A$. Therefore, if A is open in the order topology it is also open in the metric topology.

Conversely, assume that A is open in the metric topology. Then for all $u \in A$ with $u = p/q$ there exists some $r \in \mathbb{R}$ such that $B_r(u) \subseteq A$. Note that we can replace r with 2^{-N} for some $N \in \mathbb{N}$ such that $2^{-N} < r$. Then $B_{2^{-N}}(u) \subseteq A$. Let $v \in (u - 1/x^N, u + 1/x^N)$ such that $v = r/s$. Then $-1/x^N < u - v < 1/x^N$ and

$$(ps - qr)x^N < qs.$$

This implies

$$\deg(ps - qr) + N < \deg(qs)$$

so

$$\deg(ps - qr) - \deg(qs) < -N.$$

Then $|u - v| < 2^{-N}$ and so $v \in B_{2^{-N}}(u)$. Therefore $(u - 1/x^N, u + 1/x^N) \subseteq B_{2^{-N}}(u)$. Therefore if A is open in the metric topology it is also open in the order topology. $\square$