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Homework 1

Problem 1. *Let z, w be complex numbers such that $\overline{z}w \neq 1$. Prove that*

$$\left|\frac{z-w}{1-\overline{z}w}\right| < 1 \textit{ if } |z| < 1 \textit{ and } |w| < 1,$$

$$\left|\frac{z-w}{1-\overline{z}w}\right| = 1 \textit{ if } |z| = 1 \textit{ or } |w| = 1.$$

Proof. Let $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$. Then

$$\begin{aligned}
|z-w| &= |r_1(\cos(\theta_1) + i\sin(\theta_1)) - r_2(\cos(\theta_2) + i\sin(\theta_2))| \\
&= |(r_1\cos(\theta_1) - r_2\cos(\theta_2)) + i(r_1\sin(\theta_1) - r_2\sin(\theta_2))| \\
&= (r_1^2\cos^2(\theta_1) + r_2^2\cos^2(\theta_2) - 2r_1r_2\cos(\theta_1)\cos(\theta_2)) + (r_1^2\sin^2(\theta_1) + r_2^2\sin^2(\theta_2) - 2r_1r_2\sin(\theta_1)\sin(\theta_2)) \\
&= 1 - 2r_1r_2\cos(\theta_1 - \theta_2)
\end{aligned}$$

and

$$\begin{aligned}
|1-\overline{z}w| &= |1 - r_1r_2\cos(\theta_2-\theta_1) + ir_1r_2\sin(\theta_2-\theta_1)) \\
&= 1 + r_1^2r_2^2\cos^2(\theta_2-\theta_1) - 2r_1r_2\cos(\theta_2-\theta_1) + r_1^2r_2^2\sin^2(\theta_2-\theta_1) \\
&= 2 - 2r_1r_2\cos(\theta_2-\theta_1).
\end{aligned}$$

Then

$$\left|\frac{z-w}{1-\overline{z}w}\right| = \frac{1-2r_1r_2\cos(\theta_1-\theta_2)}{2-2r_1r_2\cos(\theta_2-\theta_1)}.$$

Thus if we replace θ_1 by 0 and θ_2 by $\theta_2 - \theta_1$, the norm doesn't change. Therefore we may assume that z has $\theta_1 = 0$, i.e. z is real.

Now, since $|z| < 1$ and $|w| < 1$ we have

$$\begin{aligned}
z^2 - 1 &< |w|^2(z^2-1) \\
z^2 + |w|^2 &< 1 + z^2|w|^2 \\
z^2 + |w|^2 - zw - z\overline{w} &< 1 + z^2|w|^2 - zw - z\overline{w} \\
(z-w)(z-\overline{w}) &< (1-zw)(1-z\overline{w}) \\
|z-w| &< |1-zw| \\
\left|\frac{z-w}{1-\overline{z}w}\right| &< 1.
\end{aligned}$$

On the other hand, if $|z| = 1$ or $|w| = 1$ then we can start from the second inequality $z^2 + |w|^2 = 1 + z^2|w|^2$ and proceed replacing $<$ with $=$. We arrive at the desired result. $\square$

Problem 2. *Let $f(z) = e^{2\pi i z}$. Describe the image under f of the set consisting of those points $x + iy$ with $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and $y \geq B \geq 0$.*

Proof. We have

$$f(x+iy) = e^{2\pi i(x+iy)} = e^{2\pi ix - 2\pi y} = e^{-2\pi y}e^{2\pi ix}.$$

Note that the real part of f gets mapped to $e^{-2\pi y}$ and the complex part gets mapped to $e^{2\pi x}$. Thus the real part is in the set $(0, e^{2\pi B}]$ with angle $[-\pi, \pi]$. The image is thus the disk centered at 0 with radius $e^{2\pi B}$, but without the point 0. We can write this as

$$\text{Im } f = \{re^{i\theta} \mid 0 < r \leq e^{2\pi B}, -\pi \leq \theta \leq \pi\}.$$

$\square$

Problem 3. *Consider the function* $f(z) = \frac{z+z^{-1}}{2}$. *What is the image of the set* $|z| > 1$? *The set* $|z| < 1$? *The set* $|z| = 1$? *Show that the image of any circle centered at the origin with radius* $r \neq 1$ *is an ellipse with focal points* 1 *and* -1.

Proof. We have

$$f(z) = f(x+iy) = \frac{(x+iy) + \frac{(x-iy)}{(x^2+y^2)}}{2} = \frac{x(x^2+y^2+1)}{2(x^2+y^2)} + i\frac{y(x^2+y^2-1)}{2(x^2+y^2)}.$$

If $|z|$ is close to 1 then the coefficient of the real part evaluates close to 1. As $|z|$ increases the coefficient gets closer to $1/2$. On the other hand, the coefficient is unbounded as $|z|$ approaches 0.

Likewise, if $|z|$ is close to 1 then the coefficient for the imaginary part evaluates close to 0. As $|z|$ increases the coefficient gets closer to $1/2$. As $|z|$ approaches 0, the coefficient is unbounded in the negative direction.

In the case that $|z| = 1$ we see that the real part evaluates to just x while the imaginary part drops out entirely. Therefore

$$\text{Img}(f(x+iy)) = \begin{cases} \left\{ax + iby \mid \frac{1}{2} < a < 1, 0 < b < \frac{1}{2}\right\} & |z| > 1 \\ \{ax + iby \mid 1 < a, b < 0\} & |z| < 1 \\ \{x\} & |z| = 1 \end{cases}$$

This is the same as

$$\text{Img}(f(x+iy)) = \begin{cases} \left\{x + iy \mid \frac{1}{2} < x \text{ or } x < -\frac{1}{2}\right\} & |z| > 1 \\ \mathbb{C}\backslash\{0\} & |z| < 1 \\ \mathbb{R} & |z| = 1 \end{cases}$$

We can rewrite $f(z) = f(x+iy) = \frac{1}{2|z|^2}(x(|z|^2+1) + iy(|z|^2-1))$. If we keep $|z| \neq 1$ constant, i.e., the points of a circle in $\mathbb{C}$, then this is the equation of an ellipse in $\mathbb{C}$ where the foci are at 1 and -1. $\square$

Problem 4. *What does the map* $f(z) = \overline{z}$ *do to angles at points* $z \in \mathbb{C}$? *How about* $h(z) = (g \circ f)(z)$ *if* g *is complex-differentiable at* $\overline{z}$ *with* $g'(\overline{z}) \neq 0$?

Proof. The map f has the effect of reflecting over the imaginary axis in the complex plane. Thus all angle measures remain the same, but the orientation is reversed. If we compose g with f, we effectively reverse the orientation of an angle, and then apply g to it. But g is holomorphic and holomorphic functions preserve angles. Thus, the angle measure under $g \circ f$ is preserved, but the orientation is reversed. $\square$

Problem 5. *Find a holomorphic function* f *on* $\mathbb{C}$ *such that* $\text{Re}f(x+iy) = xy$ *and* $f(0) = i$.

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$. We have the restriction that $u(x,y) = xy$. From the Cauchy-Riemann equations we have

$$\partial u/\partial x = y = \partial v/\partial y$$

and

$$\partial u/\partial y = x = -\partial v/\partial x.$$

Integrating we get $v(x,y) = \frac{y^2-x^2}{2} + C$. Using the initial value condition we get $v(x,y) = \frac{y^2-x^2}{2} + 1$. Therefore

$$f(x+iy) = u(x,y) + iv(x,y) = xy + i\left(\frac{y^2-x^2}{2} + 1\right).$$

We know that f is holomorphic because u and v are continuously differentiable and satisfy the Cauchy-Riemann equations. $\square$

Problem 6. *Let* $f(z) = \frac{az+b}{cz+d}$ *with* $a, b, c, d \in \mathbb{C}$ *be such that* $f(0) = z_1$, $f(1) = z_2$, $f(\infty) = z_3$. *Find all such* a, b, c, d, *given* $z_1, z_2, z_3 \in \mathbb{C}$. *When is there no such quadruple?*

Proof. We have $f(0) = \frac{b}{d} = z_1$ so $b = dz_1$. Also $f(1) = \frac{a+b}{c+d} = z_2$ so $(a+b) = (c+d)z_2$. Finally $f(\infty) = \frac{a}{c} = z_3$ so $a = cz_3$. Substituting the first and third equations into the third we see that $(cz_3 + dz_1) = (cz_2 + dz_2)$. Solving for d we have

$$d = \frac{c(z_3 - z_2)}{(z_2 - z_1)}.$$

Thus the set of all quadruples (a, b, c, d) is

$$\left\{ (a, b, c, d) = (cz_3, dz_1, c, d) \mid d = \frac{c(z_3 - z_2)}{(z_2 - z_1)} \right\}.$$

Note that we can't have $z_2 = z_1$ or $z_2 = z_3$, otherwise we loose the constraint on c and d. If we allow $z_i \in \mathbb{C}^*$ then we allow $d = 0$, $c = 0$ or $c = -d$. $\square$

Problem 7. *Assume the function* f *is defined on the set* $|z| > M$ *for some* M *and that* $c = \lim_{|z|\to\infty} f(z)$ *exists. If* $c \in \mathbb{C}$, *then* f *is* C^**-differentiable at* ∞ *if and only if* $f(1/z)$ *is complex-differentiable at* 0. *If* $c = \infty$, *then* f *is* C^**-differentiable at* ∞ *if and only if* $1/f(1/z)$ *is complex-differentiable at* 0. *Show that the functions* $f(z) = e^{1/z}$ *and* $g(z) = z^2 + 1$ *are* C^**-differentiable at* ∞.

Proof. We have

$$\lim_{|z|\to\infty} f(z) = \lim_{|z|\to\infty} e^{1/z} = 1.$$

Thus, f is C^*-differentiable at ∞ if $f(1/z) = e^{1/(1/z)} = e^z$ is complex-differentiable at 0. But we know that e^z is a complex-differentiable function so f is C^*-differentiable at ∞.

Now consider

$$\lim_{|z|\to\infty} g(z) = \lim_{|z|\to\infty} z^2 + 1 = \infty.$$

Thus, g is C^*-differentiable at ∞ if $1/f(1/z) = 1/(1/z^2 + 1) = z^2/(z^2 + 1)$ is complex-differentiable at 0. But note that this is the quotient of two functions which are differentiable at 0, and the denominator is not equal to 0 at 0. Thus the derivative of the quotient exists at 0. Therefore g is C^*-differentiable at ∞. $\square$