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MATH 20900
April 13, 2009

Homework 2

**** Problem 1.** *When is a locally compact group metrizable?*

Proof. Note that a topological space is metrizable if and only if there exists an embedding of the space into a metric space. We wish to show the topological product of a countable family of metric spaces is metrizable. Let X be the topological product of a sequence of metric spaces $\{X_n\}$. Define for each n and $x_n, y_n \in X_n$ the function $f_n(x_n, y_n) = \min\{1, d_n(x_n, y_n)\}$. Then f_n is a metric for X_n which generates the same topology as d_n but has the property that $f_n \leq 1$ for all points in X_n. Now we are able to define a metric for X by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x_n, y_n).$$

Then d is a metric on X which generates the topology of X. This result directly implies that the space defined as the topological product of the closed intervals $[0, 1/n]$ is metrizable. This space is the Hilbert Cube.

Let F be a family of mappings $\{f_a : X \to Y_a \mid a < y\}$ from a space X into spaces Y for each $a < y$. Let Y denote the topological product of the family $\{Y_a\}$. Let $f : X \to Y$ denote the product mapping defined by $(f(x))_a = f_a(x)$ for each $x \in X$ and $a < y$. Then f is a continuous mapping from X to Y. We wish to show that if F can distinguish points of X and can distinguish points from closed sets the $f : X \to Y$ is an embedding. Assume that F can distinguish points and distinguish points from closed sets. If $x, y \in X$ such that $x \neq y$ then there exists $a < y$ such that $f_a(x) \neq f_a(y)$ and $f(x) \neq f(y)$. Thus, f is injective. Let $U \subseteq X$ be open. Let $p \in U$ and $q = f(p)$. Since $X \backslash U$ is closed and $p \notin X \backslash U$ there exists $a < y$ such that $f_a(p) \notin \overline{f_a(X \backslash U)}$. Let

$$V = \{b \in Y \mid b_a \notin \overline{f_a(X \backslash U)}\}.$$

Then V is open in Y which means $V \cap f(X)$ is open in $f(X)$. But then $q \in V \cap f(X)$ and $V \cap f(X) \subseteq f(U)$. This shows that $f(U)$ is open in $f(X)$. This f is an injective continuos open mapping and therefore an imbedding.

Finally, we show that a regular T_1 space with a countable base can be imbedded as a subspace of the Hilbert cube. Let B be a countable base for X and let C be the subset of $B \times B$ which consists of open sets (U, V) such that $\overline{V} \subseteq U$. Note that C is countable. Since X is normal, we can obtain a countable family $F = \{f_{(U,V)} : X \to [0, 1] \mid (U, V) \in C\}$ of continuous functions which map $\overline{V}$ to 0 and $X \backslash U$ to 1. To show that F can distinguish points from closed sets let $p \in X \backslash K$ where K is closed in X. Since X is regular and B is a base, it is possible to find $U, V \in B$ such that $p \in V \subseteq \overline{V} \subseteq U \subseteq X \backslash K$. Then $f_{(U,V)}(p) = 0$ and $f_{(U,V)}(K) = 1$. Thus F can distinguish points from closed sets. This shows that F can be imbedded as a subspace of the Hilbert cube which is metrizable. Therefore X is metrizable. □

**** Problem 2.** *Show $GL_n(\mathbb{R})$ is a dense open subset of $M_n(\mathbb{R})$.*

Proof. The fact that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$ follows from the fact that the determinant map is a polynomial map. To show that $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$, consider an element $x \in M_n(\mathbb{R}) \backslash GL_n(\mathbb{R})$. Note that $\det x = 0$, but by changing an appropriate element of x by a small amount, the determinant will be nonzero. Thus we can create a sequence of matrices of this form which converges to x. Since each of the matrices in the sequence has nonzero determinant, we have every element of $M_n(\mathbb{R})$ is the limit of a sequence of elements of $GL_n(\mathbb{R})$. Therefore $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$. □

**** Problem 3.** *Show $GL_n(\mathbb{R})$ is a locally compact group that is nonabelian if and only if $n > 1$.*

Proof. If $n = 1$, then $GL_n(\mathbb{R}) = \mathbb{R}^\times$ which is clearly a locally compact, abelian group under multiplication. Conversely, suppose $n > 1$. $GL_n(\mathbb{R})$ is a set of matrices and it's known that matrix multiplication is noncommutative for $n > 1$. The group axioms are satisfied using matrix multiplication by the identity element and matrix inverses, since the determinant of an element is never 0. The set $GL_n(\mathbb{R})$ takes on the topology of $\mathbb{R}^{n^2}$ which we know is a locally compact space. Thus, $GL_n(\mathbb{R})$ is a locally compact nonabelian group. $\square$

**** Problem 4.** *Show that a closed subgroup of a locally compact group is a locally compact group.*

Proof. Let C be a compact space and let $B \subseteq C$ be closed. If B is covered by a family $\{U_\alpha\}_{\alpha \in A}$ of open sets then $C = (C \backslash B) \cup \bigcup_{\alpha \in F} U_\alpha$ and we can find a finite subset $F \subseteq A$ such that $C = (C \backslash B) \cup \bigcup_{\alpha \in F} U_\alpha$. This shows that B is compact. Therefore every closed subspace of a compact space is compact. Now let G be a locally compact group and consider a closed subgroup H. Every point in H has a neighborhood which is compact in G. Taking the intersection of this neighborhood with H produces a closed subset of a compact set, which is then compact. $\square$

**** Problem 5.** *Let V and W be real normed linear spaces and let $T : V \to W$ be an isometry such that $T(0) = 0$. Then T is linear.*

Proof. Let (X, d) be a metric space and A be a bounded subset of X. We say that a point x_0 is a center of A of the first order if $d(x_0, a) \leq (\text{diam} A)/2$ for all $a \in A$. We say x_0 is a center of A of the nth order if it is a center of the first order of the set of all centers of the $(n-1)$th order which belong to A. A point x_0 is a metric center of A if for all n it is a metric center of A of the nth order.

Now let $v_1, v_2 \in V$ and let $A = \{v \in V \mid ||v_1 - v|| = ||v_2 - v|| = ||v_1 - v_2||/2\}$. It follows that A is symmetric about $(v_1 + v_2)/2$ and so $(v_1 + v_2)/2$ is the metric center of A. Then $T((v_1 + v_2)/2)$ is the metric center of $T(A)$. Then since T is an isometry we have

$$T(A) = \{Tv \in W \mid ||v_1 - v|| = ||v_2 - v|| = ||v_1 - v_2||/2\} = \{w \in W \mid ||Tv_1 - w|| = ||Tv_2 - w|| = ||Tv_1 - Tv_2||/2\}.$$

Thus $T(A)$ is symmetric about $(Tv_1 + Tv_2)/2$ and so this is the metric center of $T(A)$. Therefore $T((v_1 + v_2)/2) = (Tv_1 + Tv_2)/2$ for all $v_1, v_2 \in V$. Setting $v_1 = 0$ and using the fact that $T(0) = 0$ gives the result $T((x_1 + x_2)/2) = T(x_1/2) + T(x_2/2)$. This shows that T is additive and the fact that T is linear follows from T being continuous. $\square$

**** Problem 6.** *What happens for complex normed linear spaces?*

Proof. The result of ** Problem 5 does not hold for complex normed linear space, as conjugation is an example of a nonlinear isometry which preserves 0. $\square$

**** Problem 7.** *Let T be a linear isometry of $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$. Show that if $Tv \cdot Tv = v \cdot v$ is equivalent to $(T^T Tv \mid v) = (v \mid v)$ then $T^T T = I$.*

Proof. We have $Tv \cdot Tv = v \cdot v = T^T Tv \cdot v$. From the last equality, the only way this can hold true for all $v \in \mathbb{R}^n$ is if $T^T T = I$. $\square$

**** Problem 8.** *Show $O(n, \mathbb{R})$ is a maximal compact subgroup of $GL_n(\mathbb{R})$.*

Proof. Let $A \subseteq GL_n(\mathbb{R})$ be a compact subset such that $O(n, \mathbb{R}) \subsetneq GL_n(\mathbb{R})$. Then there exists $x \in A \backslash O(n, \mathbb{R})$ such that $xx^T \neq I$. From the Iwasawa decomposition in ** Problem 10, we can write every element of $GL_n(\mathbb{R})$ as a product of elements in $O(n, \mathbb{R})$, A^+ and N. Since A is compact, we can write x as a product of elements from A^+ and N. Thus, we have $A = GL_n(\mathbb{R})$. $\square$

**** Problem 9.** *For $z, w \in \mathbb{C}^n$ show $|(z \mid w)| \leq |z||w|$.*

Proof. We can assume $(w \mid w)$ is nonzero since the state is trivial if $w = 0$. Let $\lambda \in \mathbb{C}$. Then

$$0 \le ||z - \lambda w||^2 = (z - \lambda w \mid z - \lambda w) = (z \mid z) - \overline{\lambda}(z \mid w) - \lambda(w \mid z) + |\lambda|^2(w \mid w).$$

Now let $\lambda = (z \mid w)(w \mid w)^{-1}$. We have

$$0 \le (z \mid z) - |(z \mid w)|^2(w \mid w)^{-1}$$

which means $|(x \mid y)|^2 \le (z \mid z)(w \mid w)$ and taking the square root gives the desired result. □

**** Problem 10.** *Every element $g \in GL_n(\mathbb{C})$ can be written uniquely as a product $g = kan$ where $k \in K$, $a \in A^+$ and $n \in N$.*

Proof. Let $e_1, \dots, e_n$ be the standard basis vectors for $GL_n(\mathbb{C})$. Let $x \in GL_n(\mathbb{C})$ and let $v_i = xe_i$. We can orthogonalize $v = (v_1, \dots, v_n)$ using a matrix $u \in N$. Let $w_1 = v_1$, $w_2 = v_2 - u_{21}w_1 \perp w_1$, $w_3 = v_3 - u_{32}w_2 - u_{31}w_1 \perp w_1, w_2$ and so on. Then $e_i' = w_i/||w_i||$ is a unit vector and we can define $a \in A^+$ such that a has $||w_i||^{-1}$ for its diagonal elements. Let $k = aux$ then $x = a^{-1}u^{-1}k$ so that k is unitary. This shows $GL_n(\mathbb{C}) = KA^+N$. Now suppose $u_1 a^T u_1 = u_2 b^T u_2$ with $u_1, u_2 \in N$ and $a, b \in A^+$. Let $u = u_2^{-1}u_1$ and we have $ua = bu^T$. Since u is upper triangular, we must have u is diagonal which shows $a = b$. Thus, we have an isomorphism between $A^+ \times N \times K$ and $GL_n(\mathbb{C})$. □

**** Problem 11.** *Let $X = \mathbb{R}$ and $S = \{(a, b) \mid a, b \in \mathbb{R}\}$. Describe $M(S)$, that is, the σ-algebra generated by S.*

Proof. The set $M(S)$ contains the following sets, as well as others. All of $\mathbb{R}$, since it is the countable union of open intervals, and $\emptyset$ since it is the intersection of disjoint intervals. All one element sets can be written as $\{a\} = \cap_{n=1}^{\infty}(a - 1/n, a + 1/n)$. This is a countable intersection of open intervals. This means every countable subset of $\mathbb{R}$ is in $M(S)$, in particular, the rationals and their complement, the irrationals are contained. All closed intervals and half open intervals as well as unbounded half closed intervals. The Cantor set, as it is a countable union of closed intervals. □