Kris Harper
CMSC 27500
June 3, 2009

Homework 7

Problem 1. *Show that:*

1) Any edge-extension of a 3*-connected cubic graph is also* 3*-connected and cubic.*

Proof. Let v and v' be the new vertices on edges xy and $x'y'$. Let f be the new edge connecting v and v'. Let G' be the edge-extention of a graph G. It's clear that G' is still cubic, since all vertices of G retain their edges and v is connected to x, y, and v'. Likewise v' is connected to x', y' and v. Without loss of generality suppose $y \neq y'$. Consider v and some other vertex of G', w. There are three internally disjoint paths from each of x, y and y' to w. We can then pick three internally disjoint paths, P, Q and R, which go from y to w, y' to w and x to w respectively. The paths Pyv, $Qy'v'v$ and Rxv are three internally disjoint paths from w to v. The same holds for v'. In the case of v and v', we can take f as one path. Then if xy and xy' are not adjacent, take one of the three internally disjoint paths between x and x' and y and y' for the other two paths. These will not intersect v'. If xy and $x'y'$ are adjacent, take vxv' or vyv' as one or both of the other paths. This shows that G' is 3-connected. □

2) Every 3*-connected cubic graph can be obtained from* K_4 *by means of a sequence of edge-extensions.*

Proof. Let G be a 3-connected cubic graph and consider some edge $e = xy$ of G. Note that since G is cubic, x and y both have two other neighbors besides each other. Suppose we delete e and merge the two remaining neighbors of x into one edge. Do the same for y. These new edges both end in vertices with precisely 3 neighbors as well, since G is cubic. Perform the same operation on each of these edges. Since G is 3-connected, we will never end up with a disconnected graph after iterating this operation. Eventually K_4 can be reached since 3-connectivity and 3-regularity is preserved. □

3) An edge-extension of an essentially 4*-edge-connected cubic graph* G *is also essentially* 4*-edge connected provided that the two edges* e *and* e' *of* G *involved in the extension are nonadjacent in* G*.*

Proof. Label all vertices, edges and graphs as in Part 1). Let $\partial(X)$ be a non-trivial edge cut of G'. Note that if X does not contain x, x', y, y', v or v' then $|\partial(X)| < 3$ since G is essentially 4-edge-connected. The same holds true if X contains precisely one of x, x', y or y' and $|\partial(X)| < 3$ if more than one of these vertices are in X since more edges have been introduced in G'. Now suppose that $v \in X$. If $x \in X$ and $y, x', y' \notin X$ then $|\partial(X) < 3$ since both vy and vv' will be cut, so one more edge is introduced from the corresponding edge cut in G. A similar statement is true if $x, x' \in X$ and $y, y' \notin X$ or $x, y' \in X$ and $x', y \notin X$. If $x, y \in X$ and $x', y' \notin X$ then vv' is cut which is one more edge than in the corresponding edge cut in G. The same is true if any two or three of x, x', y or y' are in X. Finally if only $v \in X$ and $x, x', y, y' \notin X$, then since $d(v) = 3$, we still have $|\partial(X)| > 3$. All cases hold similarly for v'. In the case that $v, v' \in X$ we have $|\partial(X)| > 3$ since xy and $x'y'$ are distinct and there are a total of four edges attached to v and v'. In all cases $|\partial(X)| > 3$ which means that all 3-edge cuts are trivial. Thus, G' is essentially 4-edge-connected. □

Problem 2. *Let* G *be a* 3*-connected graph with* $n \geq 5$*. Show that, for any edge* e*, either* G/e *is* 3*-connected or* $G\backslash e$ *can be obtained from a* 3*-connected graph by subdividing at most two edges.*

Proof. Let $e = xy$. Suppose that G/e is not 3-connected. Then there exists a vertex $z \in G$ such that $\{x, y, z\}$ is a 3-vertex cut. Note that $d(x)$ and $d(y)$ are both greater than 3 since G is 3-connected. In the case where $d(x) = 3$ or $d(y) = 3$, consider the graph $G\backslash e$ where x and y are absorbed into their neighbors. This graph remains 3-connected since any paths passing through x or y in G would have to pass through y or x and so no internally disjoint paths are broken. If $d(x) > 3$ or $d(y) > 3$ then do nothing so that internally disjoint paths from G are not broken in the new graph. Clearly, subdividing appropriate edges results in $G\backslash e$. □

Problem 3. *Let G be a simple 3-connected graph different from a wheel. Show that, for any edge e, either G/e or $G\backslash e$ is also a 3-connected simple graph.*

Proof. Let $e = xy$. Suppose that $G\backslash e$ is not a 3-connected simple graph. Then since G is 3-connected, the deletion of e must break some path between two vertices v and u. Note then that identifying x and y will preserve this path. Moreover, the two other internally disjoint paths from v to u could not have contained x or y and so this new path will be internally disjoint from them. Also, since G is simple, e is not a double edge so no loops will be created when e is contracted. Since G is different from a wheel, no double edges will be created when e is contracted. Thus G/e is simple and 3 connected. Now suppose that G/e is not a 3-connected simple graph. Then since no connections are broken, two paths which were internally disjoint, now share a vertex, namely the contracted e. If we then look at $G\backslash e$, note that the two paths must remain internally disjoint. The absence of e makes no difference since neither path contains e in G as they're internally disjoint. Additionally, since G is simple, e is not a loop or double edge and so $G\backslash e$ is also simple and moreover, 3-connected. □

Problem 4. *1) Let $\mathcal{G}$ be a family of graphs consisting of K_5, the wheels W_n, $n \geq 3$, and all graphs of the form $H \vee \overline{K}_n$, where H is a spanning subgraph of K_3 and $\overline{K}_n$ is the complement of K_n, $n \geq 3$. Show that a 3-connected simple graph G does not contain two disjoint cycles if and only if $G \in \mathcal{G}$.*

Proof. First suppose $G \in \mathcal{G}$. If $G = K_5$ or $G = W_n$ then it is easy to see G is 3 connected. If $G = H \vee \overline{K}_n$, then consider two vertices x and y in G. If x and y are both in $\overline{K}_n$ then there is a path from x to y using one edge connecting x to a vertex in H and another edge connecting this same vertex of H to y. There are two more of these paths using the other two vertices of H and they are all three internally disjoint. If x and y are both in H, a similar result holds since $n \geq 3$. If $x \in H$ and $y \in \overline{K}_n$, then the edge connecting x to y, and two paths connecting x to another vertex in H and then y make three internally disjoint paths. Thus G is 3 connected and simple. Now, if $G = K_5$ or if $G = W_n$ then it's easy to see that it does not contain two disjoint cycles. If $G = H \vee \overline{K}_n$ then the only cycle G could contain is in H. Thus G does not contain two disjoint cycles if $G \in \mathcal{G}$. Conversely, suppose that G is a 3-connected simple graph which does not contain two disjoint cycles. Suppose to the contrary that $G \notin \mathcal{G}$. Note that G must have at least 5 vertices otherwise to remain 3-connected it would have to be K_4, which is a wheel. Since G is different from a wheel we know from Problem 3 that either G/e or $G\backslash e$ is still simple and 3-connected. Either contract or delete edges from G until G has less than 5 vertices. But if it's possible to reduce G to a connected graph with 4 vertices, then G must have been of the form $H \vee \overline{K}_n$. This is a contradiction and so $G \in \mathcal{G}$. □

2) Deduce that any simple graph not containing two disjoint cycles has three vertices whose deletion results in an acyclic graph.

Proof. Let G be a simple graph not containing two disjoint cycles. Then from Part 1) we know $G \in \mathcal{G}$. If $G = K_5$ then deleting any three vertices gives an acyclic graph. If $G = W_n$ then deleting the center vertex and two adjacent vertices results in a path. If $G = H \vee \overline{K}_n$, then deleting all vertices in H gives a graph with no edges. □

Problem 5. *1) Show that if G is $2k$-edge-connected, then the graph G' obtained from G by pinching together any k edges of G is also $2k$-edge-connected.*

Proof. Since G is $2k$-edge-connected, there are $2k$ internally edge-disjoint paths from two arbitrary vertices x and y. Take k edges of G and pinch them together at a vertex v to form G'. If none of the $2k$ paths contains any of the k vertices pinched, then we're done. Suppose that some path connecting x and y contained one of the k edges. then this edge can be replaced by the two-edge path passing through v which connects the two original ends of the edge. Since there were k edges pinched, and $d(v) = 2k$, we still have $2k$ internally edge-disjoint paths from x to y. □

2) Show that, given any $2k$-edge-connected graph G, there exists a sequence $(G_1, G_2, \ldots, G_r)$ of graphs such that (i) $G_1 = K_1$, (ii) $G_r = G$ and (iii) for $1 \leq i \leq r-1$, G_{i+1} is obtained from G_i either by adding an edge or by pinching together k of its edges.

Proof. Let G be a $2k$-edge-connected graph. If we can delete an edge e from G without losing $2k$-connectivity, then delete e. Continue in this process calling the graphs successively G_{r-1}, G_{r-2} and so on. If at some point no edge can be deleted without losing $2k$-connectivity, then every edge must be used in some path. Moreover, there exists some vertex v such that $d(v) \geq 2k$ and $d(v)$ is even. We know that we can split off v and the remaining graph will still be $2k$-connected. Iterate this process k times to reverse the process of pinching k edges together to v. Index this graph with the previous integer and continue in the same process. Eventually, edge deletion will result in K_1 which we label G_1. $\square$