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MATH 25700
Quiz 3 November 4, 2009

Problem 1. *If a subgroup G of S_n contains an odd permutation, then the order of G is even and exactly half of the elements in G are odd permutations.*

Proof. Let $\sigma \in G$ be an odd permutation. Then we know the cycle decomposition of σ contains an odd number of cycles of even length. In particular, there is at least one cycle of even length. The order of σ is the least common multiple of the lengths of all the cycles in its cycle decomposition. Therefore $|\sigma|$ is even which means $|\langle\sigma\rangle|$ is even. By Lagrange's Theorem, an even number divides $|G|$ and thus $|G|$ is even.

Now note that $\epsilon : G \to Z_2$ is a homomorphism. Furthermore, since G is a subgroup, $(1) \in G$ and $\epsilon((1)) = 1$. Since $\epsilon(\sigma) = -1$ we know that $\epsilon(G) = Z_2$, that is, ϵ is surjective. Now note that $\ker\epsilon$ is the set of even permutations in G and by the First Isomorphism Theorem, $G/\ker\epsilon \cong Z_2$. That is $|G : \ker\epsilon| = 2$ which means precisely half of G is made up of even permutations. This leaves the remaining half to be odd permutations. $\square$

Problem 2. *If p is prime and G has order a power of p (p to the a, some a), and if N is a nontrivial normal subgroup of G, show that N intersects $Z(G)$ nontrivially.*

Proof. Let G act on N by conjugation. Since $N \trianglelefteq G$, this satisfies the axioms for group actions. We can then write the class equation for N as

$$|N| = |N \cap Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

where $C_G(g_i)$ are proper subgroups of G. Now from Lagrange's Theorem $|N| = p^b$ for some $1 \le b \le a$ and additionally, $|C_G(g_i)| = p^{c_i}$ for some $1 \le c_i < a$ (or such $C_G(g_i)$ don't exist for all i in the case $a = 1$). This means $p \mid |N|$ and $p \mid \sum_{i=1}^{r} |G : C_G(g_i)|$ which forces $p \mid |N \cap Z(G)|$. Noting that $N \cap Z(G)$ is nonempty since $1 \in N \cap Z(G)$, we have $N \cap Z(G) \neq 1$. $\square$