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Homework 1

In the following, A denotes a commutative ring.

Problem 1. *Let E be an A-module and $F \subseteq E$ a submodule of E such that E/F is a finite A-module. Let $I \subseteq \text{J-rad } A$ be an ideal. Suppose $E = F + IE$. Show that $E = F$.*

Proof. Since $E = F + IE$, we know $E/F = (F + IE)/F$. The right hand side is

$$\begin{aligned}\left\{\left(f + \sum a_i e_i\right) + F \mid f \in f, a_i \in I, e_i \in E\right\} &= \left\{(f + F) + \left(\sum a_i e_i + F\right) \mid f \in F, a_i \in I, e_i \in E\right\} \\ &= \left\{\sum a_i e_i + F \mid a_i \in I, e_i \in E\right\} \\ &= IE/F \\ &= I(E/F).\end{aligned}$$

Thus $E/F = I(E/F)$. Now since $I \subseteq \text{J-rad } A$ and E/F is finitely generated, we can apply Nakayama's Lemma to get $E/F = 0$. Thus $E = F$. □

Problem 2. *For an A-module E, we denote by $\operatorname{ann}(E) = \{a \in A \mid aE = 0\}$; $\operatorname{ann}(E)$ is an ideal called the annihilator of E. Let E be a finite A-module. Suppose E is a Noetherian A-module (respectively Artinian A-module). Show that $A/\operatorname{ann}(E)$ is a Noetherian ring (Artinian ring).*

Proof. Let $E = Ax_1 + \cdots + Ax_n$ and define $f : A \to E^n$ as $f : a \mapsto (ax_1, \ldots, ax_n)$ where E^n is the direct sum of n copies of E. Note that if $a \in \ker f$ then $ax_1 = \cdots = ax_n = 0$ and since the x_i generate E, $a \in \operatorname{ann}(E)$. Conversely, if $a \in \operatorname{ann}(E)$ then clearly $f(a) = 0$. But then an isomorphic copy of $A/\ker f$ sits inside E^n, which is Noetherian (Artinian). Since it's a submodule of a Noetherian (Artinian) module, it too must be Noetherian (Artinian). □

Problem 3. *Let E be an A-module. Let E_1, E_2 be submodules of E such that E/E_1 and E/E_2 are Noetherian (respectively Artinian) A-modules. Show that $E/(E_1 \cap E_2)$ is an Noetherian (Artinian) A-module.*

Proof. Consider the following exact sequence

$$0 \to E/E_1 \to E/(E_1 \cap E_2) \to E/E_2 \to 0$$

where the first map is inclusion and the second map is projection. Since the outside terms are Noetherian (Artinian), the middle term is also Noetherian (Artinian). □

Problem 4. *Let I be an ideal in A. I is called a* nil ideal *if every element of I is nilpotent, i.e. $I \subseteq \text{nil } A$. An ideal I is called* nilpotent *if $I^m = 0$ for some $m > 0$.*
(a) Five an example of a commutative ring and a nil ideal which is not nilpotent.
(b) Show that any finitely generated nil ideal is nilpotent. Thus in a Noetherian ring every nil ideal is nilpotent.

Proof. (a) Let $A = \bigoplus_{i=1}^{\infty} \mathbb{Z}/(p^i)$ for some prime p. Let I be the set of all nilpotent elements of A. Note that I is nontrivial since, for example, $(0 + (p), p + (p^2), 0 + (p^3), 0 + (p^4), \ldots)$ is an element of I. By definition, I is a nil ideal since it contains only nilpotent elements. Suppose $I^k = 0$ for some integer k. But then consider the nilpotent element

$$a = (0 + (p), 0 + (p^2), \ldots, 0 + (p^k), p + (p^{k+1}), 0 + (p^{k+2}), \ldots)$$

and note that $a^k \neq 0$, a contradiction. Thus I is not nilpotent, but is a nil ideal.

(b) Let $I = Ax_1 + \cdots + Ax_r$ be a finitely generated nil ideal. Since each x_i is nilpotent, write $x_i^{n_i} = 0$ for $1 \leq i \leq r$. An element of I^n, for a positive integer n, is of the form

$$x = \prod_{j=1}^{n} \left(\sum_{i=1}^{r} a_{ij} x_i \right).$$

If n is sufficiently large then each term in the expansion will contain a factor of x_i raised to a power greater than or equal to n_i. Each of these terms will go to 0 and so $I^n = 0$ for large enough n. Thus I is nilpotent. Since a Noetherian ring has all ideals finitely generated, every nil ideal is nilpotent. □

Problem 5. *Let K be a field and A the subring of $K[x, y]$ generated by $K \cup \{x, xy, xy^2, \dots\}$, i.e. $A = K[x, xy, xy^2, \dots] = \{f(x, y) \in K[x, y] \mid f(0, y) \in K\}$. Show that A is* not *a Noetherian ring.*

Proof. Let $I_n = (x, xy, xy^2, \dots, xy^n)$. Then $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$. Suppose that this chain terminates at I_n for some n. Then $I_{n+1} = I_n$ so we must be able to write xy^{n+1} as a sum and product of $x, xy, \dots, xy^n$ and the elements of K. Since the degree of a polynomial can't increase under addition, we must multiply two or more polynomials from I_n to get xy^{n+1}. But if we multiply two polynomials to get a y^{n+1} term, we must also get a x^2 term. There's no way to from xy^{n+1} from the generators of I_n, so this chain of ideals doesn't terminate and A is not Noetherian. □

Problem 6. *Let C denote the set of all real valued functions $f : \mathbb{R} \to \mathbb{R}$. C is a commutative ring with operations $(f \pm g)(x) = f(x) \pm g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$ for each $f, g \in C$. Show that C is not a Noetherian ring.*

Proof. Let I_n be the ideal of functions f such that $f(x) = 0$ for each $x \geq n$. This is an ideal since if g is an arbitrary element of C then $(g \cdot f)(x) = g(x) \cdot f(x) = g(x) \cdot 0 = 0$ for $x \geq n$. Also $I_1 \subseteq I_2 \subseteq I_3 \dots$ since any function which is 0 for $x \geq n$ is certainly 0 for $x \geq n+1$. Now suppose that this chain terminates for some n. Then $I_n = I_{n+1}$. But this is clearly false since I_{n+1} contains the function which is 1 for $x < n+1$ and 0 for $x \geq n$, for example. Since I_n doesn't contain this function, this is a contradiction and this chain doesn't terminate. Thus C is not Noetherian. □

Problem 7. *Let E be an A-module and E_i, $0 \leq i \leq n$ submodules such that $E = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n = 0$. Suppose each E_i/E_{i+1} is Noetherian (respectively Artinian). Show that E is Noetherian (Artinian).*

Proof. Note that E_n is trivially Noetherian and E_{n-1}/E_n is Noetherian by assumption. Thus E_{n-1} is Noetherian. Similarly, since E_{n-1} and E_{n-2}/E_{n-1} are both Noetherian, we know E_{n-2} is Noetherian. Continuing in this fashion we inductively have E_1 and E_0/E_1 are both Noetherian so $E_0 = E$ must be Noetherian. □

Problem 8. *Let A be a commutative ring and $I \subseteq A$ an ideal. Let E be an A-module. Suppose that I/I^2 and E/IE are finite A-modules (hence also finite A/I-modules).*
(a) Show that IE/I^2E is a finite A-module.
(b) Show by induction that $I^nE/I^{n+1}E$ is a finite A-module for all $n \geq 0$.
(c) Suppose further that A/I is a Noetherian ring. Show that E/I^nE is a Noetherian A-module for all $n \geq 1$.

Proof. (a) Suppose $a_i \in I$ for $1 \leq i \leq r$ and $x_j \in E$, $1 \leq j \leq m$ are such that $a_i + I^2$ and $x_j + IE$ generate I/I^2 and E/IE respectively as A-modules. Let $b = \sum_{i=1}^{n} b_i y_i + I^2E$ be an arbitrary element of IE/I^2E. Then $b = \sum_{i=1}^{n} (b_i y_i + I^2E)$. Letting i and j vary we get all possible products of elements from I/I^2 and E/IE. Since $b_i y_i + I^2E$ is of this form, we must have b in the set generated by $a_i x_j + I^2E$, $1 \leq i \leq r$, $1 \leq j \leq m$. Thus, this is a generating set for IE/I^2E.

(b) For $n = 0$ we have E/IE is finite by assumption. Now suppose $I^{n-1}E/I^nE$ is finite for some n so that $I^{n-1}E/I^nE$ is generated by $b_k y_l + I^nE$ where $b_k \in I^{n-1}$ and $y_l \in E$ for $1 \leq k \leq s$ and $1 \leq l \leq t$. Then by the same argument as in part (a), a generating set for I^n/I^{n+1} is $b_k a_i y_l x_j + I^{n+1}E$ for $1 \leq k \leq s$, $1 \leq i \leq r$, $1 \leq l \leq t$ and $1 \leq j \leq m$ where the a_i and x_j are as in part (a).

(c) Since A/I is Noetherian and $I^nE/I^{n+1}E$ is a finite A-module (and thus a finite A/I module), it's also a Noetherian A/I module. Now consider the exact sequence

$$0 \to I^{n-1}E/I^nE \to E/I^nE \to E/I^{n-1}E \to 0.$$

This sequence is exact by the third isomorphism theorem for modules which states that $(E/I^nE)/(I^{n-1}E/I^nE) \cong E/I^{n-1}E$. When $n = 1$ we know E/IE is a finite A/I-module by assumption and is thus Noetherian. Suppose that $E/I^{n-1}E$ is Noetherian for some n. Then using this hypothesis and the statement above we see that the outer two terms in the exact sequence are Noetherian, so E/I^nE must also be Noetherian. Therefore E/I^nE is Noetherian for all n. $\square$