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Homework 2

Problem 1. *Give the terms of order ≤ 3 in the power series $e^z \sin z$.*

Proof. The terms of order ≤ 3 for e^z are 1, z, $z^2/2$ and $z^3/6$. For $\sin z$ they are z and $-z^3/6$. Making these into polynomials and multiplying them we find that the terms of order ≤ 3 for $e^z \sin z$ are z, z^2 and $z^3/3$. $\square$

Problem 2. *Determine the radius of convergence for the following power series.*
(a) $\sum n^n z^n$.
(b) $\sum z^n/n^n$.
(c) $\sum 2^n z^n$.
(d) $\sum (\log n)^2 z^n$.
(e) $\sum 2^{-n} z^n$.
(f) $\sum n^2 z^n$.
(g) $\sum \frac{n!}{n^n} z^n$.
(h) $\sum \frac{(n!)^3}{(3n!)} z^n$.

Proof. (a) $r = (\limsup |n^n|^{1/n})^{-1} = (\limsup n)^{-1} = 0$.
(b) $r = (\limsup |n^{-n}|^{1/n})^{-1} = (\limsup n^{-1})^{-1} = \infty$.
(c) $r = (\limsup |2^n|^{1/n})^{-1} = (\limsup 2)^{-1} = 2$.
(d) $r = (\limsup |(\log n)^2|^{1/n})^{-1} = (\limsup (\log n)^{2/n})^{-1} = 1$.
(e) $r = (\limsup |2^{-n}|^{1/n})^{-1} = (\limsup 2^{-1})^{-1} = 2$.
(f) $r = (\limsup |n^2|^{1/n})^{-1} = (\limsup n^{2/n})^{-1} = 1$.
(g) $r = (\limsup |n!/n^n|^{1/n})^{-1} = (\limsup |e^{-n}|^{1/n})^{-1} = (\limsup e^{-1})^{-1} = e$.
(h) $r = (\limsup |(n!)^3/(3n)|^{1/n})^{-1} = 27$. $\square$

Problem 3. *Let $\sum a_n z^n$ and $\sum b_n z^n$ be two power series, with radius of convergence r and s respectively. What can you say about the radius of convergence of the series:*
(a) $\sum (a_n + b_n) z^n$.
(b) $\sum a_n b_n z^n$.

Proof. (a) When we constructed formal power series we defined $\sum (a_n + b_n) z^n = \sum a_n z^n + \sum b_n z^n$. Therefore, the set of points for which the left side converges is given by the intersection of the two sets of convergence for the right side series. That is, if t is the radius of convergence for $\sum (a_n + b_n) z^n$ then $t \leq \min(r, s)$.

(b) Let t be the radius of convergence of $\sum a_n b_n z^n$. Then

$$\begin{aligned}
t &= (\limsup |a_n b_n|^{1/n})^{-1} \\
&= (\limsup |a_n|^{1/n} |b_n|^{1/n})^{-1} \\
&\geq (\limsup |a_n|^{1/n} \cdot \limsup |b_n|^{1/n})^{-1} \\
&= (\limsup |a_n|^{1/n})^{-1} (\limsup |b_n|^{1/n})^{-1} \\
&= rs.
\end{aligned}$$

Thus the radius of convergence must be greater than or equal to the product of the two previous radii. $\square$

Problem 4. *Show that the only complex numbers z such that $\sin z = 0$ are $z = k\pi$, where k is an integer. State and prove a similar statement for $\cos z$.*

Proof. Using the power expansion of e^z we see that

$$e^{iz} = \sum i^n \frac{z^n}{n!}$$

and

$$e^{-iz} = \sum (-i)^n \frac{z^n}{n!} = \sum (-1)^n i^n \frac{z^n}{n!}.$$

Then we must have

$$\frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}\left(\sum i^n \frac{z^n}{n!} - (-1)^n i^n \frac{z^n}{n!}\right) = \sum (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sin z.$$

A similar argument proves that $\frac{e^{iz}+e^{-iz}}{2} = \cos z$. Using this formula for $\sin z$ we have $\sin z = 0$ is equivalent to $e^{iz} = e^{-iz}$ or $e^{2iz} = 1$. Letting $z = x + iy$ we have $1 = e^{2iz} = e^{2ix}e^{-2y}$. Taking the modulus of both sides reveals that $e^{-2y} = 1$ and so $y = 0$. Therefore $e^{2ix} = 1$ where $x \in \mathbb{R}$. But we already know the solutions for this equation are $x = k\pi$ for an integer k. A similar argument holds showing that the only complex numbers z for which $\cos z = 0$ are $z = k\pi/2$ for an integer k. □

Problem 5. *(a) Given an arbitrary point z_0, let C be a circle of radius $r > 0$ centered at z_0, oriented counterclockwise. Find the integral*

$$\int_C (z - z_0)^n dz$$

for all integers n, positive or negative.
(b) Suppose f has a power series expansion

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k,$$

which is absolutely convergent on a disc of radius $> R$ centered at z_0. Let C_R be the circle of radius R centered at z_0. Find the integral

$$\int_{C_R} f(z) dz.$$

Proof. (a) Let $n \neq -1$. Consider the function $g(z) = (z - z_0)^{n+1}/(n+1)$. Then we see that $g' = f$ and so f is a continuous function with a primitive. Since C is a closed path we see that

$$\int_C (z - z_0)^n dz = 0.$$

For the case $n = -1$ we can parameterize C as $C = re^{i\theta} + z_0$. Then we have

$$\int_C (z - z_0)^n dz = \int_0^{2\pi} (re^{i\theta} + z_0 - z_0)^{-1}(ire^{i\theta}) d\theta = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = \int_0^{2\pi} i d\theta = 2\pi i.$$

(b) Let $f_n(z) = \sum_{-m}^{n} a_k (z - z_0)^k$. Then we know that the sequence $\int f_n$ converges to $\int f$. Since each term is a finite sum we can take the integral term by term. By part (a) we know that all terms are 0 except for the case $k = -1$. Thus

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k = a_{-1} 2\pi i.$$

□

Problem 6. *Find the integral of each one of the following functions over each one of the curves $\gamma_1(t) = 1 + it$, $\gamma_2(t) = e^{-\pi it}$, $\gamma_3(t) = e^{i\pi t}$, $\gamma_4(t) = 1 + it + t^2$.*
(a) $f(z) = z^3$. (b) $f(z) = \overline{z}$. (c) $f(z) = 1/z$.

(a) $\gamma_1(t)$: $((1+i)^4-1)/4$, $\gamma_2(t)$: 0, $\gamma_3(t)$: 0, $\gamma_4(t)$: $((2+i)^4-1)/4$.
(b) $\gamma_1(t)$: $i+1/2$, $\gamma_2(t)$: $-\pi i$, $\gamma_3(t)$: πi, $\gamma_4(t)$: $2+2i/3$.
(c) $\gamma_1(t)$: $\log\sqrt{2}+(i\pi)/4$, $\gamma_2(t)$: $-\pi i$, $\gamma_3(t)$: πi, $\gamma_4(t)$: $\log\sqrt{5}+i\arctan(1/2)$.

Problem 7. *Let σ be a vertical line segment, say parametrized by*

$$\sigma(t) = z_0 + itc, \quad -1 \le t \le 1,$$

where z_0 is a fixed complex number, and c is a fixed real number > 0. Let $\alpha = z_0 + x$ and $\alpha' = z_0 - x$, where x is real positive. Find

$$\lim_{x\to 0}\int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right) dz.$$

Proof. We have $\sigma(t) = z_0 + itc$ and $\sigma'(t) = ic$. Therefore we have

$$\begin{aligned}
\lim_{x\to 0}\int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right) dz &= \lim_{x\to 0}\int_{-1}^{1}\left(\frac{1}{itc-x} - \frac{1}{itc+x}\right)(ic)dt \\
&= \lim_{x\to 0}(ic)\int_{-1}^{1}\frac{2x}{-(tc)^2-x^2}dt \\
&= \lim_{x\to 0}\frac{-2ic}{x}\int_{-1}^{1}\frac{1}{\left(\frac{tc}{x}\right)^2+1}dt \\
&= \lim_{x\to 0}\frac{-2ic}{x}\frac{x}{c}\arctan\left(\frac{tc}{x}\right)\Bigg|_{-1}^{1} \\
&= \lim_{x\to 0} -4i\arctan(c/x) \\
&= -4i\left(\frac{\pi}{2}\right) \\
&= -2\pi i.
\end{aligned}$$

□

Problem 8. *Let F be a continuous complex-valued function on the interval $[a.b]$. Prove that*

$$\left|\int_a^b F(t)dt\right| \le \int_a^b |F(t)|dt.$$

Proof. Let $P = [a = a_0, a_1, \dots, a_n = b]$ be a partition of $[a,b]$ such that $\max(a_{i+i} - a_i) < \delta$. Then we have

$$\left|\int_a^b F - \sum_{k=0}^{n-1} F(a_k)(a_{k+1}-a_k)\right| < \varepsilon$$

and

$$\left|\int_a^b |F| - \sum_{k=0}^{n-1} |F(a_k)|(a_{k+1}-a_k)\right| < \varepsilon.$$

Due to the triangle inequality we can write

$$\left|\int_a^b F(t)dt\right| \le \left|\sum_{k=0}^{n-1} F(a_k)(a_{k+1}-a_k)\right| + \varepsilon \le \sum_{k=0}^{n-1} |F(a_k)|(a_{k+1}-a_k) + \varepsilon.$$

Combining this with the second equation we get

$$\left|\int_a^b F(t)dt\right| \le \sum_{k=0}^{n-1} |F(a_k)|(a_{k+1}-a_k) + \varepsilon \le \int_a^b |F(t)|dt + 2\varepsilon.$$

Since this is true for arbitrary epsilon, the inequality follows. □

Problem 9. *Let $A, B \subseteq \mathbb{C}$ be such that A is compact, B is closed and $A \cap B = \emptyset$. Prove that the distance of A and B is strictly positive.*

Proof. First consider the case of the distance $d(z, B)$ between a point z and a closed set B. Suppose $d(z, B) = 0$. Then any open set containing z must contain points of B, otherwise we could find a disk around z with some radius r and this would give a nonzero distance between z and B. Therefore, z is an accumulation point of B, but B is closed, and so $z \in B$. Now consider the case for A compact and B closed. If $z \in A$ then $z \notin B$ and so $d(z, B) > 0$, by the above argument. Now for each point $z \in A$ let $r_z = (1/2)d(z, B)$ and consider the disk $D_{r_z}(z)$. Since A is compact, there are finitely many $z_k \in A$ such that $A \subseteq D_{r_{z_1}}(z_1) \cup \cdots \cup D_{r_{z_n}}(z_n)$. Now let $r = (1/2)\min(r_{z_1}, \ldots, r_{z_n})$. Now for an arbitrary point $z \in D$, $z \in D_{r_k}(z_k)$ for some k and since $D_{2r_k}(z_k)$ contains no points of B, we have $0 < r \leq (1/2)r_k \leq |z - z'|$ for some $z' \in A$. This shows that $0 < r \leq \inf_{z' \in A, z \in B} |z - z'| \leq d(A, B)$. ☐

Problem 10. *(a) Let $f : U \to \mathbb{C}$ be continuous, with $U = \mathbb{C}\backslash\{0\}$, and assume that the integral of f along the boundary of any triangle lying entirely in U is 0. Show that f has a primitive on $U\backslash\mathbb{R}^-$.*
(b) Find all such primitives F of f.
(c) Give an example of f, F as in (a) with $\lim_{\varepsilon\to 0} F(-1+i\varepsilon) \neq \lim_{\varepsilon\to 0} F(-1-i\varepsilon)$ (so F cannot be extended on all of U.

Proof. (a) Since f is defined for all $z_0 \neq 0$, and has integral of 0 around the boundary of any triangle in U, we know there exists a primitive $F(z) = \int_{z_0}^{z} f$. Note that this implies that F will not be defined for $z \in \mathbb{R}^-$.

(b) Let z_0 be a point in U. Then for $z \in U$, $F(z) = \int_{z_0}^{z} f$.

(c) Let $f = 1/z$ and $F = \log z$. Then $\lim_{\varepsilon\to 0} \log(-1 + i\varepsilon) \neq \lim_{\varepsilon\to 0}(-1 - i\varepsilon)$. ☐