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Homework 6

Exercise 1 *Show that if*
$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
is a polynomial, such that n is odd and $a_n \neq 0$ then there exists $c \in \mathbb{R}$ with $p(c) = 0$.

Proof. Suppose that $a_n > 0$. From Homework 5 we have $\lim_{x\to\infty} p(x)/(a_n x^n) = 1$. Let $\varepsilon = 1/2$. Then there exists $m \in \mathbb{R}$ such that for all $x > m$ we have $|p(x)/(a_n x^n) - 1| < 1/2$. Thus there exists $x_1 > 0$ such that $1/2 < p(x_1)/(a_n x_1^n)$. Since $x_1, a_n > 0$ and n is odd we have $0 < (a_n x_1^n)/2 < p(x_1)$. Thus $p(x_1)$ is positive. Similarly take $\lim_{x\to-\infty} p(x)/(a_n x^n) = 1$ and let $\varepsilon = 1/2$. Then there exists $m \in \mathbb{R}$ such that for all $x < m$ we have $|p(x)/(a_n x^n) - 1| < 1/2$. Then there exists $x_2 < 0$ such that $1/2 < p(x)/(a_n x^n)$. But since $x_2 < 0$ and $a_n > 0$ we have $a_n x^n < 0$ so then $p(x) < (a_n x^n)/2 < 0$. Thus $p(x_2) < 0$. Therefore there exist $x_1, x_2 \in \mathbb{R}$ with $p(x_2) < 0$ and $p(x_1) > 0$ so there must exist $c \in (x_2; x_1)$ with $p(c) = 0$ by the Intermediate Value Theorem. A very similar proof holds if $a_n < 0$ where the limits give values of opposite signs as in this proof. □

First we prove a lemma showing that for $a \in \mathbb{R}$, $a^2 \geq 0$.

Proof. Let $a \in \mathbb{R}$. If $a = 0$ then $a^2 = 0 \cdot 0 = 0$. If $a > 0$ then $a^2 = a \cdot a > 0$. If $a < 0$ then $a^2 = a \cdot a = -|a| \cdot -|a| = (-1)^2 \cdot |a| \cdot |a| = |a| \cdot |a| > 0$. In all cases $a^2 \geq 0$. □

Exercise 2 *Show that if $a, b \geq 0$ then*
$$\sqrt{ab} \leq \frac{a+b}{2}$$
and equality holds if and only if $a = b$.

Proof. Note that $0 \leq (a-b)^2 = a^2 - 2ab + b^2$ so $4ab \leq a^2 + 2ab + b^2 = (a+b)^2$. Then $ab \leq (a+b)^2/4$ and since $a, b \geq 0$ we have $\sqrt{ab} \leq (a+b)/2$. To show equality suppose $\sqrt{ab} = (a+b)/2$. Then $4ab = (a+b)^2 = a^2 + 2ab + b^2$ and so then $(a-b)^2 = 0$ which means $a - b = 0$ and $a = b$. Conversely we assume $a = b$ so $a - b = 0$ and $0 = (a-b)^2 = a^2 - 2ab + b^2$. Then $4ab = a^2 + 2ab + b^2 = (a+b)^2$ so $ab = (a+b)^2/4$ and since $ab > 0$ we have $\sqrt{ab} = (a+b)/2$. □

Exercise 3 *Show that if $a, b \in \mathbb{R}$ then*
$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$
and equality holds if and only if $a = b$.

Proof. Again note that $0 \leq (a-b)^2 = a^2 - 2ab + b^2$ so we have $2ab \leq a^2 + b^2$ and $2(a^2+b^2) \geq a^2 + 2ab + b^2 = (a+b)^2$. Then $(a+b)^2/4 \leq (a^2+b^2)/2$ and since both of these terms are positive, we have $(a+b)/2 \leq \sqrt{(a^2+b^2)/2}$. To show equality we assume $(a+b)/2 = \sqrt{(a^2+b^2)/2}$. Then $(a^2 + 2ab + b^2)/4 = (a^2+b^2)/2$ so $a^2 + 2ab + b^2 = 2(a^2+b^2)$. Thus, $0 = a^2 - 2ab + b^2 = (a-b)^2$ so $a - b = 0$ and $a = b$. Conversely assume that $a = b$. Then $0 = a - b = (a-b)^2 = a^2 - 2ab + b^2$ and $2ab = a^2 + b^2$ so $a^2 + 2ab + b^2 = 2(a^2+b^2)$. Thus $(a+b)^2/4 = (a^2+b^2)/2$. Since these terms are positive we have $(a+b)/2 = \sqrt{(a^2+b^2)/2}$. □

Exercise 4 *Show that if $a, b > 0$ then*
$$\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab}$$
and equality holds if and only if $a = b$.

Proof. Once again note that $0 \leq (a-b)^2 = a^2 - 2ab + b^2$ so $4ab \leq a^2 + 2ab + b^2 = (a+b)^2$. Then since $(a+b)^2 \neq 0$ we have $4ab/(a+b)^2 \leq 1$. Since $ab > 0$ we have $(2ab)^2/(a+b)^2 \leq ab$ and also

$$\sqrt{ab} \geq \frac{2ab}{a+b} = \frac{2}{\frac{a+b}{ab}} = \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

To show equality assume

$$\sqrt{ab} = \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

Then

$$ab = \left(\frac{2}{\frac{1}{a} + \frac{1}{b}}\right)^2 = \left(\frac{2}{\frac{a+b}{ab}}\right)^2 = \left(\frac{2ab}{a+b}\right)^2 = \frac{4a^2b^2}{a^2 + 2ab + b^2}.$$

Then $(ab), (a+b)^2 > 0$ so $1 = 4ab/(a^2 + 2ab + b^2)$ and $4ab = a^2 + 2ab + b^2$. Then $0 = (a-b)^2 = a - b$ so $a = b$. Conversely assume that $a = b$. Then $0 = a - b = (a-b)^2 = a^2 - 2ab + b^2$. Thus $4ab = a^2 + 2ab + b^2$ and since $(a^2 + 2ab + b^2) > 0$ we have $(4ab)/(a+b)^2 = 1$ and $(2ab)^2/(a+b)^2 = ab$. Then

$$ab = \frac{4a^2b^2}{a^2 + 2ab + b^2} = \left(\frac{2ab}{a+b}\right)^2 = \left(\frac{2}{\frac{a+b}{ab}}\right)^2 = \left(\frac{2}{\frac{1}{a} + \frac{1}{b}}\right)^2$$

and since both of these quantities are greater than zero we have

$$\sqrt{ab} = \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

$\square$

Exercise 5 *Show that if $a, b, c \in \mathbb{R}$ then*

$$\frac{a+b+c}{3} \leq \sqrt{\frac{a^2 + b^2 + c^2}{3}}$$

and equality holds if and only if $a = b = c$.

Proof. Note that $0 \leq (a-b)^2 + (b-c)^2 + (a-c)^2 = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac$ so $2ab + 2bc + 2ac \leq 2a^2 + 2b^2 + 2c^2$. Then $3(a^2 + b^2 + c^2) \geq a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = (a+b+c)^2$ and so $(a+b+c)^2/9 \leq (a^2 + b^2 + c^2)/3$. Since both of these values are positive we have $(a+b+c)/3 \leq \sqrt{(a^2 + b^2 + c^2)/3}$. To show equality, assume that

$$\frac{a+b+c}{3} = \sqrt{\frac{a^2 + b^2 + c^2}{3}}.$$

Then we have $3(a^2 + b^2 + c^2) = (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$. Then $2ab + 2bc + 2ac = 2a^2 + 2b^2 + 2c^2$ so $0 = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = (a-b)^2 + (b-c)^2 + (a-c)^2$. But these three terms are all greater than or equal to zero so each must be equal to zero. Then $a = b = c$. Conversely, assume that $a = b = c$. Then $0 = (a-b) = (b-c) = (a-c) = (a-b)^2 = (b-c)^2 = (a-c)^2 = (a-b)^2 + (b-c)^2 + (a-c)^2 = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac$. Thus $2a^2 + 2b^2 + 2c^2 = 2ab + 2bc + 2ac$ and $3(a^2 + b^2 + c^2) = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = (a+b+c)^2$. Then $(a+b+c)^2/9 = (a^2 + b^2 + c^2)/3$ and since both of these terms are positive we have

$$\frac{a+b+c}{3} = \sqrt{\frac{a^2 + b^2 + c^2}{3}}.$$

$\square$

Exercise 6 *Is there are real function $f : \mathbb{R} \to \mathbb{R}$ that takes on every real number an even number of times?*

Yes.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} |x| & \text{if } x \notin \mathbb{N} \\ 1 & \text{if } x = 0 \\ x+1 & \text{if } x \in \mathbb{N}. \end{cases}$$

We see that $f(x) > 0$ for all $x \in \mathbb{R}$ so for $y \leq 0$, f takes on y zero times. Consider $y > 0$. If $y \in \mathbb{N}$ and $y \neq 1$ then $y - 1 \in \mathbb{N}$ so we have $f(y-1) = (y-1)+1 = y$ and also $f(-y) = |-y| = y$. Note that by definition of absolute value, for $a \in \mathbb{R}$ with $a \neq 0$ there are only two real numbers $a, -a$ which will have an absolute value of $|a|$. Also there is only one number $z \in \mathbb{R}$ such that $z + 1 = y$. Thus, there are exactly two elements of $\mathbb{R}$ which map to y. If $y = 1$ then $f(0) = y$ and $f(-1) = |-1| = 1 = y$. We have every natural number mapping to something greater than 1, and the only other element of $\mathbb{R}$ with an absolute value of 1 is 1 and $f(1) = 2$. Thus, there are exactly two elements of $\mathbb{R}$ which map to y. Finally, if $y \notin \mathbb{N}$ then $f(y) = |y| = y$ since $y > 0$ and $f(-y) = |-y| = y$. There are no other elements of $\mathbb{R}$ with an absolute value of y so there are exactly two elements of $\mathbb{R}$ which map to y. In all cases we have f taking on every value of $\mathbb{R}$ either 0 or 2 times. $\square$