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MATH 27700
November 23, 2009

Homework 8

Problem 1. *(a) Let M be an $\mathcal{L}$-structure and let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be elements of $|M|$. Suppose there is an automorphism f of M such that $f(a_i) = b_i$ for $1 \leq i \leq n$. Show that for all $\varphi(x_1, \ldots, a_n)$, $M \models \varphi(a_1, \ldots a_n) \iff M \models \varphi(b_1, \ldots, b_n)$. In other words, $a_1, \ldots a_n$ and $b_1, \ldots b_n$ satisfy the same complete type.*

(b) Show that the converse may fail; in some models, the map $a_i \mapsto b_i$ will not extend to an automorphism.

Proof. (a) Ket $\varphi(x_1, \ldots x_n)$ be a formula. Suppose that $M \models \varphi(a_1, \ldots, a_n)$. We induct on the complexity of φ. Suppose φ is an atomic formula with n-ary relation relation R. Then $R(a_1, \ldots, a_n)$. But since f is an automorphism, $R(f(a_1), \ldots, f(a_n)) = R(b_1, \ldots, b_n)$ as well and $M \models \varphi(b_1, \ldots, b_n)$. Now Suppose that $\varphi = \theta \wedge \psi$. Then $M \models \theta(a_1, \ldots, a_n)$ and $M \models \psi(a_1, \ldots, a_n)$. From induction, we know that $M \models \theta(b_1, \ldots, b_n)$ and $M \models \psi(b_1, \ldots, b_n)$. Thus $M \models \varphi(b_1, \ldots, b_n)$. If $\varphi = \neg\theta(a_1, \ldots, a_n)$ then not $M \models \theta(a_1, \ldots, a_n)$. From induction, not $M \models \theta(b_1, \ldots, b_n)$ and so $M \models \neg\varphi(b_1, \ldots, b_n)$. In the case $\varphi = \forall x_1 \ldots \forall x_n \theta$, it's clear that $M \models \varphi(b_1, \ldots, b_n)$. Therefore $M \models \varphi(a_1, \ldots a_n) \iff M \models \varphi(b_1, \ldots, b_n)$.

(b) Let R be a relation and suppose M is a model in which a is related to countably many things and b is related to uncountably many. Then any finitary statements about a and b, so $M \models \varphi(a) \iff M \models \varphi(b)$ for all φ. But any automorphism taking a to b will force a to be related to uncountably many things. $\square$

Problem 2. *Let $\mathcal{L}$ be the language containing a single binary relation $<$ (and equality).*
(a) Let T be the theory of dense linear orders without endpoints in the language containing a single binary relation symbol $<$ and equality. You may assume that $\mathcal{M}$, $\mathcal{N}$ are countable models of T, then tuples $(a_1, \ldots, a_n) \in |\mathcal{M}|^n$ and $(b_1, \ldots, b_n) \in |\mathcal{N}|^n$ have the same type iff for $1 \leq i < j \leq n$ we have both $a_i <^{\mathcal{M}} a_j$ iff $b_i <^{\mathcal{N}} b_j$ and $a_i =^{\mathcal{M}} a_j$ iff $b_i =^{\mathcal{N}} b_j$. Show that the theory T of dense linear orders without endpoints over $\mathcal{L}$ is ω-categorical.
(b) Use the above to show that there is a unique *complete T' containing T.*
(c) Construct two nonisomorphic models of T of the same cardinality which are not isomorphic, and prove that this is the case.

Proof. (a) Let M and N be two countably infinite models of T. We inductively construct an isomorphism between M and N. Let $m_1 \in |M|$, $n_1 \in |N|$ and define $f : |M| \to |N|$ such that $f(m_1) = n_1$. Now choose $m_2 \in |M| \backslash \{m_1\}$. If $m_1 < m_2$, then since N has no endpoints we can find $n_2 \in |N| \backslash \{n_1\}$ with $n_1 < n_2$. Clearly if $m_2 < m_1$ then we can find $n_2 \in |N|$ with $n_2 < n_1$ for the same reasons. Set $f(m_2) = n_2$. Now pick $n_3 \in |N| \backslash \{n_1, n_2\}$. If n_3 is less than both n_1 and n_2 or n_3 is greater than n_1 and n_2, then we can find $m_3 \in |M|$ with the same relation to m_1 and m_2 just as we did in picking n_2. Otherwise, $n_1 < n_3 < n_2$ or $n_2 < n_3 < n_1$. In this case since $|M|$ is dense linear ordering we can find m_3 between m_1 and m_2 which has the same relations as n_1, n_2 and n_3. Hence set $f(m_3) = n_3$. This completes the base case.

Now assume that we have defined f to be an isomorphism between the two sets $\{m_1, \ldots, m_{n-1}\}$ and $\{n_1, \ldots, n_{n-1}\}$. Pick $m_n \in |M| \backslash \{m_1, \ldots, m_{n-1}\}$. Either m_n is less than every m_i, $1 \leq i \leq n-1$, m_n is greater than every m_i, $1 \leq i \leq n-1$ or $m_i < m_n < m_j$ for some $1 \leq i < j \leq n-1$. In the first two cases use the fact that N has no end points to choose n_n less than or greater than each n_i with $1 \leq i \leq n$. Otherwise use the fact that N is dense to choose n_n with $n_i < n_n < n_j$. Choosing $n_{n+1} \in |N| \backslash \{n_1, \ldots, n_n\}$ lets us use the exact same argument to find a suitable m_n. Thus f is now a bijection between $\{m_1, \ldots, m_{n+1}\}$ and $\{n_1, \ldots, n_{n+1}\}$. Since this is true for all n by induction and M and N are countable, we must have $M \cong N$. Therefore T is ω-categorical.

(b) Let S be a complete theory containing T. Extend T to a maximally consistent T'. Since T is ω-categorical it follows directly that every sentence of S must also be a sentence of T' and so S is isomorphic to T'.

(c) Let $M = \langle \mathbb{R}, < \rangle$ and N be the same model with a copy of $\mathbb{Q}$ added to the end. Define both models to have the usual interpretation of $<$. These are both dense linear orderings and clearly neither of them is countable. Also both have the cardinality of the continuum. But these models can't be isomorphic since N contains an element with countably many things greater than it while M does not. $\square$

Problem 3. *Let $\mathcal{L}$ consist of a binary relation $<$ together with constants c_n for $n \in \mathbb{N}$. Let T be the theory stating that $\mathcal{L}$ is a dense linear order without endpoints, and that for each n we have $c_n < c_{n+1}$. You may assume without proof that this theory is complete.*
(a) Show that any countable model of this theory is isomorphic to a model $\mathcal{M}$ given by $\langle \mathbb{Q}, <, c_0, c_1 \rangle$ (with the usual interpretation of $<$) in which $\{c_n^{\mathcal{M}}\}_{n\in\mathbb{N}}$ is a strictly increasing sequence of elements.
(b) There are three models (up to isomorphism) of this theory, characterized by the behavior of this sequence:
(i) $\lim_n c_n = q$ for some $q \in \mathbb{Q}$.
(ii) $\{c_n\}_{n\in\mathbb{N}}$ is bounded, but does not posses a least upper bound in $\mathbb{Q}$.
(iii) $\{c_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Q}$.
Explain (with proof) which model is saturated and which one is atomic.

Proof. (a) Let M be a countable model of T and let a_i be the interpretation of c_i in M. We will inductively construct an isomorphism between M and $N = \langle \mathbb{Q}, <, b_0, b_1 \rangle$ in which $\{b_n\}_{n\in\mathbb{N}}$ is a strictly increasing sequence of elements. Pick $m_1 \in |M| \backslash \{a_i\}$ and $n_1 \in |N| \backslash \{b_i\}$. Note that either $m_1 < a_1$ or $a_i < m_1 < a_j$ for some $i < j$. In either case, since N is dense and without endpoints, we can find n_1 sucht that $n_1 < b_1$ or $b_i < n_1 < b_j$. Hence define $f : M \to N$ such that $f(m_1) = n_1$ and $f(a_1) = b_1$. Now choose some $n_2 \in |N| \backslash (\{b_i\} \cup \{n_1\})$. Once again, we consider the relationship of n_2 with n_1 and $\{b_i\}$ and note that since M is a dense linear order without endpoints, we can find m_2 which has the same relationship to m_1 and $\{a_i\}$ and n_2 has with n_1 and $\{b_i\}$. Thus let $f(m_2) = n_2$ and $f(a_2) = b_2$. This completes the base case.

The inductive step follows in precisely the same manner as in Problem 2. The only difference is that in this case we choose m_n different from $\{a_i\} \cup \{m_1, \ldots, m_{n-1}\}$. We can find a suitable n_n for the same reasons as above. Going backwards and finding a suitable m_{n+1} for a chosen n_{n+1} distinct from $\{b_i\} \cup \{n_1, \ldots, n_n\}$ is also the same. Thus, we've inductively constructed f to be an isomorphism between M and N such that $f(m_i) = n_i$ and $f(a_i) = b_i$. Since $a_n < a_{n+1}$ for each n, it follows that $b_n < b_{n+1}$ for each n and therefore $\{b_n\}$ is a strictly increasing sequence.

(b) Model (iii) is atomic. Any n-tuple of elements $a_1, \ldots, a_n$ can be described by their relation to c_i for $1 \leq i \leq n$. Thus there exists some formula $\varphi(x_1, \ldots, x_n)$ realized by $a_1, \ldots, a_n$ which will decide every other formula by describing the relationship between x_i and c_i. Note that model (ii) can't be saturated because there exists a type which isn't realized over any set which includes the least upper bound of $\{c_i\}$. This leaves (iii) to be the saturated model, which we know exists since since T has countably many consistent n-types for each $n < \omega$ and we're assuming T is complete. $\square$

Problem 4. *Let R be the language with a single binary relation symbol R and equality. Let $T = \{\psi_n \mid n < \omega\} \cup \{\theta_n \mid n < \omega\} \cup \rho$ where ρ says that R is symmetric and irreflexive,*

$$\psi_n := \exists x_1 \ldots x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right)$$

and

$$\theta_n := \forall x_1 \ldots x_n \left(\bigwedge_{\sigma \subset n} \exists z \left(\bigwedge_{i \in \sigma} R(z, x_i) \wedge \bigwedge_{j \notin \sigma} \neg R(z, x_j) \right) \right).$$

(a) Informal, what do these axioms say?
(b) Prove that T is ω-categorical.

Proof. (a) For each $n < \omega$, ψ_n tells us that there are n elements which are all distinct from each other. This immediately says any model can't be finite, since for any finite model of size n, there are $n + 1$ distinct elements. For each $n < \omega$, θ_n tells us that for every possible subset of n, there is some point z which is related to all the x_i for i in that subset, and is not related to x_j for j not in the subset. Informally, this means that for each finite set of elements, there's an element which partitions the set in every possible way using R (although if we assume strict inclusion, $\sigma \subsetneq n$, then we can't get the trivial partition). That is, an element will either be related to z or not which forms a partition of that set.

(b) Let M and N be two countably infinite models of T. Choose $m_1 \in |M|$ and $n_1 \in |N|$ and define a function $f : M \to N$ with $f(m_1) = n_1$. If f is to be an isomorphism it must preserve all the functions, relations and constants of $\mathcal{L}$, but in this case we only need to worry about R. Choose some other point $m_2 \in |M|\backslash\{m_1\}$. Suppose first that $R(m_1, m_2)$. Then note that ψ_2 gives 2 distinct elements in $|N|$, say n_1 and n_1'. Also, using θ_2 and the subset $\{1\}$ of 2, there exists some $n_2 \in |N|$ with $R(n_1, n_2)$ and $\neg R(n_1', n_2)$. So define $f(m_2) = n_2$. Note that $n_1 \neq n_2$ since R is antisymmetric from ρ. On the other hand, if $\neg R(m_1, m_2)$ then there exists n_2' and n_2'' such that $R(n_1', n_2')$ and $\neg R(n_1', n_2'')$ while $\neg R(n_1, n_2')$ and $\neg R(n_1, n_2'')$. This all follows from using θ_2. But since this means $n_2' \neq n_2''$, at least one of them must be different from n_1. Choose this to be $n_2 \in |N|$ so that $\neg R(n_1, n_2)$.

To complete the loop, we choose $n_3 \in |N|\backslash\{n_1, n_2\}$ (using ψ_3 or the fact that N is countably infinite). We must consider the possibilities for $R(n_i, n_3)$ with $i = 1$ and $i = 2$. For each i with $R(n_i, n_3)$ let $i \in \sigma \subset 3$. Now note that using ψ_3 and θ_3 in M there is some m_3 with precisely the same relationship to m_1 and m_2 as n_3 has with n_1 and n_2. Thus let $f(m_3) = n_3$. This completes the base case for an inductively created isomorphism from M to N.

For the inductive step, choose $m_n \in |M|\backslash\{m_1, \ldots, m_{n-1}\}$. For each $1 \leq i \leq n-1$, let $i \in \sigma$ if $R(m_i, m_n)$. Now use ψ_n to pick n_n' distinct from $n_1, \ldots n_{n-1}$. Since $\sigma \subset n$, we can use θ_n to pick an element n_n which has the same relations with $n_1, \ldots, n_{n-1}$ as m_n has with $m_1, \ldots, m_{n-1}$. Call this element n_n and let $f(m_n) = n_n$. Note that we can choose n_n distinct from all the n_i with $1 \leq i < n$ using the same argument as in the base case. The argument for finding a pair of elements n_{n+1} and m_{n+1} is identical to this one. Since we've constructed f to preserve R for every element of M and N (as they are both countable, so every element is some m_i or n_i), f must be an isomorphism between M and N. Therefore $M \cong N$ and T is ω-categorical. □

Problem 5. *Give a complete proof of the statement from class that if $R = \{M_1, \ldots, M_n\}$ is a finite set of $\mathcal{L}$-structures, then [there exists a set of sentences T such that $M \models T$ iff there is $N \in R$ such that $M \cong N$] if and only if each of the M_i is finite.*

Proof. Suppose first that each M_i is finite. Let φ_{i_j} be an index of every sentence in each of the M_i. For $m > 0$ let T_m be given by the formula

$$\bigvee_{i=1}^{n} \left(\bigwedge_{j=1}^{m} \varphi_{i_j} \right)$$

and let T be the collection of these T_m. For each $1 \leq i \leq n$ and each m then, we have $M_i \models \bigwedge_{j=1}^{m} \varphi_{i_j}$. Thus every $M_i \in R$ is a model of T. On the other hand, if $N \ncong M_i$ then not $N \models \varphi_{i_k}$ for some i and k. But it must be the case that not $N \models \bigwedge_{j=1}^{m} \varphi_{i_j}$ for $m > k$. Thus R is the complete set of models of T up to isomorphism.

Conversely, suppose that there exists T such that R is the entire set of models for T up to isomorphism. Suppose that one of the models M_i were infinite. Then using upward Löwenheim-Skolem we can produce a model of every infinite cardinality. Thus R can't be finite, which is a contradiction, and so each M_i is finite. □