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MATH 25700

Quiz 2

October 14, 2009

Problem 1. *Show that every left coset of the subgroup* $\mathbb{Z}$ *of the additive group of the real numbers contains exactly one element* x *such that* $0 \leq x < 1$.

Proof. Let $H = r + \mathbb{Z}$ be a left coset of $(\mathbb{R}, +)$ where $r \in \mathbb{R}$. Let $n = \lfloor r \rfloor$ be the greatest integer less than or equal to r. Then $0 \leq r - n$ since $r \geq n$ and furthermore, $r - n < 1$. This second inequality follows from the fact that n is greater than or equal to any integer less than r. Thus $r + (-n) \in r + \mathbb{Z}$ is an element x such that $0 \leq x < 1$. Now consider an arbitrary element $s \in r + \mathbb{Z}$ such that $0 \leq s < 1$. Since all elements of $r + \mathbb{Z}$ are of the form $r + m$ for $m \in \mathbb{Z}$, we know that the difference of two elements is $(r + m) - (r + m') = m - m'$. That is, the difference is always an integer. Therefore $r + (-n) - s \in \mathbb{Z}$. But since $r + (-n)$ and s are both between 0 and 1 we must have that $r + (-n) - s = 0$ which means $r + (-n) = s$. $\square$

Problem 2. *Show by counterexample that the following is false: If a group* G *is such that every proper subgroup is cyclic, then* G *is cyclic.*

Proof. Consider the abelian Klein-4 group $G = \{1, a, b, c\}$ such that $a^2 = b^2 = c^2 = 1$ $ab = c$, $ac = b$ and $bc = a$. We've already verified that this is a group. If we consider a subset with more than 1 nonidentity element, e.g. $\{1, a, b\}$, it's clear that this set isn't closed under multiplication. That is, for any two nonidentity elements of G, their product is always the third nonidentity element. Therefore, the only possible proper subgroups are $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$. However, it's obvious that G isn't cyclic since each element has order 1 or 2, yet $|G| = 4$. $\square$