Kris Harper
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Homework 2

**Problem 1.** *Suppose the inequality (1.44) is replaced by*

$$u(x) \leq \alpha(x) + \int_a^x \beta(s)u(s)ds,$$

*where $\alpha$ and $\beta$ are continuous functions on $[a, b]$, $\beta$ nonnegative and $\alpha$ non-increasing there. Find the generalization of Gronwall's lemma to this case.*

*Proof.* Define

$$R(x) = \int_a^x \beta(s)u(s)ds.$$

Then

$$\frac{dR}{dx} = \beta(x)u(x) \leq \alpha(x)\beta(x) + \beta(x)R(x).$$

Multiplying both sides by $\exp\left(-\int_a^x \beta(t)dt\right)$ the inequality simplifies to

$$\frac{d}{dx}\left(R(x)e^{-\int_a^x \beta(t)dt}\right) \leq \alpha(x)\beta(x)e^{-\int_a^x \beta(t)dt}.$$

Integrating from $a$ to $x$ we then have

$$R(x)e^{-\int_a^x \beta(t)dt} \leq \int_a^x \alpha(x)\beta(x)e^{-\int_a^x \beta(t)dt}dx$$

and so

$$R(x) \leq e^{\int_a^x \beta(t)dt}\int_a^x \alpha(x)\beta(x)e^{-\int_a^x \beta(t)dt}dx.$$

Plugging this back in we have

$$u(x) \leq \alpha(x) + e^{\int_a^x \beta(t)dt}\int_a^x \alpha(x)\beta(x)e^{-\int_a^x \beta(t)dt}dx.$$

This is the generalized conclusion of Gronwall's lemma. Note that if $\alpha$ and $\beta$ are constants the righthand side simplifies to

$$\alpha + \alpha\beta e^{\beta x}\int_a^x e^{-\beta x}dx = \alpha + \alpha\beta e^{\beta x}\frac{e^{-\beta a} - e^{\beta x}}{\beta} = \alpha e^{\beta(x-a)}$$

which is the original statement. $\square$

**Problem 2.** *Find a Lipschitz constant $L$ such that $|f(x, y) - f(x, z)| \leq L|y - z|$ for the given function $f$ in the given domain:*
*(a) $f(x, y) = x^2 + y^2$, $x \geq 0$, $|y| \leq 2$.*
*(b) $f(x, y) = xy^2$, $0 \leq x \leq 2$, $|y| \leq 2$.*

*Proof.* (a) Note that

$$|f(x, y) - f(x, z)| = |x^2 + y^2 - x^2 - z^2| = |y^2 - z^2| = |(y + z)(y - z)| = |y + z||y - z| \leq 4|y - z|$$

since we have $|y| \leq 2$ and $|z| \leq 2$ so $|y + z| \leq |y| + |z| \leq 4$.

(b) Note that

$$|f(x, y) - f(x, z)| = |xy^2 - xz^2| = |x||y^2 - z^2| \leq 2|y^2 - z^2| \leq 8|y - z|$$

using part (a) and the fact that $|x| \leq 2$. $\square$

**Problem 3.** *Consider the initial-value problem*

$$\dot{x} = \sqrt{x}, \quad x(0) = 0.$$

- *Find two solutions on the interval* $[0, 1]$.
- *Show that the function* $f(x) = \sqrt{x}$ *cannot satisfy a Lipschitz condition on an interval of the form* $[0, a)$ *with* $a > 0$.

*Proof.* Two solutions are $x_1(t) = 0$ and $x_2(t) = x^2/4$. It's trivial that $x_1(t)$ satisfies the equation. For $x_2(t)$, note that $\dot{x_2} = x/2 = \sqrt{x^2/4} = \sqrt{x_2}$ and $x_2(0) = 0$.

Suppose there exists $L$ such that $|\sqrt{x_1} - \sqrt{x_2}| \leq L|x_1 - x_2|$ for all $0 \leq x_1, x_2 < a$ for some $a > 0$. Then in particular, if $x_2 = 0$ and $x_1 > 0$, the condition simplifies to $\sqrt{x_1} \leq Lx_1$ or $1 \leq L\sqrt{x_1}$. But we can choose $x_1$ arbitrarily close to 0. In particular, choose $x_1$ such that $\sqrt{x_1} < 1/L$. Then $1 \leq L\sqrt{x_1} < 1$, a contradiction. Therefore no such $L$ exists. □

**Problem 4.** *Consider the equation*

$$\frac{dy}{dx} = \sqrt{x^2 - y^2}.$$

*Describe the domain where this equation is defined, assuming only real values are allowed. Show that on the domain*

$$D : |x| < K, x^2 - y^2 > \Delta$$

*where* $K$ *and* $\Delta$ *are positive constants with* $\Delta < K^2$, *the function* $\sqrt{x^2 - y^2}$ *satisfies a Lipschitz condition, and estimate the Lipschitz constant* $L$.

*Proof.* The equation is defined for all values such that $x^2 - y^2 \geq 0$, or $y^2 \leq x^2$.

The region $x^2 - y^2 > \Delta$ is the set of points outside of a hyperbola with a vertical axis. The region $|x| < K$ is a vertical strip centered at 0 with width $2K$. Then condition $\Delta < K^2$ guarantees that the intersection of these two regions is nonempty.

Now using the fact that $x^2 - \Delta > y^2$ and $|x| < K$, we have

$$\begin{aligned}
|f(x, y_1) - f(x, y_2)|^2 &= \left|\sqrt{x^2 - y_1^2} - \sqrt{x^2 - y_2^2}\right|^2 \\
&= x^2 - y_1^2 - x^2 + y_2^2 - 2\sqrt{x^4 - y_1^2 y_2^2 - x^2(y_1^2 + y_2^2)} \\
&= y_2^2 - y_1^2 - 2\sqrt{x^4 - y_1^2 y_2^2 - x^2(y_1^2 + y_2^2)} \\
&\leq (y_2 - y_1)(y_2 + y_1) \\
&\leq (y_2 - y_1)(2x^2 - 2\Delta) \\
&\leq 2(K^2 - \Delta)|y_2 - y_1| \\
&\leq 2(K^2 - \Delta)|y_2 - y_1|^2.
\end{aligned}$$

Thus $\sqrt{2(K^2 - \Delta)}$ gives an approximation for $L$. Note that we have $K^2 - \Delta > 0$ by assumption. □

**Problem 5.** *For the differential equation of the preceding problem with initial data* $y = 1$ *when* $x = -2$, *show that the interval of existence of the solution cannot extend to the origin, i.e., the solution cannot exist on the interval* $(-2, 0)$.

*Proof.* Since $dy/dx = \sqrt{x^2 - y^2}$ we know $dy/dx \geq 0$. Thus $y$ is a nondecreasing function. Given the initial data $y(-2) = 1$, suppose the solution $y(x)$ exists at $x = 0$. Then $dy/dx = \sqrt{0^2 - y^2} = \sqrt{-y^2}$. This is only defined for $y(0) = 0$. But $y$ is nondecreasing and $y(-2) = 1$, a contradiction. □

**Problem 6.** *Let $u(x) = x^3$ on $[-1, 1]$ and define $v(x) = -x^3$ on $[-1, 0]$ and $v(x) = +x^3$ on $(0, 1]$. Verify that $v$ is $C^2$ on $[-1, 1]$. Calculate $W(u, v; x)$. Are these functions linearly independent on $[-1, 1]$?*

*Proof.* Clearly $v$ is $C^2$ at all nonzero points. The derivatives of the left and right parts of $v$ are $-3x^2$ and $3x^2$ respectively. These functions agree at 0 so the limit as $x \to 0^-$ and the limit as $x \to 0^+$ are the same. A similar statement can be made about the derivatives $-6x$ and $6x$. Since both derivatives are continuous we know $v$ is $C^2$ on $[-1, 1]$.

For $x \in [-1, 0]$ we have $v(x) = x^3$ and $v'(x) = -3x^2$. Thus $u(x)v'(x) - u'(x)v(x) = x^3(-3x^2) - (3x^2)(-x^3) = 0$ for $x \in [-1, 0]$. For $x \in (0, 1]$ we have $u(x)v'(x) - u'(x)v(x) = 0$ since $v = u$ on this interval. The Wronskian is thus identically 0. The functions $u$ and $v$ are linearly independent on $[-1, 1]$ since there is no $c$ such that $cv(x) = u(x)$ for all $x \in [-1, 1]$. □

**Problem 7.** *Show that the three functions $\sin(x)$, $\sin(2x)$, $\sin(3x)$ are linearly independent on any nontrivial interval of the $x$ axis.*

*Proof.* Suppose there are $c_1$, $c_2$ and $c_3$ such that

$$c_1 \sin(x) + c_2 \sin(2x) + c_3 \sin(3x) = 0.$$

Then

$$c_1 \cos(x) + 2c_2 \cos(2x) + 3c_3 \cos(3x) = 0$$

and

$$c_1 \sin(x) + 4c_2 \sin(2x) + 9c_3 \sin(3x) = 0.$$

Pick $a$ in the given interval such that $\sin(a) \neq 0$. Then we have the matrix

$$\begin{pmatrix} \sin(a) & \sin(2a) & \sin(3a) \\ \cos(a) & 2\cos(2a) & 3\cos(3a) \\ -\sin(a) & -4\sin(2a) & -9\sin(3a) \end{pmatrix}.$$

The determinant of this matrix is

$$9\cos(3a)\sin(a)\sin(2a) - 16\cos(2a)\sin(a)\sin(3a) + 5\cos(2)\sin(2a)\sin(3a)$$

which simplifies to $-16\sin^6(a) \neq 0$. Since the matrix is invertible, the system of equations has only the solution $c_1 = c_2 = c_3 = 0$, so the functions are linearly independent. □

**Problem 8.** *Find bases of solutions for the following equations:*
*(a)* $u'' = 0$.
*(b)* $u'' + 2u' = 0$.
*(c)* $u'' + xu' = 0$.

*Proof.* (a) Clearly $u(x) = 1$ and $u(x) = x$ both satisfy the equation. These are linearly independent functions since $c_1 + c_2 x = 0$ evaluated at $x = 0$ shows $c_1 = 0$ and differentiating shows $c_2 = 0$.

(b) We see that $u(x) = 1$ and $u(x) = -\frac{1}{2}e^{-2x}$ are both solutions. Suppose we have $c_1 - c_2/2e^{-2x} = 0$. Then $c_1 - c_2/2 = 0$ and $c_1 - c_2/2e = 0$. Therefore $c_2/2(e - 1) = 0$ and since $e - 1 \neq 0$, $c_2 = 0$, thus $c_1 = 0$. Hence 1 and $-\frac{1}{2}e^{-2x}$ are a basis.

(c) Clearly $u(x) = 1$ is a solution. Define

$$f(x) = \sqrt{2}\int_0^{\frac{x}{\sqrt{2}}} e^{-t^2}\,dt.$$

Then $f'(x) = \sqrt{2}e^{-x^2/2}$ and $f'' = -x\sqrt{2}e^{-x^2/2}$. So we see $f$ satisfies the equation as well. If we have $c_1 + c_2 f(x) = 0$, then putting in 0 gives $c_1 = 0$ and putting in any $x > 0$ gives $c_2 = 0$. Thus 1 and $f$ are a basis for this equation. □

**Problem 9.** *For the equation*

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{du}{dx}\right)+2u=0$$

- *On what interval does the existence theorem guarantee a solution?*
- *Verify that $u_1 = x$ is a solution.*
- *Find a second solution in an interval containing the origin. How far can this interval extend on each side of the origin?*

*Proof.* First rewrite the equation by differentiating

$$0=\frac{d}{dx}u'-\frac{d}{dx}(x^2u')+2u=u''-2xu'-x^2u''+2u=u''(1-x^2)-2xu'+2u.$$

Assuming $x \neq \pm 1$ we have

$$u''-\frac{2x}{1-x^2}u'+\frac{2}{1-x^2}u=0.$$

The existence theorem guarantees a solution on some interval if the coefficients of $u'$ and $u$ are continuous on that interval. Then $(-1,1)$ is an interval on which $2x/(1-x^2)$ and $2/(1-x^2)$ are continuous.

For $u_1(x) = x$ we have $u_1' = 1$ and $u_1'' = 0$. Then $0 - 2x/(1-x^2) + (2/(1-x^2))x = 0$ so $u_1$ is indeed a solution. Now consider the function

$$f(x)=\frac{x}{2}(\log(1+x)-\log(1-x))-1.$$

Then

$$f'(x)=\frac{x}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)+\frac{1}{2}(\log(1+x)-\log(1-x))$$

and

$$f''(x)=\frac{x}{2}\left(\frac{1}{(1-x)^2}-\frac{1}{(1+x)^2}\right)+\left(\frac{1}{1+x}+\frac{1}{1-x}\right).$$

Now

$$-f(x)\frac{2}{1-x^2}=\frac{x(\log(1-x)-\log(1+x))+2}{1-x^2}$$

and

$$f'(x)\frac{2x}{1-x^2}=x^2\left(\frac{1}{(1+x)^2(1-x)}+\frac{1}{(1-x)^2(1+x)}\right)+x\frac{\log(1+x)-\log(1-x)}{1-x^2}.$$

Adding the last two equations together results in $f''(x)$, as desired. Note that $f(x)$ exists in the interval $|x| < 1$ since for any values larger than this, $\log(1 \pm x)$ will be undefined. □

**Problem 10.** *Let $u$ and $v$ be given $C^2$ functions on an interval $[a, b]$ whose Wronskian nowhere vanishes there. Show that there is a differential equation of the form (2.7) for which $u$ and $v$ form a basis of solutions.*

*Proof.* In the case that there did exist such an equation, we would have

$$u''+p(x)u'+q(x)u=0$$

and

$$v''+p(x)v'+q(x)v=0.$$

Rewrite this as

$$p(x)u'+q(x)u=-u''$$

and

$$p(x)v'+q(x)v=v''.$$

The coefficients of this system form the transpose of the Wronskian matrix, so the determinant is unchanged. In particular, since the Wronskian is nonzero everywhere on $[a, b]$, there is a unique solution for $p(x)$ and $q(x)$ for each $x \in [a, b]$. Use these values as a definition for $p(x)$ and $q(x)$ and it follows that $u$ and $v$ are solutions to the above differential equation. Since they're linearly independent by assumption, they're a basis by definition. $\square$