Kris Harper
MATH 25900
Homework 6 June 2, 2010

Problem 1 (17.1.2). *This exercise defines the connecting map δ_n in the Long Exact Sequence of Theorem 2 and proves it is a homomorphism. In the notation of Theorem 2 let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes, where for simplicity the cochain maps for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are all denoted by the same d.*
(a) If $c \in C^n$ represents the class $x \in H^n(\mathcal{C})$ show that there is some $b \in B^n$ with $\beta_n(b) = c$.
(b) Show that $d_{n+1}(b) \in \ker \beta_{n+1}$ and conclude that there is a unique $a \in A^{n+1}$ such that $\alpha_{n+1}(a) = d_{n+1}(b)$.
(c) Show that $d_{n+2}(a) = 0$ and conclude that a defines a class $\overline{a}$ in the quotient group $H^{n+1}(\mathcal{A})$.
(d) Prove that $\overline{a}$ is independent of the choice of b, i.e., if b' is another choice and a' is its unique preimage in A^{n+1} then $\overline{a} = \overline{a'}$, and that $\overline{a}$ is also independent of the choice of c representing the class x.
(e) Define $\delta_n(x) = \overline{a}$ and prove that δ_n is a group homomorphism from $H^n(\mathcal{C})$ to $H^{n+1}(\mathcal{A})$.

Proof. (a) Since our sequences are exact, we know β_n is surjective. Thus there exists $b \in B^n$ with $\beta_n(b) = c$.

(b) Note that $c \in \ker d_{n+1}$ by assumption so $d_{n+1}(c) = 0$. From the commutativity of the diagram we have $\beta_{n+1}d_{n+1}(b) = d_{n+1}\beta_n(b) = d_{n+1}(c) = 0$. Thus $d_{n+1}(b) \in \ker \beta_{n+1}$. Since $\ker \beta_{n+1} = \operatorname{im} \alpha_{n+1}$ we can write $d_{n+1}(b) = \alpha_{n+1}(a)$. By exactness, α_{n+1} is injective, so this a is unique.

(c) Note that by commutativity $\alpha_{n+2}d_{n+2}(a) = d_{n+2}\alpha_{n+1}(a) = d_{n+2}d_{n+1}(b) = 0$. Since α_{n+2} is injective, we must have $d_{n+2}(a) = 0$. Thus $a \in \ker d_{n+2}$ so it gives a class $\overline{a} \in \ker d_{n+2}/\operatorname{im} d_{n+1}$, or $H^{n+1}(\mathcal{A})$.

(d) Suppose we choose $b' \in B^n$ such that $\beta_n(b') = c$. then $\beta_n(b - b') = \beta_n(b) - \beta_n(b') = c - c = 0$ so $b - b' \in \ker \beta_n$. By exactness we can write $b - b' = \alpha_n(p)$ for some $p \in A^n$. Then by commutativity we know $\alpha_{n+1}d_{n+1}(p) = d_{n+1}\alpha_n(p) = d_{n+1}(b - b') = d_{n+1}(b) - d_{n+1}(b') = \alpha_{n+1}(a) - \alpha_{n+1}(a') = \alpha_{n+1}(a - a')$. Thus $d_{n+1}(p) = a - a'$ so $a - a' \in \operatorname{im} d_{n+1}$ showing that $\overline{a} = \overline{a'}$.

A different choice of c has the form $c + d_n(c')$ for some $c' \in C^{n-1}$. We know $c' = \beta_{n-1}(b')$ for some $b' \in B^{n-1}$. Then by commutativity $c + d_n(c') = c + d_n\beta_{n-1}(b') = \beta_n d_n(b') = \beta_n(b) + \beta_n d_n(b') = \beta_n(b + d_n(b'))$. Thus b gets replaced with $b + d_n(b')$ leaving $d_{n+1}(b)$ unchanged since $d_n d_{n+1}(b') = 0$. Thus a is also unchanged.

(e) Suppose $\delta_n(x_1) = \delta_n(\overline{c}) = \overline{a_1}$ and $\delta_n(x_2) = \delta_n(\overline{c}) = \overline{a_2}$ through elements b_1 and b_2 in B^n. Then $\beta_n(b_1 + b_2) = \beta_n(b_1) + \beta_n(b_2) = c_1 + c_2$ and $\alpha_{n+1}(a_1 + a_2) = \alpha_{n+1}(a_1) + \alpha_{n+1}(a_2) = d_{n+1}(b_1) + d_{n+1}(b_2) = d_{n+1}(b_1 + b_2)$. Thus we have $\delta_n(x_1 + x_2) = \overline{a_1} + \overline{a_2}$. □

Problem 2 (17.1.3). *Suppose*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \end{tikzcd}$$

is a commutative diagram of R-modules with exact rows.
(a) If $c \in \ker h$ and $\beta(b) = c$ prove that $g(b) \in \ker \beta'$ and conclude that $g(b) = \alpha'(a')$ for some $a' \in A'$.
(b) Show that $\delta(c) = a' \bmod \text{image } f$ is a well defined R-module homomorphism from $\ker h$ to the quotient $A'/\text{image } f$.
*(c) (*The Snake Lemma*) Prove there is an exact sequence*

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

where $\operatorname{coker} f$ *(the* cokernel *of f) is $A'/(\text{image} f)$ and similarly for* $\operatorname{coker} g$ *and* $\operatorname{coker} h$.
(d) Show that if α is injective and β' is surjective (i.e., the two rows in the commutative diagram above can be extended to short exact sequences) then the exact sequence in (c) can be extended to the exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0\,.$$

Proof. (a) By commutativity and the fact that $c \in \ker h$ we know $\beta' g(b) = h\beta(b) = h(c) = 0$. Thus $g(b) \in \ker \beta'(b)$. By exactness $\ker \beta' = \operatorname{im} \alpha'$ so we can write $\beta'(b) = \alpha'(a')$ for some $a' \in A'$.

(b) We need to show the class represented by a' in the quotient by $\operatorname{im} f$ doesn't depend on the choice of $b \in B$ and that this map is actually a homomorphism. Both of these proofs are nearly identical to the corresponding statements in parts (d) and (e) from Problem 1.

(c) Let's label the maps as follows

$$\ker f \xrightarrow{\gamma} \ker g \xrightarrow{\epsilon} \ker h \xrightarrow{\delta} \operatorname{coker} f \xrightarrow{\zeta} \operatorname{coker} g \xrightarrow{\xi} \operatorname{coker} h\,.$$

Note that if $a \in \ker f$ then $g\alpha(a) = \alpha' f(a) = \alpha'(0) = 0$ so $\alpha(a) \in \ker g$. Thus α restricted to $\ker f$ gives our map $\gamma : \ker f \to \ker g$. A similar argument holds to show that β restricted to $\ker g$ gives $\epsilon : \ker g \to \ker h$. Now suppose $\overline{a'} \in \operatorname{coker} f$ so that $\alpha'(a') \in B'$. Note that if $a' \in \operatorname{im} f$ then $\alpha'(a') \in \operatorname{im} g$ by commutativity. Thus we have a map $\zeta : \operatorname{coker} f \to \operatorname{coker} g$ and similarly a map $\xi : \operatorname{coker} g \to \operatorname{coker} h$.

Let $b \in \operatorname{im} \gamma$ so that $\gamma(a) = b$. By how γ is defined we know $b \in \operatorname{im} \alpha$ as well so $b \in \ker \beta$ by exactness. Since ϵ is a restriction of β, we must have $b \in \ker \epsilon$ as well so that $\operatorname{im} \gamma \subseteq \ker \epsilon$. Conversely, suppose $b \in \ker \epsilon$. Then $b \in \ker \beta$ as well so $b \in \operatorname{im} \alpha$ by exactness. Since $b \in \ker g$ we see that $b \in \operatorname{im} \gamma$ as well so $\operatorname{im} \gamma = \ker \epsilon$. This shows the sequence is exact at $\ker g$.

Now let $\overline{b'} \in \operatorname{im} \zeta$. Then $b' \in \operatorname{im} \alpha'$ by how we defined ζ, so by exactness we know $b' \in \ker \beta'$. But then by the definition of ξ we must have $\overline{b'} \in \ker \xi$. Thus $\operatorname{im} \zeta \subseteq \ker \xi$. Conversely, suppose $\overline{b'} \in \ker \xi$. Then $b' \in \ker \beta'$ and $b' \in \operatorname{im} \alpha'$ as well. This means $\overline{b'} \in \operatorname{im} \zeta$ because ζ is induced from α'. Therefore $\ker \zeta = \operatorname{im} \xi$ and the sequence is exact at $\operatorname{coker} g$.

Let $c \in \operatorname{im} \epsilon$ so that $\epsilon(b) = c$ for $b \in \ker g$. From definition of δ we know $\delta(c)$ is the unique class $\overline{a'} \in \operatorname{coker} f$ such that $\alpha'(a') = g(b)$. But since $b \in \ker g$ we know $\alpha'(a') = 0$ and since α' is injective by exactness, we know $a' = 0$. Thus $c \in \ker \delta$ and $\operatorname{im} \epsilon \subseteq \ker \delta$. Conversely, suppose $c \in \ker \delta$ so that $\delta(c) = \overline{a'}$ with $\alpha'(a') = 0$ (since α' is injective). Then note that $\beta(b) = c$ and $\alpha'(a') = g(b) = 0$, so $b \in \ker g$ and we also have $\epsilon(b) = c$. Thus $c \in \operatorname{im} \epsilon$ and we have $\operatorname{im} \epsilon = \ker \delta$ so the sequence is exact at $\ker h$.

Finally suppose $\overline{a'} \in \operatorname{im} \delta$ with $\delta(c) = \overline{a'}$. Then we know $\alpha'(a') = g(b)$ for some $b \in B$ with $\beta(b) = c$. Since α' induces ζ we see that $\zeta(\overline{a'}) = \overline{g(b)}$ so $\zeta(\overline{a'}) = 0$ since it's in the image of g. Thus $\operatorname{im} \delta \subseteq \ker \zeta$. Conversely, suppose $\overline{a'} \in \ker \zeta$ so that $\zeta(\overline{a'}) = \overline{b'}$ with $b' \in \operatorname{im} g$. From the definition of δ and ζ we know $\alpha'(a') = g(b)$ where $g(b) = b'$, and furthermore if we take $\beta(b) = c$ then we must have $\delta(c) = \overline{a'}$. Thus $\overline{a'} \in \operatorname{im} \delta$ and $\operatorname{im} \delta = \ker \zeta$ so the sequence is exact at $\operatorname{coker} f$.

(d) Using part (c) all we need to show is that γ is injective and ξ is surjective. But since these maps are induced by α and β' which are now assumed to be injective and surjective respectively, we get this immediately. □

Problem 3. *Let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes. Prove that if any two of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are exact, then so is the third.*

Proof. Given the exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ we know we get a long exact sequence on cohomology groups $0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \to H^1(\mathcal{A}) \to \dots$. Assuming two of the cochain complexes are exact we know all but every third term in this sequence is 0, which forces every term to be 0 by exactness. Thus all three cochain complexes have trivial homology groups so $\ker \delta_n = \operatorname{im} \delta_{n-1}$ and all three cochain complexes are exact. □

Problem 4. *Assume you have a commutative ring R and two chain complexes A, B of R-modules. We wish to construct a new chain complex $A \otimes B$ with constituent R-modules $(A \otimes B)_k = \bigoplus_{i=0}^{k} (A_i \otimes B_{k-i})$. As a matter of convention, let $a \in A_i$, $b \in B_j$.*

- *i) Defining the differential: Show that the map given on generators by $a \otimes b \mapsto da \otimes b + a \otimes db$ need not give the structure of a chain complex on $A \otimes B$, but that the map given by $a \otimes b \mapsto da \otimes b + (-1)^i a \otimes db$ does.*

Kris Harper
MATH 25900

- *ii) The twist map: With the differential as above, check that the map given by $a \otimes b \mapsto b \otimes a$ need not be map of chain complexes, but that $a \otimes b \mapsto (-1)^{ij} b \otimes a$ gives rise to an isomorphism $t : A \otimes B \to B \otimes A$ of chain complexes.*
- *iii) Chain homotopy: Let I denote the chain complex $0 \to R \to R \to 0$, where the two R's are in degrees 1, 0. First check that $I \otimes A$ at level i is given by $A_i \oplus A_{i-1}$. Next, check that a map $I \otimes A \to B$ of chain complexes is given by maps $h_i : A_i \to B_i$ and $s_i : A_i \to B_{i+1}$ such that the h_i give a map of chain complexes $A \to B$, and that the formula $d \circ s_i + s_{i-1} \circ d = h_i$ holds (here we are using h_i to represent the difference $f_i - g_i$ of two chain-homotopic maps).*

Proof. i) We have

$$d^2(a \otimes b) = d(da \otimes b + (-1)^i a \otimes db) = (d^2 a \otimes b) + ((-1)^{i-1} da \otimes db) + ((-1)^i da \otimes db) + (a \otimes d^2 b) = 0 + ((-1)^{i-1} da + (-1)^i da) \otimes db + 0 = 0$$

Without the sign term we have

$$d(da \otimes b + a \otimes db) = (d^2 a \otimes b) + (da \otimes db) + (da \otimes db) + (a \otimes d^2 b) = 2(da \otimes db).$$

In particular, if $da \neq 0$ or $db \neq 0$ then this is not a differential.

(ii) We have

$$d((-1)^{ij} b \otimes a) = ((-1)^{ij} db \otimes a) + ((-1)^j (-1)^{ij} b \otimes da) = ((-1)^{ij} db \otimes a) + ((-1)^{(i+1)j} b \otimes da)$$

and

$$t(da \otimes b + (-1)^i a \otimes db) = ((-1)^{(i-1)j} b \otimes da) + ((-1)^{i(j-1)} db \otimes (-1)^i a) = ((-1)^{(i+1)j} b \otimes da) + ((-1)^{ij} db \otimes a).$$

Since the two terms are equal we have $dt = td$ so that t is a chain map. We see that t is an isomorphism between $A \otimes B$ and $B \otimes A$ because it takes a generator to plus or minus a generator. If the sign term isn't present we have

$$d(b \otimes a) = db \otimes a + (-1)^j b \otimes da$$

and

$$t(da \otimes b + (-1)^i a \otimes db) = b \otimes da + db \otimes (-1)^i a.$$

So we need $(-1)^i a = a$ and $(-1)^j b = b$, for the maps to commute which won't happen for $i > 0$.

iii) We have

$$(I \otimes A)_i = \bigoplus_{k=0}^{i} I_k \otimes A_{i-k} = (A_i \otimes R) \oplus (A_{i-1} \otimes R) \oplus 0 \oplus \cdots \oplus 0 = A_i \oplus A_{i-1}.$$

Let $\varphi : I \otimes A \to B$ be a chain map. Note that $\varphi_i : (I \otimes A)_i \to B_i$ can be defined in terms of its components so that we have maps $h_i : A_i \to B_i$ and $s_i : A_{i-1} \to B_i$. By assumption $d\varphi = \varphi d$ and from this we have $ds_i + s_{i-1} d = h_i$. □