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# Sheet 1: Basics

**Definition 1 (Empty Set)** *The empty set is denoted by $\emptyset$; it contains no elements*

**Definition 2 (Element)** *Instead of saying "$A$ contains $a$," we say that $a$ is an element of $A$, and write this as $a \in A$. For the converse statement, that $a$ is not an element of $A$, we write $a \notin A$.*

**Exercise 3** *Is it true that every element of the empty set is a whistling, flying purple cow?*

Yes. Since the empty set has no elements, it is vacuously true that all elements are whistling, flying purple cows.

**Definition 4 (Subset)** *Let $A$ and $B$ be sets. If each element of $A$ is also an element of $B$, we say that $A$ is a subset of $B$. In symbols $A \subseteq B$.*

**Exercise 5** *How many subsets does the empty set have?*

*Proof.* Let $A \subseteq \emptyset$. Then every element of $A$ is in $\emptyset$. But $\emptyset$ has no elements so $A$ must have no elements. Therefore $A = \emptyset$. So the empty set has only one subset, itself. $\square$

**Exercise 6** *Let $A_n = \{1, 2, ..., n\}$. How many subsets does $A_n$ have?*

$A_n$ has $2^n$ subsets.

*Proof.* We use induction on $n$. We see that the statement is true for $n = 1$ since $\{1\}$ has one element and its only subsets are $\{1\}$ and $\emptyset$ and $2 = 2^1$. Let $S_k$ be the set $\{1, 2, 3, ..., k\}$. Then we assume $S_k$ has $2^k$ subsets and show that $S_{k+1}$ has $2^{k+1}$ subsets. Consider a set $A$ such that $A \subseteq S_{k+1}$. Then we see that either $(k + 1) \in A$ or $(k + 1) \notin A$. Let $(k + 1) \notin A$. Then $A \subseteq S_k$. But then there exists a set $A \cup \{k + 1\} \subseteq S_{k+1}$ for every $A \subseteq S_k$. Therefore, for every subset $A$ of $S_k$, $A$ and $A \cup \{k + 1\}$ are subsets of $S_{k+1}$. Thus, $S_{k+1}$ has at least $2 \cdot 2^k = 2^{k+1}$ subsets since there are $2^k$ subsets of $S_k$.

But suppose there are more than $2^{k+1}$ subsets of $S_{k+1}$ Then there exists a subset $B$ of $S_{k+1}$ such that $B \nsubseteq S_k$ and $B \backslash \{k + 1\} \nsubseteq S_k$. Then there exists a $b \in B$ such that $b \notin S_k$ and $b \neq k + 1$. Since $B \subseteq S_{k+1}$, $b \in S_{k+1}$. But $S_k \cap S_{k+1} = S_k$ which means that $S_{k+1}$ contains every element in $S_k$ as well as $k + 1$. Therefore, $b \in S_k$ or $b = k + 1$. This is a contradiction. Therefore, $S_{k+1}$ must have exactly $2^{k+1}$ subsets. $\square$

**Exercise 7** *What is the number of subsets of $A_n$ that contain exactly 2 elements?*

$\binom{n}{2} = \frac{n(n-1)}{2}$ subsets.

*Proof.* We note that in $A_1 = \{1\}$ there is only one element and so there are no subsets with 2 elements. So the theorem holds for $n = 1$ since $\frac{1(1-1)}{2} = 0$. We now assume that $A_n$ has $\frac{n(n-1)}{2}$ subsets of size 2 and use induction on $n$. The set $A_{n+1}$ has $n + 1$ elements and $A_{n+1} \backslash A_n = \{n + 1\}$. So for every $k \in A_n$ there exists a subset $\{k, n + 1\} \subseteq A_{n+1}$. Since there are $n$ elements in $A_n$ we have $n$ more subsets of size 2 in $A_{n+1}$.

Now suppose there are more than $n$ subsets of size 2 added to $A_{n+1}$. Then there exists some subset $\{a, b\} \subseteq A_{n+1}$ which we have not yet considered. But then it is not the case that $\{a, b\} \subseteq A_n$ and it is not the case that one element of $\{a, b\}$ is in $A_n$ and the other is $n + 1$. Thus, because $A_{n+1} = A_n \cup \{n + 1\}$, we see that $a, b \notin A_{n+1}$ and this is a contradiction. Therefore there are exactly $n$ subsets of size 2 added to $A_{n+1}$.

So there are $\frac{n(n-1)}{2} + n = \frac{n^2-n+2n}{2} = \frac{n(n+1)}{2} = \frac{(n+1)(n+1-1)}{2}$ subsets of $A_{n+1}$ of size 2. So the statement holds for $n = 1$ and $n + 1$ when it holds for $n$ so it must hold for all $n \in \mathbb{N}$. $\square$

**Definition 8 (Union, Intersection, Difference and Direct Product)** *If $A$ and $B$ are sets then:*

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

*the union of $A$ and $B$;*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

*the intersection of $A$ and $B$ and*

$$A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}$$

*the difference of $A$ and $B$. If $B = \{b\}$ is the set consisting of a single element $b$, we will write $A\backslash b$ rather than $A\backslash\{b\}$. Finally*

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

*the set of ordered pairs from $A$ and $B$. The set $A \times B$ is called the direct product of $A$ and $B$.*

**Definition 9 (Union and Intersection of Many Sets)** *Let $S$ be a set consisting of sets. Then the intersection and union of $S$ is defined as follows:*

$$\bigcup_{A \in S} A = \{x \mid \text{there exists } A \in S \text{ such that } x \in A\}$$

*and*

$$\bigcap_{A \in S} A = \{x \mid \text{for all } A \in S \text{ we have } x \in A\}.$$

**Theorem 10** *Let $X$ be a set and let $S$ be a set consisting of subsets of $X$. Then*

$$X \backslash \left( \bigcup_{A \in S} A \right) = \bigcap_{A \in S} (X \backslash A)$$

*and*

$$\bigcup_{A \in S} (X \backslash A) = X \backslash \left( \bigcap_{A \in S} A \right).$$

*Proof.* Let $X$ be a set and let $S$ be a set consisting of subsets of $X$.

Let $a \in X \backslash \left(\bigcup_{A\in S} A\right)$. Then $a \in X$ and $a \notin \left(\bigcup_{A\in S} A\right)$. That is, $a \notin A$ for all sets $A \in S$. Thus, for all sets $A \in S$, $a \in X\backslash A$. Since this is true for all sets $A \in S$ we may write $x \in \bigcap_{A\in S} (X\backslash A)$. Therefore $X\backslash \left(\bigcup_{A\in S} A\right) \subseteq \bigcap_{A\in S} (X\backslash A)$.

Let $a \in \bigcap_{A\in S} (X\backslash A)$. Thus, $a \in X\backslash A$ for every set $A \in S$. Since $a \notin A$ for all $A \in S$ we can state that $a \notin \bigcup_{A\in S} A$. But since $a \in X$, we can now write $a \in X\backslash \left(\bigcup_{A\in S} A\right)$. Therefore $\bigcap_{A\in S} (X\backslash A) \subseteq X\backslash \left(\bigcup_{A\in S} A\right)$.

Let $a \in \bigcup_{A\in S}(X\backslash A)$. Thus, $a \in X\backslash A$ for at least one $A \in S$. Since $a \notin A$ for at least one $A \in S$, then we can write $a \notin \bigcap_{A\in S} A$. But since $a \in X$, we can now write $a \in X\backslash\left(\bigcap_{A\in S} A\right)$. Therefore $\bigcup_{A\in S}(X\backslash A) \subseteq X\backslash\left(\bigcap_{A\in S} A\right)$.

Let $a \in X\backslash\left(\bigcap_{A\in S} A\right)$. Thus, $a \in X$, but $a \notin A$ for at least one $A \in S$. In other words, $a \in X\backslash A$ for at least one $A \in S$. Thus we may write $a \in \bigcup_{A\in S}(X\backslash A)$. Therefore $X\backslash\left(\bigcap_{A\in S} A\right) \subseteq \bigcup_{A\in S}(X\backslash A)$.

Since we have shown both inclusions for both of De Morgan's Laws, we can conclude the result is true. □

**Definition 11 (Function)** *A function $f : A \to B$ is defined as a subset of $A \times B$ such that for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$. Instead of $(a, b) \in f$ we will use the notation $f(a) = b$.*

**Definition 12 (Domain and Range)** *The domain of $f$ is $A$. The range of $f$ (image under $f$) is*

$$f(A) = \{f(a) \mid a \in A\}.$$

**Definition 13 (Surjective, Injective and Bijective)** *A function $f : A \to B$ is surjective (onto) if $f(A) = B$. It is injective (1 to 1) if for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$. It is bijective if it is surjective and injective.*

**Definition 14 (Inverse Function)** *Let $f : A \to B$ be a bijection. Then the inverse of $f$, $f^{-1} : B \to A$ is defined by*

$$(b, a) \in f^{-1} \text{ if and only if } (a, b) \in f.$$

**Definition 15 (Image and Preimage)** *Let $f : A \to B$ be a function. Let $X \subseteq A$. Then the image of $X$ under $f$ is*

$$f(X) = \{f(x) \mid x \in A\}.$$

*Let $Y \subseteq B$. Then the preimage of $Y$ under $f$ is*

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

**Exercise 16** *$f^{-1}(f(X)) = X$ for all $X \subseteq A$.*

False. Let $A = \{1, 2\}$ and $B = \{3\}$ such that $f(1) = f(2) = 3$. Then $f(\{1\}) = \{3\}$. But $f^{-1}(\{3\}) = \{1, 2\} \neq \{1\}$. We can prove the statement if we assume $f$ is injective.

*Proof.* Let $f : A \to B$ be an injective function. Let $X \subseteq A$. Suppose $a \in f^{-1}(f(X))$. Then $a \in A$ and $f(a) \in f(X)$ which means there exists some $b \in X$ such that $f(b) = f(a)$. But $f$ is injective and so $a = b$ and $a \in X$. Therefore $f^{-1}(f(X)) \subseteq X$. Now suppose $a \in X$. Then $f(a) \in f(X)$ and $a \in f^{-1}(f(X))$. Thus $X \subseteq f^{-1}(f(X))$. Since both sets are subsets of each other, they are equal. □

**Exercise 17** *$f(f^{-1}(Y)) = Y$ for all $Y \subseteq B$.*

False. Let $A = \{1, 2\}$ and $B = \{3, 4\}$ such that $f(1) = f(2) = 3$. Then $f^{-1}(B) = \{1, 2\}$. But $f(\{1, 2\}) = \{3\} \neq B$. We can prove the statement if we assume $f$ is surjective.

*Proof.* Let $f : A \to B$ be a surjective function. Let $Y \subseteq B$. Suppose $b \in f(f^{-1}(Y))$. Then there exists an $a \in f^{-1}(Y)$ such that $f(a) = b$. Then $a \in f^{-1}(Y)$ and so $f(a) \in Y$. Thus $b \in Y$ and so $f(f^{-1}(Y)) \subseteq Y$. Now let $b \in Y$. Since $f$ is surjective, $f(A) = B$ and so there exists an element $a \in A$ such that $f(a) = b$. Thus $f(a) \in Y$. Then $a \in f^{-1}(Y)$ and $f(a) \in f(f^{-1}(Y))$. Thus, $Y \subseteq f(f^{-1}(Y))$. Again, since both sets are subsets of each other, they must be equal. □

**Exercise 18** $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$ *for all* $X_1$ *and* $X_2 \subseteq A$.

False. Let $f : A \to B$ be a function such that $X_1 = \{1,3\}$ and $X_2 = \{2,3\}$ are subsets of $A$. Let $f(1) = f(2) = 10$ and $f(3) = 11$. Then we see that $f(X_1) = \{10,11\}$ and $f(X_2) = \{10,11\}$ and so $f(X_1) \cap f(X_2) = \{10,11\}$. But $X_1 \cap X_2 = \{3\}$ and so $f(X_1 \cap X_2) = \{11\}$. We can prove the statement if we assume $f$ is injective.

*Proof.* Let $f : A \to B$ be an injective function and let $X_1$ and $X_2$ be subsets of $A$. Suppose $b \in f(X_1 \cap X_2)$. Then there exists an $a \in X_1 \cap X_2$ such that $f(a) = b$. Thus, $a \in X_1 \cap X_2$ and so $a \in X_1$ and $a \in X_2$. Therefore $f(a) \in f(X_1)$ and $f(a) \in f(X_2)$ and so $f(a) \in f(X_1) \cap f(X_2)$ and $b \in f(X_1) \cap f(X_2)$. Thus, $f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$.

Now suppose $a \in f(X_1) \cap f(X_2)$. Then $a \in f(X_1)$ and $a \in f(X_2)$. Then there exists a $b \in X_1$ such that $f(b) = a$ and a $c \in X_2$ such that $f(c) = a$. But since $f$ is injective and $f(b) = f(c)$, then $b = c$ and so $b \in X_1$ and $b \in X_2$. Thus $b \in X_1 \cap X_2$ and so $f(b) \in f(X_1 \cap X_2)$ and $a \in f(X_1 \cap X_2)$. Therefore $f(X_1) \cap f(X_2) \subseteq f(X_1 \cap X_2)$. Since both sets are subsets of each other, they are equal. $\square$

**Exercise 19** $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$ *for all* $Y_1$ *and* $Y_2 \subseteq B$.

*Proof.* Let $f : A \to B$ be a function and let $Y_1$ and $Y_2$ be subsets of $B$. Let $b \in f^{-1}(Y_1 \cap Y_2)$. Then $f(b) \in Y_1 \cap Y_2$ and so $f(b) \in Y_1$ and $f(b) \in Y_2$. Thus $b \in f^{-1}(Y_1)$ and $b \in f^{-1}(Y_2)$ and so $b \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Therefore $f^{-1}(Y_1 \cap Y_2) \subseteq f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

Now let $b \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Then $b \in f^{-1}(Y_1)$ and $b \in f^{-1}(Y_2)$. Thus $f(b) \in Y_1$ and $f(b) \in Y_2$ and so $f(b) \in Y_1 \cap Y_2$. Therefore $b \in f^{-1}(Y_1 \cap Y_2)$. Thus $f^{-1}(Y_1) \cap f^{-1}(Y_2) \subseteq f^{-1}(Y_1 \cap Y_2)$. Since both sets are subsets of each other, they are equal. $\square$

**Exercise 20** $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$ *for all* $X_1, X_2 \subseteq A$.

*Proof.* Let $f : A \to B$ be a function and let $X_1$ and $X_2$ be subsets of $A$. Let $a \in f(X_1 \cup X_2)$. Then there exists a $b \in X_1 \cup X_2$ such that $f(b) = a$. Then $b \in X_1 \cup X_2$ which means $b \in X_1$ or $b \in X_2$. Thus $f(b) \in f(X_1)$ or $f(b) \in f(X_2)$ and so $f(b) \in f(X_1) \cup f(X_2)$ and $a \in f(X_1) \cup f(X_2)$. Therefore $f(X_1 \cup X_2) \subseteq f(X_1) \cup f(X_2)$.

Now let $a \in f(X_1) \cup f(X_2)$. Then $a \in f(X_1)$ or $a \in f(X_2)$. So there exists a $b \in X_1$ and $c \in X_2$ such that $a = f(b) = f(c)$ and $f(b) \in f(X_1)$ or $f(c) \in f(X_2)$. Thus $b \in X_1$ or $c \in X_2$. Therefore $b, c \in X_1 \cup X_2$ and so $f(b), f(c) \in f(X_1 \cup X_2)$ and $a \in f(X_1 \cup X_2)$. Thus $f(X_1) \cup f(X_2) \subseteq f(X_1 \cup X_2)$. Since both sets are subsets of each other, they are equal. $\square$

**Exercise 21** $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$ *for all* $Y_1, Y_2 \subseteq B$.

*Proof.* Let $f : A \to B$ be a function and let $Y_1$ and $Y_2$ be subsets of $B$. Let $a \in f^{-1}(Y_1 \cup Y_2)$. Then $f(a) \in Y_1 \cup Y_2$ which means $f(a) \in Y_1$ or $f(a) \in Y_2$. Thus $a \in f^{-1}(Y_1)$ or $a \in f^{-1}(Y_2)$ and so $a \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$. Therefore $f^{-1}(Y_1 \cup Y_2) \subseteq f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

Now let $a \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$. Thus $a \in f^{-1}(Y_1)$ or $a \in f^{-1}(Y_2)$ which means $f(a) \in Y_1$ or $f(a) \in Y_2$. Therefore $f(a) \in Y_1 \cup Y_2$ and so $a \in f^{-1}(Y_1 \cup Y_2)$. Thus $f^{-1}(Y_1) \cup f^{-1}(Y_2) \subseteq f^{-1}(Y_1 \cup Y_2)$. Since both sets are subsets of each other, they are equal. $\square$

**Theorem 22** $f(f^{-1}(f(X))) = f(X)$ *for all* $X \subseteq A$.

*Proof.* Let $f : A \to B$ be a function and let $X \subset A$. Let $a \in f(f^{-1}(f(X)))$. Then there exists some $b \in f^{-1}(f(X))$ such that $a = f(b)$. Since $b \in f^{-1}(f(X))$ we have $f(b) \in f(X)$ and so $a \in f(X)$. Thus $f(f^{-1}(f(X))) \subseteq f(X)$. Now let $a \in f(X)$. Then there exists a $b \in X$ such that $f(b) = a$. Since $f(b) \in f(X)$, $b \in f^{-1}(f(X))$. But then $f(b) \in f(f^{-1}(f(X)))$. Thus $X \subseteq f(f^{-1}(f(X)))$. Since the sets are subsets of each other, they are equal. □

**Theorem 23** $f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y)$ *for all* $Y \subseteq B$.

*Proof.* Let $f : A \to B$ be a function and let $Y \subseteq B$. Let $a \in f^{-1}(f(f^{-1}(Y)))$. Then $f(a) \in f(f^{-1}(Y))$ and so there exists a $b \in f^{-1}(Y)$ such that $f(a) = f(b)$. But then $f(b) \in Y$ and so $f(a) \in Y$ and $a \in f^{-1}(Y)$. Thus $f^{-1}(f(f^{-1}(Y))) \subseteq f^{-1}(Y)$. Now let $a \in f^{-1}(Y)$. Then $f(a) \in f(f^{-1}(Y))$ and $a \in f^{-1}(f(f^{-1}(Y)))$. Thus, $f^{-1}(Y) \subseteq f^{-1}(f(f^{-1}(Y)))$ and since the sets are subsets of each other, they must be equal. □

**Problem 24** *Try to define the direct product of infinitely many sets.*

For the sets $A_1, A_2, A_3 \dots$, we define the direct product of the sets as

$$A_1 \times A_2 \times A_3 \cdots = \{(a_1, a_2, a_3, \dots) \mid a_i \in A_i \text{ for every } i \in \mathbb{N}\}$$

**Definition 25 (Equivalence Relation)** *Let $A$ be a set. Then $\sim \subseteq A \times A$ is an equivalence relation if the following hold:*
*1) For all $a \in A$ we have $a \sim a$ (reflexivity);*
*2) For all $a, b \in A$, if $a \sim b$ then $b \sim a$ (reflexivity);*
*3) For all $a, b, c \in A$, if $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity).*

**Exercise 26** *$L$ is the set of lines on the plane. For $a$, $b \in L$ let $a \sim b$ if $a$ and $b$ are parallel.*

*Proof.* We see that $a \sim b$ is reflexive because every line $a$ has the same slope as itself and is therefore parallel to itself. We also see that it is symmetric as two lines which are parallel have the same slope and so $a \sim b$ implies $b \sim a$. Additionally, if we take three lines $a$, $b$ and $c$ such that $a$ has the same slope as $b$ and $b$ has the same slope as $c$ then $a$ must have the same slope as $c$ and so $a \sim b$ and $b \sim c$ implies $a \sim c$. Since all three conditions are met, the relation is an equivalence relation. □

**Exercise 27** *For $a$, $b \in L$ let $a \sim b$ if $a$ and $b$ intersect each other.*

This is not an equivalence relation since it fails the transitive property. If we take two lines $a$ and $c$ to be parallel, then a line $b$ may intersect both of them, but it doesn't imply that $a$ intersects $c$.

**Exercise 28** *For $a$, $b \in \mathbb{Z}$ let $a \sim b$ if $a - b$ is even.*

*Proof.* We see that $\sim$ is reflexive since for some $a \in \mathbb{Z}$ $a - a = 0$ which is even so $a \sim a$. Also, if $a \sim b$ for $a, b \in \mathbb{Z}$ then $a - b = 2k$ for some $k \in \mathbb{Z}$ and so $b - a = -2k = 2(-k)$. Since $-k \in \mathbb{Z}$, $b - a$ is even and so $b \sim a$. Finally, if $a \sim b$ and $b \sim c$, then $a - b = 2k$ and $b - c = 2l$ for some $k, l \in \mathbb{Z}$. Then adding the equations we have $a - c = 2k + 2l = 2(k + l)$. Since $k + l \in \mathbb{Z}$, $a - c$ is even and $a \sim c$. Since the relation satisfies all three properties, it is an equivalence relation. □

**Exercise 29** *For $a$, $b \in \mathbb{Z}$ let $a \sim b$ if $a - b$ is odd.*
This relation fails the reflexive test since for all $a \in \mathbb{Z}$, $a - a = 0$ which is not odd and so $a \nsim a$.

**Exercise 30** *For $a$, $b \in \mathbb{Z}$ let $a \sim b$ if $a - b$ is divisible by 7.*

*Proof.* We see that $\sim$ is reflexive since for some $a \in \mathbb{Z}$ $a - a = 0$ which is divisible by 7 so $a \sim a$. Also, if $a \sim b$ for $a, b \in \mathbb{Z}$ then $a - b = 7k$ for some $k \in \mathbb{Z}$ and so $b - a = -7k = 7(-k)$. Since $-k \in \mathbb{Z}$, $b - a$ is divisible by 7 and so $b \sim a$. Finally, if $a \sim b$ and $b \sim c$, then $a - b = 7k$ and $b - c = 7l$ for some $k, l \in \mathbb{Z}$. Then adding the equations we have $a - c = 7k + 7l = 7(k + l)$. Since $k + l \in \mathbb{Z}$, $a - c$ is divisible by 7 and $a \sim c$. Since the relation satisfies all three properties, it is an equivalence relation. $\square$

**Exercise 31** *Do 2) and 3 imply 1)?*

No. Take the closed interval $[m; n]$ and for $a, b \in [m; n]$ define $a \sim b$ if there exists a region $R \subseteq [m; n]$ such that $a, b \in R$. Then $a \nsim a$ for some $a \in [m; n]$ because there are no points $p \in [m; n]$ such that $p < m$. Thus every region $(p; q)$ such that $m \in (p; q)$ contains a point $x$ such that $p < x < m$ because regions are nonempty. And so $x \notin [m; n]$ and $(p; q) \nsubseteq [m; n]$. If $a \sim b$ then there exists a region $R \subseteq [m; n]$ such that $a, b \in R$ and so $b, a \in R$ and $b \sim a$. And for $a, b, c \in [m; n]$ if $a \sim b$ and $b \sim c$, then there exist regions $R_1$ and $R_2$ such that $a, b \in R_1$ and $b, c \in R_2$. Then let $R_3 = R_1 \cup R_2$. We see that the region $R_3 \subseteq [m; n]$ and contains $a$ and $c$ so $a \sim c$ (5.5). The proof to show $R_3$ is a region containing $a$ and $c$ is found on Sheet 5.

**Definition 32 (Equivalence Class)** *Let $A$ be a set and let $\sim$ be an equivalence relation on $A$. Then for $a \in A$, the $\sim$-equivalence class of $a$ is defined as:*

$$\overline{a} = \{x \in A \mid a \sim x\}.$$

**Theorem 33** *We have*

$$\bigcup_{a \in A} \overline{a} = A$$

*Furthermore, for all $a, b \in A$ we have*

$$\overline{a} = \overline{b} \text{ or } \overline{a} \cap \overline{b} = \emptyset.$$

*Proof.* Let $A$ be a set and let $\sim$ be an equivalence relation on $A$. Let $x \in \bigcup_{a \in A} \overline{a}$. Then $x \in \overline{a}$ for some $a \in A$. By definition, $x \in \{m \in A \mid a \sim m\}$ and so $x \in A$. Therefore, $\bigcup_{a \in A} \overline{a} \subseteq A$. Now suppose $x \in A$. Since $\sim$ is an equivalence relation on $A$, we know $x \sim x$ and so $x \in \overline{x}$. Thus $x \in \bigcup_{a \in A} \overline{a}$ and $A \subseteq \bigcup_{a \in A} \overline{a}$.

Suppose that there exist $a, b \in A$ such that $\overline{a} \neq \overline{b}$ and $\overline{a} \cap \overline{b} \neq \emptyset$. Then there exists an $x$ such that $x \in \overline{a}$ and $x \in \overline{b}$ and so $a \sim x$ and $b \sim x$. But then $x \sim b$ and so $a \sim b$ and $b \sim a$. Now if we choose an element $c \in \overline{a}$ we see that $a \sim c$. But also $c \sim a$, $c \sim b$ and $b \sim c$. This implies that $c \in \overline{b}$ and so $\overline{a} \subseteq \overline{b}$. A similar argument is used to show that $\overline{b} \subseteq \overline{a}$. Thus, $\overline{a} = \overline{b}$ which is a contradiction. $\square$

**Exercise 34** *Try to write a formal definition of a partition.*

We define a partition of a set $A$ to be a set $P$ consisting of $n$ subsets of $A$ such that

$$\bigcup_{S \in P} S = A,$$

where $S$ is a subset of A, and $S_i \cap S_j = \emptyset$ for all $1 \leq i, j \leq n$.

**Exercise 35** *How does a partition naturally define an equivalence relation on a set $A$?*

A partition will divide a set into separate equivalence classes $\overline{a_i}$. If $x \in A$ and $x \in \overline{a_i}$ then $a_i \sim x$. Thus, the equivalence relation $\sim$ is defined by which equivalence class $x$ falls into.

**Exercise 36** *How many equivalence classes are there in Exercise 30?*

There are 7 equivalence classes. For the proof, we first prove a lemma showing that every $x \in \mathbb{Z}$ can be written as $x = 7n + k$ where $k \in \{0, 1, 2, 3, 4, 5, 6\}$.

*Proof.* Let $x \in \mathbb{N} \cup \{0\}$ and let $S = \{0, 1, 2, 3, 4, 5, 6\}$. Then let $T = \{k \in \mathbb{N} \cup \{0\} \mid \text{there exists } n \in \mathbb{Z} \text{ such that } x = 7n + k\}$. Then we see that $T \neq \emptyset$ since $x = 7(0) + x$ and $x \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z}$. Then we see there exists a least element $m$ of $T$ and so $x = 7n + m$ for some $n \in \mathbb{Z}$. If $m \in S$ then we are done. If $m \notin S$ then $m > 7$ and so $m - 7 > 0$. Therefore we can write $x = 7(n + 1) + (m - 7)$ and so $(m - 7) \in T$. But $m - 7 < m$ and since $m$ is the least element of $T$ this is a contradiction so $m \in S$. Therefore every $x \in \mathbb{N} \cup \{0\}$ can be written as $7n + k$ for some $n \in \mathbb{Z}$ and $k \in S$. We now consider the case where $x \in \mathbb{Z} \backslash (\mathbb{N} \cup \{0\})$. We see that $-x = -7n - k = 7(-n - 1) + (-k + 7)$. But if $k \neq 0$ then $-k + 7 \in S$ and if $k = 0$ then $x = 7n$ and so $-x = 7(-n)$ and so we see that for $x \in \mathbb{Z}$ we can write $x = 7n + k$ for $n \in \mathbb{Z}$ and $k \in S$. $\square$

Now we prove the original result.

*Proof.* Let $x \in \mathbb{Z}$ and let $S = \{0, 1, 2, 3, 4, 5, 6\}$. Then we see that $x = 7n + k$ and $x - k = 7n$ for some $n \in \mathbb{Z}$ and $k \in S$. But then $x \sim k$ and so $x \in \overline{k}$. Since there are only 7 possible values for $k$, we see that there are at most 7 equivalence classes. If we choose two elements $p, q \in S$ such that $p \neq q$ then without loss of generality we can assume $p > q$ and so $(p - q) \in S$. But then $p - q \neq 7n$ for some $n \in \mathbb{Z}$ and so $p \nsim q$ and $\overline{p} \neq \overline{q}$. So no two equivalence classes are the same. Additionally, for every $p \in S$ we see that $p = 7(0) + p$ and since $0 \in \mathbb{Z}$ and $p \in S$, we see every element of $p$ is in an equivalence class. So we see that there are at least 7 and at most 7 equivalence classes so there must be exactly 7 equivalence classes. $\square$

**Problem 37** *Try to find a way to multiply and add the equivalence classes from Exercise 30.*

It seems that if $a \in \overline{a}$ and $b \in \overline{b}$ then $(a + b) \in \overline{a + b}$ and $ab \in \overline{ab}$. Also, for a scalar $c$, $ca \in \overline{ca}$ for all $a \in \overline{a}$.

**Definition 38 (Ordering)** *Let $A$ be a set. Then $< \subseteq A \times A$ is an ordering if the following hold:*
*1) For all $x, y \in A$ such that $x \neq y$ we have $x < y$ or $y < x$;*
*2) For all $x, y \in A$ if $x < y$ then $x \neq y$;*
*3) For all $x, y, z \in A$ if $x < y$ and $y < z$ then $x < z$.*

**Theorem 39** *If $x, y \in A$, then it cannot be true that both $x < y$ and $y < x$.*

*Proof.* Suppose that $x < y$ and $y < x$. Then $x < x$ and so $x \neq x$. This is a contradiction. $\square$

## Sheet 2: The Phantom Continuum

**Axiom 1** *There is at least one point in $C$.*

**Axiom 2** *The relation $<$ is an ordering on $C$.*

**Definition 1 (First and Last Point)** *If $A \subseteq C$ is a subset, then a point $a \in A$ is called a first point of $A$ if for all $x \in A$, either $x = a$ or $a < x$. The last point is defined analogously.*

**Lemma 2** *Every nonempty finite set of points of $C$ has a first and a last point*

*Proof.* A subset of $C$ with one element, $a$, has a first point and a last point since for every element $x$ in the set, $a = x$ or $a < x$ and for every element $x$ in the set, $a = x$ or $a > x$. So $a$ is the first point and the last point. We now consider a set $S \subseteq C$ which has $n$ elements and since $n \in \mathbb{N}$ we can use induction on $n$. We assume every subset of $C$ with $n$ elements has a first point and a last point. Now let $T$ be a set with $n+1$ elements. Consider the set $T\backslash\{a\}$ for some $a \in T$. $T\backslash\{a\}$ has $n$ elements so it has a first point $x$ and a last point $y$. Since $a \notin T\backslash\{a\}$ we see that $a \neq x$ and $a \neq y$. Because $<$ is an ordering on $C$, we have four cases:

*Case 1:* Let $a < x$ and $a < y$. We know that $x \leq p$ for all points $p \in T$ such that $p \neq a$. But $a < x$ and so $a \leq p$ for all $p \in T$. Likewise, $y \geq p$ for all points $p \in T$ such that $p \neq a$. But $a < y$ and so $y \geq p$ for all $p \in T$. Then $a$ is less than or equal to every point in $T$ and $y$ is greater than or equal to every point in $T$ so the first and last points are $a$ and $y$.

*Case 2:* Let $x < a$ and $a < y$. Using a similar argument as in *Case 1* we see that $a$ is between $x$ and $y$ and so $x$ is the first point of $T$ and $y$ is the last point.

*Case 3:* Let $x < a$ and $y < a$. Using the argument from *Case 1* we have $x$ is less than or equal to every point in $T$ and $a$ is greater than or equal to every point in $T$ so the first and last points are $x$ and $a$.

*Case 4:* Let $a < x$ and $a > y$. Since we know that $x < y$ (because $x$ is the first point of $T\backslash\{a\}$), then we have $a < x$ and $x < y$ so $a < y$. But this is a contradiction so this case is impossible.

We see that in all possible cases $T$ has $n+1$ elements and has a first point and a last point. Thus, by induction, every nonempty finite subset of $C$ has a first and a last point. □

**Theorem 3** *Let $A$ be a nonempty finite set of points of $C$. If $A$ contains $n$ points, then we can label them as $a_1, a_2, \ldots, a_n$ in such a way that $a_1 < a_2, a_2 < a_3, \ldots, a_{n-1} < a_n$. (In other words, $a_i < a_{i+1}$ for $i = 1, 2, \ldots n-1$.)*

*Proof.* A subset of $C$ with one element can be indexed by labeling it $a_1$. Now let $S$ be a subset of $C$ with $n$ elements. Since $n \in \mathbb{N}$ we can use induction on $n$. We assume that every subset of $C$ with $n$ elements can be indexed as $a_1, a_2, \ldots, a_n$ such that $a_i < a_{i+1}$ for $i = 1, 2, \ldots n-1$. Now consider a set $T$ with $n+1$ elements. By Lemma 2 we know that $T$ has a first point and a last point (2.2). Let $x$ be the last point. We see that $T\backslash\{x\}$ has $n$ elements and so it can be indexed $a_1, a_2, \ldots, a_n$ so that $a_i < a_{i+1}$ for $i = 1, 2, \ldots, n-1$. We now call $x$, $a_{n+1}$. We see that $T$ can be indexed since $a_n < a_{n+1}$ and so $a_i < a_{i+1}$ for $i = 1, 2, \ldots n$. Since a set with one element can be indexed, and a set with $n+1$ elements can be indexed whenever a set with $n$ elements can be indexed, we see that all nonempty finite sets can be indexed. □

**Definition 4 (Betweenness)** *Let $x, y, z$ be points of $C$. We say that $y$ lies between $x$ and $z$ if $x < y$ and $y < z$.*

**Corollary 5** *Of three distinct points, one always lies between the other two.*

*Proof.* By Theorem 3, we can label three distinct points in a set $a_1$, $a_2$ and $a_3$ such that $a_1 < a_2$ and $a_2 < a_3$ (2.3). Thus $a_2$ will always be between $a_1$ and $a_3$. □

**Axiom 3** *The continuum $C$ has no first point and no last point.*

**Corollary 6** *$C$ is infinite.*

*Proof.* Suppose, to the contrary, that $C$ is finite. Then by Axiom 1 $C$ has at least one point and by Lemma 2 $C$ has a first and last point (A2.1, 2.2). But this defies Axiom 3 (2.3). This is a contradiction. □

**Definition 7 (Region)** *Let $a, b$ be points in $C$ such that $a < b$. The set of all points that lie between $a$ and $b$ is called the region $(a; b)$.*

**Theorem 8** *For every point $p \in C$, there exists a region containing $p$.*

*Proof.* Since $C$ has no first and last points then for every $p \in C$ there exist $a, b \in C$ such that $a < p$ and $p < b$ (A2.3). Therefore, since $p$ is between $a$ and $b$, $p$ is contained in the region $(a; b)$. □

**Definition 9 (Limit Point)** *Let $A \subseteq C$ be a subset. A point $p$ is called a limit point of $A$ if for every region $R$ that contains $p$, $R$ contains at least one point in $A$ other than $p$. In other words, for every region $R$ such that $p \in R$ we have*

$$R \cap (A \backslash p) \neq \emptyset.$$

**Theorem 10** *Let $A \subseteq B \subseteq C$ be subsets. If some point $p$ is a limit point of $A$, then it is also a limit point of $B$.*

*Proof.* Let $A \subseteq B \subseteq C$ be subsets. If $p$ is a limit point of $A$, then for every region $R$ containing $p$, it also contains at least one point in $A$. But since $A \subseteq B$, all points in $A$ are also in $B$. Thus, if $R$ contains at least one point in $A$, then it also contains at least one point in $B$. Therefore $p$ must also be a limit point for $B$. □

**Definition 11 (Exterior)** *Let $(a; b)$ be a region. Then $C \backslash (a; b) \backslash a \backslash b$ is called the exterior of $(a; b)$; the symbol is $\mathrm{ext}(a; b)$.*

**Lemma 12** *For any region $(a; b)$, we have*

$$C = \{x \mid x < a\} \cup \{a\} \cup (a; b) \cup \{b\} \cup \{x \mid b < x\}.$$

*Proof.* Let $(a; b)$ be a region in $C$. Let $S = \{x \mid x < a\} \cup \{a\} \cup (a; b) \cup \{b\} \cup \{x \mid b < x\}$. Suppose there is an element $k \in C$ such that $k \notin S$. Then $k \notin (a; b)$ and so $k$ is not between $a$ and $b$. Thus $k \leq a$ or $k \geq b$ (A2.2). But $k \neq a$ and $k \neq b$ and so $k < a$ or $k > b$. But we assumed that $k \notin \{x \mid x < a\}$ and $k \notin \{x \mid b < x\}$. This is a contradiction and so it is not the case that $k \in C$ and $k \notin S$. Therefore $C \subseteq S$. Since every point in $S$ is in $C$, we see that $C = \{x \mid x < a\} \cup \{a\} \cup (a; b) \cup \{b\} \cup \{x \mid b < x\}$. □

**Lemma 13** *For any region $(a; b)$, we have*

$$C = \mathrm{ext}(a; b) \cup (a; b) \cup \{a\} \cup \{b\}$$

*Proof.* Let $(a;b)$ be a region in $C$ and let $p \in C$ be a point. Then either $p < a$, $p = a$ or $p > a$ and $p < b$, $p = b$ or $p > b$. Thus there are nine possibilities for $p$ in relation to $a$ and $b$, but since $a < b$ some are impossible. Since $a < b$ it's not the case that $p < a$ and $p = b$ or $p < a$ and $p > b$ and it's not the case that $p = a$ and $p = b$ or $p = a$ and $p > b$ (A2.2, 1.39). This leaves us with five cases.

*Case 1:* If $p = a$ and $p < b$ then $p \in \{a\}$.

*Case 2:* If $p > a$ and $p < b$ then $p \in (a;b)$.

*Case 3:* If $p > a$ and $p = b$ then $p \in \{b\}$.

*Case 4:* If $p < a$ and $p < b$ then $p \notin \{a\}$, $p \notin (a;b)$ and $p \notin \{b\}$ (A2.2, 1.39). But since $p \in C$, we have $p \in C \backslash (a;b) \backslash \{a\} \backslash \{b\}$ and therefore $p \in \text{ext}(a;b)$.

*Case 5:* If $p > a$ and $p > b$ then by a similar argument from *Case 4* we have $p \in \text{ext}(a;b)$.

In all possible cases we have $p \in \{a\}$, $p \in \{b\}$, $p \in (a;b)$ or $p \in \text{ext}(a;b)$. Therefore $p \in \text{ext}(a;b) \cup (a;b) \cup \{a\} \cup \{b\}$. Thus $C \subseteq \text{ext}(a;b) \cup (a;b) \cup \{a\} \cup \{b\}$. Likewise, since every element of $\text{ext}(a;b) \cup (a;b) \cup \{a\} \cup \{b\}$ is an element of $C$, we see that the set is a subset of $C$. Hence the two sets are equal. $\square$

**Lemma 14** *No point of the exterior of a region is a limit point of the region. No point of a region is a limit point of the exterior of the region.*

*Proof.* Let $A = (a;b)$ be a region in $C$ and let $p \in \text{ext}(A)$. Suppose $p < a$. Then there exists a point $x \in \text{ext}(A)$ such that $x < p$ because $C$ has no first point (A2.3). So we have $(x;a)$ is a region containing $p$, which contains no points in $A$. We can make a similar statement if $p > b$ because $C$ has no last point (A2.3). Likewise, if $p \in A$, then $A$ is a region containing $p$, but no points in $A$ are in $\text{ext}(A)$ so $p$ cannot be a limit point of $\text{ext}(A)$. $\square$

**Theorem 15** *If two regions $A$ and $B$ have a point $x$ in common, then $A \cap B$ is also a region containing $x$.*

*Proof.* Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be regions such that $x \in A$ and $x \in B$. Then we see that $x \in A \cap B$. Without loss of generality, let $a_1 \leq b_1$. Then we see that $a_2 > b_1$, otherwise $A \cap B = \emptyset$. Thus there are two cases.

*Case 1:* Let $a_1 \leq b_1$ and $a_2 < b_2$ Then we have $a_1 \leq b_1 < a_2 < b_2$. Every element in $A$ which is also in $B$ must be both greater than $a_1$ and $b_1$ and less than $a_2$ and $b_2$. Since $a_1 \leq b_1$ and $a_2 < b_2$, $A \cap B = (b_1, a_2)$.

*Case 2:* Let $a_1 \leq b_1$ and $a_2 \geq b_2$. Then we have $a_1 \leq b_1 < b_2 \leq a_2$. Every element in $A$ which is also in $B$ must be both greater than $a_1$ and $b_1$ and less than $a_2$ and $b_2$. Since $a_1 \leq b_1$ and $b_2 \leq a_2$, $A \cap B = (b_1, b_2)$.

We see that in all cases, $A \cap B$ is a region which contains $x$. $\square$

**Corollary 16** *If $n$ regions $R_1, R_2, \ldots, R_n$ have a point $x$ in common, then their intersection $R_1 \cap R_2 \cap \cdots \cap R_n$ is also a region containing $x$.*

*Proof.* We use induction on $n$. Note that if $x \in R_1$ then $R_1$ is a region containing $x$ and so the statement holds for the base case $n = 1$. We now assume that for $n$ regions $R_1, R_2, \ldots, R_n$ which all contain a point $x$, the intersection $R_1 \cap R_2 \cap \cdots \cap R_n$ is a region containing $x$. Consider $n+1$ regions $R_1, R_2, \ldots, R_{n+1}$. Which all contain a point $x$. We know that the intersection $R_1 \cap R_2 \cap \cdots \cap R_n$ is a region containing $x$ by the inductive hypothesis. But then by Theorem 15 we know that $R_1 \cap R_2 \cap \cdots \cap R_n \cap R_{n+1}$ is also a region containing $x$ (2.15). Since this is true for $n = 1$ and for $n + 1$ when it's true for $n$, it must be true for all $n \in \mathbb{N}$. $\square$

**Theorem 17** *Let $A$ and $B$ be subsets of $C$. We have $p$ is a limit point of $A \cup B$ if and only if $p$ is a limit point of $A$ or $p$ is a limit point of $B$.*

*Proof.* Let $p \in C$ be a limit point of $A \cup B$. Then suppose to the contrary that $p$ is a not a limit point of $A$ and $p$ is not a limit point of $B$. Then there exist regions $R_1$ and $R_2$ such that $p \in R_1$, $p \in R_2$, $R_1 \cap (A \backslash p) = \emptyset$ and $R_2 \cap (B \backslash p) = \emptyset$. Then there exists a region $R_3 = R_1 \cap R_2$ which also contains $p$ (2.15). And since $R_3 \cap (A \backslash p) = \emptyset$ and $R_3 \cap (B \backslash p) = \emptyset$ we see that $R_3 \cap ((A \cup B) \backslash p) = \emptyset$ which means $p$ is not a limit point of $A \cup B$. This is a contradiction. Conversely, if $p$ is a limit point of $A$ or $p$ is a limit point of $B$ then by Theorem 10, $p$ must be a limit point of $A \cup B$ since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ (2.10). $\square$

**Corollary 18** *Let $A_1, A_2, \ldots, A_n$ be subsets of $C$. We have $p$ is a limit point of the union $A_1 \cup A_2 \cup \cdots \cup A_n$ if and only if $p$ is also a limit point of at least one of the sets $A_k$.*

*Proof.* We use induction on $n$. Note that if $p$ is a limit point of one set $A_1$, then the inductive base case holds for $n = 1$. We now assume that if $p$ is a limit point of the union of the $n$ sets $A_1, A_2, \ldots, A_n$, then it is a limit point of at least one of the sets $A_k$. Now consider the $n+1$ sets $A_1, A_2, \ldots, A_n, A_{n+1}$ and let $p$ be a limit point of their union. Using Theorem 17 we know that $p$ is a limit point of the union $A_1 \cup A_2 \cup \cdots \cup A_n$ or $p$ is a limit point of $A_{n+1}$ (2.17). Additionally, by the inductive hypothesis, if $p$ is a limit point the union $A_1 \cup A_2 \cup \cdots \cup A_n$ then $p$ is a limit point of at least one set $A_k$. So we see that $p$ is a limit point of at least one of the sets $A_1, A_2, \ldots, A_{n+1}$. Since this is true for one set and for $n+1$ sets when it's true for $n$ sets, it must be true for every natural number of sets.

For the converse suppose that $A_1, A_2, \ldots, A_n$ are subsets of $C$ and suppose that $p$ is a limit point of at least one of the sets $A_k$. Then $A_k \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$ and so by Theorem 10, since $p$ is a limit point of a subset of the union, then $p$ must be a limit point of the union of all $n$ sets (2.10). $\square$

**Exercise 19** *Find realizations of $(C, <)$, that is, concrete sets endowed with a relation $<$ that satisfies all the axioms so far. Are they the same? What does this question mean?*

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are two sets with an ordering $<$ which satisfy the axioms so far. There are other sets such as the set of odd integers or even integers which also satisfy the axioms and have an ordering. The set of even integers is essentially the same as $\mathbb{Z}$ in this case though because there is a bijection between them which preserves the ordering.

## Sheet 3: Attack of the Continuum

**Definition 1 (Disjoint)** *Two sets $A$ and $B$ are disjoint if $A \cap B = \emptyset$. A set of sets $S$ is pairwise disjoint if for all sets $A, B \in S$ we have $A = B$ or $A \cap B = \emptyset$.*

**Theorem 2** *If $p, q \in C$ and $p < q$, then there exist disjoint regions containing $p$ and $q$.*

*Proof.* Let $a, c, p, q \in C$ such that $a < p$, $p < q$ and $q < c$ (A2.1, A2.2, A2.3). Then there are two possibilities. There may be another point $b \in C$ which is between $p$ and $q$. We see that this implies $p < b$ and $b < q$ and so the region $(a; b)$ contains $p$ and the region $(b; c)$ contains $q$ but $(a; b) \cap (b; c) = \emptyset$. There is also the possibility that there are no points between $p$ and $q$. Then the region $(a; q)$ contains $p$ but not $q$ and the region $(p; c)$ contains $q$ but not $p$ and $(a; q) \cap (p; c) = (p; q) = \emptyset$. $\square$

**Corollary 3** *A set consisting of one point has no limit points.*

*Proof.* Let $A \subseteq C$ be a set with one point $x$. If $p \in C$ is to be a limit point of $A$, then every region which contains $p$ must also contain a point in $A$ which is not $p$. So we see $p \neq x$. But then $p < x$ or $p > x$. In either case, Theorem 2 shows that there are disjoint regions containing $p$ and $x$ which means there exist regions containing $p$, but no points in $A$ so $p$ cannot be a limit point of $A$ (3.2). $\square$

**Theorem 4** *A nonempty finite set of points has no limit points.*

*Proof.* By Corollary 3 we see that a set with one point has no limit points (3.3). Use induction on $n$ and assume that a subset of $C$ with $n \in \mathbb{N}$ points has no limit points. Consider the set $S$ where $|S| = n + 1$ and let $a \in S$. We know that $|S \backslash a| = n$ and so $S \backslash a$ has no limit points. But $S = (S \backslash a) \cup \{a\}$ and so we know that a limit point of $S$ is a limit point of $S \backslash a$ or a limit point of $\{a\}$ (2.17). By the inductive hypothesis and Corollary 3 we know that both $S \backslash a$ and $\{a\}$ have no limit points (3.3). Therefore $S$ has no limit points. Thus, by induction, all nonempty finite sets have no limit points. $\square$

**Corollary 5** *If $A \subseteq C$ is a finite set and $x \in A$, then there exists a region $R$ such that $A \cap R = \{x\}$.*

*Proof.* Let $A \subseteq C$ be a finite set of $n$ elements such that $x \in A$. We know that $x$ cannot be a limit point of $A$ by Theorem 4 and so there exists a region $R$ such that $x \in R$ and $R \cap (A \backslash x) = \emptyset$ (3.4). But then we have $R \cap A = \{x\}$. $\square$

**Theorem 6** *If $p$ is a limit point of a set $A$ and $R$ is a region containing $p$, then the set $R \cap A$ is infinite.*

*Proof.* Assume that $p$ is a limit point of a set $A \subseteq C$ and $R$ is a region containing $p$. Assume to the contrary that $R \cap A$ is finite. Then $p$ is not a limit point of $R \cap A$ by Theorem 4 (3.4). But since $(A \backslash (R \cap A)) \cup (R \cap A) = A$, and $p$ is a limit point of the union, we see that $p$ must be a limit point of $A \backslash (R \cap A)$ (2.17). We also have $R \cap (A \backslash (R \cap A)) = \emptyset$ and $p \in R$ so $p$ is not a limit point of $A \backslash (R \cap A)$. This is a contradiction and so $R \cap A$ must be infinite. $\square$

**Definition 7 (Closed Set)** *A set $A \subseteq C$ is closed if it contains all its limit points.*

**Corollary 8** *Finite sets are closed.*

*Proof.* Finite sets have no limit points and so they vacuously contain all of their limit points (3.4). $\square$

**Definition 9 (Closure)** *Let $M \subseteq C$ be a set. Let $\overline{M}$, the closure of $M$, be the set consisting of $M$ and all of its limit points:*

$$\overline{M} = M \cup \{x \in C \mid x \text{ is a limit point of } M\}.$$

**Theorem 10** *A set is $M \subseteq C$ is closed if and only if $M = \overline{M}$.*

*Proof.* We see that if $M \subseteq C$ is closed, then it contains all its limit points. That is $\{x \in C \mid x \text{ is a limit point of } M\} \subseteq M$. So we have $M = M \cup \{x \in C \mid x \text{ is a limit point of } M\} = \overline{M}$. Conversely, if $M = \overline{M}$ then $M = M \cup \{x \in C \mid x \text{ is a limit point of } M\}$. Therefore $\{x \in C \mid x \text{ is a limit point of } M\} \subseteq M$, and so $M$ contains all its limit points. Thus $M$ is closed. $\square$

**Theorem 11** *For all $M \subseteq C$ we have $\overline{M} = \overline{\overline{M}}$*

*Proof.* We wish to show that the set of limit points of $\overline{M}$ is a subset of $\overline{M}$ for $M \subseteq C$. Consider a limit point $p \in C$ of $\overline{M}$. Since $\overline{M} = M \cup \{x \mid x \text{ is a limit point of } M\}$ we see that $p$ is a limit point of $M$ or $p$ is a limit point of the set of limit points of $M$ because $p$ is a limit point of their union (2.17). If $p$ is a limit point of $M$, then $p \in \overline{M}$. If $p$ is a limit point of the set of limit points of $M$, then every region containing $p$ contains a limit point of $M$. But every region containing a limit point of $M$ contains a point in $M$ and so for all regions $R \subseteq C$ such that $p \in R$ we have $R \cap M \neq \emptyset$. But then either $p$ is in $M$ or $p$ is a limit point of $M$ and so $p \in \overline{M}$. Thus we see $\{x \mid x \text{ is a limit point of } \overline{M}\} \subseteq \overline{M}$ and so

$$\overline{M} = M \cup \{x \mid x \text{ is a limit point of } M\} \cup \{x \mid x \text{ is a limit point of } \overline{M}\} = \overline{\overline{M}}.$$

$\square$

**Corollary 12** *If $M$ is a set of points, then $\overline{M}$ is closed.*

*Proof.* Let $M \subseteq C$ be a set of points. By Theorem 11 we know $\overline{M} = \overline{\overline{M}}$ and so by Theorem 10, $\overline{M}$ is closed (3.10, 3.11). $\square$

**Theorem 13** *The sets $C$ and $\emptyset$ are closed.*

*Proof.* All limit points are elements of $C$ and so $C$ must contain all its limit points and is closed. The empty set can have no limit points since there are no regions which contain a point in $\emptyset$. Therefore it vacuously contains all its limit points and is closed. $\square$

**Definition 14 (Open Set)** *A set of points $M$ is open if the complement $C \backslash M$ is closed.*

**Theorem 15** *The sets $C$ and $\emptyset$ are open.*

*Proof.* We see that the complement $C \backslash C = \emptyset$ and $\emptyset$ is closed so $C$ is open (3.13). Likewise $C \backslash \emptyset = C$ and $C$ is closed so $\emptyset$ is open (3.13). $\square$

**Theorem 16** *Every region is open and its complement is closed.*

*Proof.* We wish to show that for all regions $R$, the complement $C \backslash R$ is closed. So we assume that for some region $R$ there exists a limit point $p$ of $C \backslash R$ such that $p \notin C \backslash R$. Thus, $p \in R$. But then, since $R \cap (C \backslash R) = \emptyset$, $p$ is not a limit point of $C \backslash R$ and this is a contradiction. Thus, $C \backslash R$ contains all its limit points and so it is closed which means $R$ is open for all $R \subseteq C$. $\square$

**Theorem 17 (Open Condition)** *A set $A \subseteq C$ is open if and only if for all $x \in A$, there exists a region $R \subseteq A$ such that $x \in R$.*

*Proof.* Let $A \subseteq C$ be open. Then $C \backslash A$ is closed. Assume there exists $x \in A$ such that for all regions $R$ containing $x$, $R$ is not a subset of $A$. Then for all regions $R$ containing $x$, $R$ contains at least one point in $C \backslash A$ and so $x$ is a limit point of $C \backslash A$. But $x$ is in $A$ and $C \backslash A$ is closed and so we have a contradiction. Thus for all $x \in A$, there exists a region $R \subseteq A$ such that $x \in R$.

Conversely, let $A \subseteq C$ be a subset such that for all $x \in A$ there exists a region $R \subseteq A$ such that $x \in R$. Assume $A$ is not open. Then $C \backslash A$ is not closed and so it doesn't include all its limit points. But then there exists a limit point $p$ of $C \backslash A$ such that $p \in A$. And so there exists a region $R \subseteq A$ which contains $p$ and so $p$ is not a limit point of $C \backslash A$. This is a contradiction and so $A$ must be open. $\square$

**Corollary 18** *Every nonempty open set is the union of regions.*

*Proof.* Let $R_a \subseteq A$ denote a region containing an element $a \in A$ for some open set $A$. Then we see that $\bigcup_{a \in A} R_a$ is a union of subsets of $A$ which contains every element of $A$ so it must be equal to $A$. $\square$

# Sheet 4: Revenge of $\mathbb{Q}$

Let $\mathbb{Z}$ denote the integers. Let

$$P = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$$

and let the relation $\sim$ be defined on $P$ by

$$(a_1, b_1) \sim (a_2, b_2) \text{ if } a_1 b_2 = a_2 b_1$$

**Theorem 1** *$\sim$ is an equivalence relation on $P$.*

*Proof.* Let $(a, b) \in P$. Then $ab = ab$ and so $(a, b) \sim (a, b)$. Hence, reflexivity applies to $\sim$. Now let $(a_1, b_1), (a_2, b_2) \in P$ such that $(a_1, b_1) \sim (a_2, b_2)$. Then $a_1 b_2 = a_2 b_1$ and so $a_2 b_1 = a_1 b_2$. Thus $(a_2, b_2) \sim (a_1, b_1)$ and so symmetry holds for $\sim$. Now suppose $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in P$ such that $(a_1, b_1) \sim (a_2, b_2)$ and $(a_2, b_2) \sim (a_3, b_3)$. Then $a_1 b_2 = a_2 b_1$ and $a_2 b_3 = a_3 b_2$. Multiplying the first equation by $b_3$ we have $a_1 b_2 b_3 = a_2 b_1 b_3$. But then since $a_2 b_3 = a_3 b_2$ we have $a_1 b_2 b_3 = a_3 b_1 b_2$ and dividing by $b_2 \neq 0$ we have $a_1 b_3 = a_3 b_1$. Therefore $(a_1, b_1) \sim (a_3, b_3)$ implying transitivity and since all three conditions have been met, $\sim$ is an equivalence relation on $P$. $\square$

Now let $\mathbb{Q}$ denote the set of $\sim$-equivalence classes of $P$. We now define two operators, $+$ and $\cdot$ as follows. For $X, Y \in \mathbb{Q}$ let $(a_1, b_2) \in X$ and $(a_2, b_2) \in Y$. Let

$$X + Y = \overline{(a_1 b_2 + a_2 b_1, b_1 b_2)}$$

and let

$$X \cdot Y = \overline{(a_1 a_2, b_1 b_2)}.$$

We now show that these definitions are well-defined.

**Theorem 2** *If $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$ then*

$$(a_1 b_2 + a_2 b_1, b_1 b_2) \sim (c_1 d_2 + c_2 d_1, d_1 d_2)$$

*and*

$$(a_1 a_2, b_1 b_2) \sim (c_1 c_2, d_1 d_2).$$

*Proof.* Let $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$. Then we have $a_1 d_1 = b_1 c_1$ and $a_2 d_2 = b_2 c_2$. We multiply the first equation by $b_2 d_2$ so we have $a_1 b_2 d_1 d_2 = b_1 b_2 c_1 d_2$ and we multiply the second equation by $b_1 d_1$ so we have $a_2 b_1 d_1 d_2 = b_1 b_2 c_2 d_1$. Now we add the two new equations together so we have $a_1 b_2 d_1 d_2 + a_2 b_1 d_1 d_2 = b_1 b_2 c_1 d_2 + b_1 b_2 c_2 d_1$ and so $(a_1 b_2 + a_2 b_1) d_1 d_2 = (c_1 d_2 + c_2 d_1) b_1 b_2$ which implies $(a_1 b_2 + a_2 b_1, b_1 b_2) \sim (c_1 d_2 + c_2 d_1, d_1 d_2)$. Similarly, if we multiply $a_1 d_1 = b_1 c_1$ and $a_2 d_2 = b_2 c_2$ together we have $a_1 a_2 d_1 d_2 = b_1 b_2 c_1 c_2$ and so $(a_1 a_2, b_1 b_2) \sim (c_1 c_2, d_1 d_2)$. $\square$

**Theorem 3 (Associativity of Addition)** *For all $p, q, r \in \mathbb{Q}$ we have $(p + q) + r = p + (q + r)$.*

*Proof.* Let $p, q, r \in \mathbb{Q}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $(r_1, r_2) \in r$. Then we see that

$$\begin{aligned}
(p + q) + r &= \left(\overline{(p_1, p_2)} + \overline{(q_1, q_2)}\right) + \overline{(r_1, r_2)} \\
&= \overline{(p_1 q_2 + p_2 q_1, p_2 q_2)} + \overline{(r_1, r_2)} \\
&= \overline{((p_1 q_2 + p_2 q_1) r_2 + p_2 q_2 r_1, p_2 q_2 r_2)} \\
&= \overline{(p_1 q_2 r_2 + p_2 q_1 r_2 + p_2 q_2 r_1, p_2 q_2 r_2)} \\
&= \overline{((q_1 r_2 + q_2 r_1) p_2 + p_1 q_2 r_2, p_2 q_2 r_2)} \\
&= p + \overline{(q_1 r_2 + q_2 r_1, q_2 r_2)} \\
&= p + (q + r).
\end{aligned}$$

□

**Theorem 4 (Commutativity of Addition)** *For all $p, q \in \mathbb{Q}$ we have $p + q = q + p$.*

*Proof.* Let $p, q \in \mathbb{Q}$ such that $(p_1, p_2) \in p$ and $(q_1, q_2) \in q$. Then we have $p + q = \overline{(p_1, p_2)} + \overline{(q_1, q_2)} = \overline{(p_1q_2 + p_2q_1, p_2q_2)} = \overline{(q_1p_2 + q_2p_1, q_2p_2)} = \overline{(q_1, q_2)} + \overline{(p_1, p_2)} = q + p$. □

**Theorem 5 (Additive Identity)** *There exists an $n \in \mathbb{Q}$ such that for all $p \in \mathbb{Q}$ we have $n + p = p$. Show that $n$ is unique.*

*Proof.* We see that if we let $n \in \mathbb{Q}$ such that $n = \overline{(0, 1)}$ and if we let $(p_1, p_2) \in p$ for some $p \in \mathbb{Q}$ then we have $n + p = \overline{(0, 1)} + \overline{(p_1, p_2)} = \overline{((0)p_2 + (1)p_1, (1)p_2)} = \overline{(p_1, p_2)} = p$. Now suppose there exist two additive identities such that for all $p \in \mathbb{Q}$ we have $n_1 + p = p$ and $n_2 + p = p$. Then we have $n_2 = n_1 + n_2 = n_2 + n_1 = n_1$ and so $n_1 = n_2$. Thus, the additive identity is unique. □

From now on we will call the additive identity 0.

**Theorem 6 (Additive Inverse)** *For all $p \in \mathbb{Q}$ there exists $q \in \mathbb{Q}$ such that $p + q = 0$. Show that $q$ is unique.*

*Proof.* Let $p \in \mathbb{Q}$ such that $(p_1, p_2) \in p$. Then we choose $q = \overline{(-p_1, p_2)}$ for $q \in \mathbb{Q}$. Then we have $p + q = \overline{(p_1, p_2)} + \overline{(-p_1, p_2)} = \overline{(p_1p_2 + -p_1p_2, p_2p_2)} = \overline{(0, p_2p_2)} = \overline{(0, 1)} = 0$ since $(0)p_2p_2 = (0)(1)$. Now suppose there exist two additive inverses so that $p + n_1 = 0$ and $p + n_2 = 0$. Then we have $p + n_1 = p + n_2$ and adding $\overline{(-p_1, p_2)}$ to both sides we have

$$\overline{(-p_1, p_2)} + \overline{(p_1, p_2)} + n_1 = \overline{(-p_1p_2 + p_1p_2, p_2p_2)} + n_1 = 0 + n_1 = n_1$$

on the left and

$$\overline{(-p_1, p_2)} + \overline{(p_1, p_2)} + n_2 = \overline{(-p_1p_2 + p_1p_2, p_2p_2)} + n_2 = 0 + n_2 = n_2$$

on the right. So $n_1 = n_2$ and the additive inverse is unique. □

From now on we will call the additive inverse for $p$, $-p$.

**Theorem 7 (Associativity of Multiplication)** *For all $p, q, r \in \mathbb{Q}$ we have $(p \cdot q) \cdot r = p \cdot (q \cdot r)$.*

*Proof.* Let $p, q, r \in \mathbb{Q}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $(r_1, r_2) \in r$. Then we have $(p \cdot q) \cdot r = \left(\overline{(p_1, p_2)} \cdot \overline{(q_1, q_2)}\right) \cdot \overline{(r_1, r_2)} = \overline{(p_1q_1, p_2q_2)} \cdot \overline{(r_1, r_2)} = \overline{(p_1q_1r_1, p_2q_2r_2)} = p \cdot \overline{(q_1r_1, q_2r_2)} = p \cdot (q \cdot r)$. □

**Theorem 8 (Commutativity of Multiplication)** *For all $p, q \in \mathbb{Q}$ we have $p \cdot q = q \cdot p$.*

*Proof.* Let $p, q \in \mathbb{Q}$ such that $(p_1, p_2) \in p$ and $(q_1, q_2) \in q$. Then $p \cdot q = \overline{(p_1, p_2)} \cdot \overline{(q_1, q_2)} = \overline{(p_1q_1, p_2q_2)} = \overline{(q_1p_1, q_2p_2)} = \overline{(q_1, q_2)} \cdot \overline{(p_1, p_2)} = q \cdot p$. □

**Theorem 9 (Multiplicative Identity)** *There exists $e \in \mathbb{Q}$ such that for all $p \in \mathbb{Q}$ we have $e \cdot p = p$.*

*Proof.* Let $p \in \mathbb{Q}$ such that $(p_1, p_2) \in p$ and let $e \in \mathbb{Q}$ such that $e = \overline{(1, 1)}$. Then we have $e \cdot p = \overline{(1, 1)} \cdot \overline{(p_1, p_2)} = \overline{(p_1(1), p_2(1))} = p$. Suppose there exist two multiplicative identities $e_1$ and $e_2$ such that for all $p \in \mathbb{Q}$, $e_1 \cdot p = p$ and $e_2 \cdot p = p$. Then we have $e_1 = e_2 \cdot e_1$ and $e_2 = e_1 \cdot e_2 = e_2 \cdot e_1$. So we have $e_1 = e_2$ and so the multiplicative identity is unique. □

From now on we will call the multiplicative identity 1.

**Theorem 10 (Multiplicative Inverse)** *For all $p \in \mathbb{Q}$ with $p \neq 0$ there exists $q \in \mathbb{Q}$ such that $p \cdot q = 1$.*

*Proof.* Let $p \in \mathbb{Q}$ such that $(p_1, p_2) \in p$ and since $p_1 \neq 0$ let $q \in \mathbb{Q}$ such that $(p_2, p_1) \in q$. Then we see that $p \cdot q = \overline{(p_1, p_2)} \cdot \overline{(p_2, p_1)} = \overline{(p_1p_2, p_1p_2)} = \overline{(1,1)} = 1$. Now suppose there are two multiplicative inverses for some $p \in \mathbb{Q}$ such that $p \cdot q_1 = 1$ and $p \cdot q_2 = 1$. Then, multiplying both equations by $\overline{(p_2, p_1)}$, we have $q_1 = \overline{(1,1)} \cdot q_1 = \overline{(p_1p_2, p_1p_2)} \cdot q_1 = \overline{(p_2, p_1)} \cdot \overline{(p_1, p_2)} \cdot q_1 = \overline{(p_2, p_1)} \cdot \overline{(p_1, p_2)} \cdot q_2 = \overline{(p_1p_2, p_1p_2)} \cdot q_2 = \overline{(1,1)} \cdot q_2 = q_2$. So the multiplicative inverse is unique. □

From now on we will call the multiplicative inverse for $p$, $p^{-1}$.

**Theorem 11 (Distributivity)** *For all $p, q, r \in \mathbb{Q}$ we have $p \cdot (q + r) = p \cdot q + p \cdot r$.*

*Proof.* Let $p, q, r \in \mathbb{Q}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $(r_1, r_2) \in r$. Then we have

$$\begin{aligned}
p \cdot (q + r) &= \overline{(p_1, p_2)} \cdot \left(\overline{(q_1, q_2)} + \overline{(r_1, r_2)}\right) \\
&= \overline{(p_1, p_2)} \cdot \overline{(q_1r_2 + q_2r_1, q_2r_2)} \\
&= \overline{(p_1q_1r_2 + p_1q_2r_1, p_2q_2r_2)} \\
&= \overline{(p_1q_1r_2 + p_1q_2r_1, p_2q_2r_2)} \cdot \overline{(p_2, p_2)} \\
&= \overline{(p_1p_2q_1r_2 + p_1p_2q_2r_1, p_2p_2q_2r_2)} \\
&= \overline{(p_1q_1, p_2q_2)} + \overline{(p_1r_1, p_2r_2)} \\
&= \overline{(p_1, p_2)} \cdot \overline{(q_1, q_2)} + \overline{(p_1, p_2)} \cdot \overline{(r_1, r_2)} \\
&= p \cdot q + p \cdot r.
\end{aligned}$$

□

**Theorem 12** *The function $f : \mathbb{Z} \to \mathbb{Q}$ where $f(n) = \overline{(n, 1)}$ is injective.*

*Proof.* Let $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$. Then we have $\overline{(a, 1)} = \overline{(b, 1)}$ and so $(a, 1) \sim (b, 1)$ which implies $a = b$. □

**Theorem 13** *For all $m, n \in \mathbb{Z}$ we have*

$$f(m + n) = f(m) + f(n) \text{ and } f(mn) = f(m) \cdot f(n).$$

*Proof.* Let $m, n \in \mathbb{Z}$. Then we have $f(m + n) = \overline{(m + n, 1)} = \overline{(m(1) + n(1), (1)(1))} = \overline{(m, 1)} + \overline{(n, 1)} = f(m) + f(n)$. Additionally we see that $f(mn) = \overline{(mn, (1)(1))} = \overline{(m, 1)} \cdot \overline{(n, 1)} = f(m) \cdot f(n)$. □

**Theorem 14** *For every rational number $r \in \mathbb{Q}$ there exist $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $r = mn^{-1}$.*

*Proof.* Let $r \in \mathbb{Q}$ such that $(m, n) \in r$ (since $r$ is nonempty). Then we see $m, n \in \mathbb{Z}$. Thus we can write $m = \overline{(m, 1)}$ and $n = \overline{(n, 1)}$. And so $n^{-1} = \overline{(1, n)}$ since $n \neq 0$ and we have $m \cdot n^{-1} = \overline{(m, 1)} \cdot \overline{(1, n)} = \overline{(m, n)} = r$. □

**Lemma 15** *Any element in $\mathbb{Q}$ can be written as $\overline{(a, b)}$ with $b > 0$.*

*Proof.* Let $\overline{(a, b)} \in \mathbb{Q}$. There are two cases:

*Case 1:* If $b > 0$ then we are done.

*Case 2:* If $b < 0$ then we have $a(-b) = -ab = (-a)b$ and so $(a, b) \sim (-a, -b)$. Thus $\overline{(a, b)} = \overline{(-a, -b)}$ and $-b > 0$. □

We now define a relation $<$ on $\mathbb{Q}$. For $p, q \in \mathbb{Q}$ let $(a_1, b_1) \in p$ such that $b_1 > 0$ and let $(a_2, b_2) \in q$ such that $b_2 > 0$. Then we define

$$p < q \text{ if } a_1 b_2 < a_2 b_1$$

**Theorem 16** *Show that $<$ is a well-defined relation on $\mathbb{Q}$.*

*Proof.* Let $\overline{(a_1, b_1)}, \overline{(a_2, b_2)}, \overline{(c_1, d_1)}, \overline{(c_2, d_2)} \in \mathbb{Q}$ such that $\overline{(a_1, b_1)} < \overline{(a_2, b_2)}$ and $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$. We take $b_1$, $b_2$, $d_1$ and $d_2$ to all be greater than 0 by Lemma 15. Then we have $a_1 b_2 < a_2 b_1$ and so $a_1 b_2 d_1 d_2 < a_2 b_1 d_1 d_2$. But we also know that $a_1 d_1 = b_1 c_1$ and $a_2 d_2 = b_2 c_2$. Making the appropriate substitutions we see $b_1 b_2 c_1 d_2 < b_1 b_2 c_2 d_1$. Since $b_1 b_2 > 0$ we have $c_1 d_2 < c_2 d_1$ and so $\overline{(c_1, c_2)} < \overline{(d_1, d_2)}$. This shows that $<$ is well-defined. ☐

**Theorem 17** *The relation $<$ is an ordering on $\mathbb{Q}$.*

*Proof.* Let $p, q, r \in \mathbb{Q}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $(r_1, r_2) \in r$. By Lemma 15 we let $p_2$, $q_2$ and $r_2$ all be greater than 0. If $p \neq q$ then we see that $(p_1, p_2) \nsim (q_1, q_2)$ and so $p_1 q_2 \neq p_2 q_1$. Then we have have either $p_1 q_2 < p_2 q_1$ and so $p < q$ or $p_2 q_1 < p_1 q_2$ and so $q < p$. Secondly if $p < q$ then we have $p_1 q_2 < p_2 q_1$ and so $p_1 q_2 \neq p_2 q_1$. Therefore $(p_1, p_2) \nsim (q_1, q_2)$. Thus $p \neq q$. Finally, if $p < q$ and $q < r$ then $p_1 q_2 < p_2 q_1$ and $q_1 r_2 < q_2 r_1$. Multiplying the first inequality by $r_2$ and the second by $p_2$ we have $p_1 q_2 r_2 < p_2 q_1 r_2$ and $p_2 q_1 r_2 < p_2 q_2 r_1$ since $p_2 > 0$ and $r_2 > 0$. This implies $p_1 q_2 r_2 < p_2 q_2 r_1$ and since $q_2 > 0$ we have $p_1 r_2 < p_2 r_1$ and so $p < r$. Since all three conditions are satisfied, we see that $<$ is an ordering on $\mathbb{Q}$. ☐

**Exercise 18** *Is $(\mathbb{Q}, <)$ a model of $C$? That is, which axioms does it satisfy?*

*Proof.* Since the integers are a subset of $\mathbb{Q}$ and there exists at least one integer and since we showed that $<$ was and ordering on $\mathbb{Q}$, we see that axioms 1 and 2 are satisfied. Theorem 20 shows that there is no last point of $\mathbb{Q}$. To show that there is no first point we use a similar argument. Let $\overline{(a, b)} \in \mathbb{Q}$ such that $b > 0$. We consider three cases:

*Case 1:* Let $a > 0$. Then $a(1) > (0)b$ and so $\overline{(a, b)} > \overline{(0, 1)} = 0$.

*Case 2:* Let $a < 0$. Then since $b > 0$, $a > ab - b$ which means $\overline{(a, b)} > \overline{(a - 1, 1)} = a - 1$.

*Case 3:* Let $a = 0$ then $\overline{(a, b)} = \overline{(0, b)} = 0$ and since $-1 < 0$ we see $\overline{(a, b)} > \overline{(-1, 1)} = -1$.

So we see that for any element of $\mathbb{Q}$ there is always an element greater than it and an element less than it which means it can have no first or last point and so it satisfies axiom 3. ☐

**Theorem 19** *For every $p, q \in \mathbb{Q}$ such that $p < q$ there exists $r \in \mathbb{Q}$ such that $p < r < q$.*

*Proof.* Let $p, q, r \in \mathbb{Q}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $r = \overline{(p_1 q_2 + p_2 q_1, 2 p_2 q_2)}$. Let $p < q$ and by Lemma 15 let $p_2 > 0$ and $q_2 > 0$. Then we have $p_1 q_2 < p_2 q_1$ and so $p_1 p_2 q_2 < p_2 p_2 q_1$ which implies $2 p_1 p_2 q_2 < p_1 p_2 q_2 + p_2 p_2 q_1$. We see that this implies $\overline{(p_1, p_2)} < \overline{(p_1 q_2 + p_2 q_1, 2 p_2 q_2)}$ which means $p < r$. Similarly, we have $p_1 q_2 < p_2 q_1$ which means $p_1 q_2 q_2 < p_2 q_1 q_2$ and $p_1 q_2 q_2 + p_2 q_1 q_2 < 2 p_2 q_1 q_2$. This implies $\overline{(p_1 q_2 + p_2 q_1, 2 p_2 q_2)} < \overline{(q_1, q_2)}$ which means $r < q$. Thus $p < r < q$. ☐

**Theorem 20 (Archimedean Property)** *For every $p \in \mathbb{Q}$ there exists $n \in \mathbb{Z}$ such that $p < n$.*

*Proof.* Let $p \in \mathbb{Q}$ such that $(a, b) \in p$. Let $b > 0$ by Lemma 15. We have to consider three cases:

*Case 1:* Let $a > 0$. Then $a < ab + b$ and so $\overline{(a, b)} < \overline{(a + 1, 1)} = a + 1$.

*Case 2:* Let $a < 0$. Then $a(1) < b(0)$ and so $\overline{(a, b)} < \overline{(0, 1)} = 0$.

*Case 3:* Let $a = 0$. Then $\overline{(a, b)} = \overline{(0, b)} = \overline{(0, 1)} = 0$ and since $0 < 1$ we see $\overline{(a, b)} < \overline{(1, 1)} = 1$. ☐

# Sheet 5: A New Continuum

**Theorem 1 (Intersections)** *The intersection of any set of closed sets is closed and the intersection of a finite number of open sets is open.*

*Proof.* Consider the set $S$ of closed sets $A \subseteq C$. Then let $p$ be a limit point of $\bigcap_{A \in S} A$. Then since $\bigcap_{A \in S} \subseteq A$ for all $A \in S$ we see that $p$ is a limit point of $A$ for all $A \in S$ (2.10). But all $A \in S$ are closed so $p \in A$ for all $A \in S$. And so $p \in \bigcap_{A \in S} A$ and we have $\bigcap_{A \in S} A$ is closed.

To show that an intersection of finitely many open sets is open, use induction on the number of sets, $n$. For the base case we have a single open set. Assume that the intersection of any $n$ open sets is open. Then consider the set of $n+1$ open sets $S = \{A_1, A_2, \ldots, A_{n+1}\}$. We see that the intersection $\bigcap_{A_i \in S \setminus A_{n+1}} A_i$ is open and $A_{n+1}$ is open. Then for all $x \in \bigcap_{A_i \in S} A_i$ , we have $x \in \bigcap_{A_i \in S \setminus A_{n+1}}$ and $x \in A_{n+1}$. By the open condition, for all $x \in \bigcap_{A_i \in S} A_i$ there exist regions $R_1 \subseteq \bigcap_{A_i \in S \setminus A_{n+1}} A_i$ and $R_2 \subseteq A_{n+1}$ such that $x \in R_1$ and $x \in R_2$ (3.17). But then $x$ is in the region $R_3 = R_1 \cap R_2$ and $R_3 \subseteq \bigcap_{A_i \in S} A_i$ (2.15). So for all $x \in \bigcap_{A_i \in S} A_i$ there exists a region $R \subseteq \bigcap_{A_i \in S} A_i$ such that $x \in R$. Thus the intersection is open by the open condition (3.17). By mathematical induction, this must be true for all $n \in \mathbb{N}$. $\square$

**Theorem 2 (Unions)** *The union of any set of open sets is open, and the union of a finite set of closed sets is closed.*

*Proof.* Consider the set $S$ of open sets $A \subseteq S$. By the open condition, for every $x \in A$ for some $A \in S$, there exists a region $R \subseteq A$ such that $x \in R$ (3.17). But if $x \in A$, then $x \in \bigcup_{A \in S} A$ and so there exists a region $R \subseteq A \subseteq \bigcup_{A \in S} A$ and $x \in R$ so the union must be open (3.17).

Now we use induction on a finite number of closed sets $n$. For the base case we have one closed set. Assume that the union of any $n$ closed sets is closed. Consider the set of $n+1$ closed sets $S = \{A_1, A_2, \ldots, A_{n+1}\}$. We see $\bigcup_{A_i \in S \setminus A_{n+1}} A_i$ is closed and $A_{n+1}$ is closed. Then if $p$ is a limit point of $\bigcup_{A_i \in S} A_i$ then it is a limit point of $\bigcup_{A_i \in S \setminus A_{n+1}} A_i$ or it is a limit point of $A_{n+1}$ (2.17). And since $\bigcup_{A_i \in S \setminus A_{n+1}} A_i$ and $A_{n+1}$ are closed, then we have $p \in \bigcup_{A_i \in S \setminus A_{n+1}} A_i$ or $p \in A_{n+1}$. Thus $p \in \bigcup_{A_i \in S} A_i$ and so it is closed. So by mathematical induction we see that this is true for any $n \in \mathbb{N}$. $\square$

**Axiom 1 (Connectedness)** *The only point sets which are both closed and open are $C$ and $\emptyset$.*

**Exercise 3** *Show that Theorem 1 does not hold for the intersection of an infinite number of open sets.*

*Proof.* We see that for all $a \in C$ we have $\{a\} = C \setminus (C \setminus a)$ is closed since $\{a\}$ is a finite set and so $C \setminus a$ must be open (2.8). Now consider a point $p \in C$ and consider the intersection

$$\bigcap_{a \in C, a \neq p} C \setminus a = \{p\}.$$

Since $C \setminus p$ is infinite, this is an intersection of an infinite number of open sets. But their intersection is $\{p\}$ which is not open (2.8, A5.1). $\square$

**Exercise 4** *Show that Theorem 2 does not hold for the union of an infinite number of closed sets.*

*Proof.* Similarly, we take a point $p \in C$ and then consider all the sets containing a single point other than $p$. Then we have

$$\bigcup_{a \in C, a \neq p} \{a\} = C \setminus p.$$

Since $\{a\}$ is finite, it is closed for all $a \in C$ (2.8). From Exercise 3 and Axiom 1 we know $C \setminus p$ is not closed (A5.1, 5.3). So we have a union of an infinite number of closed sets which equals a set that is not closed. $\square$

Let $O$ be an open subset of $C$. Let us define the relation $\sim$ on $O$ as follows: $a \sim b$ if there exists a region $R \subseteq O$ containing both $a$ and $b$.

**Theorem 5** *$\sim$ is an equivalence relation.*

First we prove a lemma showing that if two regions contain a common element $x$, then their union is also a region containing all points in either region.

*Proof.* Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be regions such that $x \in A$ and $x \in B$. Then we see that $x \in A \cup B$. Without loss of generality, let $a_1 \leq b_1$. Then we see that $a_2 > b_1$, otherwise $A$ and $B$ would not both contain $x$. Thus there are two cases.

*Case 1:* Let $a_1 \leq b_1$ and $a_2 < b_2$ Then we have $a_1 \leq b_1 < a_2 < b_2$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $a_1 < x < a_2$. But $a_2 < b_2$ so $a_1 < x < b_2$ and $x \in (a_1, b_2)$. Likewise, if $x \in B$ then $b_1 < x < b_2$. But $a_1 \leq b_2$ so $a_1 < x < b_2$ and $x \in (a_1, b_2)$. Therefore $A \cup B \subseteq (a_1, b_2)$. Additionally, if $x \in (a_1, b_2)$ then $x < a_2$ or $x \geq a_2$. If $x < a_2$ then $a_1 < x < a_2$ and $x \in A$. If $x \geq a_2$ then $b_1 < x < b_2$ and $x \in B$. Therefore $x \in A$ or $x \in B$ and $x \in A \cup B$. Thus $(a_1, b_2) \subseteq A \cup B$ and so $A \cup B = (a_1, b_2)$.

*Case 2:* Let $a_1 \leq b_1$ and $a_2 \geq b_2$. Then we have $a_1 \leq b_1 < b_2 \leq a_2$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \in (a_1, a_2)$. Likewise, if $x \in B$ then $b_1 < x < b_2$. But $a_1 \leq b_2$ and $b_2 \leq a_2$ so $a_1 < x < a_2$ and $x \in (a_1, a_2)$. Therefore $A \cup B \subseteq (a_1, a_2)$. Additionally, if $x \in (a_1, a_2)$ then either $x > b_1$ and $x < b_2$ and so $x \in (b_1, b_2)$ or $x \leq b_1$ or $x \geq b_2$. If $x \in (b_1, b_2)$ then $x \in B$. If $x \leq b_1$ or $x \geq b_2$ then $a_1 < x < a_2$ and $x \in A$. Therefore $x \in A$ or $x \in B$ and $x \in A \cup B$. Thus $(a_1, a_2) \subseteq A \cup B$ and so $A \cup B = (a_1, a_2)$.

We see that in either case, $A \cup B$ is a region which contains every point in either $A$ or $B$. $\square$

We now prove the original result.

*Proof.* Let $O$ be an open subset of $C$. We see that if a $a \in O$, then by the open condition there exists a region $R \subseteq O$ such that $a \in R$ and so $a \sim a$ so we have reflexivity (3.17). Also if $a \sim b$ then $a, b \in R$ for a region $R \subseteq O$ and so $b, a \in R$ and $b \sim a$. So we have symmetry. Finally, if $a \sim b$ and $b \sim c$, then we have $a, b \in R_1$ and $b, c \in R_2$ where $R_1, R_2 \subseteq O$ are regions. But by the previous lemma $R_3 = R_1 \cup R_2 \subseteq O$ is a region and since $a, b, c \in R_3$ we have $a \sim c$ so we have transitivity. $\square$

**Theorem 6** *For all $a \in C$ the sets $\{x \mid x < a\}$ and $\{x \mid a < x\}$ are open.*

*Proof.* Let $a, p \in C$ such that $p \in \{x \mid x < a\}$. Then there exists some point $q \in C$ such that $q < p$ since $C$ has no first point and so $p \in (q; a)$ (A2.3). Since $(q; a) \subseteq \{x \mid x < a\}$ we see that there exists a region containing $p$ which is a subset of $\{x \mid x < a\}$. So $\{x \mid x < a\}$ must be open by the open condition (3.17). A similar proof holds for $\{x \mid a < x\}$ because $C$ has no last point (A2.3). $\square$

**Corollary 7** *If $A, B \subseteq C$ are open subsets, $A \cap B = \emptyset$ and $A \cup B = C$, then $A = \emptyset$ or $B = \emptyset$.*

*Proof.* We have $A \cap B = \emptyset$ and so $B \subseteq C \backslash A$. But additionally we have $A \cup B = C$ and so $C \backslash A \subseteq B$. Then $B = C \backslash A$ and since $A$ is open, $C \backslash A$ is closed and so $B$ is both open and closed. But then either $B = C$ or $B = \emptyset$ by Axiom 1 (A5.1). If $B = \emptyset$ then we're done and if $B = C$ then $A = \emptyset$ because $A \cap B = \emptyset$. So either $A$ or $B$ is empty. $\square$

**Theorem 8 (Regions are Nonempty)** *For all $a < b$ there exists $c$ such that $a < c < b$.*

*Proof.* Consider $a, b \in C$ such that $a < b$. Then the sets $\{x \mid x < b\}$ and $\{x \mid a < x\}$ are both open by Theorem 6 (5.6). For every $p \in C$ we have $p < a$, $p = a$ or $p > a$ and so $\{x \mid x < b\} \cup \{x \mid a < x\} = C$. But $\{x \mid x < b\} \cap \{x \mid a < x\} = (a; b)$ and using Corollary 7 and the fact that $C$ has no first or last point we see that this intersection cannot be empty since $\{x \mid x < b\} \neq \emptyset$ and $\{x \mid x > a\} \neq \emptyset$ (A2.3, 5.7). $\square$

**Corollary 9** *For all $a < b$ both $a$ and $b$ are limit points of the region $(a;b)$.*

*Proof.* Let $(p;q)$ be a region such that $a \in (p;q)$. Then $q \geq b$ or $q < b$. If $q \geq b$ then $(a;b) \subseteq (p;q)$ and because regions are nonempty there exists $c \in (a;b)$ such that $c \in (p;q)$ (5.8). If $q < b$ then there exists a point $c \in C$ such that $a < c < q$ and so $c \in (a;b)$ and $c \in (p;q)$ (5.8). We see that all regions containing $a$ also contain a point in $(a;b)$ so $a$ must be a limit point of $(a;b)$. A similar proof holds for $b$. $\square$

**Corollary 10** *Every point of a region is a limit point of that region.*

*Proof.* Let $A$ be a region and let $p \in A$. Then for all regions $R$ such that $p \in R$, we have $R \cap A = (a;b) \neq \emptyset$ where $(a;b)$ is a region containing $p$ (2.15). Thus there exists a $c \in (a;b)$ such that $a < c < p$ (5.8). But then for all regions $R$ containing $p$ we have $R \cap (A \backslash p) \neq \emptyset$ and so $p$ is a limit point of $A$. $\square$

**Corollary 11** *Every nonempty region contains infinitely many points*

*Proof.* Suppose to the contrary that a nonempty region contains a finite number of points. Then it has no limit points (3.4). But by Corollary 10 we know that every point is a limit point and so this is a contradiction (5.10). $\square$

**Corollary 12** *Every point in $C$ is a limit point of $C$*

*Proof.* Let $p \in C$. Then we see that every region $R$ which contains $p$ contains infinitely many points and so for all regions $R$ which contain $p$, we have $R \cap (C \backslash p) \neq \emptyset$ (5.11). $\square$

# Sheet 6: The Continuum Strikes Back

**Definition 1 (Upper and Lower Bound)** *Let $A \subseteq C$ be a set. We say that $u \in C$ is an upper bound of $A$ if for all $a \in A$ we have $a \leq u$. We say that $l \in C$ is a lower bound of $A$ if for all $a \in A$ we have $a \geq l$.*

**Exercise 2** *Show that $C$ has no upper or lower bounds.*

*Proof.* Since $C$ has no last point, for every point $u \in C$, there exists another point $u' \in C$ such that $u' > u$ (A2.3). Similarly, since $C$ has no first point, for every $l \in C$, there exists another point $l' \in C$ such that $l' < l$ (A2.3). Thus, $C$ can have no upper or lower bounds. $\square$

**Definition 3 (Bounded Sets)** *A set $A \subseteq C$ is bounded above if there exists an upper bound of $A$. A set $A \subseteq C$ is bounded below if there exists a lower bound of $A$. A set $A \subseteq C$ is bounded if it is bounded above and bounded below.*

**Definition 4 (Least Upper Bound)** *Let $A \subseteq C$ be a set. We say that $u \in C$ is the least upper bound of $A$, or $u = \sup A$, if $u$ is an upper bound of $A$ and for all $u'$ that are upper bounds of $A$ we have $u \leq u'$.*

**Definition 5 (Greatest Lower Bound)** *Let $A \subseteq C$ be a set. We say that $l \in C$ is the greatest lower bound of $A$, or $l = \inf A$, if $l$ is a lower bound of $A$ and for all $l'$ that are lower bounds of $A$ we have $l' \leq l$.*

**Exercise 6** *Show that if $\sup A$ exists then it is unique.*

*Proof.* Let $A \subseteq C$ be a set and let $u$ and $u'$ be least upper bounds of $A$. Then for all $a \in C$ such that $a$ is an upper bound of $A$, we have $u \leq a$ and $u' \leq a$. But $u$ and $u'$ and upper bounds of $A$ so we have $u \leq u'$ and $u' \leq u$. Thus we have $u' = u$ and $\sup A$ is unique. $\square$

**Theorem 7** *For all $a < b$ we have $\sup(a; b) = b$ and $\inf(a; b) = a$.*

*Proof.* Let $a, b \in C$ such that $a < b$. We see $b$ is an upper bound of $(a; b)$ because $b > p$ for all $p \in (a; b)$. Suppose to the contrary that there exists $u \in C$ such that $u$ is an upper bound of $(a; b)$ and $u < b$. Then for all $p \in (a; b)$ we have $a < p$ and $p \leq u < b$ and so we see that $u \in (a; b)$. But there exists a $u' \in C$ such that $u < u' < b$ because regions are nonempty (5.8). Since $a < u' < b$, we see $u' \in (a; b)$. Thus, since $u < u'$, this is a contradiction and so there are no upper bounds of $(a; b)$ which are less than $b$. Therefore $b = \sup(a; b)$. A similar proof holds to show that $a = \inf(a; b)$. $\square$

**Theorem 8** *Let $A$ be a point set that has a least upper bound $s = \sup A$. Show that if $s \notin A$ then $s$ is a limit point of $A$.*

*Proof.* Let $A \subseteq C$ such that $s = \sup A$ and let $s \notin A$. Consider the case where $A$ has a last point $x$. Then $x \geq a$ for all $a \in A$ so $x$ is an upper bound of $A$. Likewise, since $x$ is the largest element of $A$, any other upper bound of $A$ must be greater than $x$. Then $x = s$ and so $A$ has no last point. Consider a region $(a; b)$ such that $s \in (a; b)$. Since $s = \sup A$ and $A$ has no last point there exists $c \in A$ such that $a < c < s$. But then $c \in A$ and $c \in (a; b)$. Since every region containing $s$ contains a point in $A$, $s$ must be a limit point of $A$. $\square$

**Theorem 9** *Let $A \subseteq C$ be a set. Show that the set*

$$N(A) = \{x \in C \mid x \text{ is not an upper bound of } A\}$$

*is open.*

*Proof.* Let $p \in N(A)$ for some $A \subseteq C$. Then $p$ is not an upper bound of $A$ and so there exists $b \in A$ such that $p < b$. But $C$ has no first point so there exists $a \in C$ such that $a < p$ and since $a < b$, $a$ is not an upper bound of $A$ (A2.3). But then $p \in (a; b)$ and $(a; b) \subseteq N(A)$ and so $N(A)$ must be open by the open condition (3.17). □

**Theorem 10** *Let $A \subseteq C$ be a set. Show that the set*

$$U(A) = \{x \in C \mid x \text{ is an upper bound of } A \text{ but not a least upper bound}\}$$

*is open.*

*Proof.* $U(A)$ can have no first point. To show this we assume the first point of $U(A)$ is $x$ and consider two possibilities. First, if $\sup A$ exists, then the region $(\sup A; x)$ is empty because there are no non-least upper bounds of $A$ which are less than the first point $x$. But this is a contradiction because regions are nonempty (5.8). Similarly, if $\sup A$ does not exist, then $x$ is an upper bound of $A$ which is less than or equal to all upper bounds of $A$ so $x = \sup A$. But this is a contradiction as well since $\sup A \notin U(A)$.

Let $p \in U(A)$ for some $A \subseteq C$. Then $p$ is an upper bound of $A$ but $p \neq \sup A$. $C$ has no last point so there exists $b \in C$ such that $p < b$ and so $b$ is an upper bound of $A$ since it is greater than every point in $A$ (2.3). Since $U(A)$ has no first point, there exists another upper bound $a$ of $A$ such that $a < p$. But then $p \in (a; b)$ and $(a; b) \subseteq U(A)$ so $U(A)$ must be open by the open condition (3.17). □

**Theorem 11 (Nonempty Bounded Sets Have Least Upper Bounds)** *Let $A$ be a nonempty point set that is bounded above. Show that* $\sup A$ *exists.*

*Proof.* Let $A$ be a nonempty set which is bounded above such that $\sup A$ doesn't exist. The sets $N(A)$ and $U(A)$ are two open sets that share no common points by definition. That is $N(A) \cap U(A) = \emptyset$. But also, since there is no least upper bound of $A$, every point in $C$ is either in $N(A)$ or $U(A)$ and so $N(A) \cup U(A) = C$. But $A$ is bounded above so $U(A)$ is not empty. Also $A$ is nonempty and $C$ has no first point so there exists some point which is less than a point in $A$ so $N(A)$ is nonempty (A2.3). Then this is a contradiction because $N(A) \neq \emptyset$ and $U(A) \neq \emptyset$ (5.17). So $\sup A$ must exist. □

**Theorem 12 (Nonempty Bounded Sets Have Greatest Upper Bounds)** *Let $A$ be a nonempty point set that is bounded below. Show that* $\inf A$ *exists.*

*Proof.* We can make analogous proofs for Theorems 9 and 10 about lower bounds of a set $A \subseteq C$. Using the two sets defined in these proofs for lower bounds we can make another analogous proof for Theorem 11 about $\inf A$. □

**Exercise 13** *Do the above two theorems hold in* $(\mathbb{Q}, <)$*?*

No.

*Proof.* Let $(\mathbb{Q}, <)$ be a model of the continuum and consider the set $S = \{x \in C \mid x^2 < 2\}$. For $x \in S$ we have $x^2 < 2$ and so $x < \sqrt{2}$ or $x > -\sqrt{2}$. Thus $\sqrt{2}$ is an upper bound of $S$. Suppose that $p$ is an upper bound of $S$ such that $p < \sqrt{2}$. We know that $1^2 < 2$ and so $1 \in S$ which means $1 < p < \sqrt{2}$. But then $1 < p^2 < 2$. Consider the set $T = \left\{1 + \frac{2n+1}{n^2} \mid n \in \mathbb{N}\backslash\{1, 2\}\right\}$. This set is based on the reciprocals of the natural numbers and so it reverses their ordering. That is $1 + \frac{1}{n^2} > 1 + \frac{1}{(n+1)^2}$ for $n \in \mathbb{N}$. Using the Archimedean Property we know that there exists an element of $T$ such that this element is less than $p^2$ (4.20). But using the Well Ordering Principle we know that there exists a greatest such element $1 + \frac{2q+1}{q^2}$. But then $p^2 < 1 + \frac{2(q-1)+1}{(q-1)^2}$. We see that $1 + \frac{2(q-1)+1}{(q-1)^2} = \frac{q^2}{(q-1)^2}$ and so $\sqrt{\frac{q^2}{(q-1)^2}} = \pm\frac{q}{q-1}$. But then we have $p < \frac{q}{q-1} < \sqrt{2}$ and so there exists an element of $S$ which is greater than an upper bound of $S$. This is a contradiction and so $\sqrt{2} = \sup S$. But $\sqrt{2} \notin C$ and so $(\mathbb{Q}, <)$ is not a model of $C$. □

# Sheet 7: Return of the Continuum

**Definition 1 (Open Cover)** *Let $X \subseteq C$ be a set and let $\mathcal{A}$ be a set of subsets of $C$. We say that $\mathcal{A}$ is an open cover for $X$ if for all $A \in \mathcal{A}$ the set $A$ is open and*

$$X \subseteq \bigcup_{A \in \mathcal{A}} A.$$

**Exercise 2** *Let $p \in C$ be a point and let*

$$\mathcal{A} = \{\text{ext}(a; b) \mid p \in (a; b)\}.$$

*Show that $\mathcal{A}$ is an open cover for $C \backslash p$.*

*Proof.* Let $x \in C \backslash p$. Then $x \in C$ and $x \neq p$ and so $x < p$ or $p < x$. Suppose $x < p$. Since regions are nonempty there exists $a \in C$ such that $x < a < p$ (5.8). And because $C$ has no last point there exists $b \in C$ such that $p < b$ (A2.3). But then $p \in (a; b)$ and since $x < a$, $x \in \text{ext}(a; b)$. Because this is true for some region $(a; b)$, we see $x \in \bigcup_{A \in \mathcal{A}} A$. Therefore, $C \backslash p \subseteq \bigcup_{A \in \mathcal{A}} A$. From Exercise 12 we see that $\text{ext}(a; b)$ is open and so $\mathcal{A}$ is an open cover for $C \backslash p$ (7.12). A similar argument holds if $p < x$ because $C$ has no first point (A2.3). Note that Exercise 12 does not depend on this exercise. □

**Definition 3 (Subcover)** *Let $\mathcal{A}$ be an open cover for $X$. A subset $\mathcal{B} \subseteq \mathcal{A}$ is a subcover if*

$$X \subseteq \bigcup_{B \in \mathcal{B}} B.$$

**Exercise 4** *Show that the set*

$$A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}.$$

*is closed.*

*Proof.* Let $p \in C$ be point such that $p \notin A$. Then there are three cases.

*Case 1:* Let $p < 0$. Then since $C$ has no first point there exists a point $x \in C$ such that $x < p$ and so the region $(x; 0)$ contains $p$ but no points in $A$ (A2.3).

*Case 2:* Let $p > 1$. Then since $C$ has no last point there exists a point $y \in C$ such that $p < y$ and so the region $(1; y)$ contains $p$ but no points in $A$ (A2.3).

*Case 3:* Let $p \in (0; 1)$. Then $p = \frac{a}{b}$ for some $a, b \in \mathbb{N}$ and since $0 < \frac{a}{b} < 1$, we have $a < b$. Since $0 < \frac{b}{a}$, by the Archimedean Property there exists a natural number $k$ such that $\frac{b}{a} < k$ (4.20). But since $k \in \mathbb{N}$, by the Well Ordering Principle there exists a least such element $n$. Since $p \notin A$, $a \neq 1$ and so $\frac{b}{a} \notin \mathbb{N}$. But then $n - 1 < \frac{b}{a} < n$ and so $\frac{1}{n} < p < \frac{1}{n-1}$. Therefore $p \in \left(\frac{1}{n}; \frac{1}{n-1}\right)$ which doesn't contain any elements of $A$.

In all three cases there exists a region containing $p$ which contains no elements of $A$ and so $p$ cannot be a limit point of $A$. Therefore if $A$ has any limit points, they must be in $A$. Since $A$ contains all its limit points, it is closed. □

**Exercise 5** *Prove that every open cover of $A$ has a finite subcover.*

*Proof.* Let $\mathcal{A}$ be a cover of $A$. Then for every element of $A$, there exists an open set in $\mathcal{A}$ which contains that element. But then there exists an open set $B$ in $\mathcal{A}$ containing 0. And so there exists a region $(a;b) \subseteq B$ such that $0 \in (a;b)$ by the open condition (3.17). There are three cases.

*Case 1:* Let $1 < b$. Then $A \subseteq B$ and so the set containing $B$ is a finite subcover of $\mathcal{A}$.

*Case 2:* Let $b = 1$. Then the region $(a;b)$ contains all the elements of $A$ except for 1. Thus the set containing $B$ and a set from $\mathcal{A}$ containing 1 is a finite subcover of $A$.

*Case 2:* Let $b < 1$. Then $b = \frac{p}{q}$ for some $p, q \in \mathbb{N}$ and since $0 < \frac{p}{q} < 1$, we have $p < q$. Since $0 < \frac{q}{p}$, by the Archimedean Property there exists a natural number $k$ such that $\frac{q}{p} < k$ (4.20). But since $k \in \mathbb{N}$, by the Well Ordering Principle there exists a least such element $n$. There are a finite number of natural numbers less than $n$ and since every element of $A$ is a reciprocal of a natural number, there are a finite number of elements $x$ of $A$ such that $\frac{1}{n} < x$. All the other elements of $A$ are less than $b$ so they are contained in $(a;b)$. For each element of $A$ greater than $\frac{1}{n}$ there exists a set in $\mathcal{A}$ containing that element. There are finitely many of these elements so there exist finitely many sets of $\mathcal{A}$ containing them. So those sets and $B$ form a finite subcover of $\mathcal{A}$. $\square$

**Definition 6 (Compact Set)** *A set $X$ is compact if every open cover of $X$ has a finite subcover.*

**Exercise 7** *Let $\mathcal{A}$ be the set of all regions. Show that no finite subset of $\mathcal{A}$ covers $C$.*

*Proof.* Let $\mathcal{B}$ be a finite subset of $\mathcal{A}$. If $\mathcal{B} = \emptyset$ then it is clear that it is not an open cover for $C$. Then $\mathcal{B} = \{(a_1;b_1), (a_2;b_2), \ldots, (a_n;b_n)\}$. But since there are a finite number of lower boundary points $a_i$ for regions in $\mathcal{B}$, we can order them so that $x$ is a lower boundary point and $x \leq a_i$ for all regions in $\mathcal{B}$. Then $x$ is less than every point in every region in $\mathcal{B}$. But since $C$ has no first point there exists a point $p \in C$ such that $p < x$ and so $C \nsubseteq \bigcup_{(a;b)\in\mathcal{B}}(a;b)$ (A2.3). $\square$

**Exercise 8** *Let $p \in C$ be a point and let $\mathcal{A} = \{\text{ext}(a;b) \mid p \in (a;b)\}$. Show that no finite subset of $\mathcal{A}$ covers $C \backslash p$.*

*Proof.* Let $\mathcal{B}$ be a finite subset of $\mathcal{A}$. Clearly if $\mathcal{B} = \emptyset$ then it is not an open cover for $C \backslash p$. Then $\mathcal{B} = \{\text{ext}(a_1;b_1), \text{ext}(a_2;b_2), \ldots, \text{ext}(a_n;b_n)\}$ such that $p \in (a;b)$ for all $\text{ext}(a;b) \in \mathcal{B}$. Consider the finite set of values of $a_i$ for exteriors in $\mathcal{B}$. Since this set is finite there exists a last point $x$ so that $x \geq a_i$ for all exteriors in $\mathcal{B}$ (2.2). Since regions are nonempty there exists a point $y \in C$ such that $x < y < p$ and so $y \notin \text{ext}(a_i;b_i)$ for any exterior in $T$ (5.8). But then $C \backslash p \nsubseteq \bigcup_{B\in\mathcal{B}} B$. $\square$

**Theorem 9 (Compact Sets Are Bounded)** *If $X \subseteq C$ is not bounded, then $X$ is not compact.*

*Proof.* Let $X \subseteq C$ be a set which is not bounded below and let $\mathcal{A}$ be the set of all regions. Consider a finite subset of $\mathcal{A}$, $\mathcal{B}$. Since $\emptyset$ is bounded below, $X \neq \emptyset$. So in the case where $\mathcal{B} = \emptyset$ we see that $\mathcal{B}$ is not an open cover for $X$. Then $\mathcal{B} = \{(a_1;b_1), (a_2;b_2), \ldots, (a_n;b_n)\}$. But since there are a finite number of lower boundary points $a_i$ for regions in $\mathcal{B}$, we can order them so that $x$ is a lower boundary point and $x \leq a_i$ for all regions in $\mathcal{B}$ (2.2). Then $x$ is less than every point in every region in $\mathcal{B}$. But since $X$ has no lower bound, for all $p \in C$ there exists $q \in X$ such that $q < p$. Therefore there exists a $q \in X$ such that $q < x$ and so $X \nsubseteq \bigcup_{B\in\mathcal{B}} B$. A similar proof holds if $X$ is a set which is not bounded above. $\square$

**Theorem 10 (Compact Sets Are Closed)** *If $X \subseteq C$ is not closed, then $X$ is not compact.*

*Proof.* Let $X \subseteq C$ be a set which is not closed and $p \notin X$ be a limit point of $X$. Let $\mathcal{A} = \{\text{ext}(a;b) \mid p \in (a;b)\}$. Since $p \notin X$ we see that $\mathcal{A}$ covers $X$. Suppose that $\mathcal{B}$ is a finite subset of $\mathcal{A}$. We see that $X \neq \emptyset$ because $\emptyset$ is closed (3.13). So in the case where $\mathcal{B} = \emptyset$ we see that $\mathcal{B}$ does not cover $X$. Then $\mathcal{B} = \{\text{ext}(a_1;b_1), \text{ext}(a_2;b_2), \ldots, \text{ext}(a_n;b_n)\}$. But then the set of lower boundary points $a_i$ and the set of upper boundary points $b_i$ for exteriors in $\mathcal{B}$ are finite. Thus there exists a last point $x$ such that $x$ is a lower boundary point of some exterior in $\mathcal{B}$ and $x \geq a_i$ for all exteriors in $\mathcal{B}$. Likewise there exists a smallest upper boundary point $y$ for exteriors in $\mathcal{B}$. Note that $x$ and $y$ need not define the same exterior in $\mathcal{B}$. But then the region $(x; y)$ must contain $p$ because all lower boundary points are less than $p$ and all upper boundary points are greater than $p$. Since $p$ is a limit point of $X$, then $(x; y)$ also contains a point in $X$. But $(x; y)$ is defined so that $(x; y) \nsubseteq \bigcup_{B \in \mathcal{B}} B$. Therefore $X \nsubseteq \bigcup_{B \in \mathcal{B}} B$ and so $\mathcal{B}$ is not a finite subcover for $\mathcal{A}$ and $X$ is not compact. ☐

**Definition 11** *For $a < b$ let the closed interval $[a; b]$ be defined as*

$$[a; b] = (a; b) \cup \{a\} \cup \{b\}.$$

**Exercise 12** *Closed intervals are closed*

*Proof.* Let $a, b, p \in C$ be points such that $a < b$ and $p \notin [a; b]$. Then $p < a$ or $p > b$. Let $p < a$. Since $C$ has no first point there exists a point $x \in C$ such that $x < p$ (A2.3). But then the region $(x; a)$ contains $x$ but no points in $[a; b]$. A similar argument holds for $b < p$ and so $p$ cannot be a limit point of $[a; b](A2.3)$. But then any limit points of $[a; b]$ must be in $[a; b]$ and so $[a; b]$ is closed. ☐

**Definition 13 (Chain of Regions)** *Let $a < b$. A chain of regions going from $a$ to $b$ is defined as a finite sequence $R_1, R_2, \ldots, R_n$ of regions such that $a \in R_1$, $b \in R_n$ and for $1 \leq i \leq n-1$ we have $R_i \cap R_{i+1} \neq \emptyset$.*

**Exercise 14** *A chain of regions from $a$ to $b$ covers the the closed interval $[a; b]$.*

*Proof.* Let $R_1, R_2, \ldots R_n$ be a chain of regions going from $a$ to $b$ such that $R_i = (p_i; q_i)$. Let $x \in [a; b]$. Then $x$ is greater than a finite number of upper boundary points $q_i$. Consider the set of indexes for these points. If the set is empty then $x \in R_1$. If the set is not empty then we can take the last point of the set $k$ (2.2). By definition $R_k \cap R_{k+1} \neq \emptyset$ and so $p_{k+1} < q_k$. But $q_k < x$ and $x < q_{k+1}$ and so $x \in (p_{k+1}; q_{k+1}) = R_{k+1}$. Therefore, if $x \in [a; b]$ then $x$ is in one of the regions $R_1, R_2, \ldots, R_n$. Thus $[a; b] \subseteq R_1 \cup R_2 \cup \cdots \cup R_n$. Since all regions are open, the chain of regions covers $[a; b]$ (3.16). ☐

**Theorem 15** *Let $a < b$ and let $\mathcal{A}$ be a set of regions that covers $[a; b]$. Let $X = \{x \in [a; b] \mid$ there is a chain of regions $R_1, R_2, \ldots, R_n \in S$ going from $a$ to $x\}$. Then $\sup X = b$. Moreover $b \in X$.*

*Proof.* Since $X \subseteq [a; b]$ we see that $X$ is bounded above by $b$. Therefore $\sup X$ exists (6.11). Let $u = \sup X$. If $u > b$ then we have $b \geq x$ for all $x \in [a; b]$ and thus $b \geq x$ for all $x \in X$. Therefore $b$ is an upper bound of $X$ which is less than $u$. This is a contradiction and so $u \leq b$. Additionally, $X \neq \emptyset$ because there exists a region $R_1 \in \mathcal{A}$ which contains $a$ and so there is a finite chain of regions going from $a$ to all points in $R_1$ greater than or equal to $a$. So we have $a < u$ and $u < b$ so $u \in [a; b]$. Since $\mathcal{A}$ is an open cover of $[a; b]$ there exists a region $R_i \in \mathcal{A}$ such that $u \in R_i$. Because $u = \sup X$ and because regions are nonempty there exists a point $c \in X$ such that $c \in R_i$ and $a < c < u$ (5.8). But then $u \in X$ because $c$ and $u$ are in the same region from $\mathcal{A}$. Now assume to the contrary that $u < b$. Then there exists another point $u' \in R_i$ such that $u < u' < b$ (5.8). But then $u' \in X$ and $u < u'$. This is a contradiction since $u = \sup X$. Therefore $\sup X = b$ and $b \in X$. ☐

**Theorem 16 (Closed Intervals Are Compact With Respect To Regions)** *Let $a < b$. Then any set of regions that covers $[a; b]$ has a finite subcover.*

*Proof.* This follows from Theorem 15 and Exercise 14. Because $b \in X$ we see that there exists a finite chain of regions going from $a$ to $b$ (7.15). Since regions are open sets, this chain forms a finite subcover for $[a; b]$ (7.14). ☐

**Theorem 17** *Let $a < b$ be points in $C$ and let $\mathcal{A}$ be an open cover for $[a; b]$. Let*

$$S = \{(c; d) \mid c < d, \text{ there exists } A \in \mathcal{A} \text{ with } (c; d) \subseteq A\}.$$

*We have*

$$[a; b] \subseteq \bigcup_{(c;d)\in S} (c; d).$$

*Proof.* We know that $\mathcal{A}$ is an open cover for $[a; b]$. Thus, for all $x \in [a; b]$ there exists $A \in \mathcal{A}$ such that $A$ is open and $x \in A$. But by the open condition there exists a region $(c; d) \subseteq A$ such that $x \in (c; d)$ (3.17). Then $x \in \bigcup_{(c;d)\in S}(c; d)$ because $x \in \bigcup_{A\in\mathcal{A}} A$ and $(c; d) \subseteq A$ for all $A \in \mathcal{A}$. Therefore $[a; b] \subseteq \bigcup_{(c;d)\in S}(c; d)$. □

**Corollary 18** *For $(c; d) \in S$ let $A_{(c;d)} \in \mathcal{A}$ such that $(c; d) \subseteq A_{(c;d)}$. We have*

$$[a; b] \subseteq \bigcup_{(c;d)\in S} A_{(c;d)}.$$

*Proof.* From Theorem 17 we have $[a; b] \subseteq \bigcup_{(c;d)\in S}(c; d)$ (7.17). For all $(c; d) \in S$ we have $(c; d) \subseteq A_{(c;d)}$. Therefore $\bigcup_{(c;d)\in S}(c; d) \subseteq \bigcup_{(c;d)\in S} A_{(c;d)}$. And so $[a; b] \subseteq \bigcup_{(c;d)\in S} A_{(c;d)}$. □

**Theorem 19 (Closed Intervals Are Compact)** *For $a < b$ the closed interval $[a; b]$ is compact*

*Proof.* Let $\mathcal{A}$ be an open cover for $[a; b]$ for $a, b \in C$. Define

$$S = \{(c; d) \mid c < d, \text{ there exists } A \in \mathcal{A} \text{ with } (c; d) \subseteq A\}.$$

From Theorem 17 we know that $S$ is a cover for $[a; b]$ (7.17). Since $S$ is composed entirely of regions, by Theorem 16 there exists a finite subcover of $S$ for $[a; b]$. So there exists finitely many regions from $S$ which will form an open cover of $[a; b]$. Call this set $T$. Then for $(c; d) \in T$ let $B_{(c;d)} \in \mathcal{A}$ such that $(c; d) \subseteq B_{(c;d)}$. From Corollary 18 we know that the set of all $A_{(c;d)}$ for $(c; d) \in S$ is an open cover for $[a; b]$ (7.18). But $T \subseteq S$ and so the set of all $B_{(c;d)}$ is a subset of the set of all $A_{(c;d)}$. And because $(c; d) \subseteq B_{(c;d)}$ for all $(c; d) \in T$, and $T$ is an open cover for $[a; b]$ we have $[a; b] \subseteq \bigcup_{(c;d)\in T} B_{(c;d)} \subseteq \bigcup_{(c;d)\in S} A_{(c;d)}$. So the set of all $B_{(c;d)}$ is a finite open subcover for $[a; b]$ because $T$ is finite and $B_{(c;d)} \in \mathcal{A}$ for all $(c; d) \in T$. □

**Theorem 20** *Let $X \subseteq C$ be a closed set and let $\mathcal{A}$ be an open cover of $X$. Then $\mathcal{A} \cup \{C\backslash X\}$ is an open cover of $C$.*

*Proof.* We know that $X = C\backslash(C\backslash X)$ is closed and so $C\backslash X$ is open. Then let $p \in C$. Then $p \in X$ or $p \notin X$. If $p \in X$ then $p \in \bigcup_{A\in\mathcal{A}} A$. If $p \notin X$ then $p \in C\backslash X$. Therefore $p \in \bigcup_{A\in\mathcal{A}} \cup (C\backslash X)$. Thus $C \subseteq \bigcup_{A\in\mathcal{A}} A \cup (C\backslash X)$. Since all the sets in $\mathcal{A} \cup \{C\backslash X\}$ are open, $\mathcal{A} \cup \{C\backslash X\}$ is an open cover for $C$. □

**Theorem 21** *Let $X \subseteq C$ be a set and let $\mathcal{B}$ be an open cover of $X$ such that $C\backslash X \in \mathcal{B}$. Then $\mathcal{B}\backslash\{C\backslash X\}$ is an open cover of $X$.*

*Proof.* There are no points of $X$ which are in $C\backslash X$. Therefore, since $X \subseteq \bigcup_{B\in\mathcal{B}} B$, we also have $X \subseteq \bigcup_{B\in\mathcal{B}, B\neq(C\backslash X)} B$. And so $\mathcal{B}\backslash\{C\backslash X\}$ is an open cover for $X$. □

**Theorem 22 (Bounded Closed Sets Are Compact)** *Let $X \subseteq C$ be a bounded closed set. Then $X$ is compact.*

*Proof.* Let $\mathcal{A}$ be an open cover of $X$. Then from Theorem 20 we have $\mathcal{A} \cup \{C \backslash X\}$ is an open cover of $C$ (7.20). Since $X$ is bounded we see that $\inf X$ and $\sup X$ exist (6.11, 6.12). But then $[\inf X; \sup X] \subseteq C$ and so $\mathcal{A} \cup \{C \backslash X\}$ is an open cover for $[\inf X; \sup X]$. But from Theorem 19 we know that $[\inf X; \sup X]$ is compact and so we let $\mathcal{B} \subseteq \mathcal{A} \cup \{C \backslash X\}$ be a finite subset which covers $[\inf X; \sup X]$ (7.19). Then $X \subseteq [\inf X; \sup X]$ by definition and so $\mathcal{B}$ is an open cover for $X$. But then we know that $\mathcal{B} \subseteq \mathcal{A} \cup \{C \backslash X\}$ and so $\mathcal{B} \backslash \{C \backslash X\} \subseteq \mathcal{A}$. From Theorem 21 we know that $\mathcal{B} \backslash \{C \backslash X\}$ is an open cover for $X$ because $\mathcal{B}$ is an open cover for $X$ (7.21). Since $\mathcal{B} \backslash \{C \backslash A\} \subseteq \mathcal{A}$ is finite we now have a finite open subset of $\mathcal{A}$ which covers $X$ so $X$ is compact. $\square$