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Homework 7

Problem 1. *Let $U \subseteq \mathbb{R}^n$. A path $\phi : I \to U$ is called* piecewise-linear *if there exist $0 = x_0 < x_1 < \cdots < x_n = 1$ such that on every interval $[x_i, x_{i+1}]$, ϕ has the form*

$$\phi(t) = \mathbf{a}_i t + \mathbf{b}_i$$

for some $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{R}^n$. (Note that $\mathbf{a}_i$, $\mathbf{b_i}$ need not lie in U.)

Let U be a connected open subset of $\mathbb{R}^n$. Use the Local-to-Global Lemma to show that there is a piecewise-linear path in U between any two points.

Proof. Define a relation on the points of U where $\mathbf{x} \sim \mathbf{y}$ if and only if there is a piecewise-linear path between $\mathbf{x}$ and $\mathbf{y}$. This relation is reflexive since the constant path is piecewise-linear. The relation is symmetric since reversing the direction of any path from $\mathbf{x}$ to $\mathbf{y}$ is a path from $\mathbf{y}$ to $\mathbf{x}$. The relation is transitive because a path from ψ from $\mathbf{x}$ to $\mathbf{y}$ can be composed with a path ψ from $\mathbf{y}$ to $\mathbf{z}$. This composed path will still be piecewise-linear as the line segments in $\mathbb{R}^n$ remain the same and the intervals in I become $[x_i/2, x_{i+1}/2]$ for ϕ and $[x_i/2 + 1/2, x_{i+1}/2 + 1/2]$ for ψ. Thus $\sim$ is an equivalence relation.

Let $\mathbf{x} \in U$ and consider an ε-ball B around $\mathbf{x}$ contained in U. But all the points $\mathbf{y} \in B$ have a piecewise-linear path connecting them to $\mathbf{x}$. Namely, $\phi(t) = (\mathbf{y} - \mathbf{x})t + \mathbf{x}$. Thus, every point of U has a neighborhood of equivalent points. By the Local-to-Global Lemma there is a piecewise-linear path between any two points in U. $\square$

Problem 2. *(a) Show that every connected proper open set of $\mathbb{R}$ is either an open interval or an open ray.*
(b) Let U be an open subset of $\mathbb{R}^n$. Show that the components of U are open.
(c) Show that every proper open subset of $\mathbb{R}$ is a countable disjoint union of open intervals and (at most two) open rays.

Proof. (a) Let A be a connected open subset of $\mathbb{R}$. Suppose that there exists $a, b \in A$ such that $a < b$ and $c \in (a, b)$ such that $c \notin A$. Then $\{(-\infty, c), (c, \infty)\}$ forms a separation of A. Thus $c \in A$ and every connected subset of $\mathbb{R}$ is convex. Note that if A is not bounded above or below, but is a proper subset of $\mathbb{R}$ then there exists $c \notin A$ and $(-\infty, c)$ and (c, ∞) form a separation of A. Thus A must be bounded above or below.

Suppose first that A is bounded above and below by u and v respectively. Note that v must be a limit point of A because otherwise there would be some neighborhood V of v which didn't intersect A except at v. This set and $\mathbb{R}\backslash V$ would form a separation of A. Likewise, u must be a limit point of A. Then every open neighborhood of v intersects A and every open neighborhood of u intersects A. In particular, points arbitrarily close to u and v are in A and since A is convex, every point between u and v must be in A as well. Therefore $A = (u, v)$. If A isn't bounded above or below, a similar argument holds to show that $A = (u, \infty)$ or $A = (-\infty, v)$ respectively.

(b) Let U be an open subset of $\mathbb{R}^n$ and let C be a component of U. Let $\mathbf{x} \in C$ and consider all the paths which go from $\mathbf{x}$ to points less than ε away from $\mathbf{x}$. Each of these paths form a connected set, so $\mathbf{x}$ is connected to each of these points. But the union of these points is simply the ε-ball around $\mathbf{x}$. This is then contained in C so C is open.

(c) Let U be an open proper subset of $\mathbb{R}$. Using part (b), the components of U are open connected subsets and these are either intervals or open rays by part (a). Note that the components of U are disjoint by definition and since each interval or open ray contains a rational number, there can be at most countably many components. Also, if three of the components of U are rays, then one necessarily contains the other, so there can only be two rays. Therefore U is a countable disjoint union of open intervals and at most two open rays. $\square$

Problem 3. *Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup A_n$ is connected.*

Proof. For each $n \in \mathbb{N}$ there exists some point p_n such that $p_n \in A_n \cap A_{n+1}$. We use induction on n to show that $\bigcup_{i=1}^n A_n$ is connected for every n. For the $n = 1$ case we're done since A_1 is connected by assumption. Suppose $\bigcup_{i=1}^n A_n$ is connected but $\{C, D\}$ is a separation of $\bigcup_{i=1}^{n+1} A_n$. Note that $p_n \in \bigcup_{i=1}^{n+1} A_n$ so without loss of generality suppose $p_n \in C$. Then since $\bigcup_{i=1}^n A_n$ is connected, this entire set must also be in C. But also $p_n \in A_{n+1}$ so $A_{n+1} \subseteq C$ as well. Then $D = \emptyset$ and $\{C, D\}$ isn't a separation. Therefore $\bigcup_{i=1}^n A_n$ is connected for all natural numbers n. Note that $\bigcup_n A_n$ is the union of each of these sets and each set in this union contains some point in A_1. Therefore $\bigcup_n A_n$ is connected as well. $\square$

Problem 4. *Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that*

$$(X \times Y)\backslash(A \times B)$$

is connected.

Proof. Let $Z = (X \times Y)\backslash(A \times B)$. Choose $a \in X\backslash A$ and $b \in Y\backslash B$ and form the sets $\{a\} \times Y$ and $X \times \{b\}$. Each of these sets is homeomorphic to a connected set so they're both connected. Let $T = (\{a\} \times Y) \cup (X \times \{b\})$ and note that T is the union of two connected sets intersecting in (a, b) so T is connected. Now choose an arbitrary point $(x, y) \in Z$ and note either $x \notin A$ or $y \notin B$. If $x \notin A$ then note that $A_x = \{x\} \times Y$ is a subset of Z. Otherwise, if $y \notin B$ then note that $A_y = X \times \{y\}$ is a subset of Z. But each A_x and A_y is homeomorphic to Y or X and is thus connected. Each A_x and A_y intersects T at (x, b) or (a, y). Therefore the collection $\{(A_x \cup T), (A_y \cup T) \mid (x, y) \in Z\}$ is a set of connected sets which all intersect at the point (a, b). Their union must then be connected. But this union is Z. $\square$

Problem 5. *(a) Show that no two of the spaces $(0, 1)$, $(0, 1]$, and $[0, 1]$ are homeomorphic.*
(b) Suppose that there exist imbeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.
(c) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Proof. (a) Note that removing any point x from $(0, 1)$ results in a disconnected space with separation $\{(-\infty, x) \cap (0, 1), (x, \infty) \cap (0, 1)\}$. But if we remove 1 from $(0, 1]$ or $[0, 1]$ we get connected spaces since these are intervals in $\mathbb{R}$. Thus $(0, 1)$ is not homeomorphic to $(0, 1]$ or $[0, 1]$. Furthermore, removing any two points from $(0, 1]$ results in a disconnected space since at least one of them must be some $x \in (0, 1)$ and $\{(-\infty, x) \cap (0, 1], (x, \infty) \cap (0, 1]\}$ is a separation of this space. But we can remove the points 0 and 1 from $[0, 1]$ and still have a connected space, so $(0, 1]$ cannot be homeomorphic to $[0, 1]$. Thus, no two of these spaces are homeomorphic.

(b) Let $f : (0, 1) \to [0, 1]$ be the identity and $g : [0, 1] \to (0, 1)$ be given by $g(x) = 1/4 + x/2$. We see that f is clearly a homeomorphism onto it's image as is g since it simply scales open intervals to make them smaller, but still open. But by part (a) we know $(0, 1)$ and $[0, 1]$ are not homeomorphic.

(c) We know $\mathbb{R}^n\backslash\{0\}$ for $n > 1$ is a connected space. It follows that $\mathbb{R}^n$ without any single point x is still connected. On the other hand, $\mathbb{R}\backslash\{0\}$ is disconnected. So $\mathbb{R}^n$ is connected after removing one point and $\mathbb{R}$ is not. Thus the two spaces can't be homeomorphic. $\square$

Problem 6. *Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.*

Proof. Note that S^1 is connected set since it's clearly path connected. Consider the function $g(x) = f(x) - f(-x)$ and let $a \in S^1$. Note that $g(x) = -g(-x)$. If $g(a) = 0$ then we're clearly done. Suppose that $g(a) > 0$. Then $g(-a) = -g(a) < 0$. On the other hand, if $g(a) < 0$ then $g(-a) = -g(a) > 0$. In both cases since S^1 is connected there must exist some $b \in S^1$ such that $g(b) = 0$. Thus $f(b) = f(-b)$ and we're done. $\square$

Problem 7. *Assume that $\mathbb{R}$ is uncountable. Show that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2\backslash A$ is path connected.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. There are two cases to consider. If $\mathbf{x}$ and $\mathbf{y}$ are not collinear with some point $\mathbf{a} \in A$, then we're done since the line connecting $\mathbf{x}$ and $\mathbf{y}$ serves as a path from $\mathbf{x}$ to $\mathbf{y}$. Otherwise, note that there are uncountably many lines in $\mathbb{R}^2$ intersecting $\mathbf{x}$ and only countably many points of A. Therefore, at least one of these lines passing through $\mathbf{x}$ is not collinear with any point of A. Call it l. Likewise, at least two distinct lines m and n passing through $\mathbf{y}$ contain no points of A. Note that only one of m or n is possibly parallel to l, so we can assume m is not parallel to l. Thus l and m intersect in some point $\mathbf{z}$. Then the line from $\mathbf{x}$ to $\mathbf{z}$ composed with the line from $\mathbf{z}$ to $\mathbf{y}$ is a path in $\mathbb{R}^2$ from $\mathbf{x}$ to $\mathbf{y}$ which doesn't intersect A. Therefore it's a path in $\mathbb{R}^2 \backslash A$ and this set is path connected. $\square$