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Homework 1

**Problem 1.** $\dot{x} = 0.05x$, $x(0) = 100$. *This is the result of "continuous compounding" as described in Example 1.1.5 above. Evaluate the solution numerically at $t = 1$, i.e., at the end of one year, and compare this with monthly compounding as given in equation (1.7).*

Since $dx/dt = .05x$, $dx/dt(1/x) = .05$. Integrating both sides gives

$$.05t - .05t_0 = \int_{t_0}^{t} \frac{dx}{dt}\frac{1}{x}dt = \int_{t_0}^{t} \frac{d}{dt}\ln(x)dt = \ln(x(t)) - \ln(x(t_0)).$$

Then $x(t)/x(t_0) = e^{.05(t-t_0)}$ and $x(t) = x(t_0)e^{.05(t-t_0)}$. Setting $t_0 = 0$ we have $x(t) = 100e^{.05t}$. At $t = 1$ this is $x(1) = 100e^{.05} \approx 105.13$. This is nearly the same as the monthly compounding in equation (1.7).

**Problem 2.** $\dot{x} = -tx + 1$, $x(1) = 0$.

We multiply both sides by $P(t) = \exp\left(-\int_{t_0}^{t}(-s)ds\right) = \exp\left(t^2/2 - t_0^2/2\right)$. This gives

$$P(t) = \frac{dx}{dt}P(t) + P(t)tx = \frac{d}{dt}(P(t)x).$$

Now integrate from $t_0$ to $t$ and multiply each side by $1/P(t)$ to get

$$\begin{aligned} x(t) &= \exp\left(\int_{t_0}^{t} -sds\right)\left(x(t_0) + \int_{t_0}^{t}\exp\left(-\int_{t_0}^{s} -udu\right)ds\right) \\ &= e^{-(t^2-t_0^2)/2}\left(x(t_0) + \int_{t_0}^{t} e^{(s^2-t_0^2)/2}ds\right). \end{aligned}$$

Putting in $t_0 = 1$ we have

$$x(t) = e^{(1-t^2)/2}\int_{1}^{t} e^{(s^2-1)/2}ds.$$

**Problem 3.** *Suppose that $x_1(t)$ and $x_2(t)$ are both solutions of equation (1.16). Show that $x(t) = c_1x_1(t) + c_2x_2(t)$ is then also a solution, for arbitrary values of the constants $c_1$ and $c_2$.*

*Proof.* Since $x_1(t)$ and $x_2(t)$ are both solutions of equation (1.16), we have

$$\frac{dx_1}{dt} - k(t)x_1 = \frac{dx_2}{dt} - k(t)x_2 = 0.$$

Then since the derivative is a linear operator,

$$\begin{aligned} \frac{dx}{dt} - k(t)x &= \frac{d}{dt}(c_1x_1(t) + c_2x_2(t)) - k(t)(c_1x_1(t) + c_2x_2(t)) \\ &= c_1\frac{dx_1}{dt} + c_2\frac{dx_2}{dt} - c_1k(t)x_1 - c_2k(t)x_2 \\ &= c_1\left(\frac{dx_1}{dt} - k(t)x_1\right) + c_2\left(\frac{dx_2}{dt} - k(t)x_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0. \end{aligned}$$

Thus $x(t)$ also satisfies equation (1.16). ☐

**Problem 4.** *An* equilibrium *solution is one that is constant, i.e. does not depend on the independent variable. Therefore if $\tilde{x}$ is such a solution, $\dot{\tilde{x}} = 0$ for all $t$. Find an equilibrium solution $\tilde{x}$ of the equation of Problem 7 above. Show that, if $x(t)$ is* any *solution of this equation, $x(t) \to \tilde{x}$ as $t \to +\infty$.*

*Proof.* As in Problem 2 above, the general solution to Problem 7 is of the form

$$\begin{aligned}
x(t) &= \exp\left(\int_{t_0}^{t} -\alpha ds\right)\left(x(t_0) + \int_{t_0}^{t} \exp\left(-\int_{t_0}^{s} -\alpha du\right)\gamma ds\right) \\
&= e^{\alpha(t_0-t)}\left(x(t_0) + \gamma\int_{t_0}^{t} e^{\alpha(s-t_0)} ds\right) \\
&= e^{\alpha(t_0-t)}\left(x(t_0) + \frac{\gamma}{\alpha}\left(e^{\alpha(t-t_0)} - 1\right)\right) \\
&= e^{-\alpha t}\left(1 + \frac{\gamma}{\alpha}\left(e^{\alpha t} - 1\right)\right) \\
&= e^{-\alpha t} + \frac{\gamma}{\alpha} - e^{-\alpha t}\frac{\gamma}{\alpha}.
\end{aligned}$$

A equilibrium solution is them $\tilde{x}(t) = \gamma/\alpha$ since then

$$\frac{dx}{dt} + \alpha x - \gamma = 0 + \alpha\frac{\gamma}{\alpha} - \gamma = 0.$$

Furthermore, for general solution for $x(t)$, we must have $x(t) \to \tilde{x}$ as $t \to +\infty$ since both terms $e^{-\alpha t} \to 0$ as $t \to +\infty$ and this only leaves the $\gamma/\alpha = \tilde{x}(t)$ term. □

**Problem 5.** *For the partial differential equation*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \alpha u,$$

*where $\alpha$ is a constant, introduce polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$. With $u(x,y) = v(r,\varphi)$, deduce that the equation takes the form*

$$r\frac{\partial v}{\partial r} = \alpha v.$$

*Give the general solution of this equation, noting that it involves not an arbitrary constant, but an arbitrary function.*

*Proof.* Using the chain rule we have

$$r\frac{\partial v}{\partial r} = r\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right) = r\frac{\partial v}{\partial x}\cos\varphi + r\frac{\partial v}{\partial y}\sin\varphi = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}.$$

The general solution is given by $v(r,\varphi) = f(\varphi)r^{\alpha}$ where $f(\varphi)$ is an arbitrary function with no dependence on $r$. □

**Problem 6.** *Apply the result of the preceding problem to the case*

$$k \equiv 1, \quad a(t) = \sin t, \quad (T = 2\pi)$$

*to find an initial value $x(0) = x_0$ such that $x(T) = x_0$, and verify that the solution is periodic in this case and only this case.*

*Proof.* We have the equation $dx/dt = x + \sin(t)$ which has the general solution

$$\begin{aligned} x(t) &= x(t_0)e^{t-t_0} + e^{t-t_0}\int_{t_0}^{t} \sin(s)e^{t_0-s}ds \\ &= x(t_0)e^{t-t_0} + \frac{1}{2}\left(e^{t-t_0}\sin(t_0)+\cos(t_0)\right) - \frac{1}{2}(\sin(t)+\cos(t)) \\ &= e^{t-t_0}\left(x(t_0) + \frac{1}{2}(\sin(t_0)+\cos(t_0))\right) - \frac{1}{2}(\sin(t)+\cos(t)). \end{aligned}$$

Putting in $t_0 = 0$ and $t_0 = T$ we get

$$e^t\left(x_0 + \frac{1}{2}\right) - \frac{1}{2}(\sin(t)+\cos(t)) = e^{t-2\pi}\left(x_0+\frac{1}{2}\right) - \frac{1}{2}(\sin(t)+\cos(t))$$

which simplifies to

$$x_0 + \frac{1}{2} = e^{-2\pi}\left(x_0 + \frac{1}{2}\right).$$

The only possible solution to this equation is $x_0 = -1/2$. The particular solution is then

$$\begin{aligned} x(t) &= e^t\left(-\frac{1}{2} + \frac{1}{2}(\sin(0)+\cos(0))\right) - \frac{1}{2}(\sin(t)+\cos(t)) \\ &= -\frac{\sin(t)+\cos(t)}{2}. \end{aligned}$$

This solution is clearly periodic since $\sin(t)$ and $\cos(t)$ are periodic. Suppose $x(t)$ is periodic for some other initial value so that $x_0 = x(0) = x(P)$. Then we have

$$\left(x_0 + \frac{1}{2}\right) = e^{-P}\left(x_0 + \frac{1}{2}(\sin(P)+\cos(P))\right).$$

Assuming $x_0 \neq -1/2$, solving for $x_0$ gives

$$x_0 = \frac{\frac{1}{2}(\sin(P)+\cos(P)-1)}{1-e^{-P}}.$$

Plugging this back in for $x(t)$ and simplifying gives $x(0) = x_0$ and

$$x(P) = \frac{(\sin(P)+\cos(P))(e^P - e^{-P}+1)-1}{2(1-e^{-P})}.$$

The only way this is equal to $x_0$ is if $P = 0$. Thus $x(t)$ is only periodic for the above value of $x_0$. ☐

**Problem 7.** *A variant of the simple model of population growth in Example 1.2.1 asserts that birth rates get higher with overcrowding, so that the equation (1.18), where $k$ is constant, should be replace by the equation $\dot{x} = k_0 x^{1+\epsilon}$, i.e., $k$ should be replaced by $k_0 x^\epsilon$ to allow for enhanced birthrates with increasing population. Here $k_0$ and $\epsilon$ are positive constants. Solve the initial-value problem for this equation explicitly and show that it predicts infinite population in finite time.*

*Proof.* Simplifying we get

$$k_0 = \frac{dx}{dt}\frac{1}{x^{1+\epsilon}} = -\frac{1}{\epsilon}\frac{d}{dt}\left(\frac{1}{x^\epsilon}\right).$$

Integrating from $t_0$ to $t$ we have

$$-k_0\epsilon(t-t_0) = \frac{1}{(x(t))^\epsilon} - \frac{1}{(x(t_0))^\epsilon}.$$

Solving for $x(t)$ gives

$$(x(t))^\epsilon = \frac{(x(t_0))^\epsilon}{(x(t_0))^\epsilon k_0\epsilon(t_0 - t) + 1}.$$

Note that this function has a vertical asymptote at $t = t_0 + 1/((x(t_0))^\epsilon k_0 \epsilon)$, so after this amount time, the population will reach infinity. □

**Problem 8.** *Let $\phi$ be an arbitrary function defined and continuously differentiable ($C^1$) on an interval $I$ of the $x$ axis, and suppose that it is not constant there. Show that, at least on some subinterval $I'$ of $I$, there is a continuous function $f$ such that the differental equation $y' = f(y)$ possesses the solution $y = \phi(x)$ on $I'$.*

*Proof.* Since $\phi$ is nonconstant, there is some point $a \in I$ for which $\phi'(a) > 0$. (Here we're choosing greater than over less than without loss of generality). Then since $\phi'$ is continuous there exists an interval $I'$ such that $a \in I'$ and $\phi'(x) > 0$ for all $x \in I'$. Then $\phi$ is increasing on $I'$, so it has a well defined inverse function, $\phi^{-1}$ on this interval. Define $f = \phi' \circ \phi^{-1}$. Since $\phi$ and $\phi'$ are both continuous, we see that $f$ is continuous. Then on $I'$ we have $y' = \phi' = \phi' \circ \phi^{-1} \circ \phi = f \circ \phi = f(y)$. □

**Problem 9.** *Suppose that $p(x, y)$ and $q(x, y)$ are both integrating factors for the form $M(x, y, )dx+N(x, y)dy$. Show that $\alpha p + \beta q$ is also an integrating factor, for arbitrary constants $\alpha$ and $\beta$.*

*Proof.* Since $p(x, y)$ and $q(x, y)$ are integrating factors, we know

$$\frac{\partial(pM)}{\partial y} - \frac{\partial(pN)}{\partial x} = \frac{\partial(qM)}{\partial y} - \frac{\partial(qN)}{\partial x} = 0.$$

Now, since the partial derivative is a linear operator,

$$\begin{aligned}
\frac{\partial}{\partial y}((\alpha p + \beta q)M) - \frac{\partial}{\partial x}((\alpha p + \beta q)N) &= \alpha\frac{\partial(pM)}{\partial y} + \beta\frac{\partial(qM)}{\partial y} - \alpha\frac{\partial(pN)}{\partial x} - \beta\frac{\partial(qN)}{\partial x} \\
&= \alpha\left(\frac{\partial(pM)}{\partial y} - \frac{\partial(pN)}{\partial x}\right) + \beta\left(\frac{\partial(qM)}{\partial y} - \frac{\partial(qN)}{\partial x}\right) \\
&= \alpha \cdot 0 + \beta \cdot 0 \\
&= 0.
\end{aligned}$$

□

**Problem 10.** *Let $M(x, y) = yf(xy)$ and $N(x, y) = xg(xy)$, where $f(v)$ and $g(v)$ are functions of a single real variable $v$, defined and continuously differentiable for all real values of $v$. Under what conditions on $f$ and $g$ is the form $Mdx + Ndy$ exact for all values of $x$, $y$ in the plane? In that case, find the function $u(x, y)$ such that equation (1.33) holds, and use that information to infer the general solution to the equation $y' = -(M(x, y)/N(x, y))$.*

*Proof.* We have

$$\frac{\partial M}{\partial y} = xyf_y(xy) + f(xy)$$

and

$$\frac{\partial N}{\partial x} = xyg_x(xy) + g(xy).$$

If the form is exact, then

$$f(xy) + xyf_y(xy) = g(xy) + xyg_x(xy)$$

or

$$f(xy) - g(xy) = -xy(f_y(xy) - g_x(xy)).$$

Assuming this condition on exactness, we know there exists a function $u(x, y)$ such that

$$du = M(x, y)dx + N(x, y)dy = yf(xy)dx + xg(xy)dy.$$

Then $\frac{\partial u}{\partial x} = yf(xy)$ so $u(x, y) = \int f(xy)dx + c_1(y)$ where $c_1$ depends only on $y$. Then

$$\frac{\partial u}{\partial y} = xf(xy) + c_1'(y) = xg(xy)$$

but since we don't know if $xg(xy)$ depends on $x$, we can't claim $c_1'(y) = 0$. Thus $u(x, y) = \int f(xy)dx + \int g(xy)dy + c_1(y) + c_2(x)$ where $c_1$ and $c_2$ depend only on $x$ and $y$. Since the equation was exact, we now have $\int f(xy)dx + \int g(xy)dy = c_1(y) + c_2(x) + c_3$, where $c_3$ is a constant and we've relabeled $c_1$ and $c_2$. This gives an implicit equation for $y$. □

**Problem 11.** *Use the technique of the preceding problem to reduce the Riccati initial-value problem*

$$y' - y + 2x^{-3}y^2 = x^2, \quad y(1) = 0$$

*to a specific linear, inhomogeneous problem.*

Make the substitution $y = -x^2 + 1/u$. Then $y' = -2x - u'/u^2$ and $y^2 = x^4 + 1/u^2 - 2x^2/u$. The equation now reduces to

$$\begin{aligned} 0 &= -2x - \frac{u'}{u^2} + x^2 - \frac{1}{u} + 2x^{-3}\left(x^4 + \frac{1}{u^2} - \frac{2x^2}{u}\right) - x^2 \\ &= -2x - \frac{u'}{u^2} - \frac{1}{u} + 2x + \frac{2}{x^3u^2} - \frac{4}{xu} \\ &= u' + u - \frac{2}{x^3} + \frac{4u}{x} \end{aligned}$$

and then

$$u' = -u\left(1 + \frac{4}{x}\right) + \frac{2}{x^3}, \quad u = \frac{1}{y - x^2}, \quad y(1) = 0$$

is a linear, inhomogeneous problem.

**Problem 12.** *Consider the linear, second-order, differential equation*

$$u'' + a(x)u' + b(x)u = 0,$$

*and put*

$$u = \exp\left\{\int^x y(s)ds\right\}.$$

*Show that $y$ satisfies a Riccati equation.*

*Proof.* Note that

$$u' = y(x)\exp\left(\int^x y(x)ds\right) = y(x)u$$

and

$$u'' = y'(x)\exp\left(\int^x y(x)ds\right) + (y(x))^2\exp\left(\int^x y(x)ds\right) = y'(x)u + (y(x))^2u.$$

Then the equation reduces to

$$0 = y'(x)u + (y(x))^2u + a(x)y(x)u + b(x)u = y' + y^2 + a(x)y + b(x)$$

or

$$y' + a(x)y + y^2 = -b(x)$$

which is the form of a Riccati equation. □