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MATH 25800
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Problem 1 (11.3.3)**.** *Let S be any subset of V^* for some finite dimensional space V. Define* $\mathrm{Ann}(S) = \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}$. *(*$\mathrm{Ann}(S)$ *is called the* annihilator *of S in V).*
(a) Prove that $\mathrm{Ann}(S)$ *is a subspace of V.*
(b) Let W_1 and W_2 be subspaces of V^*. Prove that $\mathrm{Ann}(W_1 + W_2) = \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$ *and* $\mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.
(c) Let W_1 and W_2 be subspaces of V^*. Prove that $W_1 = W_2$ if and only if $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$.

Proof. (a) Note that $0 \in \mathrm{Ann}(S)$ so $\mathrm{Ann}(S) \neq \emptyset$. Let $v, u \in \mathrm{Ann}(S)$ and let $r \in F$. Let $f \in V^*$. Then $f(rv + u) = rf(v) + f(u) = 0$ because f is linear. Thus $rv + u \in \mathrm{Ann}(S)$ as well and $\mathrm{Ann}(S)$ is a subspace of V.

(b) Let $v \in \mathrm{Ann}(W_1 + W_2)$. Then for each $f \in W_1$ and $g \in W_2$ we have $0 = (f + g)(v) = f(v) + g(v)$. In particular, if g is the zero function, then $f(v) = 0$ necessarily. The same is true for g and so $f(v) = g(v) = 0$. Thus $v \in \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$. Conversely, suppose $v \in \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$. Then for each $f \in W_1$ and $g \in W_2$ we have $f(v) = g(v) = 0$. But then $f(v) + g(v) = (f + g)(v) = 0$ and $v \in \mathrm{Ann}(W_1 + W_2)$.

Now note that

$$\begin{aligned}
\mathrm{Ann}(W_1 \cap W_2) &= \{v \in V \mid f(v) = 0 \text{ for all } f \in W_1 \cap W_2\} \\
&= \{v \in V \mid (f + g)(v) = 0 \text{ for all } f \in W_1 \text{ and } g \in W_2\} \\
&= \{v + u \in V \mid f(v) = 0 \text{ for all } f \in W_1 \text{ and } g(u) = 0 \text{ for all } g \in W_2\} \\
&= \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2).
\end{aligned}$$

(c) Suppose $W_1 = W_2$. Let $v \in \mathrm{Ann}(W_1)$ and let $f \in W_2$. Since $W_1 = W_2$, $f \in W_1$ as well, so $f(v) = 0$. Since f was arbitrary, $v \in \mathrm{Ann}(W_2)$. The second inclusion holds similarly. Conversely, suppose $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$. Let $f \in W_1$. Since W_1 and W_2 are subspace of V^* and f agrees with every function of W_2 on the vectors in $\mathrm{Ann}(W_1)$, it follows that $f \in W_2$ as well. The second inclusion follows similarly. □

Problem 2. *(a) Prove that the elementary row operations have the following effect on determinants:*
(i) Interchanging two rows changes the sign of the determinant.
(ii) Add a multiple of a row to another does not alter the determinant.
(iii) Multiplying any row by a nonzero element u from F multiplies the determinant by u.
(b) Prove that $\det A$ *is nonzero if and only if A is row equivalent to the $n \times n$ identity matrix. Suppose A can be row reduced to the identity matrix using a total of s row interchanges as in (i) and by multiplying rows by the nonzero elements $u_1, u_2, \dots, u_t$ as in (iii). Prove that* $\det A = (-1)^s (u_1 u_2 \dots u_t)^{-1}$.

Proof. (a) These all follow from the fact that det is alternating and multilinear and that $\det(A) = \det(A^t)$. In particular, if A is a matrix with columns $A_1, \dots A_n$, then we know

$$\det(A_1, \dots, A_i, \dots, A_j, \dots, A_n) = -\det(A_1, \dots, A_j, \dots, A_i, \dots, A_n).$$

Also,

$$\det(A_1, \dots, A_i + kA_j, \dots, A_n) = \det(A_1, \dots, A_i, \dots, A_n) + k\det(A_1, \dots, A_j, \dots, A_j, \dots, A_n) = \det(A) + 0.$$

Finally, note

$$\det(A_1, \dots, kA_i, \dots, A_n) = k\det(A_1, \dots, A_i, \dots, A_n).$$

Looking at A^t instead of A will translate all of these statements about columns to rows.

(b) Suppose A is row equivalent to the identity. Then a finite number of the operations (i), (ii) and (iii) will result in the identity matrix. Part (a) states that the determinant of A will then be a finite number of constants multiplied together with a possible sign change. Conversely, if $\det A$ is nonzero then the rows $A_1, \dots, A_n$ must be linearly independent. This means we can perform row operations on them until we end up with the identity matrix. In particular, if A is reduced to the identity in s row interchanges, then part (a) tells us that $\det(A)$ will change by $(-1)^s$ from $\det(I)$. If A is reduced by multiplying by the elements $u_1, \dots, u_t$, then part (a) tells us $\det(A)$ will change by $u_1 \dots u_t$ from $\det(I)$. Thus, in total $\det(A) = (-1)^s (u_1 \dots u_t)^{-1} \det(I) = (-1)^s (u_1 \dots u_t)^{-1}$. □

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Problem 3. *Compute the determinants of the following matrices using row reduction.*

Proof. First we consider A. Interchange the first and third rows and add the second row to the first.

$$\begin{pmatrix} 1 & 4 & 0 \\ -2 & 0 & 2 \\ 5 & 4 & -6 \end{pmatrix}$$

Add twice the first row to the second row and -5 times the first row to the third row.

$$\begin{pmatrix} 1 & 4 & 0 \\ 0 & 8 & 2 \\ 0 & -16 & -6 \end{pmatrix}$$

Add twice the second row to the third row. Multiply the third row by $-1/2$. Add -2 times the third row to the second row.

$$\begin{pmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply the second row by $1/8$ and add -4 times this row to the first row.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In all, we've interchanged rows once ance multiplied by $-1/2$ and $1/8$. Thus, $\det(A) = 16$. Now consider B. Interchange the first and third rows. Add -2 times the first row to the second row and -1 times the first row to the third row.

$$\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & -1 & 2 & -4 \\ 0 & 2 & -5 & 6 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

Add the second row to the fourth row and two times the second row to the third row. Multiply the second row by -1.

$$\begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Multiply the last row by -1, add twice this to the third row. Multiply the third row by -1. Add twice this to the second row and -4 times the last row to the second row. Add -1 times the third row and two times the fourth row to the first row.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In all, we've interchanged rows once and multiplied by -1 three times. Thus, $\det(B) = 1$. $\square$