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Homework 3

Problem 1. *Let V be a 3-dimensional vector space over a field k. Show that the projective plane $\mathbb{P}(V)$ satisfies our axioms (1) and (3) but not the parallel postulate (2); instead, show (2'): any two lines intersect in a unique point.*

Proof. Let $L_1 = \langle v_1 \rangle$ and $L_2 = \langle v_2 \rangle$ be distinct. Then these two 1-dimensional subspaces of V are two points in $\mathbb{P}(V)$. Since L_1 and L_2 are distinct, v_1 and v_2 are independent so the space $\langle v_1, v_2 \rangle = \langle v_1 \rangle + \langle v_2 \rangle$ is a 2-dimensional subspace of V containing L_1 and L_2. Suppose L_1 and L_2 are contained in some other 2-dimensional subspace different from $L_1 + L_2$. Then there is some vector $w \in L_1 + L_2$ which is not in this subspace. Writing $w = \alpha v_1 + \beta v_2$ we see that this subspace cannot possibly contain L_1 and L_2. Thus L_1 and L_2 form a unique line in $\mathbb{P}(V)$ so axiom (1) is satisfied.

Since V is 3-dimensional we can take b_1, b_2 and b_3 to be a basis for V. Then these three vectors are independent and $L_1 = \langle b_1 \rangle$, $L_2 = \langle b_2 \rangle$ and $L_3 = \langle b_3 \rangle$ are three distinct points of $\mathbb{P}(V)$. We know that $L_3 \nsubseteq L_1 + L_2$ because b_1, b_2 and b_3 are a basis so no proper subset of them can span V. Thus we cannot express points in L_3 as a linear combination of points in L_1 and L_2. Thus there exist three distinct points forming three distinct lines so axiom (3) is satisfied.

Take any two distinct lines $P_1 = \langle v_1, v_2 \rangle$ and $P_2 = \langle w_1, w_2 \rangle$. These are two distinct 2-dimensional subspaces of V so their intersection has dimension at most 1. Note that v_1 and v_2 are independent and so are w_1 and w_2, but we can't have four linearly independent vectors in a 3-dimensional vector space. Without loss of generality then we can write $v_1 = \alpha w_1 + \beta w_2$. But then $\langle \alpha w_1 + \beta w_2 \rangle \subseteq P_1 \cap P_2$ so P_1 and P_2 intersect in a single point. Thus axiom (4) is satisfied.

Now note that if P is a line in $\mathbb{P}(V)$ and L is some point off of P then any line containing L will be distinct from P and intersect P in a unique point, hence cannot be parallel to P. Thus axiom (2) is not satisfied. $\square$

Problem 2. *Show that the only field automorphism of $\mathbb{R}$ is the identity.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be an automorphism. Note that for $r \in \mathbb{R}$, $f(r) = f(0 + r) = f(0) + f(r)$ so $f(0) = 0$. Likewise, $f(r) = f(1 \cdot r) = f(1)f(r)$ so $f(1) = 1$. It immediately follows that $f(\mathbb{Z}) = \mathbb{Z}$ since any integer $n = 1 + \cdots + 1 = f(1) + \cdots + f(1) = f(1 + \cdots + 1) = f(n)$ where there are n terms in the sum. Now let $p \in \mathbb{Q}$ such that $p = a/b$. Note that $f(p) = f(a/b) = f(a)f(1/b) = a(1/f(b)) = a(1/b) = a/b = p$. Thus $f(\mathbb{Q}) = \mathbb{Q}$.

Let $a \leq b$ in $\mathbb{R}$ so that $b - a \geq 0$. But then we know $(b - a) = c^2$ for some nonnegative real number c. Thus $f(b - a) = f(c^2) = f(c)^2 \geq 0$. Thus f must preserve order on $\mathbb{R}$. Now let $r \in \mathbb{R}$ and suppose $r < f(r)$. Choose $p \in \mathbb{Q}$ so that $r < p < f(r)$. But then $f(r) < f(p) = p$ contrary to our assumption. A similar proof holds if $f(r) < r$. Thus $r = f(r)$ and f is the identity. $\square$

Problem 3. *In our classifications of the collineations of $\mathbb{P}(V)$ when V has dimension at least 3, we defined a map $\theta : k \to k$ by the requirement that*

$$\sigma(\langle e_1 + x_2 e_2 \rangle) = (\langle f_1 + \theta(x_2) f_2 \rangle).$$

We could have defined θ_i for any $i > 2$ in the same way, but replacing e_2 and f_2 by e_i and f_i. Show that $\theta_i = \theta$.

Proof. Consider the line $\langle xe_2 - xe_i \rangle$. This lies in $\langle e_2 \rangle + \langle e_i \rangle$ and also in $\langle e_1 + xe_2 \rangle + \langle e_1 + xe_i \rangle$. Under σ then it is spanned by some vector of $\langle f_2 \rangle + \langle f_i \rangle$ and also by some vector of $\langle f_1 + \theta(x) f_2 \rangle + \langle f_1 + \theta_i(x) f_i \rangle$. Thus the image line must be $\langle \theta(x) f_2 - \theta_i(x) f_i \rangle$. On the other hand $\langle xe_2 - xe_i \rangle = \langle e_2 - e_i \rangle$ and the image of this line must be $\langle \theta(1) f_2 - \theta_i(1) f_i \rangle = \langle f_2 - f_i \rangle$. Thus we have $\langle \theta(x) f_2 - \theta_i(x) f_i \rangle = \langle f_2 - f_i \rangle$ so $\theta(x) = \theta_i(x)$. $\square$

Problem 4. *Show that*

$$\sigma(\langle e_1 + x_2e_2 + \cdots + x_ne_n\rangle) = (\langle f_1 + \theta(x_2)f_2 + \cdots + \theta(x_n)f_n\rangle)$$

and

$$\sigma(\langle x_2e_2 + \cdots + x_ne_n\rangle) = (\langle \theta(x_2)f_2 + \cdots + \theta(x_n)f_n\rangle).$$

Proof. We proceed by induction with the case $n = 2$ being done already. Suppose $\sigma(\langle e_1 + x_2e_2 + \cdots + x_{n-1}e_{n-1}\rangle) = \langle f_1 + \theta(x_2)f_2 + \cdots + \theta(x_{n-1})f_{n-1}\rangle$. The line $\langle e_1 + x_2e_2 + \cdots + x_ne_n\rangle$ lies in $\langle e_1 + x_2e_2 + \cdots + x_{n-1}e_{n-1}\rangle + \langle e_n\rangle$. Since it's distinct from $\langle e_n\rangle$ it's image is the span of some vector of the form $f_1 + \theta(x_2)f_2 + \cdots + \theta(x_{n-1})f_{n-1} + yf_n$. But note that this line is also in $\langle e_1 + x_ne_n\rangle + \langle e_2\rangle + \cdots + \langle e_{n-1}\rangle$ so it has image in $\langle f_1 + \theta(x_n)f_n\rangle + \langle f_2\rangle + \cdots + \langle f_{n-1}\rangle$. Since the image includes f_1 we must have $y = \theta(x_n)$ so that

$$\sigma(\langle e_1 + x_2e_2 + \cdots + x_ne_n\rangle) = (\langle f_1 + \theta(x_2)f_2 + \cdots + \theta(x_n)f_n\rangle).$$

The image of $\langle x_2e_2 + \cdots + x_ne_n\rangle$ lies in $\langle f_2\rangle + \cdots + \langle f_n\rangle$. But note this line also lies in $\langle e_1 + x_2e_2 + \cdots + x_ne_n\rangle + \langle e_1\rangle$ so its image is also in $\langle f_1 + \theta(x_2)f_2 + \cdots + \theta(x_n)f_n\rangle + \langle f_1\rangle$. It then follows that $\sigma(\langle x_2e_2 + \cdots + x_ne_n\rangle) = (\langle \theta(x_2)f_2 + \cdots + \theta(x_n)f_n\rangle)$. □