Kris Harper
MATH 20900
June 2, 2009

Homework 9

**** Problem 1.** *For a Hilbert space V, show that the norm defined by $||v|| = (v|v)^{1/2}$ is an actual norm.*

Proof. We already know $(v|v) \geq 0$ and $(v|v) = 0$ if and only if $v = 0$. Thus, $||v|| = (v|v)^{1/2} \geq 0$ and $||v|| = 0$ means $(v|v) = 0$ and so $v = 0$. Now consider $\alpha \in \mathbb{C}$. We have

$$||\alpha v|| = (\alpha v|\alpha v)^{\frac{1}{2}} = (\alpha(v|\alpha v))^{\frac{1}{2}} = (\alpha\overline{\alpha}(v|v))^{\frac{1}{2}} = |\alpha|(v|v)^{\frac{1}{2}} = |\alpha|||v||.$$

Finally, for $w \in V$, we have

$$||v+w||^2 = (v+w|v+w) = ||v||^2 + ||w||^2 + (v|w) + (w|v) = ||v||^2 + ||w||^2 + 2\mathrm{Re}(v|w).$$

The triangle inequality follows using Cauchy-Schwartz. □

**** Problem 2.** *Show $\widehat{(\mathbb{R},+)} = \{\chi_t \mid t \in \mathbb{R}, \chi_t(x) = e^{itx}\}$.*

Proof. Given $\chi \in \widehat{(\mathbb{R},+)}$ we want to show there exists $t \in \mathbb{R}$ such that $\chi = \chi_t$. Let H be the kernel of χ and note that H is a closed subgroup of $\mathbb{R}$ under addition. Either $H = \mathbb{R}$, $H = \{0\}$ or there exists $b \in \mathbb{R}^+$ such that $H = \{nb \mid n \in \mathbb{Z}\}$. In the case $H = \mathbb{R}$ we know $\chi = 1$ and $t = 0$ suffices. The case $H = \{0\}$ is impossible since $\chi(0) = \chi(2n\pi)$. Consider the third case. Note $\chi(b/2)^2 = \chi(b) = 1$ and since $b/2 < b$, $\chi(b/2) = -1$. Now note $\chi(b/4)^2 = \chi(b/2) = -1$ and so $\chi(b/4) = \pm i$ and without loss of generality we can choose $\chi(b/4) = i$. We show by induction on n that for $n \geq 2$, $\chi(b/2^n) = e^{i\pi/2^{n-1}}$. We have shown this for the base case, $n = 2$, so now assume that for some $n \geq 2$ the result holds. Consider $\chi(b/2^{n+1}$. Note that $\chi(b/2^{n+1})^2 = \chi(b/2^n) = e^{i\pi/2^{n-1}}$. Then we have $\chi(b/2^{n+1}) = \pm e^{i\pi/2^n}$. Note that $\chi((-b/4, b/4))$ must map to $\{e^{i\theta} \mid -\pi/2 < \theta < \pi/2\}$ from continuity. Therefore $\chi(b/2^{n+1} = e^{i\pi/2^n}$. Now consider $x \in [0,b]$ and create a sequence a_m such that a_m converges to x and $c_m = kb/2^n$ for some $k \in \mathbb{R}$. Then $\chi(c_m) = \chi(b/2^n)^k = e^{ik\pi/2^{n-1}}$ and since χ is continuous this must converge to $\chi(x) = e^{2\pi ix/b}$. Therefore $t = 2/b$. □

**** Problem 3.** *Show $\widehat{\mathbb{T}} = \{\chi_n \mid n \in \mathbb{Z}, \chi_n(e^{i\theta} = e^{in\theta}\}$.*

Proof. Note that $\mathbb{T}$ is the quotient of $(\mathbb{R},+)$ by the subgroup $[0, 2\pi)$. We can thus use the proof in ** Problem 2 where $b = 2\pi$. □