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MATH 16300
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Homework 4

Exercise 11 *Show that* $(1+i)/(2+3i) = (4-i)/13$.

Proof. We have
$$\frac{1+i}{2+3i} = \frac{(1+i)(2-3i)}{(2+3i)(2-3i)} = \frac{1-i+3}{13} = \frac{4-i}{13}.$$
□

Exercise 22 *Where is the mistake in the following?*
$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = -1.$$

The square root function is only defined for non-negative real numbers. It makes no sense to say $\sqrt{-1 \cdot -1} = \sqrt{-1} \cdot \sqrt{-1}$ because $\sqrt{-1}$ is meaningless.

Exercise 23 *Let u, w be complex numbers. Find the complex numbers z such that u, w, z form a equilateral triangle. Express the centers of these triangles.*

Proof. Given the three points u, w, z, the centroid of the triangle formed by them should be
$$x = \frac{u+w+z}{3}.$$
Given this and the two points u and w we want the condition each of u w and z are a distance L from the center, x, and are separated by an angle of $2\pi/3$. Thus
$$u - x = L\left(\cos(\alpha) + i\sin(\alpha)\right),$$
$$z - x = L\left(\cos\left(\alpha - \frac{2\pi}{3}\right) + i\sin\left(\alpha - \frac{2\pi}{3}\right)\right)$$
and
$$w - x = L\left(\cos\left(\alpha + \frac{2\pi}{3}\right) + i\sin\left(\alpha + \frac{2\pi}{3}\right)\right)$$
for some angle α. This implies that $(u-x)(w-x) = (z-x)^2$ which after substituting for x and expanding gives us
$$u^2 + w^2 + z^2 = uw + uz + wz.$$
Using the quadratic formula to solve for z we end up with
$$z = \frac{u + w \pm i\sqrt{3}(u-w)}{2}.$$
The center of the triangle is then at
$$\frac{u+w}{2} \pm \frac{i\sqrt{3}(u-w)}{6}.$$
□

Exercise 24 *Take an arbitrary and draw an equilateral triangle on all sides looking outward. Prove that the centers of these triangles forms an equilateral triangle.*

Proof. Let a, b and c be vertices of an equilateral triangle and x, y and z be the centers of the outer equilateral triangles formed. Then

$$x = \frac{a+b}{2} \pm \frac{i\sqrt{3}(a-b)}{6},$$

$$y = \frac{b+c}{2} \pm \frac{i\sqrt{3}(b-c)}{6}$$

and

$$z = \frac{c+a}{2} \pm \frac{i\sqrt{3}(c-a)}{6}.$$

Then we can verify that

$$x^2 + y^2 + z^2 = xy + yz + xz$$

which is the condition we had earlier for an equilateral triangle. □

Exercise 25 *Compute* $(1+i)^{2006}$.

Let $z = 1 + i$. Note that $|z| = \sqrt{z\overline{z}} = \sqrt{2}$. Then let $\alpha = \pi/4$ so that

$$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = |z|(\cos\alpha + i\sin\alpha).$$

Then

$$z^{2006} = \sqrt{2}^{2006}\left(\cos\left(\frac{1003\pi}{2}\right) + i\sin\left(\frac{1003\pi}{2}\right)\right) = -i2^{1003}$$

Exercise 26 *What is the sum of the nth roots of unity?*

Proof. Note that the kth root of unity is given by

$$\varepsilon_{n,k} = \cos\left(k\frac{2\pi}{n}\right) + i\sin\left(k\frac{2\pi}{n}\right).$$

Let $n > 1$ and let $k = 1$. Then

$$\varepsilon_{n,1} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \neq 1$$

and the arguments of $\varepsilon_{n,k}$ are $(2\pi)/n$. But then

$$\begin{aligned}\varepsilon_{n,1}^k &= |\varepsilon_{n,1}|\left(\cos\left(k\frac{2\pi}{n}\right) + i\sin\left(k\frac{2\pi}{n}\right)\right)\\ &= \cos\left(k\frac{2\pi}{n}\right) + i\sin\left(k\frac{2\pi}{n}\right)\\ &= \varepsilon_{n,k}\end{aligned}$$

by Corollary 17 (25.17). Thus if we have one nontrivial root of unity we can find the rest by taking powers of the first for powers $0 \leq k \leq n-1$. But then

$$\sum_{k=0}^{n-1} \varepsilon_{n,k} = \sum_{k=0}^{n-1} \varepsilon_{n,1}^k = \frac{1-\varepsilon_{n,1}^n}{1-\varepsilon_{n,1}} = 0$$

because $\varepsilon_{n,1} = 1$. □

Exercise 27 *What is the product of the nth roots of unity?*

Proof. Similarly

$$\prod_{k=0}^{n-1} \varepsilon_{n,k} = \prod_{k=0}^{n-1} \varepsilon_{n,1}^{k} = \varepsilon_{n,1}^{\frac{n(n-1)}{2}} = \left(\varepsilon_{n,1}^{n}\right)^{\frac{n-1}{2}} = 1.$$

□

Exercise 28 *What is the sum of the squares of the nth roots of unity?*

Proof. We have

$$\sum_{k=0}^{n-1} \varepsilon_{n,k}^{2} = \sum_{k=0}^{n-1} \varepsilon_{n,1}^{2k} = \varepsilon_{n,1}^{0} + \varepsilon_{n,1}^{2} + \cdots + \varepsilon_{n,1}^{2n-2}.$$

If we multiply both sides of this equation by $1 - \varepsilon_{n,1}^{2}$ we have

$$\sum_{k=0}^{n-1} \varepsilon_{n,1}^{2k} = \frac{1 - \varepsilon_{n,1}^{2n}}{1 - \varepsilon_{n,1}^{2}} = \frac{1 - \left(\varepsilon_{n,1}^{n}\right)^{2}}{1 - \varepsilon_{n,1}} = 0.$$

□