Kris Harper
MATH 20700
December 1, 2008

Homework 9

Lemma 1. *Suppose that $p, q \in \mathbb{R}$ such that $1/p + 1/q = 1$. Then*

$$rs \leq \frac{r^p}{p} + \frac{s^q}{q}$$

for all nonnegative real numbers, r and s.

Proof. Suppose that $0 < \alpha < 1$ and let $f(t) = t^\alpha - \alpha t$ when $t > 0$. Then f takes on its maximum value when $t = 1$ so $t^\alpha - \alpha t \leq 1 - \alpha$ when $t > 0$. Now let $u, v \in \mathbb{N}$ and let $t = u/v$. Multiplying by v we obtain

$$u^\alpha v^{1-\alpha} \leq \alpha u + (1 - \alpha)v.$$

The inequality also holds when $u, v \geq 0$. Finally, for nonnegative reals r and s, let $u = r^p$, $v = s^q$, $\alpha = p^{-1}$ so that $1 - \alpha = q^{-1}$ and make the appropriate substitutions to see the result. $\square$

Lemma 2. *Let $1 \leq p \leq \infty$ and let $q \in \mathbb{R}$ such that $1/p + 1/q = 1$. Let $a = (a_n)$ and $b = (b_m)$ be sequences in $\mathbb{R}$ or $\mathbb{C}$. Then*

$$||ab||_1 \leq ||a||_p ||b||_q.$$

Proof. We can assume that neither (a_n) nor (b_m) is the zero sequence and therefore that $||a||_p$ and $||b||_q$ are nonzero. We can also assume these two terms are finite and so $||a||_p$ and $||b||_q$ are both positive reals. Let $\alpha = ||a||_p$ and $\beta = ||b||_q$. Assume, for the moment that the inequality holds for $\alpha = \beta = 1$. We see that

$$||\alpha^{-1}a||_p = ||\beta^{-1}b||_q = 1$$

and from this we have

$$\alpha^{-1}\beta^{-1}||ab||_1 = ||\alpha^{-1}a\beta^{-1}b||_1 \leq ||\alpha^{-1}a||_p||\beta^{-1}b||_q = 1.$$

Multiplying by $||a||_p||b||_q$ gives the desired result. Thus, we can assume that $||a||_p = ||b||_q = 1$ and so we must show that $||ab||_1 \leq 1$. First suppose that $p = 1$. Then $q = \infty$. Since $||b||_\infty = 1$ it must be the case that $|b_m| \leq 1$ for all m and thus

$$||ab||_1 = \sum_{n=1}^{\infty} |a_n b_n| \leq \sum_{n=1}^{\infty} |a_n| = ||a||_1 = 1.$$

Now let $1 \leq p \leq \infty$. From Lemma 1 we have

$$||ab||_1 = \sum_{n=1}^{\infty} |a_n b_n| \leq \sum_{n=1}^{\infty} \left(\frac{|a_n|^p}{p} + \frac{|b_n|^q}{q} \right) = \frac{||a||_p^p}{p} + \frac{||b||_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

$\square$

**** Problem 1.** *Show that $|| \cdot ||_p$ is a norm on $\ell^p(F)$ for $1 \leq p \leq \infty$ where $F = \mathbb{R}$ or $\mathbb{C}$.*

Proof. First let $1 \leq p < \infty$. For $x = (x_n)$ where $(x_n) \in \ell^p(F)$ we have

$$||x||_p = \left(\sum_{i \in \mathbb{N}} |x_i|^p \right)^{\frac{1}{p}}.$$

It is thus clear that $||x||_p \geq 0$. Suppose that $||x||_p = 0$. Then since $|x_i| \geq 0$ for all $i \in \mathbb{N}$ it must be the case that $|x_i| = 0$ for all $i \in \mathbb{N}$. Now suppose that $(x_n) = 0$, that is, $|x_i| = 0$ for all $i \in \mathbb{N}$. Then it must be the case that $||x||_p = 0$. Next for some constant $a \in F$ consider

$$||a \cdot x||_p = \left(\sum_{i \in \mathbb{N}} |a \cdot x_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i \in \mathbb{N}} |a|^p |x_i|^p \right)^{\frac{1}{p}} = |a| \left(\sum_{i \in \mathbb{N}} |x_i|^p \right)^{\frac{1}{p}} = |a| \cdot ||x||_p.$$

For the case where $p = \infty$ we have

$$||x||_p = \sup_{n \in \mathbb{N}} |x_n|$$

so that clearly $||x||_p \geq 0$. Assuming that $||x||_p = 0$ implies that the absolute value of the greatest term of (x_n) is 0 and so all the terms must then be 0. Conversely, if each term is zero then the greatest term must also be zero. Additionally, for $a \in F$ we have

$$||a \cdot x||_p = \sup_{n \in \mathbb{N}} |a \cdot x_n| = |a| \sup_{n \in \mathbb{N}} |x_n| = |a| \cdot ||x||_p.$$

Finally, suppose that $1 \leq p \leq \infty$. Then from Lemma 2 we can apply Hölder's Inequality to infinite sequences in the same way we applied it to finite ones for $\mathbb{R}^n$ and $\mathbb{C}^n$. That is, we note that

$$||x + y||_p^p = \sum_{n=1}^{\infty} |x_n + y_n|^p \leq \sum_{n=1}^{\infty} |x_n + y_n|^{p-1} |x_n| + \sum_{n=1}^{\infty} |x_n + y_n|^{p-1} |y_n|.$$

Letting $q = p/(p-1)$ we can apply Hölder's Inequality on the right so that we have

$$||x + y||_p^p \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} \right)^{\frac{1}{q}}.$$

Now multiply both sides by

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^{(p-1)q} \right)^{-\frac{1}{q}}$$

and note that $1 - 1/q = 1/p$ so that we have

$$||x + y||_p^p = \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}}.$$

This shows the triangle inequality for $|| \cdot ||_p$. □

**** Problem 2.** *Show that if A and B are compact then $d(A, B)$ is assumed.*

Proof. Suppose that the $p = d(A, B)$ is not assumed. Then we can choose $a \in A$ and $b \in B$ such that $d(a, b)$ is arbitrarily close to p. We know that A is closed, because it's compact. So arbitrarily take $b_1 \in B$ and then $d(b, A)$ is assumed (since A is compact). Now take $b_2 \in B$ such that $d(b_2, A) > d(b_1, A)$. Inductively, choose $b_k \in B$ such that $d(b_k, A) > d(b_{k-1}, A)$. But since B is compact, then it is sequentially compact and so this sequence has a convergent subsequence to some element $b \in B$. Since $d(b, A) < d(b_k, A)$ for all k, we see that $d(b, A) = p$. But then we can't choose points from B and A which have a distance arbitrarily close to p. This is a contradiction and so $d(A, B)$ must be assumed. □

Problem 1. *A sequentially compact metric space is totally bounded.*

Proof. Let X be a sequentially compact metric space and let $\varepsilon > 0$. Suppose that we need an infinite number of balls of radius ε to cover X. Then create a sequence by taking one point form each of the balls. This is an infinite sequence, but the distance between any two points is always greater than ε and so there can never be a convergent subsequence. This is a contradiction and so the space must be totally bounded. □

Problem 2. *Let X be a metric space. If $A \subseteq X$ has the property that every infinite subset of A has an accumulation point in X, then there exists a countable collection of open sets $\{U_i \mid i \in \mathbb{N}\}$ such that, if V is any open set in X and $x \in A \cap V$, then there is some U_i such that $x \in U_i \subseteq V$.*

Proof. Suppose, to produce a contradiction, that for some $n \in \mathbb{N}$ there is no finite collection of balls with radius $\frac{1}{n}$ centered at points in A which cover A. Then for every $k \in \mathbb{N}$, assume that A is infinite and then we can create a sequence of points in A as follows. For $y_1 \in A$ the ball $B_{\frac{1}{n}}(y_1)$ does not cover A. Choose $y_2 \in A \backslash B_{\frac{1}{n}}(y_1)$. Then $B_{\frac{1}{n}}(y_1) \cup B_{\frac{1}{n}}(y_2)$ doesn't cover A and $d(y_1, y_2) \geq \frac{1}{n}$. Inductively, we choose $y_1, y_2, \ldots, y_k$ such that $B_k = B_{\frac{1}{n}}(y_1) \cup B_{\frac{1}{n}}(y_2) \cup \cdots \cup B_{\frac{1}{n}}(y_k)$ doesn't cover A, and $d(y_i, y_j) \geq \frac{1}{n}$ for all $i \neq j$. Choose $y_{k+1} \in A \backslash B_k$. But since the distance between every point in (y_k) is greater than or equal to $\frac{1}{n}$ the sequence doesn't have an accumulation point in X, which is a contradiction. Thus for every $n \in \mathbb{N}$ there exist finitely many points in A such that the set of open balls of radius $\frac{1}{n}$ centered at these points covers A. These balls form the required collection of sets. □

Problem 3. *Verify that the collection mentioned in Problem 2 satisfies the conclusion of the Problem.*

Proof. Let $V \subseteq X$ be an open set and let $x \in A \cap V$. From the previous problem we know that the collection of sets covers A and so it must be the case that x is contained in one of the sets. We then need to verify the condition that this set is also a subset of V. Since V is open there exists $r \in \mathbb{R}$ such that $B_r(x) \subseteq V$. So now simply choose n large enough such that $1/n < r$. Then the set of balls of radius $1/n$ which cover A will also contain a set which is a subset of V. □

Problem 4. *Let X be a metric space. If $A \subseteq X$ has the property that every infinite subset of A has an accumulation point in A, show that for any open cover of A, there exists a countable subcover.*

Proof. Let $\{V_i\}_{i \in I}$ be an open covering of A. We apply Problem 2 and note that there exists a finite collection of sets $\{U_1, U_2, \ldots, U_n\}$ such that if $x \in A \cap V_i$ then there is some U_j such that $x \in U_j \subseteq V_i$. Thus for any open cover there is a finite subcover. □

Problem 5. *1) Show that a compact metric space is complete.*
2) Show that a totally bounded complete metric space is compact.

Proof. 1) Let (X, d) be a compact metric space and suppose that (X, d) is not complete. Then there exists some Cauchy sequence $(a_n) \in X$ such that (a_n) does not converge. Therefore for all $x \in X$ there exists some ball $B_\varepsilon(x)$ such that there are infinitely many n with $a_n \notin B_\varepsilon(x)$. Let $\mathcal{A}$ be the set of all such balls and let $\mathcal{A}' = \{B_{\varepsilon/2}(x) \mid B_{\varepsilon/2}(x) \in \mathcal{A}\}$. Then $\mathcal{A}'$ is an open cover for X and X is compact so let $\mathcal{B}$ be a finite subcover for $\mathcal{A}'$. Take $B_{\varepsilon/2}(x) \in \mathcal{B}$. Note that there are infinitely many n such that $a_n \notin B_\varepsilon(x)$ so there are infinitely many n such that $a_n \notin B_{\varepsilon/2}(x)$. We have (a_n) is Cauchy so there exists N such that for all $n, m > N$ we have $d(a_n, a_m) < \varepsilon/2$. Suppose that there are infinitely many n with $a_n \in B_{\varepsilon/2}(x)$. Since there are infinitely many n with $a_n \in B_{\varepsilon/2}(x)$ and $a_n \notin B_{\varepsilon/2}(x)$, choose $n, m > N$ with $a_n \in B_{\varepsilon/2}(x)$ and $a_m \notin B_{\varepsilon/2}(x)$. But then $d(x, a_m) \leq d(x, a_n) + d(a_n, a_m) < \varepsilon$. Thus there are infinitely many n with $a_n \notin B_\varepsilon(x)$ which is a contradiction. Therefore there are finitely many n with $a_n \in B_{\varepsilon/2}(x)$. But this is true for all $B_{\varepsilon/2}(x) \in \mathcal{B}$ and there are finitely many elements of $\mathcal{B}$ which is an open cover for X. So we have finitely many n with $a_n \in X$ which is a contradiction. Therefore (X, d) is complete.

2) Let (X, d) be a totally bounded, complete metric space and consider a sequence $(a_n) \in X$. Since X is totally bounded, cover the set with finitely many balls of radius 1. One of these must contain infinitely

many points of (a_n). Inductively, for each $k \in \mathbb{N}$, define a ball $B_{1/k}$ of radius $1/k$ such that $B_{1/k}$ contains infinitely many points of (a_n), all of which are contained in the ball of radius $1/(k-1)$. Then choose one distinct point of (a_n) from each of these balls so that we have a Cauchy subsequence of (a_n). But since X is complete, this sequence is convergent. Therefore X is sequentially compact and thus compact. □

Problem 6. *Suppose that X and X' are metric space with X separable. Let $f : X \to X'$ be a continuous surjection. Show that X' is separable.*

Proof. Let A be a countable subset of X which is dense in X. Then we can create a sequence $(a_n) \in A$ such that every nonempty open subset of X must contain a term of (a_n). Then note that for some open set $B \subseteq X'$ we have $f^{-1}(B)$ is open in X because f is continuous. But then there exists n such that $a_n \in f^{-1}(B)$ and so $f(a_n) \in B$. Since this is true for all open sets in X', the images of the points in (a_n) form an infinite sequence such that at least one term must be in any open set in X'. Thus $f(A)$ is a dense countable subset of X' and so X' is separable. □

Problem 7. *Determine the conditions, if they exist, for which the following metric spaces are separable:*
1) $\mathbb{R}$
2) $\mathcal{B}(X, F)$
3) $\mathcal{BC}(X, F)$.

Proof. 1) $\mathbb{R}$ with the usual metric is separable because $\mathbb{Q}$ is a dense, countable subset.

2) If X is separable then $\mathcal{B}(X, F)$ is separable.

3) If X is a compact metric space then $\mathcal{BC}(X, F)$ is separable. This follows from the fact that every element of $\mathcal{BC}(X, F)$ is a uniformly continuous function from X to F. □

Problem 8. *1) Show that an open ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ with the usual metric is a connected set.*
2) Show that a closed ball in $\mathbb{R}^n$ or $\mathbb{C}^n$ with the usual metric is a connected set.
3) Show that $GL(2, \mathbb{R})$ with the metric inherited from $M_2(\mathbb{R})$ is not a connected set.
4) Show that $GL(2, \mathbb{C})$ with the metric inherited from $M_2(\mathbb{C})$ is a connected set.

Proof. 1) Let F be $\mathbb{R}^n$ or $\mathbb{C}^n$. Let B be an open ball in F and suppose that B is disconnected. Then there exist open sets U and V such that $U \cap B \neq \emptyset$, $V \cap B \neq \emptyset$, $(U \cap B) \cap (V \cap B) = \emptyset$ and $B = (U \cap B) \cup (V \cap B)$. Since B, U and V are all open, we can replace U and V with $U \cap B$ and $V \cap B$ so that we have two sets U and V such that $U \cup V = B$ and $U \cap V = \emptyset$. But then note that $F \backslash U$ and $F \backslash V$ are both closed since U and V are open. Then $U \cup V$ is closed and $U \cup V = B$. This is a contradiction since B is open. Therefore B is a connected set.

2) Let F be $\mathbb{R}^n$ or $\mathbb{C}^n$. Let B be an open ball of radius m in F and suppose that B is disconnected. Consider the distance function $d : F \to \mathbb{R}$ where $d(x) = d(\mathbf{0}, x)$. Then f is continuous and maps to an interval $[0, m]$ in $\mathbb{R}$. But since B is disconnected it must be the case that one of these values is not mapped to. This is a contradiction and so B must be connected.

3) Assume that $GL(2, \mathbb{R})$ is connected. Then for any continuous function $f : X \to \mathbb{R}$, $f(GL(2, \mathbb{R}))$ is an interval with the condition that if $x \in f(GL(2, \mathbb{R})$ then there exists $a \in GL(2, \mathbb{R})$ such that $f(a) = x$. Note that the determinant function is continuous, but that no elements of $GL(2, \mathbb{R})$ have a determinant of 0. Since some determinants have negative values and positive values, this violates the Intermediate Value Theorem since 0 will be in the interval which the determinant function maps $GL(2, \mathbb{R})$ to.

4) Assume that $GL(2, \mathbb{C})$ is not connected. Then consider $f : GL(2, \mathbb{C}) \to \mathbb{R}$ where f is the absolute value of the determinant function. Then f is continuous, but since $GL(2, \mathbb{C})$ is disconnected, there must be some element of $\mathbb{R}$ which is between two other elements in the image of f but is not in the image of f. But this

is not the case because the only nonnegative value of f which is not taken on is 0. Thus $f(GL(2,\mathbb{C}))$ is an interval $(0,c)$ for some constant $c \in \mathbb{R}$ and $GL(2,\mathbb{C})$ is connected. $\square$