Kris Harper
MATH 20800
January 21, 2009

Homework 3

**** Problem 1.** *Are $c(F)$ and $c_0(F)$ complete?*
Yes.

Proof. Let $c = c(F)$ and $c_0 = c_0(F)$. Let (a_{jk}) be a Cauchy sequence of elements in c where a_{jk} is the kth term in the sequence and the jth term in that term. Then for all $\varepsilon > 0$ there exists N such that for all $n, m > N$ we have

$$||a_{jn} - a_{jm}|| = \sup_{j \in \mathbb{N}} |a_{jn} - a_{jm}| < \varepsilon.$$

Now create a subsequence (a_{jk_i}) such that $||a_{jk_i} - a_{jk_{i+1}}|| < 2^{-i}$. Then this subsequence must converge since the series $\sum_{i=1}^{\infty} 2^{-i}$ converges. But because (a_{jk}) is a Cauchy sequence with a convergent subsequence, it must be convergent as well. Therefore c is complete. The same proof holds for c_0 since 2^{-i} goes to 0 as i goes to infinity. Thus if $(a_{jk}) \in c_0$, there's a subsequence which converges to a sequence which converges to 0. This proves (a_{jk}) is convergent, and that c_0 is complete. $\square$

**** Problem 2.** *Which of the following are separable?*
1) $\mathcal{BC}(X, F)$, where X is infinite.
2) $\ell^p(F)$.
3) $\mathcal{B}(X, F)$, where X is infinite.
4) $L^p([a, b])$.
5) $c(F)$.
6) $c_0(F)$.

Proof. 1) $\mathcal{BC}(X, F)$ is separable if and only if X is compact. To show this, first suppose X is a compact metric space. Then we can apply the Stone-Weierstrass Theorem to $\mathcal{BC}(X, F)$ so that any subset of $\mathcal{BC}(X, F)$ which contains a constant function and separates points is dense in $\mathcal{BC}(X, F)$. The set of rational valued polynomials is a countable set which satisfies this. Therefore $\mathcal{BC}(X, F)$ is separable. Conversely, suppose that $\mathcal{BC}(X, F)$ is separable. Then there exists a countable dense subset, $A \subseteq \mathcal{BC}(X, F)$. It may be assumed that A contains a nonzero constant function. Then by the Stone-Weirstrass Theorem, A separates points. Let $\mathcal{A}$ be an open cover for X. But the existence of a countable dense subset of continuous functions from X to F shows that $\mathcal{A}$ has a finite subcover, which shows that X is compact.

2) For $1 \le p < \infty$ the space is separable. To see this, consider the set, A, of all sequences where each term is rational (if $F = \mathbb{C}$ then the real and imaginary parts are rational) and all but finitely many terms are 0. Each of these sequences is in $\ell^p(F)$, since the associated series is finite. Also, A is countable since it can be associated with finitely many products of $\mathbb{Q}$. It is also dense for the same reason that $\mathbb{Q}$ is dense in $\mathbb{R}$. If $p = \infty$ then the space is not separable. To see this, consider the set, B, of sequences in which terms are either 0 or 1. It is clear that B is uncountable because it corresponds to the unit interval of the real line. Additionally, for two distinct elements in B, the distance between them is 1. Now consider the set of all open balls of radius $1/2$ around elements of B. These balls must all be disjoint, but any dense subset must have at least one point in each of them, which means no dense subset is countable.

3) If X is countable then this is a special case of part 2) where $p = \infty$. This is because we can map each element of X to an element of $\mathbb{N}$ and then $\mathcal{B}(X, F)$ just becomes sequences. If X is uncountable, the the same proof will hold. Simply take a countable subset of X and make a sequence out of it as in the case where X is countable, then let every other element map to 0. These elements are a subset of $\mathcal{B}(X, F)$ but they are enough to show that any dense subset must be uncountable.

4) This space is separable. The space of integrable step functions is dense in $L^p([a,b])$. If we consider only rational step functions then we have a countable dense subset of $L^p([a,b])$.

5) The space $c = c(F)$ is separable. The set, A, of sequences where each term is rational and all but finitely many terms are 0 is a countable dense subset. This set is in the space and is countable for the same reasons it was in part 2). To see that it's dense, note that an element $(x_n) \in c$ must converge to 0. Then for the same reason that $\mathbb{Q}$ is dense in $\mathbb{R}$, finitely many terms of (x_n) are arbitrarily close to corresponding nonzero terms from some element of A. The rest of the terms of (x_n) get arbitrarily close to 0, and thus to the remaining terms of this element of A. Thus (x_n) is arbitrarily close to some element of A, and so A is dense in X.

6) This space is separable for the exact same reasons as in part 5). □

** **Problem 3.** *For $u \in V/V_0$, is $||u||$ necessarily assumed?*

No.

Proof. Consider $u \in V/V_0$ such that $u = u + V_0$. Let $\varepsilon > 0$. Then there must exist some element $v \in V_0$ such that $||u - v|| < ||u + V_0|| + \varepsilon$. Since ε is arbitrary we have $||w|| \leq ||u||$ where $w = u - v$, $w \in V$ and $u = u + V_0$. □

** **Problem 4.** *If V is a complete vector space and V_0 a closed subspace of V. Show that V/V_0 is complete.*

Proof. Let (u_n) be a Cauchy sequence in V/V_0 where u_n is the coset $u_n + V_0$. Since (u_n) is Cauchy, we can choose a subsequence (u_{n_k}) such that $||u_{n_k} - u_{n_{k+1}}|| < 2^{-k}$. Now create a sequence (v_k) such that $||v_k - v_{k+1}|| < 2||u_{n_k} - u_{n_{k+1}}||$. We can do this because of the definition of the norm. It is then clear that (x_k) is Cauchy and so it converges to $v \in V$ since V is complete. Let $u = v + V_0$. Then using the definition of a norm we see that $||u_{n+k} - u|| < ||x_k - x||$ so that (u_{n_k}) converges to u. Since (u_n) is Cauchy, this implies that (u_n) converges to u as well. □

** **Problem 5.** *If V/V_0 is complete, is V necessarily complete?*

No.

Proof. Suppose that the result is true. Then consider some incomplete vector space V, and note that V is a closed subspace of itself. Then if V/V is complete, it should directly imply V is complete, but this is clearly not the case. □