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MATH 25800
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Problem 1. *If N is a submodule of M, the* annihilator of N in R *is defined to be* $\{r \in R \mid rn = 0$ *for all* $n \in N\}$. *Prove that annihilator of N in R is a 2-sided ideal of R.*

Proof. Let N be a submodule of M and let A be the annihilator of N in R. Let $a, b \in A$ and let $n \in N$. Then $(a-b)n = an - bn = 0 - 0 = 0$ so $a - b \in A$ and $(ab)n = a(bn) = a \cdot 0 = 0$ since $0 \in N$ so $ab \in A$. Finally, let $r \in R$ and note that $(ra)n = r(an) = r \cdot 0 = 0$ and $(ar)n = a(rn) = 0$ since $rn \in N$. (Note that $r \cdot 0 = 0$ follows from the usual proof—$r \cdot 0 = r(0+0) = r \cdot 0 + r \cdot 0$ so $0 = r \cdot 0$). Since a, b, n and r are arbitrary, this shows that A is subring of R which is closed under left and right multiplication by elements in R. Therefore A is 2-sided ideal of R. □

Problem 2. *Let N be a submodule of M. Prove that if both M/N and N are finitely generated then so is M.*

Proof. Suppose $M/N = RA$ and $N = RB$ where A and B are finite with $|A| = i$ and $|B| = j$. Let $m \in M$ and note that

$$\begin{aligned}
\{m + n \mid n \in N\} &= m + N \\
&= r_1(a_1 + N) + r_2(a_2 + N) + \cdots + r_i(a_i + N) \\
&= ((r_1a_1) + N) + ((r_2a_2) + N) + \cdots + ((r_ia_i) + N) \\
&= (r_1a_1 + r_2a_2 + \cdots + r_ia_i) + N \\
&= \{(r_1a_1 + r_2a_2 + \cdots + r_ia_i) + n \mid n \in N\}.
\end{aligned}$$

So for some n and n' and elements of R we have $m + n = (r_1a_1 + r_2a_2 + \cdots + r_ia_i) + n'$. But N is finitely generated by B, so for some other coefficients from R we have

$$m + (s_1b_1 + s_2b_2 + \cdots + s_jb_j) = (r_1a_1 + r_2a_2 + \cdots + r_ia_i) + (t_1b_1 + t_2b_2 + \cdots + t_jb_j)$$

and

$$m = r_1a_1 + r_2a_2 + \cdots + r_ia_i + (t_1 - s_1)b_1 + (t_2 - s_2)b_2 + \cdots + (t_j - s_j)b_j.$$

So an arbitrary element of m can be written as a finite sum of elements with coefficients from R. Thus M is finitely generated. □