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Homework 8

Problem 1. *(a) Let u be real harmonic. Show that u^2 is subharmonic.*
(b) Let u be real harmonic, $u = u(x, y)$. Show that

$$(\text{grad}u)^2 = (\text{grad}u) \cdot (\text{grad}u)$$

is subharmonic.
(c) Show that the function $u(x, y) = x^2 + y^2 - 1$ is subharmonic.
(d) Let u_1, u_2 be subharmonic, and c_1, c_2 positive numbers. Show that $c_1u_1 + c_2u_2$ is subharmonic.

Proof. We have

$$\frac{\partial u^2}{\partial x} = 2u\frac{\partial u}{\partial x}$$

and

$$\frac{\partial u^2}{\partial x^2} = 2\frac{\partial^2 u}{\partial x^2}u + 2\left(\frac{\partial u}{\partial x}\right)^2.$$

Likewise

$$\frac{\partial u^2}{\partial y^2} = 2\frac{\partial^2 u}{\partial y^2}u + 2\left(\frac{\partial u}{\partial y}\right)^2.$$

Adding these equations and noting that $\Delta u = 0$ we obtain

$$\Delta u^2 = \frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial x^2} = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial x}\right)^2 \geq 0.$$

(b) Let $v = (\text{grad}u)^2$. Note that

$$v = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

Differentiating twice we find

$$\frac{\partial^2 v}{\partial x^2} = 2\left(\frac{\partial^3 u}{\partial x^3}\frac{\partial u}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^3 u}{\partial y^2 \partial x}\frac{\partial u}{\partial x} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2\right)^2$$

and a similar expression for y. Adding these two equations and again using the fact that $\Delta u = 0$, we find that $\Delta v \geq 0$ as in part (a).

(c) It's immediate that $\Delta u = 4 \geq 0$.

(d) Since differentiation is linear, we have $\Delta(c_1u_1 + c_2u_2) = c_1\Delta u_1 + c_2\Delta u_2$. Since c_1 and c_2 are positive and u_1 and u_2 are subharmonic, it's immediate that this sum is greater than or equal to 0. □

Problem 2. *Suppose that φ is defined on an open set U and is subharmonic on U. Prove the maximum principal, that no point $a \in U$ can be a strict maximum for φ, i.e. that for every disk of radius r centered at a with r sufficiently small, we have*

$$\varphi(a) \leq \max \varphi(z) \quad \textit{for} \quad |z - a| = r.$$

Proof. Let $a \in U$ and choose r small enough that $\overline{D}_r(a) \subseteq U$. Now since φ is continuous on the circle $\partial\overline{D}_r(a)$ which is compact, it obtains a maximum on that set. Therefore $\varphi(a + re^{i\theta}) \leq \max_{z\in\partial\overline{D}_r(a)} \varphi(z)$ for every θ. Now integrate over θ from 0 to 2π.

$$\varphi(a) \leq \int_0^{2\pi} \varphi(a + re^{i\theta})\frac{d\theta}{2\pi} \leq \max_{z\in\partial\overline{D}_r(a)} \varphi(z).$$

This is precisely the statement. □

Problem 3. *Let φ be subharmonic on an open set U. Assume that the closure $\overline{U}$ is compact, and that φ extends to a continuous function on $\overline{U}$. Show that a maximum for φ occurs on the boundary.*

Proof. Since φ is continuous and $\overline{U}$ is compact, we know φ obtains a maximum on $\overline{U}$. Suppose this maximum is at a in the interior of U. Choose r small enough so that $\overline{D}_r(a) \subseteq U$. Define a function f on $\partial\overline{D}_r(a)$ as $f(\theta) = \varphi(a) - \varphi(a + re^{i\theta})$. Note that $f \geq 0$. Suppose there exists $0 \leq \theta_0 \leq 2\pi$ such that $f(\theta_0) > 0$. Then

$$\int_0^{2\pi} f(\theta)d\theta > 0$$

since f is continuous. But then

$$\varphi(a) \leq \int_0^{2\pi} \varphi(a + re^{i\theta}\frac{d\theta}{2\pi} < \varphi(a)$$

and so f must be constantly 0. Therefore φ is locally constant.

For each connected open set $V \subseteq U$ we see that $\varphi(z) = \varphi(a)$ for $z \in V$. Suppose that V is the largest such connected open set and suppose that $\partial V \nsubseteq U$. Then there exists $z \in \partial V$ with $z \in U$ and so there exists some $D_\varepsilon(z) \subseteq U$. But then $V \subseteq D_\varepsilon(z)$ is open, connected and contained in U which contradicts the maximality of V. Thus $\partial V \subseteq \partial U$. Now using continuity, it must be that $\varphi(z) = \varphi(a)$ for $z \in \partial V$ and thus there exists $z \in \partial U$ such that $\varphi(z) = \varphi(a)$. Therefore, φ attains a maximum on ∂U. □

Problem 4. *Let U be a bounded open set. Let u, v be continuous functions on $\overline{U}$ such that U is harmonic on U, v is subharmonic on U and $u = v$ on the boundary of U. Show that $v \leq u$ on U. Thus a subharmonic function lies below the harmonic function having the same boundary value, whence its name.*

Proof. The function $v - u$ is subharmonic by linearity of differentiations. That is

$$\Delta(v - u) = \Delta v - \Delta u = \Delta v \leq 0.$$

Note that on ∂U we have $v - u \leq 0$ and so using Problem 3 we must have $v - u \leq 0$ on all of U. Thus $v \leq u$ on all of U. □

Problem 5. *Define*

$$P_{R,r}(\theta) = \frac{1}{2\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}$$

for $0 \leq r < R$. Prove the inequalities

$$\frac{R-r}{R+r} \leq 2\pi P_{R,r}(\theta - \varphi) \leq \frac{R+r}{R-r}$$

for $0 \leq r < R$.

Proof. Note that

$$-2rR \leq -2rR\cos(\theta - \varphi) \leq 2rR$$

which means

$$(R-r)^2 \leq R^2 - 2rR\cos(\theta - \varphi) + r^2 \leq (R+r)^2.$$

Now using $0 \le r < R$ we have

$$\frac{R^2 - r^2}{(R+r)^2} \le 2\pi P_{R,r}(\theta - \varphi) \le \frac{R^2 - r^2}{(R-r)^2}$$

Now expand $R^2 - r^2 = (R+r)(R-r)$ to obtain

$$\frac{R-r}{R+r} \le 2\pi P_{R,r}(\theta - \varphi) \le \frac{R+r}{R-r}.$$

$\square$

Problem 6. *Let f be analytic on the closed disk $\overline{D}(\alpha, R)$ and let $u = \mathrm{Re}(f)$. Assume that $u \ge 0$. Show that for $0 \le r < R$ we have*

$$\frac{R-r}{R+r} u(\alpha) \le u(\alpha + re^{i\theta}) \le \frac{R+r}{R-r} u(\alpha).$$

Proof. We can assume $\alpha = 0$ by applying a translation of the disk. Note that u is harmonic because it is the real part of an analytic function and therefore

$$u(re^{i\varphi}) = \int_0^{2\pi} u(r3^{i\theta}) P_r(\theta - \varphi) d\theta.$$

Now use Problem 5 and the fact that $u \ge 0$ to obtain

$$\int_0^{2\pi} u(re^{i\theta}) \frac{R-r}{R+r} \frac{d\theta}{2\pi} \le u(re^{i\varphi}) \le \int_0^{2\pi} u(re^{i\theta}) \frac{R+r}{R-r} \frac{d\theta}{2\pi}.$$

But since

$$u(0) = \int_0^{2\pi} u(re^{i\theta}) \frac{d\theta}{2\pi}$$

the result follows. Shifting back by α will finish the general case. $\square$

Problem 7. *Prove that if $v(z) = \mathrm{Im}\left(\left(\frac{1+z}{1-z}\right)^2\right)$, then v is harmonic on $\mathbb{D}$ and $\lim_{r \uparrow 1} v(re^{i\theta}) = 0$ for all $\theta \in [0, 2\pi)$. Why does this not contradict the maximum principal?*

Proof. Let $z = x + iy$. Then

$$\begin{aligned}
\mathrm{Im}\left(\left(\frac{1+z}{1-z}\right)^2\right) &= \mathrm{Im}\left(\left(\frac{x+iy+1}{-x-iy+1}\right)^2\right) \\
&= \mathrm{Im}\left(\frac{1+2x+x^2+2iy+2ixy-y^2}{1-2x+x^2-2iy+2ixy-y^2}\right) \\
&= \mathrm{Im}\left(\frac{1+2x+x^2+2iy+2ixy-y^2}{1-2x+x^2-2iy+2ixy-y^2} \frac{1-2x+x^2+2iy-2ixy-y^2}{1-2x+x^2+2iy-2ixy-y^2}\right) \\
&= \mathrm{Im}\left(\frac{1-2x^2+x^4+4iy-4ix^2y-6y^2+2x^2y^2-4iy^3+y^4}{1-4x+6x^2-4x^3+x^4+2y^2-4xy^2+2x^2y^2+y^4}\right) \\
&= \frac{1-2x^2+x^4-6y^2+2x^2y^2+y^4}{1-4x+6x^2-4x^3+x^4+2y^2-4xy^2+2x^2y^2+y^4}.
\end{aligned}$$

But also

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\frac{1-2x^2+x^4-6y^2+2x^2y^2+y^4}{1-4x+6x^2-4x^3+x^4+2y^2-4xy^2+2x^2y^2+y^4} \\
&= \frac{8(2-2x^4+x^5-16y^2+6y^4+x^2(8-12y^2)-2x^3(1+y^2)+x(-7+30y^2-3y^4))}{(1-2x+x^2+y^2)^4} \\
&= -\frac{\partial^2}{\partial y^2}\frac{1-2x^2+x^4-6y^2+2x^2y^2+y^4}{1-4x+6x^2-4x^3+x^4+2y^2-4xy^2+2x^2y^2+y^4} \\
&= -\frac{\partial^2 v}{\partial y^2}.
\end{aligned}$$

Thus $v(z)$ is harmonic.

Note that $\frac{1+z}{1-z}$ maps $\mathbb{D}$ to the right half plane. Thus $\left(\frac{1+z}{1-z}\right)^2$ maps $\mathbb{D}$ to $\mathbb{C}$ without the negative imaginary axis and v maps $\mathbb{D}$ to the upper half plane. Furthermore, note that $\frac{1+z}{1-z}$ takes $\partial\mathbb{D}$ to the imaginary axis and squaring this line results in the negative real axis. The imaginary part of this is obviously 0 which shows why $\lim_{r\uparrow 1} v(re^{i\theta}) = 0$ for all $\theta \in [0, 2\pi)$. This doesn't contradict the maximum principle because $v(z)$ is not continuous on $\partial\mathbb{D}$ at 1. $\square$