Sheet 18: Convergence of Functions

Definition 1 *For $a < b$ with $a, b \in \mathbb{R}$ let*

$$B[a;b] = \{f \ : \ [a;b] \to \mathbb{R} \mid f \text{ is bounded on } [a;b]\}$$

be the set of bounded real functions on $[a;b]$.

Definition 2 *We say that f is the pointwise limit of (f_n), or*

$$\overset{\bullet}{\lim_{n\to\infty}} f_n = f$$

if for all $x \in [a;b]$ we have

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Definition 3 *For $f, g \in B$ let*

$$d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)|.$$

Theorem 4 *d is a metric on B.*

Proof. Let $f, g, h \in B$. We have $|f(x) - g(x)| \geq 0$ for all $x \in [a;b]$ so then $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| \geq 0$. Also if $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = 0$ then $|f(x) - g(x)| = 0$ for all $x \in [a;b]$ because $d(f,g)$ is an upper bound. But then $f(x) = g(x)$ for $x \in [a;b]$. Conversely suppose that $f(x) = g(x)$ for all $x \in [a;b]$. Then $|f(x) - g(x)| = 0$ for all $x \in [a;b]$ and so $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = 0$. Also $d(f,g) = \sup_{x\in[a;b]} |f(x) - g(x)| = \sup_{x\in[a;b]} |g(x) - f(x)| = d(g,f)$. Finally from the triangle inequality we have $|f(x) - g(x)| + |g(x) - h(x)| \geq |f(x) - h(x)|$ for all $x \in [a;b]$ so $|f(x) - g(x)| + |g(x) - h(x)| \geq \sup_{x\in[a;b]} |f(x) - h(x)|$ for all $x \in [a;b]$. But then $d(f,g) + d(g,h) = \sup_{x\in[a;b]} |f(x) - g(x)| + \sup_{x\in[a;b]} |g(x) - h(x)| \geq |f(x) - g(x)| + |g(x) - h(x)| \geq \sup_{x\in[a;b]} |f(x) - h(x)| = d(f,h)$ for all $x \in [a;b]$. □

Definition 5 *We say that f is the uniform limit of (f_n), or*

$$\lim_{n\to\infty} f_n = f$$

if $\lim_{n\to\infty} f_n = f$ in the metric d.

Theorem 6 *W have $\lim_{n\to\infty} f_n = f$ if and only if for all $\varepsilon > 0$ there exists N such that for all $n > N$ and for all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$.*

Proof. Suppose that $\lim_{n\to\infty} f_n = f$. Then $\lim_{n\to\infty} f_n = f$ in the metric d. Thus $\lim_{n\to\infty} d(f, f_n) = 0$ which means $\lim_{n\to\infty} \sup_{x\in[a;b]} |f(x) - f_n(x)| = 0$ (17.1). Then for all $\varepsilon > 0$ there exists N such that for all $n > N$ we have $|\sup_{x\in[a;b]} |f(x) - f_n(x)|| < \varepsilon$. But then for all $\varepsilon > 0$ there exists N such that for all $n > N$ and for all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$.

Conversely suppose that for all $\varepsilon > 0$ there exists N such that for all $n > N$ and for all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$. Since this is true for all $x \in [a;b]$ then for all $\varepsilon > 0$ there exists N such that for all $n > N$ we have $\sup_{x\in[a;b]} |f(x) - f_n(x)| = |\sup_{x\in[a;b]} |f(x) - f_n(x)| - 0| = |d(f, f_n) - 0| < \varepsilon$. But then $\lim_{n\to\infty} d(f, f_n) = 0$ and so $\lim_{n\to\infty} f_n = f$ (17.1). □

Theorem 7 *If* $\lim_{n\to\infty} f_n = f$ *then* $\lim^{\bullet}_{n\to\infty} f_n = f$.

Proof. We have $\lim_{n\to\infty} f_n = f$ and so for all $\varepsilon > 0$ there exists N such that for all $n > N$ and all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$. But then for all $x \in [a;b]$ and all $\varepsilon > 0$ there exists N such that for all $n > N$ we have $|f(x) - f_n(x)| < \varepsilon$. Thus $\lim^{\bullet}_{n\to\infty} f_n = f$. $\square$

Theorem 8 *The sequence* $f_n(x) = x^n$ *on the interval* $[0;1]$ *converges pointwise but not uniformly.*

Proof. Let

$$f = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

and let $x \in [0;1)$. Since $0 \le x < 1$ we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0 = f(x)$. If $x = 1$ then $x^n = 1$ for all n and so $\lim_{n\to\infty} x^n = 1 = f(x)$. Thus, (f_n) converges pointwise.

Suppose that (f_n) converges uniformly. Then for all $\varepsilon > 0$ there exists an N such that for all $n > N$ and for all $x \in [0;1]$ we have $|f(x) - f_n(x)| < \varepsilon$. Let $x \in [0;1)$. Note that since $0 \le x < 1$ we have $\lim_{n\to\infty} f_n(x) = 0$ which means $\lim_{n\to\infty} |f(x) - f_n(x)| = |f(x)| < \varepsilon$. Since this is true for arbitrarily small ε, we have $f(x) = 0$ for $x \in [0;1)$. For $x = 1$ note that $f_n(x) = 1$ for all n so we have $|f(1) - 1| < \varepsilon$ for arbitrarily small ε which means that $f(1) = 1$. Thus (f_n) must converge to the function above.

Now let $1 > \varepsilon > 0$. Since $f(x) = 0$ for $x \in [0;1)$ we can choose x large enough such that $x^{N+1} \ge \varepsilon < 1$. Thus there exists $\varepsilon > 0$ such that for all N there exists $n > N$ and $x \in [0;1]$ such that $|f(x) - f_n(x)| \ge \varepsilon$ and so (f_n) doesn't converge uniformly. $\square$

Theorem 9 *Let* (f_n) *be a sequence of continuous functions on* $[a;b]$ *that uniformly converges to* f *on* $[a;b]$. *Then* f *is continuous on* $[a;b]$.

Proof. Let $\varepsilon > 0$ and consider $\varepsilon/3$. We know (f_n) uniformly converges to f so there exists N such that for all $n > N$ and for all $x, y \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon/3$ and $|f(y) - f_n(y)| < \varepsilon/3$. Also f_n is continuous for all n so for all $n > N$ and for all $x \in [a;b]$ there exists $\delta_n > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta_n$ we have $|f_n(x) - f_n(y)| < \varepsilon/3$. Consider δ_{N+1}. Then for all $x \in [a;b]$ there exists $\delta_{N+1} > 0$, which may depend on x, such that for all $y \in [a;b]$ with $|x - y| < \delta_{N+1}$ we have $|f_{N+1}(x) + f_{N+1}(y)| < \varepsilon/3$. By the triangle inequality we have $|f(x) - f_{N+1}(y)| \le |f_{N+1}(x) - f_{N+1}(y)| + |f(x) - f_{N+1}(x)| < 2\varepsilon/3$ and then $|f(x) - f(y)| < |f(x) - f_{N+1}(y)| + |f(y) - f_{N+1}(y)| < \varepsilon$. Thus for all $x \in [a;b]$ there exists some $\delta > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Therefore f is continuous on $[a;b]$. $\square$