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MATH 25900
Quiz 1 April 7, 2010

Problem 1. *Find the rational canonical form of*

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show R_θ is similar to R_ϕ in $M_2(\mathbb{R})$ iff $\theta = \pm\phi$. Find eigenvalues of R_θ.

Proof. First we restrict the values of θ to the interval $[-\pi, \pi)$. We find the characteristic polynomial of R_θ, $c_{R_\theta}(x) = (x - \cos\theta)^2 + \sin^2\theta = x^2 - 2x\cos\theta + \cos^2\theta + \sin^2\theta = x^2 - 2x\cos\theta + 1$. Using the quadratic formula we find that the solutions to $c_{R_\theta}(x)$ are $\cos\theta \pm \sqrt{\cos^2\theta - 1}$. These are then the eigenvalues for R_θ. Note that they're only real-valued if $\cos^2\theta = 1$ so $\theta \in \{-\pi, 0\}$ and in this case the eigenvalues simplify to either -1 or 1.

Note that $(R_\theta - (\cos\theta \pm \sqrt{\cos^2\theta - 1})I) \neq 0$ so the minimal polynomial $m_{R_\theta}(x) = c_{R_\theta}(x)$. Thus the rational canonical form is simply the 2×2 companion matrix for $c_{R_\theta}(x)$

$$\begin{pmatrix} 0 & -1 \\ 1 & 2\cos\theta \end{pmatrix}.$$

Finally note that R_θ is similar to R_ϕ if and only if R_θ and R_ϕ have the same rational canonical form. Thus, they're similar if and only if $2\cos\theta = 2\cos\phi$ which is true if and only if $\theta = \pm\phi$ since $\theta, \phi \in [-\pi, \pi)$. □

Problem 2. *Find all possible Jordan forms for all 8×8 matrices having $x^2(x-1)^3$ as a minimal polynomial.*

Proof. If $x^2(x-1)^3$ is the minimal polynomial for a matrix then the elementary divisors are powers of x and $(x-1)$ such that x^2 and $(x-3)^3$ appear at least once and the product of all the elementary divisors must be an eighth degree polynomial. This generates the following possible lists of elementary divisors. We have

$$\begin{array}{ccc} x,\ x,\ x,\ x^2,\ (x-1)^3 & x,\ x^2,\ x^2,\ (x-1)^3 & x,\ x,\ x^2,\ (x-1),\ (x-1)^3 \\ x^2,\ x^2,\ (x-1),\ (x-1)^3 & x,\ x^2,\ (x-1),\ (x-1),\ (x-1)^3 & x,\ x^2,\ (x-1)^2,\ (x-1)^3 \\ x^2,\ (x-1),\ (x-1),\ (x-1),\ (x-1)^3 & x^2,\ (x-1),\ (x-1)^2,\ (x-1)^3 & x^2,\ (x-1)^3,\ (x-1)^3 \end{array}$$

These respectively have the following Jordan forms up to a permutation in their Jordan blocks. We have

$$\begin{pmatrix} 1&1&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 1&1&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 1&1&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{0} & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\qquad
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{0} & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$\square$

Problem 3. *Show that the matrix A has only one Jordan block of size 3 iff $ab \neq 0$, where*

$$
A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.
$$

Proof. Note that the characteristic polynomial is $c_A(x) = (x-1)^3$. We will produce a Jordan block of size 3 if and only if the minimal polynomial is $c_A(x)$ since this ensures that there will only be one invariant factor, $(x-1)^3$, and thus one elementary divisor. But this will happen if and only if $(A-I)^2 \neq 0$. Note that

$$
(A-I)^2 = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

So $(A-I)^2 \neq 0$ if and only if $ab \neq 0$. $\square$

Problem 4. *Find characteristic polynomial, minimal polynomial, invariant factors, elementary divisors, rational canonical form and Jordan canonical form form the following matrix over $\mathbb{Q}$:*

$$
A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.
$$

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Proof. Note that A is already in rational canonical form It has a companion matrix of size 2 in the upper left corner and two companion matrices of size 1 following. The corresponding invariant factors are $x^2 - 2x - 3$, x and $x - 2$. The characteristic polynomial is then $c_A(x) = (x^2 - 2x - 3)x(x - 2)$. As a check we can take the determinant of $xI - A$. This is

$$\begin{aligned} c_A(x) = \det(xI - A) = \det\left(\begin{pmatrix} x & -3 & 0 & 0 \\ -1 & x-2 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x-2 \end{pmatrix}\right) \\ = x^4 - 4x^3 + x^2 + 6x = (x^2 - 2x - 3)x(x - 2) \\ = x(x+1)(x-2)(x-3). \end{aligned}$$

Since the characteristic polynomial is composed of relatively prime factors, the minimal polynomial must be equal to the characteristic polynomial so $m_A(x) = c_A(x) = (x^2 - 2x - 3)x(x - 2)$. As stated earlier, the invariant factors are $x^2 - 2x - 3$, x and $x - 2$. The elementary divisors are the prime powers of these factors, so they must be x, $(x + 1)$, $(x - 2)$ and $(x - 3)$. We've already stated that A is in rational canonical form as given, and the list of elementary divisors shows that the Jordan form is

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\square$