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Midterm Problems

Problem 1a *Prove that for $\mathbb{N}$, the Well-Ordering Principle implies the Induction Axiom.*

Proof. Let $S \subseteq \mathbb{N}$ be a subset such that $1 \in S$ and if $n \in S$ then $n' \in S$. Suppose that $S \neq \mathbb{N}$. Then $\mathbb{N} \backslash S \neq \emptyset$. Let k be the least element of $\mathbb{N} \backslash S$. We know that $k = l'$ for some $l \in \mathbb{N}$. Note that $l < l' = k$ and so $l \notin \mathbb{N} \backslash S$. Then $l \in S$. But then $l' \in S$ and $l' = k$. This is a contradiction, hence $S = \mathbb{N}$. $\square$

Problem 1b *Prove that for $\mathbb{N}$, the Induction Axiom implies the Well-Ordering Principle.*

Proof. Let $A \subseteq \mathbb{N}$ be a nonempty subset. Suppose that A has no least element. Then let B be the set of elements of $\mathbb{N}$ which are not in A. We see that $1 \in B$ because otherwise it would be the least element of A. Also, if $n \in B$ then $n' \in B$ because otherwise n' would be the least element of A. Then by Induction, $B = \mathbb{N}$. But then A must be empty which is a contradiction. $\square$

Problem 2a *Show that the sequence*

$$a_n = \sum_{k=1}^{n} \frac{1}{10^{\frac{(k^2+k)}{2}}}$$

is a Cauchy sequence.

Proof. Let $\varepsilon > 0$. Then note that there exists $N \in \mathbb{N}$ such that

$$\frac{1}{10^{\frac{N^2+N}{2}}} < \varepsilon.$$

Then for all $n, m > N$ with $n > m$ we have

$$|a_n - a_m| = \left| \sum_{k=m}^{n} \frac{1}{10^{\frac{k^2+k}{2}}} \right| < \frac{1}{10^{\frac{N^2+N}{2}}} < \varepsilon.$$

$\square$

Problem 2b *Show that $(a_n)_{n=1}^{\infty}$ does not converge to a rational number.*

Proof. Note that

$$\lim_{n\to\infty} = 0.1.0100100010000100000 1\ldots.$$

We see that this number will never terminate or repeat. Thus it cannot be rational. $\square$

Problem 3 *For $p(x), q(x) \in \mathbb{R}(x)$, we define the open interval*

$$(p(x), q(x)) = \{f(x) \in \mathbb{R}(x) \mid p(x) < f(x) < q(x)\}.$$

For any $a \in \mathbb{R}$ show that there exist $p(x), q(x) \in \mathbb{R}(x)$, such that $(p(x), q(x)) \cap \mathbb{R} = \{a\}$.

Proof. Let $a \in \mathbb{R}$ and consider the interval $S = (a - 1/x, a + 1/x)$. Suppose that $b \in S \cap \mathbb{R}$ such that $b \neq a$. Then we have

$$0 < b - \frac{ax-1}{x} = \frac{(b-a)x-1}{x}$$

which means $(b-a)x^2 - x > 0$ and so $b - a > 0$. Thus $b > a$. Similarly,

$$0 < \frac{ax+1}{x} - b = \frac{(a-b)x-1}{x}$$

which means $(a-b)x^2 - x > 0$ and so $a - b > 0$. Thus $a > b$. This is a contradiction and so $b = a$. $\square$

Problem 4a *Let* $(a_n)_{n=1}^{\infty}$ *be a bounded sequence in* $\mathbb{R}$. *Define the sequence* $(b_n)_{n=1}^{\infty}$ *as*

$$b_n = \sup\{a_k \mid k \geq n\}.$$

Show that $\lim_{n\to\infty}$ *exists.*

Proof. We have
$$b_n = \sup\{a_k \mid k \geq n\} \leq \sup\{a_k \mid k \geq n+1\} = b_{n+1}$$
which means that (b_n) is a monotonically decreasing sequence. Thus it is convergent. □

Problem 4b *Suppose that a subsequence* $(a_{n_j})_{n=1}^{\infty}$ *of* (a_n) *converges to* x. *Show that* $x \leq \lim_{n\to\infty} b_n$.

Proof. Note that $b_n \geq a_{n_j}$ for all n and j. Thus

$$\lim_{n\to\infty} b_n \geq x.$$

□

Problem 4c *Prove that there exists a convergent subsequence* $(a_{n_j})_{j=1}^{\infty}$ *of* (a_n) *such that*

$$\lim_{j\to\infty} a_{n_j} = \lim_{n\to\infty} b_n.$$

Proof. Let $\lim_{n\to\infty} b_n = b$. There exists N_1 such that for all $n > N_1$ we have

$$b - 1 < b_n < b + 1.$$

Since there exists some $n > N_1$, there exists n_1 such that

$$b - 1 < a_{n_1} < b_n < b + 1.$$

Similarly, there exists some $N_2 > N_1$ such that for all $n > N_2$

$$b - \frac{1}{2} < b_n < b + \frac{1}{2}.$$

Then there exists n_2 such that
$$b - \frac{1}{2} < a_{n_2} < b_n < b + \frac{1}{2}.$$
Then in general, there exists n_j such that

$$b - \frac{1}{j} < a_{n_j} < b + \frac{1}{j}$$

which means that a_{n_j} will converge to b. □