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Homework 9

Problem 1. *Let $f : X \to Y$ be a continuous open map. Show that if X satisfies the first or second countability axiom, then $f(X)$ satisfies the same axiom.*

Proof. Suppose first that X is first countable and let $x \in X$ with countable basis $\mathcal{B}$. Let V be a neighborhood of $f(x) \in f(X)$. Then since f is continuous there exists a neighborhood U of x such that $f(U) \subseteq V$. Note that U contains some $B \in \mathcal{B}$ since X is first countable. But then $f(B) \subseteq f(U) \subseteq V$ and $f(B)$ is open since f is an open map. Thus, the collection $\{f(B) \mid B \in \mathcal{B}\}$ serves as a countable basis for $f(x) \in X$ showing that $f(X)$ is also first countable.

Now suppose that X is second countable with countable basis $\mathcal{B}$. Let U be open in $f(X)$ and note that $f^{-1}(U)$ is open in X since f is continuous. Then $f^{-1}(U) = \bigcup B_i$ is the countable union of basis elements $B_i \in \mathcal{B}$. Since f is surjective onto its image, we have $U = f(f^{-1}(U)) = f\left(\bigcup B_i\right) = \bigcup f(B_i)$. Since f is open the sets $f(B_i)$ are open and therefore form a countable basis for $f(X)$. □

Problem 2. *Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.*

Proof. Let $\mathcal{A}$ be an open covering of $X \times Y$. For $x \in X$, the set $x \times Y$ is compact, and therefore can be covered by finitely many $A_i \in \mathcal{A}$. Let $N = \bigcup A_i$ be an open set in $X \times Y$ and note that $x \times Y \subseteq N$. By the tube lemma, we know there exists an open set $W_x \subseteq X$ containing x such that $W_x \times Y \subseteq N$. Note that $W_x \times Y$ is covered by finitely many $A_i \in \mathcal{A}$. Now the sets W_x form an open cover of X and since X is Lindelöf, only countably many of them $W_1, W_2, \ldots$ cover X. Since each $W_i \times Y$ can be covered by finitely many A_i, and the sets $W_i \times Y$ cover $X \times Y$, we see that $X \times Y$ is Lindelöf. □

Problem 3. *Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.*

Proof. Let A and B be disjoint closed subsets of X. Since X is normal, there exist disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. But then, again since X is normal, there exist open sets C and D such that $A \subseteq C$, $B \subseteq D$, $\overline{C} \subseteq U$ and $\overline{D} \subseteq V$. Since U and V are disjoint, the sets $\overline{C}$ and $\overline{D}$ satisfy the statement. □

Problem 4. *Let $p : X \to Y$ be a closed continuous surjective map. Show that if X is normal then so is Y.*

Proof. First we show that if U is an open set in X containing $p^{-1}(y)$ for $y \in Y$, then there exists a neighborhood of y, W, such that $p^{-1}(W) \subseteq U$. Note that $X \backslash U$ is closed and so $p(X \backslash U)$ is also closed. Let $W = Y \backslash p(X \backslash U)$. Then we have $y \in W$ and $p^{-1}(W) \subseteq U$. Now suppose B is a subspace in Y such that $p^{-1}(B) \subseteq U$ for some open set U of X. For each $b \in B$ there exists some neighborhood W_b of b such that $p^{-1}(W_b) \subseteq U$. Let $W = \bigcup_{b \in B} W_b$. Then $B \subseteq W$ and $p^{-1}(W) = p^{-1}(\bigcup_{b \in B} W_b) = \bigcup_{b \in B} p^{-1}(W_b) \subseteq U$.

Since points in X are closed and p is closed and surjective, all points in Y are closed. Let A and B be closed sets in Y and note that since p is continuous $p^{-1}(A)$ and $p^{-1}(B)$ are closed sets. Since X is normal there exist open disjoint sets U and V containing $p^{-1}(A)$ and $p^{-1}(B)$ respectively. Use the above result to pick open neighborhoods C and D of Y containing A and B respectively so that $p^{-1}(C) \subseteq U$ and $p^{-1}(D) \subseteq V$. Then C and D are disjoint since U and V are disjoint and Y is normal. □

Problem 5. *Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a* perfect map.*)*
(a) Show that if X is Hausdorff then so is Y.
(b) Show that if X is regular then so is Y.
(c) Show that if X is locally compact then so is Y.
(d) Show that if X is second countable then so is Y.

Proof. (a) Let $a, b \in Y$. Then $p^{-1}(a)$ and $p^{-1}(b)$ are disjoint compact subspaces of X. We know that there exist disjoint open sets U and V containing $p^{-1}(a)$ and $p^{-1}(b)$ respectively since these sets are compact in a Hausdorff space. Using the result from Problem 4 we can find neighborhoods A and B of a and b respectively such that $p^{-1}(A) \subseteq U$ and $p^{-1}(B) \subseteq V$. Thus, A and B must be disjoint showing that Y is Hausdorff.

(b) Let $a \in Y$ and B be a closed subset of Y not containing a. Then $p^{-1}(a)$ is a compact subspace of X and $p^{-1}(B)$ is closed. Since X is regular, for each $x \in p^{-1}(a)$ there exist disjoint open sets U_x and V_x such that $x \in U_x$ and $p^{-1}(B) \subseteq V_x$. The collection of the open sets U_x clearly cover $p^{-1}(a)$ and so finitely many of them, $U_1, \ldots, U_n$ also cover it since $p^{-1}(a)$ is compact. Taking $U = \bigcup_{i=1}^n U_i$ and $V = \bigcap_{i=1}^n V_i$ we have disjoint sets U and V such that $p^{-1}(a) \subseteq U$ and $p^{-1}(B) \subseteq V$. Now using the proof of Problem 4 again we can find open sets C and D such that $a \in C$, $B \subseteq D$, $p^{-1}(C) \subseteq U$ and $p^{-1}(D) \subseteq V$. Thus C and D must be disjoint and Y must be regular.

(c) Let $a \in Y$ and so that $p^{-1}(a)$ is compact in X. Since X is locally compact, for each $x \in p^{-1}(a)$ we can find a neighborhood U_x of x such that there exists a compact set C_x containing U_x. These sets C_x cover $p^{-1}(a)$ so finitely many of them $U_1, \ldots, U_n$ also cover. Then $U = \bigcup_{i=1}^n U_i$ is an open set containing $p^{-1}(a)$. Note that $C = \bigcup_{i=1}^n C_x$ is still compact since it is a finite union of compact sets (namely, any open cover of C is an open cover of C_x for each x and the corresponding finite subcovers will only constitute finitely many open sets). Thus $p^{-1}(y) \subseteq U \subseteq C$ where U is open and C is compact. Now using the proof of Problem 4 there exists an open neighborhood W of y such that $p^{-1}(W) \subseteq U$. Since p is continuous, $p(C)$ is compact in Y and this must contain W. Therefore $y \in W$ and $W \subseteq p(C)$ which is compact. Thus Y is locally compact.

(d) Let $\mathcal{B}$ be a countable basis for X with index $B_1, B_2, \ldots$. For each finite subset $J \subseteq \mathbb{N}$ let U_J be the union of all sets of the form $p^{-1}(W)$ where W is open in Y and $p^{-1}(W) \subseteq \bigcup_{j \in J} B_j$. This shows that the collection of U_J is countable since it is a union of finite subsets of a countable set. Note that since p is surjective $p(U_J)$ is a union of open sets in Y and is thus open. Let U be an open set in Y. Note that $p^{-1}(U) = \bigcup_{y \in U} p^{-1}(y)$ where each $p^{-1}(y)$ is compact. This means it can be covered by finitely many basis elements contained in $p^{-1}(U)$. That is, $p^{-1}(y) = \bigcup_{j \in J_y} B_j$. From the proof of Problem 4 there is an open set $W \subseteq Y$ such that $p^{-1}(y) \subseteq p^{-1}(W) \subseteq \bigcup_{j \in J_y} B_j$. Then taking the union of all such sets W we have $p^{-1}(y) \subseteq U_{J_y} \subseteq \bigcup_{j \in J_y} B_j \subseteq p^{-1}(U)$. This shows that $p^{-1}(U) = \bigcup_{y \in U} U_{J_y}$. But then $U = \bigcup_{y \in U} p(U_{J_y})$ is a union of sets from the collection of sets of the form $p(U_J)$. Since this collection is countable, it follows that Y is second countable. □

Problem 6. *A space X is said to be* completely normal *if every subspace of X is normal. Show that X is completely normal if and only if for every pair A, B of separated sets in X (that is, sets such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$), there exist disjoint open sets containing them.*

Proof. Suppose X is completely normal and let A and B be a pair of separated sets in X. Note that A and B are completely contained in $Y = X \backslash (\overline{A} \cap \overline{B})$ because if a point $a \in A$ is in $\overline{A} \cap \overline{B}$ then $a \in A \cap \overline{B}$ but A and B are separated. Since X is completely normal, Y is a normal subspace of Y. Thus there exist disjoint open sets U and V of Y containing A and B respectively. But also note that $\overline{A} \cap \overline{B}$ is necessarily closed, so Y is open. Thus U and V are open in X as well and contain A and B.

Conversely, suppose that for any two separated sets A and B there exist open sets U and V containing them. Let Y be a subspace of X and let A and B be two disjoint closed subsets of Y. Note that if $\overline{A}$ is the closure of A in X, then $\overline{A} \cap Y$ is the closure of A in Y. Thus $\overline{A} \cap Y$ and $\overline{B} \cap Y$ are disjoint. Now $\overline{A} \cap B = \overline{A} \cap (Y \cap B) = (\overline{A} \cap Y) \cap (B \cap Y) = \emptyset$ and similarly $A \cap \overline{B} = \emptyset$. Then there exist disjoint open sets U and V containing A and B respectively. If we assume that one-point sets are closed it follows that Y is normal. □

Problem 7. *Which of the following spaces are completely normal? Justify your answers.*
(a) A subspace of a completely normal space.
(b) The product of two completely normal spaces.
(c) A well ordered set in the order topology.
(d) A metrizable space.
(e) A compact Hausdorff space.

(f) A regular space with a countable basis.
(g) The space $\mathbb{R}_\ell$.

Proof. (a) Let X be a completely normal space, let Y be a subspace of X and let A be a subspace of Y. Then we know the topology on A as a subspace of Y is the same as the topology on A as a subspace of X. Thus, A is normal and so Y must be completely normal as well.

(b) Using part (c), we know that S_Ω and $\overline{S_\Omega}$ are both completely normal. But their product isn't even normal.

(c) Let X be a well ordered set in the order topology and let Y be any subspace of X. Note that Y is necessarily well ordered in the order topology as well as any subset of Y will be a subset of X and have a least element. Since all well ordered sets in the order topology are normal, we have that Y is normal and X is completely normal.

(d) All metrizable spaces are normal and any subspace of a metrizable space is metrizable, therefore normal. Thus all metrizable spaces are completely normal.

(e) The product $\overline{S_\Omega} \times \overline{S_\Omega}$ is a product of two compact Hausdorff spaces so it's compact Hausdorff. But the subspace $\overline{S_\Omega} \times S_\Omega$ is not normal.

(f) A regular space with a countable basis is normal. A subspace of a regular space is regular and a subspace of a second countable space is second countable. Therefore all subspaces of a regular second countable space are normal and such a space is completely normal.

(g) We use Problem 6. Let A and B be two separated sets in $X = \mathbb{R}_\ell$. For each $a \in X \backslash \overline{B}$ there exists some open set $[a, x_a) \subseteq X \backslash \overline{B}$ containing a and for each $b \in X \backslash \overline{A}$ there exists some open set $[b, y_b) \subseteq X \backslash \overline{A}$. Let $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, y_b)$. Note that U and V are open and contain A and B respectively. Suppose they are not disjoint. Then some $[a, x_a)$ intersects some $[b, y_b)$ and $a \neq b$ since A and B are disjoint. If $a < b$ then $b < x_a$ and $b \in [a, x_a) \cap B$ which is a contradiction since $[a, x_a) \subseteq X \backslash \overline{B}$. If $b < a$ then we have a similar contradiction. Thus U and V must be disjoint and X is completely normal by Problem 6. $\square$