Kris Harper
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Homework 5

**** Problem 1.** *Is $C_c(X,F)$ a closed subspace of $\mathcal{BC}(X,F)$?*
No.

Proof. Let $X = \mathbb{N}$ with the discrete topology. Then every function in $\mathcal{BC}(X,F)$ is continuous. Consider the sequence of functions (f_n) where $f_n = (1, 1/2, 1/3, \dots, 1/n, 0, 0, \dots)$. That is f_n has the reciprocals of the indices for the first n terms and then 0s following. Then (f_n) is clearly Cauchy and converges to the harmonic sequence. But each f_n is certainly compactly supported, since the set $\{1, 1/2, 1/3, \dots, 1/n\}$ is compact in X. Yet X is not compact in itself, take the set of balls of radius $1/2$ around each point to see this. Therefore (f_n) doesn't converge, and $C_c(X,F)$ is not complete. This shows that it is not a closed subset of $\mathcal{BC}(X,F)$. $\square$

**** Problem 2.** *For $1 < p < \infty$ and q such that $1/p + 1/q = 1$, show that $(\ell^p(F))^* = \ell^q(F)$. Also show $(\ell^p_n(F))^* = \ell^q_n(F)$.*

Proof. Let $f \in (\ell^p(F))^*$ and define the sequence $(f(e_j))$ where (e_j) is the sequence in $\ell^p(F)$ with a 1 in the jth spot and 0s elsewhere. Then this sequence is an element of $\ell^q(F)$. Conversely, let $a = (a_n) \in \ell^q(F)$ and define $f(b) = \sum_{n=1}^{\infty} a_n b_n$ for all $b = (b_n)$ in ℓ^p. We can use Hölder's Inequality to show that this result holds and that f is indeed a linear functional on $\ell^p(F)$. $\square$

**** Problem 3.** *Find an element of $(\ell^\infty(\mathbb{R}))^*$ that is not in $\ell^1(\mathbb{R})$.*

Proof. We will show that if a normed linear space V is not separable, then the dual of V is not separable. We use the contrapositive and assume that V^* is separable. Then The unit sphere in V^* is separable so there exists a countable dense subset $\{f_n\}$ of the unit sphere each with norm $||f_n|| = \sup_{||x||=1} f_n(x) = 1$. Then using the definition of supremum there must exist a point $x_n \in V$ for each f_n with $||x_n|| = 1$ such that $|f_n(x_n)| \geq 1/2$. Let W be the closure of the space spanned by $\{x_n\}$. Then W is separable because the space of linear combinations of $\{x_n\}$ using rational coefficients is a countable dense subset of W. Suppose that $W \neq V$. Since W is closed we can find a functional f on W with $||f|| = 1$ such that $f(w) = 0$ for all $w \in W$. Since $x_n \in W$ we have $f(x_n) = 0$. Then for all n we have

$$\frac{1}{2} \leq |f_n(x_n)| = |f_n(x_n) - f(x_n)| = |(f_n - f)(x_n)| \leq ||f_n - f||||x||$$

and $||x_n|| = 1$. Then $||f_n - f|| \geq 1/2$ which contradicts the fact that $\{f_n\}$ is dense in the unit sphere since $||f|| = 1$. Thus $W = V$ and so V is separable. Note that $\ell^1(\mathbb{R})$ is separable, but $\ell^\infty(\mathbb{R})$ is not separable. Thus $(\ell^\infty(\mathbb{R}))^*$ is not separable and so it must contain an element which is not in $\ell^1(\mathbb{R})$. $\square$

**** Problem 4.** *Show that following are equivalent for Banach spaces V and W:*
1) If $T \in \mathcal{BL}(V,W)$ is surjective, then T is open.
2) If $T \in \mathcal{BL}(V,W)$ is a bijection, then $T^{-1} \in \mathcal{BL}(V,W)$.
3) Suppose $T : V \to W$ is linear. If the graph of T is closed in $V \times W$, then T is bounded.
4) Suppose $T : V \to W$ is linear. If T is bounded then the graph of T is closed in $V \times W$.

Proof. Suppose 1) is true. Let $T \in \mathcal{BL}(V,W)$ be a bijection. Then T is an open map, which means T^{-1} is continuous, and therefore bounded.

Suppose 2) is true. Suppose A as defined earlier is closed. Let $p_1 : V \times W \to V$ and $p_2 : V \times W \to W$ be the projections from $V \times W$ to V and W respectively. It is clear that these functions are continuous. Define

$T' : V \to A$ as $T'x = (x, Tx)$. Note that $A \subseteq V \times W$ and so let p_1', be the function p_1 restricted to A. Then p_1' is a bijection since T is linear. But then $p_1'^{-1}$ is continuous by 2) and moreover, $p_1'^{-1}$ is simply T'. Then $T = p_2 \circ T'$ is continuous since it is the composition of two continuous functions. Therefore $T \in \mathcal{BL}(V, W)$.

Suppose 3) is true. Let $T \in \mathcal{BL}(V, W)$. Then T is continuous. Let $A = \{(x, Tx) \mid x \in V\}$ be the graph of T in $V \times W$. Consider a sequence in A, (x_n, Tx_n), which converges to some $(x, y) \in V \times W$. Since T is continuous we must have $\lim_{n\to\infty} Tx_n = Tx = y$. Then $(x, y) \in A$ so A is closed.

Suppose 4) is true. Let $T \in \mathcal{BL}(V, W)$ be surjective and let $U \subseteq V$ be an open set. Suppose that $T(U)$ is not open in W. Then there exists $x \in U$ such that for all $r > 0$, $B_r(Tx) \nsubseteq T(U)$. Thus for every $r > 0$ we can choose a point in $B_r(Tx)$ which is not in $T(U)$. Since T is surjective, we can call these points Tx_n where $x_n \in V$. By continually choosing r small enough, we can create a sequence (Tx_n) which converges to Tx in W. Since the graph of T is closed, the sequence (x_n) converges to x in V. But since U is open, there are infinitely many points of (x_n) in U, but none of the points of (Tx_n) are in $T(U)$. This is a contradiction and so $T(U)$ must be open. □

** **Problem 5.** *Let V be a nonzero vector space over F. Find three linear functionals on V.*

Proof. 1) Take the 0 function which maps every element of V to 0.

2) Let B be a basis for V with $v_1 \in B$. Define $f(v_1) = c$ for some $c \neq 0$ in F. Then define $f(\alpha v_1) = \alpha c$ for $\alpha \in F$. Now for all v_i with $i \neq 1$, define $f(v_i) = 0$. Then for $v, w \in V$ we have $v = \sum_i \alpha_i v_i$ and $w = \sum_i \beta_i v_i$ where arbitrarily many of the v_i terms are 0 in each sum and $\alpha_i, \beta_i \in F$. But then

$$\begin{aligned}
f(v + w) &= f\left(\sum_i (\alpha_i v_i + \beta_i v_i)\right) \\
&= f\left(\sum_i (\alpha_i + \beta_i) v_i\right) \\
&= f((\alpha_1 + \beta_1) v_1) \\
&= (\alpha_1 + \beta_1) c \\
&= \alpha_1 c + \beta_1 c \\
&= f(\alpha_1 v_1) + f(\beta_1 v_1) \\
&= f\left(\sum_i \alpha_i v_i\right) + f\left(\sum_i \beta_i v_i\right) \\
&= f(v) + f(w).
\end{aligned}$$

Also

$$f(\alpha v) = f\left(\sum_i \alpha(\alpha_i v_i)\right) = f\left(\alpha \sum_i \alpha_i v_i\right) = f(\alpha(\alpha_1 v_1)) = \alpha\alpha_1 c = \alpha f(\alpha_1 v_1) = \alpha f\left(\sum_i (\alpha_i v_i)\right) = \alpha f(v).$$

Thus f is a linear functional on V.

3) Consider the case in 2) and assign a constant value in F to as many terms of the basis as needed. Then map all the other basis elements to 0, as before. This is still a linear functional for the same reasons as in 2). □

** **Problem 6.** *Show that a subspace of a finite dimensional vector space is finite dimensional.*

Proof. Let V be an n-dimensional vector space and let W be a subspace. Let $\{v_1, v_2, \ldots, v_m\}$ be a linearly independent set in $W \subseteq V$. Note that since $\dim V = n$ the maximal linearly independent set has n vectors, thus $m \leq n$. But then since W is a vector space it has some basis $\{B\}$ which has at most n linearly independent vectors. Therefore W is finite dimensional. $\square$

**** Problem 7.** *Let V be a vector space over $\mathbb{R}$ and let W be a subspace of V. Show that a linear functional $f : W\mathbb{R}$ can be extended to a linear functional $F : V \to \mathbb{R}$.*

Proof. Let $\mathcal{A}$ be the set of all linear extensions of f to subspaces of V. Note that $\mathcal{A} \neq \emptyset$ since $f \in \mathcal{A}$. We can partially order $\mathcal{A}$ by defining $g \leq h$ if and only if h is a linear extension of g. If T is a totally ordered subset of $\mathcal{A}$ then $\bigcup_{g \in T} g$ is a an upper bound for T. Thus Zorn's Lemma applies and so let F be the maximal element of $\mathcal{A}$. The proof will be finished if we can show the domain of F is V. Suppose there exists x in V which is not in the domain of F. Let V' be the subspace spanned by the domain of F and x. Then for $y \in V'$ there is a unique representation of y as $y = m + rx$ where m is in the domain of F and $r \in \mathbb{R}$. If $s \in \mathbb{R}$ then we can define $F'(y) = F'(m + rx) = F(m) + rs$ a linear functional on V'. But this is clearly an extension of F which contradicts the maximality of F. Therefore the domain of F is V and so F is a linear functional on V. $\square$