Sheet 8: Proving Connectedness

Theorem 1 *Let A be a bounded infinite set. Then A has a limit point.*

Proof. Assume that A has no limit points. Then A is closed because it vacuously contains all its limit points. Because A is closed and bounded it is compact. Since A has no limit points, for all $a \in A$ there exists some region R_a with $a \in R_a$ such that $R_a \cap (A \backslash a) = \emptyset$. Let $\mathcal{A} = \{R_a \mid a \in A\}$ be an open cover for A. Since A is compact there exists a finite subcover, $\mathcal{B}$, of $\mathcal{A}$ for A. But then $\mathcal{B}$ is a finite open cover for A containing only regions which contain one point of A and A is an infinite set. This is a contradiction and so A must have a limit point. $\square$

Theorem 2 *Let O be a nonempty, bounded, open set. Then* $\sup O$ *is a limit point of both O and the complement of O.*

Proof. We have O is nonempty and bounded so $\sup O$ exists. Let $(a;b)$ be a region containing $\sup O$. Suppose there are no points of O in $(a;\sup O)$. We know that regions are nonempty and since $\sup O$ is greater than every point in O, there exists a point in $(a;\sup O)$ which is greater than every point in O, but is less than $\sup O$. This is a contradiction and so there exists a point in $(a;\sup O)$ which is also in O and in $(a;b)$. Thus, any region containing $\sup O$ must also contain an additional point from O and so $\sup O$ is a limit point of O. Additionally if we take the region $(a;b)$ containing $\sup O$ then $(\sup O;b)$ is a nonempty region containing at least one point greater than $\sup O$ which is therefore not in O. Thus $(a;b)$ contains a point which is in the complement of O and so $\sup O$ is a limit point of the complement of O. $\square$

Theorem 3 *Let A be a set which is open and closed such that $A \neq \emptyset$ and $A \neq C$. Let B be the complement of A. Let $a \in A$ and $b \in B$ and without loss of generality assume that $a < b$. Now let $s = \sup(A \cap (a;b))$. Then s is a limit point of A and B.*

Proof. Suppose that s is not a limit point of A. Then there exists some region $(p;q)$ which contains s but no other points of A. Thus $(p;q)$ contains no points of $(A \cap (a;b)) \backslash s$ as well. But $s \geq x$ for all $x \in A \cap (a;b)$ there exists an element of $(p;s)$ which is greater than or equal to every element of $A \cap (a;b)$ but less than s. This is a contradiction and so s must be a limit point of A.

To show s is a limit point of B consider two cases.

Case 1: Suppose $s \neq b$. First suppose $s > b$. Then there exists $c \in (b;s)$. For all $x \in A \cap (a;b)$ we have $x < b < c$ so c is an upper bound for $A \cap (a;b)$ which is less than s. This is a contradiction so $s < b$. Suppose that $s < a$. Then for all $x \in A \cap (a;b)$ we have $s < a < x$ which is a contradiction since $s \geq x$ for all $x \in A \cap (a;b)$. Thus, $s \in (a;b)$. Consider the region $(s;b)$. Suppose there exists $x \in (s;b)$ such that $x \in A$. Then $x \in A \cap (s;b) \subseteq A \cap (a;b)$. But this is a contradiction since $x > s$. So for all $x \in (s;b)$ we have $x \in B$ so $(s;b) \subseteq B$. But then every region containing s will contain a point in $(s;b) \subseteq B$ so s is a limit point of B.

Case 2: Suppose $s = b$. Then A is closed and so B is open. Thus there exists some region $R \subseteq B$ such that $s \in R$. But then every region containing s will intersect $R \backslash s$ nontrivially so s must be a limit point of B. $\square$

Theorem 4 *If every bounded nonempty point set has a least upper bound and regions are nonempty, then the only sets that are both open and closed are C and $\emptyset$.*

Proof. Suppose to the contrary that there exists an open and closed set A such that $A \neq \emptyset$ and $A \neq C$. Then we can construct B, $(a;b)$ and s as in Theorem 3 so that s is a limit point of both A and B. But A is closed so $s \in A$. But A is open so B is closed and since s is a limit point of B we have $s \in B$ which is a contradiction. Therefore the only sets which are open and closed must be $\emptyset$ and B. $\square$