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MATH 25800

Quiz 1

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Problem 1. *An element in a ring is an idempotent if $a^2 = a$. Show that a division ring has exactly 2 idempotent elements.*

Proof. Let a be a nonzero element of a division ring R. Then R contains some a^{-1}. Thus if $a^2 = a$ we can multiply on the left by a^{-1} to obtain $a = 1$. Furthermore, $0 \cdot b = 0$ for any element $b \in R$ so in particular, $0^2 = 0$. Thus, the only two elements of R which are idempotent are 0 and 1. $\square$

Problem 2. *Find a polynomial of degree greater than 0 in $(\mathbb{Z}/4\mathbb{Z})[x]$ that is a unit.*

Proof. Consider the element $2x+1$. If we square this element we obtain $4x^2+2x+2x+1 = 4x^2+4x+1 = 1$. $\square$

Problem 3. *Find the sum and product of $f(x) = 4x - 5$ and $g(x) = 2x^2 - 4x + 2$ in $(\mathbb{Z}/8\mathbb{Z})[x]$.*

Proof. The sum is carried out component wise so we have

$$f(x) + g(x) = 4x - 5 + 2x^2 - 4x + 2 = 2x^2 + 5.$$

For the product we have

$$\begin{aligned} f(x)g(x) &= (4x - 5)(2x^2 - 4x + 2) \\ &= 4x(2x^2 - 4x + 2) - 5(2x^2 - 4x + 2) \\ &= 8x^3 - 16x^2 + 8x - 10x^2 + 20x - 10 \\ &= 0 + 0 + 0 + 6x^2 + 4x + 6 \\ &= 6x^2 + 4x + 6. \end{aligned}$$

$\square$

Problem 4. *In the group ring $\mathbb{Z}D_8$ compute the product $\beta\alpha$ for $\alpha = r + r^2 - 2s$ and $\beta = -3r^2 + rs$.*

Proof. We have

$$\begin{aligned} \beta\alpha &= (-3r^2 + rs)(r + r^2 - 2s) \\ &= -3r^2(r + r^2 - 2s) + rs(r + r^2 - 2s) \\ &= -3r^3 - 3r^4 + 6r^2s + rsr + rsr^2 - 2rs^2 \\ &= -3 - 3r + 6r^2s + s + r^2s - 2r \\ &= -3 - 5r + 6r^2s + s + r^2s. \end{aligned}$$

$\square$