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Homework 8

Exercise 2 *Let* $p \in C$ *be a point and let*

$$S = \{\text{ext}(a; b) \mid p \in (a; b)\}.$$

Show that S *is an open cover for* $C \backslash p$.

Proof. Let $x \in C \backslash p$. Then $x \in C$ and $x \neq p$ and so $x < p$ or $p < x$. Suppose $x < p$. Then by Theorem 5.8 there exists $a \in C$ such that $x < a < p$. And by Axiom 2.3 there exists $b \in C$ such that $p < b$. But then $p \in (a; b)$ and since $x < a$, $x \in \text{ext}(a; b)$. Because this is true for some region $(a; b)$, we see $x \in \bigcup_{A \in S} A$. Therefore, $C \backslash p \subseteq \bigcup_{A \in S} A$ and by Exercise 7.12 we know that $\text{ext}(a; b)$ is open so S is an open cover for $C \backslash p$. A similar argument holds if $p < x$. $\square$

Exercise 4 *Show that the set*

$$A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}.$$

is closed.

Proof. Let $p \in C$ be point such that $p \notin A$. Then there are three cases.

Case 1: Let $p < 0$. Then by Axiom 2.3 there exists a point $x \in C$ such that $x < p$ and so the region $(x; 0)$ contains p but no points in A.

Case 2: Let $p > 1$. Then by Axiom 2.3 there exists a point $y \in C$ such that $p < y$ and so the region $(1; y)$ contains p but no points in A.

Case 3: Let $p \in (0; 1)$. Then $p = \frac{a}{b}$ and since $0 < \frac{a}{b} < 1$, we have $a < b$. Since $0 < \frac{b}{a}$, by the Archimedean Property there exists a natural number k such that $\frac{b}{a} < k$. But since $k \in \mathbb{N}$, by the Well Ordering Principle there exists a least such element n. Since $p \notin A$, $a \neq 1$ and so $\frac{b}{a} \notin \mathbb{N}$. But then $n - 1 < \frac{b}{a} < n$ and so $\frac{1}{n} < p < \frac{1}{n-1}$. But then $p \in \left(\frac{1}{n}; \frac{1}{n-1}\right)$ which doesn't contain any elements of A.

In all three cases there exists a region containing p which contains no elements of A and so p cannot be a limit point of A. Therefore if A has any limit points, they must be in A. Since A contains all its limit points, it is closed. $\square$

Exercise 5 *Prove that every open cover of* A *has a finite subcover.*

Proof. Let S be a cover of A. Then for every element of A, there exists an open set in S which contains that element. But then there exists an open set B in S containing 0. And so there exists a region $(a; b) \subseteq B$ such that $0 \in (a; b)$. There are two cases.

Case 1: Let $1 \leq b$. Then $A \subseteq B$ and so the set containing B is a finite subcover of S.

Case 2: Let $b < 1$. Then $b = \frac{p}{q}$ and since $0 < \frac{p}{q} < 1$, we have $p < q$. Since $0 < \frac{q}{p}$, by the Archimedean Property there exists a natural number k such that $\frac{q}{p} < k$. But since $k \in \mathbb{N}$, by the Well Ordering Principle there exists a least such element n. There are a finite number of natural numbers less than n and since every element of A is a reciprocal of a natural number, there are a finite number of elements a of A such that $\frac{1}{n} < a$. All the other elements of A are less than b so they are contained in $(a; b)$. Then the sets B and the sets of S which contain the elements of A which are greater than $\frac{1}{n}$ form a finite subcover of S. $\square$

Exercise 7 *Let S be the set of all regions. Show that no finite subset of S covers C.*

Proof. Let T be a finite subset of S. Then $T = \{(a_1; b_1), (a_2; b_2), \ldots, (a_n; b_n)\}$. But since there are a finite number of lower boundary points a_i for regions in T, by Theorem 2.3 we can order them so that x is a lower boundary point and $x \leq a_i$ for all regions in T. Then x is less than every point in every region in T. But by Axiom 2.3 there exists a point $p \in C$ such that $p < x$ and so $C \nsubseteq \bigcup_{(a;b)\in T}(a; b)$. $\square$

Exercise 8 *Let $p \in C$ be a point and let $S = \{\text{ext}(a; b) \mid p \in (a; b)\}$. Show that no finite subset of S covers $C \backslash p$.*

Proof. Let T be a finite subset of S. Then $T = \{\text{ext}(a_1; b_1), \text{ext}(a_2; b_2), \ldots, \text{ext}(a_n; b_n)\}$ such that $p \in (a; b)$ for all $\text{ext}(a; b) \in T$. Consider the finite set of values of a_i for exteriors in T. By Theorem 2.3 there exists a last point x so that $x \geq a_i$ for all exteriors in T. By Theorem 5.8 there exists a point $y \in C$ such that $x < y < p$ and so $y \notin \text{ext}(a_i; b_i)$ for any exterior in T. But then $C \backslash p \nsubseteq \bigcup_{A\in T} A$. $\square$

Exercise 12 *Closed intervals are closed*

Proof. Let $a, b, p \in C$ be points such that $a < b$ and $p \notin [a; b]$. Then $p < a$ or $p > b$. Let $p < a$. Then by Axiom 2.3 there exists a point $x \in C$ such that $x < p$. But then the region $(x; a)$ contains x but no points in $[a; b]$. A similar argument holds for $b < p$ and so p cannot be a limit point of $[a; b]$. But then any limit points of $[a; b]$ must be in $[a; b]$ and so $[a; b]$ is closed. $\square$

Lemma *If two regions share a common point, then their union is a region which contains every point in either region.*

Proof. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be regions such that $x \in A$ and $x \in B$. Then we see that $x \in A \cup B$. Without loss of generality, let $a_1 \leq b_1$. Then we see that $a_2 > b_1$, otherwise A and B would not both contain x. Thus there are three cases.

Case 1: Let $a_1 \leq b_1$ and $a_2 < b_2$ Then we have $a_1 \leq b_1 < a_2 < b_2$ and so every element of A is less than b_2 and every element of B is greater than a_1. But then every element of A or B is between a_1 and b_2 so $A \cup B = (a_1, b_2)$.

Case 2: Let $a_1 \leq b_1$ and $a_2 > b_2$. Then we have $a_1 \leq b_1 < b_2 < a_2$ and so every element of A is less than a_2 and every element of B is greater than a_1. But then every element of A or B is between a_1 and a_2 so $A \cup B = (a_1, a_2)$.

Case 3: Let $a_1 \leq b_1$ and $a_2 = b_2$. Then we have $a_1 \leq b_1 < b_2 = a_2$ and so every element of A is less than a_2 and every element of B is greater than a_1. But then every element of A or B is between a_1 and a_2 so $A \cup B = (a_1, a_2)$.

We see that in all cases, $A \cup B$ is a region which contains every element of A and B. $\square$

Exercise 14 *A chain of regions from a to b covers the the closed interval $[a; b]$.*

Proof. Let $a < b$ be points in C and let $R_1, R_2, \ldots, R_n$ be a chain of n regions going from a to b. In the case where $n = 1$ we have a region R_1 which contains both a and b. Now use induction on n and suppose that the union of a chain of n regions such that for $1 \leq i \leq n - 1$ we have $R_i \cap R_{i+1} \neq \emptyset$ is a region containing every point in each of the n regions. Consider the case for $n + 1$. We know that $R_1 \cup R_2 \cup \cdots \cup R_n$ is a region containing every element in R_1 through R_n. Because $R_n \cap R_{n+1} \neq \emptyset$, by the previous lemma we know that the union of this region with R_{n+1} is a region containing every element of the regions $R_1, R_2, \ldots, R_{n+1}$. So for any natural number n regions such that for $1 \leq i \leq n - 1$ we have $R_i \cap R_{i+1} \neq \emptyset$ we see their union is a region containing every element in each of the regions. But if $a \in R_1$ and $b \in R_n$ the union of this chain of regions will contain a, b and every element between a and b. Since each region is open, the chain covers $[a; b]$. $\square$