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Homework 6

Problem 1. *Let (V,ω) be a symplectic vector space and let $T \leq Sp(V,\omega)$ be the subgroup generated by the symplectic transvections. Suppose (e_1, f_1) and (e_2, f_2) are hyperbolic pairs. Let $\tau \in T$ satisfy $\tau(e_1) = e_2$ and put $f_3 = \tau(f_2)$. Suppose $\omega(f_3, f_2) = 0$. Show that we can find $\sigma \in T$ such that $\sigma(e_2) = e_2$ and $\sigma(f_3) = f_2$.*

Proof. Note that (e_2, e_2+f_2) is also a hyperbolic pair and furthermore $\omega(f_2, e_2+f_2) = \omega(f_2, e_2)+\omega(f_2, f_2) = -1$. Thus we can find some σ_1 which takes (e_2, f_2) to (e_2, e_2, f_2). But now $\omega(e_2 + f_2, f_3) = \omega(e_2, f_3) + \omega(f_2, f_3) = \omega(e_2, f_3) = 1$ and we can thus find σ_2 taking $(e_2, e_2 + f_2)$ to (e_2, f_3). Set $\sigma = \sigma_2\sigma_1$. □

Problem 2. *Let V be a two-dimensional vector space and ω a symplectic form on V. Show that $Sp(V,\omega) = SL(V)$.*

Proof. We already know $Sp(V,\omega) \subseteq SL(V)$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(V)$$

and let $(x, y) \in V$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Now since $\det(A) = ad - bc = 1$ we have

$$\begin{aligned} \omega(ax + by, cx + dy) &= \omega(ax, cx) + \omega(ax, dy) + \omega(by, cx) + \omega(by, dy) \\ &= ac\omega(x, x) + bd\omega(y, y) + ad\omega(x, y) + bc\omega(y, x) \\ &= ad\omega(x, y) - bc\omega(x, y) \\ &= (ad - bc)\omega(x, y) \\ &= \omega(x, y). \end{aligned}$$

Thus $A \in Sp(V,\omega)$ and we have the reverse containment as well. □