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Homework 3

**** Problem 1.** *Let R be an integral domain. Show that $(\widetilde{R}, +, \cdot)$ is a field.*

**** Problem 1.1** *Show that $+$ and $\cdot$ are well-defined. That is if $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$ then*

$$(a_1b_2 + a_2b_1, b_1b_2) \sim (c_1d_2 + c_2d_1, d_1d_2)$$

and

$$(a_1a_2, b_1b_2) \sim (c_1c_2, d_1d_2)$$

.

Proof. Let $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$. Then we have

$$a_1d_1 = b_1c_1$$

and

$$a_2d_2 = b_2c_2.$$

We multiply the first equation by b_2d_2 so we have

$$a_1b_2d_1d_2 = b_1b_2c_1d_2$$

and we multiply the second equation by b_1d_1 so we have

$$a_2b_1d_1d_2 = b_1b_2c_2d_1.$$

Now we add the two new equations together so we have

$$a_1b_2d_1d_2 + a_2b_1d_1d_2 = b_1b_2c_1d_2 + b_1b_2c_2d_1$$

and so

$$(a_1b_2 + a_2b_1)d_1d_2 = (c_1d_2 + c_2d_1)b_1b_2$$

which implies

$$(a_1b_2 + a_2b_1, b_1b_2) \sim (c_1d_2 + c_2d_1, d_1d_2).$$

Similarly, if we multiply $a_1d_1 = b_1c_1$ and $a_2d_2 = b_2c_2$ together we have

$$a_1a_2d_1d_2 = b_1b_2c_1c_2$$

and so

$$(a_1a_2, b_1b_2) \sim (c_1c_2, d_1d_2).$$

□

**** Problem 1.2 (Associativity of Addition)** *For all $p, q, r \in \widetilde{R}$ we have $(p + q) + r = p + (q + r)$.*

Proof. Let $p, q, r \in \widetilde{R}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $(r_1, r_2) \in r$. Then we see that

$$\begin{aligned}
(p+q)+r &= \left(\overline{(p_1,p_2)} + \overline{(q_1,q_2)}\right) + \overline{(r_1,r_2)} \\
&= \overline{(p_1q_2 + p_2q_1, p_2q_2)} + \overline{(r_1,r_2)} \\
&= \overline{((p_1q_2 + p_2q_1)r_2 + p_2q_2r_1, p_2q_2r_2)} \\
&= \overline{(p_1q_2r_2 + p_2q_1r_2 + p_2q_2r_1, p_2q_2r_2)} \\
&= \overline{((q_1r_2 + q_2r_1)p_2 + p_1q_2r_2, p_2q_2r_2)} \\
&= p + \overline{(q_1r_2 + q_2r_1, q_2r_2)} \\
&= p + (q+r).
\end{aligned}$$

□

**** Problem 1.3 (Commutativity of Addition)** *For all $p, q \in \widetilde{R}$ we have $p + q = q + p$.*

Proof. Let $p, q \in \widetilde{R}$ such that $(p_1, p_2) \in p$ and $(q_1, q_2) \in q$. Then we have

$$p + q = \overline{(p_1,p_2)} + \overline{(q_1,q_2)} = \overline{(p_1q_2 + p_2q_1, p_2q_2)} = \overline{(q_1p_2 + q_2p_1, q_2p_2)} = \overline{(q_1,q_2)} + \overline{(p_1,p_2)} = q + p.$$

□

**** Problem 1.4 (Additive Identity)** *There exists an $n \in \widetilde{R}$ such that for all $p \in \widetilde{R}$ we have $n + p = p$.*

Proof. We see that if we let $n \in \widetilde{R}$ such that $n = \overline{(0,1)}$ and if we let $(p_1, p_2) \in p$ for some $p \in \widetilde{R}$ then we have

$$n + p = \overline{(0,1)} + \overline{(p_1,p_2)} = \overline{((0)p_2 + (1)p_1, (1)p_2)} = \overline{(p_1,p_2)} = p.$$

□

**** Problem 1.5 (Additive Inverse)** *For all $p \in \widetilde{R}$ there exists $q \in \widetilde{R}$ such that $p + q = 0$.*

Proof. Let $p \in \widetilde{R}$ such that $(p_1, p_2) \in p$. Then we choose $q = \overline{(-p_1, p_2)}$ for $q \in \widetilde{R}$. Then we have

$$p + q = \overline{(p_1,p_2)} + \overline{(-p_1,p_2)} = \overline{(p_1p_2 + -p_1p_2, p_2p_2)} = \overline{(0, p_2p_2)} = \overline{(0,1)} = 0$$

since $(0)p_2p_2 = (0)(1)$. □

**** Problem 1.6 (Associativity of Multiplication)** *For all $p, q, r \in \widetilde{R}$ we have $(p \cdot q) \cdot r = p \cdot (q \cdot r)$.*

Proof. Let $p, q, r \in \widetilde{R}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $(r_1, r_2) \in r$. Then we have

$$(p \cdot q) \cdot r = \left(\overline{(p_1,p_2)} \cdot \overline{(q_1,q_2)}\right) \cdot \overline{(r_1,r_2)} = \overline{(p_1q_1, p_2q_2)} \cdot \overline{(r_1,r_2)} = \overline{(p_1q_1r_1, p_2q_2r_2)} = p \cdot \overline{(q_1r_1, q_2r_2)} = p \cdot (q \cdot r).$$

□

**** Problem 1.7 (Commutativity of Multiplication)** *For all $p, q \in \widetilde{R}$ we have $p \cdot q = q \cdot p$.*

Proof. Let $p, q \in \widetilde{R}$ such that $(p_1, p_2) \in p$ and $(q_1, q_2) \in q$. Then

$$p \cdot q = \overline{(p_1,p_2)} \cdot \overline{(q_1,q_2)} = \overline{(p_1q_1, p_2q_2)} = \overline{(q_1p_1, q_2p_2)} = \overline{(q_1,q_2)} \cdot \overline{(p_1,p_2)} = q \cdot p.$$

□

**** Problem 1.8 (Multiplicative Identity)** *There exists $e \in \widetilde{R}$ such that for all $p \in \widetilde{R}$ we have $e \cdot p = p$.*

Proof. Let $p \in \widetilde{R}$ such that $(p_1, p_2) \in p$ and let $e \in \widetilde{R}$ such that $e = \overline{(1,1)}$. Then we have

$$e \cdot p = \overline{(1,1)} \cdot \overline{(p_1, p_2)} = \overline{(p_1(1), p_2(1))} = p.$$

$\square$

**** Problem 1.9 (Multiplicative Inverse)** *For all $p \in \widetilde{R}$ with $p \neq 0$ there exists $q \in \widetilde{R}$ such that $p \cdot q = 1$.*

Proof. Let $p \in \widetilde{R}$ such that $(p_1, p_2) \in p$ and since $p_1 \neq 0$ let $q \in \widetilde{R}$ such that $(p_2, p_1) \in q$. Then we see that

$$p \cdot q = \overline{(p_1, p_2)} \cdot \overline{(p_2, p_1)} = \overline{(p_1 p_2, p_1 p_2)} = \overline{(1,1)} = 1.$$

$\square$

**** Problem 1.10 (Distributivity)** *For all $p, q, r \in \widetilde{R}$ we have $p \cdot (q + r) = p \cdot q + p \cdot r$.*

Proof. Let $p, q, r \in \widetilde{R}$ such that $(p_1, p_2) \in p$, $(q_1, q_2) \in q$ and $(r_1, r_2) \in r$. Then we have

$$\begin{aligned}
p \cdot (q + r) &= \overline{(p_1, p_2)} \cdot \left(\overline{(q_1, q_2)} + \overline{(r_1, r_2)}\right) \\
&= \overline{(p_1, p_2)} \cdot \overline{(q_1 r_2 + q_2 r_1, q_2 r_2)} \\
&= \overline{(p_1 q_1 r_2 + p_1 q_2 r_1, p_2 q_2 r_2)} \\
&= \overline{(p_1 q_1 r_2 + p_1 q_2 r_1, p_2 q_2 r_2)} \cdot \overline{(p_2, p_2)} \\
&= \overline{(p_1 p_2 q_1 r_2 + p_1 p_2 q_2 r_1, p_2 p_2 q_2 r_2)} \\
&= \overline{(p_1 q_1, p_2 q_2)} + \overline{(p_1 r_1, p_2 r_2)} \\
&= \overline{(p_1, p_2)} \cdot \overline{(q_1, q_2)} + \overline{(p_1, p_2)} \cdot \overline{(r_1, r_2)} \\
&= p \cdot q + p \cdot r.
\end{aligned}$$

$\square$

Since all the field axioms have been met for $(\widetilde{R}, +, \cdot)$ we see that it is a field.

**** Problem 2.** *Show that the ordering axioms hold for $<$ on an integral domain R.*

Proof. Let $P \subseteq R$ be a set such that for $a \in R$ exactly one of $a \in P$, $a = 0$, $-a \in P$ holds and for $a, b \in P$ we have $a + b, ab \in P$. Let $a, b \in R$. Suppose first that $a < b$. Then $(b - a) \in P$ and $(b - a) \neq 0$. Therefore $b \neq a$. Also, we know $-(b - a) = a - b$ is not in P so b is not less than a. If $a = b$, then $a - b = b - a = 0$ so $(a - b), (b - a) \notin P$ and b is not less than a nor is a less than b. Finally, if $b < a$ then $(a - b) \in P$ and so $a - b \neq 0$ so $b \neq a$. Also $-(a - b) = b - a$ is not in P. Thus a is not less than b. Therefore either $a < b$, $a = b$ or $a > b$.

Let $a, b, c \in R$ such that $a < b$ and $b < c$. Then $(b - a), (c - b) \in P$. Since P is closed under addition, $(b - a) + (c - b) = c - a$ is in P. Thus $a < c$.

Suppose again that $a < b$. Then $(b - a) \in P$. Note that

$$b - a = b - a + c - c = (b + c) - (a + c)$$

so $(b + c) - (a + c) \in P$. Then $a + c < b + c$.

Finally let $a < b$ and $c > 0$. Then $(b - a) \in P$ and since P is closed under multiplication, $(b - a)c = bc - ac$ is in P. Thus $ac < bc$. $\square$

**** Problem 3.** *On $\widetilde{R}$ define $\overline{(a,b)} \in P$ if $ab \in P$ in R. Show that this is well defined and gives an ordering on $\widetilde{R}$.*

Proof. Let $\overline{(a,b)}, \overline{(c,d)} \in \widetilde{R}$ such that $(a,b) \sim (c,d)$ and $\overline{(a,b)} \in P$. Then $ab \in P$ in and $ad = bc$ in R. Multiplying both sides by ac we have

$$a^2cd = abc^2.$$

Since $ab > 0$ and $c^2 > 0$ we know that $abc^2 > 0$ so $a^2cd > 0$. Also, since $a^2 > 0$, we see that $cd > 0$ so $cd \in P$ and $\overline{(c,d)} \in P$. This shows that the definition is well defined. An ordering on $\widetilde{R}$ is defined by $\overline{(a_1,b_1)} < \overline{(a_2,b_1)}$ if $\overline{(a_2,b_2)} + \overline{(-a_1,b_1)} \in P$. We now show the ordering axioms are met for this relation and elements $a = \overline{(a_1,a_2)}$, $b = \overline{(b_1,b_2)}$, $c = \overline{(c_1,c_2)}$ in $\widetilde{R}$.

First let $a < b$. Then

$$\overline{(a_2b_1 - a_1b_2, a_2b_2)} \in P$$

so

$$(a_2b_1 - a_1b_2)a_2b_2 \in P$$

and

$$(a_2b_1 - a_1b_2)a_2b_2 \neq 0.$$

Since $a_2b_2 \neq 0$, we see that

$$a_2b_1 \neq a_1b_2$$

so

$$\overline{(a_1,a_2)} \neq \overline{(b_1,b_2)}.$$

Also,

$$-((a_2b_1 - a_1b_2)a_2b_2) = (a_1b_2 - a_2b_1)a_2b_2$$

is not in P so

$$\overline{(a_1b_2 - a_2b_1, a_2b_2)} = \overline{(a_1,a_2)} + \overline{(-b_1,b_2)}$$

is not in P in $\widetilde{R}$. Thus b is not less than a. If $b < a$ it follows similarly that $a \neq b$ and a is not less than b. Finally, if $a = b$ then $(a_1,a_2) \sim (b_1,b_2)$ and

$$a_1b_2 = a_2b_1.$$

Thus

$$(a_1b_2 - a_2b_1) = 0$$

and

$$(a_1b_2 - a_2b_1)a_2b_2 = 0$$

which implies

$$\overline{(a_1b_2 - a_2b_1, a_2b_2)} = \overline{(b_1,b_2)} + \overline{(-a_1,a_2)}$$

is not in P. Thus a is not less than b. A similar argument shows that b is not less than a.

Suppose now that $a < b$ and $b < c$. Then

$$\overline{(a_2b_1 - a_1b_2, a_2b_2)} \in P$$

and

$$\overline{(b_2c_1 - b_1c_2, b_2c_2)} \in P.$$

Thus

$$(a_2b_1 - a_1b_2)a_2b_2 > 0$$

and

$$(b_2c_1 - b_1c_2)b_2c_2 > 0.$$

Multiply the first equation by c_2^2 and the second by a_2^2 and add them to obtain

$$0 < (a_2^2b_2^2c_1c_2 - a_2^2b_1b_2c_2^2) + (a_2^2b_1b_2c_2^2 - a_1a_2b_2^2c_2^2) = a_2^2b_2^2c_1c_2 - a_1a_2b_2^2c_2^2.$$

Then

$$(a_2c_1 - a_1c_2)a_2c_2 > 0$$

which means

$$a = \overline{(a_1, a_2)} < \overline{(c_1, c_2)} = c.$$

Still supposing that $a < b$, we have again

$$(a_2b_1 - a_1b_2)a_2b_2 > 0.$$

Multiplying both sides by c_2^4 we can write

$$c_2^2(a_2b_1 - a_1b_2)a_2b_2c_2^2 = (a_2b_1c_2^2 + a_2b_2c_1c_2 - (a_1b_2c_2^2 + a_2b_2c_1c_2))a_2b_2c_2^2 > 0$$

which simplifies to

$$((b_1c_2 + b_2c_1)a_2c_2 - (a_1c_2 + a_2c_1)b_2c_2)a_2b_2c_2^2 > 0.$$

Thus

$$a + c = \overline{(a_1, a_2)} + \overline{(c_1, c_2)} = \overline{(a_1c_2 + a_2c_1, a_2c_2)} < \overline{(b_1c_2 + b_2c_1, b_2c_2)} = \overline{(b_1, b_2)} + \overline{(c_1, c_2)} = b + c.$$

Finally, assume that $a < b$ and $c > 0$. Then

$$(a_2b_1 - a_1b_2)a_2b_2 > 0$$

and $c_1c_2 > 0$. Then we have

$$0 < (a_2b_1 - a_1b_2)a_2b_2(c_1c_2)(c_2^2) = (a_2b_1c_1c_2 - a_1b_2c_1c_2)a_2b_1c_2^2$$

which means

$$ac = \overline{(a_1, a_2)} \cdot \overline{(c_1, c_2)} = \overline{(a_1c_1, a_2c_2)} < \overline{(b_1c_1, b_2c_2)} = \overline{(b_1, b_2)} \cdot \overline{(c_1, c_2)} = bc.$$

□

** **Problem 4.** *Show that for a polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ the definition $p(x) \in P$ if $a_n > 0$ holds for the above definition.*

Proof. Note that if $a_n > 0$ in R then $a_n \neq 0$ and $-a_n > 0$. Thus $p(x) \neq 0$ and $-p(x) \in P$. Also, if we let $q(x) = b_mx^m + b_{m-1}x^{m-1} + \ldots b_1x + b_0$ such that $q(x) \in P$, then $p(x) + q(x) \in P$ because $p(x) + q(x)$ either has leading term $\max(a_n, b_m)$ or $a_n + b_m$. Likewise $p(x)q(x) \in P$ since it has leading term $a_nb_m > 0$. □

Lemma 1. *Let $a \in \mathbb{Q}$ such that $0 < a < 1$. Then $a^2 < a$. Likewise, if $a > 1$, then $a^2 > a$.*

Proof. Let $0 < a < 1$ such that $a = \overline{(a_1, a_2)}$. Then $a^2 = \overline{(a_1^2, a_2^2)}$ and

$$a - a^2 = \overline{(a_1, a_2)} + \overline{(-a_1^2, a_2^2)} = \overline{(a_1 a_2^2 - a_1^2 a_2, a_2^3)}.$$

Since $a > 0$, we can assume that both $a_1, a_2 > 0$. Then $a_2^3 > 0$. Also, since $a < 1$ we have $1 - a > 0$ so $a_2 - a_1 > 0$ and $a_1 < a_2$. Then $a_1(a_1 a_2) < a_2(a_1 a_2)$. This shows that

$$(a_1 a_2^2 - a_1^2 a_2)(a_2^3) > 0$$

which means $a^2 < a$. A similar proof is used to show that for $a > 1$, $a^2 > a$. $\square$

Problem 1. *Let a be a positive rational number. Let $A = \{x \in \mathbb{Q} \mid x^2 < a\}$. Show that A is bounded in $\mathbb{Q}$.*

Proof. Let $x \in A$. Note that if $x \leq 0$ then $x \leq 0 < a < a + 1$. If $0 < x < 1$ then by Lemma 1, $x^2 < x < 1 < a + 1$ since $a > 0$. If $x \geq 1$ then by Lemma 1, $x \leq x^2 < a < a + 1$. In all cases $a + 1$ serves as an upper bound for A. $\square$

Problem 2. *Show that the least upper bound of a set is unique, if it exists.*

Proof. Let A be set such that u and v are least upper bounds for A. Then u and v are upper bounds for A and each one is less than every other upper bound of A. Thus, it is not the case that $u < v$ or $v < u$. Therefore $u = v$. $\square$

Problem 3. *Show that any two ordered fields with the least upper bound property are order isomorphic.*

Proof. Let F and F' be two ordered fields with the least upper bound property. We already know that F and F' contain the rationals as a subfield. Thus there exist injective maps $q_1 : \mathbb{Q} \to Q$ and $q_2 : \mathbb{Q} \to Q'$ where $Q \subseteq F$ and $Q' \subseteq F'$. Since both Q and Q' are both order isomorphic to $\mathbb{Q}$, we know there is an order isomorphism from Q to Q'. Thus there is an injective order homomorphism $f : Q \to F'$. Now let $A_r = \{x \in Q \mid x < r\}$ for $r \in F$. Since A_r is nonempty and bounded in F, it follows that $f(A_r)$ is nonempty and bounded in F'. Now define $g : F \to F'$ such that $g(x) = \sup(A_x)$. Define

$$A_{x+y} = \{a + b \in Q \mid a \in A_x, b \in A_y\}.$$

For multiplication define sets $P = \{p \in Q \mid p > 0\}$, $N = \{p \in Q \mid p \leq 0\}$ and the product of two sets A and B as $A * B = \{ab \mid a \in A, b \in B\}$. Then for $x, y > 0$ we have

$$A_{xy} = N \cup ((A \cap P) * (B \cap P))$$

and in general

$$A_{xy} = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \\ A_{|x||y|} & \text{if } x > 0 \text{ and } y > 0 \text{ or } x < 0 \text{ and } y < 0 \\ -A_{|x||y|} & \text{if } x < 0 \text{ and } y > 0 \text{ or } x > 0 \text{ and } y < 0 \end{cases}$$

Here $-A_x = \{a \in Q \mid a < -x\}$. Using Problem 5 we can see that

$$g(x + y) = \sup(A_{x+y}) = \sup(A_x) + \sup(A_y) = g(x) + g(y)$$

and

$$g(xy) = \sup(A_{xy}) = \sup(A_x)\sup(A_y) = g(x)g(y).$$

Additionally, since f is an order preserving map from Q to F' we see that g is order preserving. Thus there exists an order homomorphism from Q to F'. Similarly, there exists an order homomorphism from Q' to F. Using the Schrder-Berstein Theorem, we can say that there is an order preserving isomorphism from F to F'. $\square$

Problem 4. *Let n be a positive integer that is not a perfect square. Let $A = \{x \in \mathbb{Q} \mid x^2 < n\}$. Show that A is bounded in $\mathbb{Q}$, but has neither a greatest lower bound nor a least upper bound in $\mathbb{Q}$. Conclude that $\sqrt{n}$ exists in $\mathbb{R}$, that is, there exists a real number a such that $a^2 = n$.*

Proof. Problem 1 Shows that A is bounded in $\mathbb{Q}$ by $n+1$. Suppose that u is an upper bound for A. Note that since $0 \in A$, we have $u > 0$. Then $u^2 > n$ and $u^2 - n > 0$. But then

$$\frac{u^2 - n}{u + n} > 0$$

and letting

$$v = u - \frac{u^2 - n}{u + n} = \frac{nu + n}{u + n}$$

we see that $u - v > 0$ so $v < u$. But

$$v^2 = \frac{n^2u^2 + 2n^2u + n^2}{u^2 + 2nu + n^2} > \frac{n^2u^2 + 2n^2u + n^2}{\frac{1}{n}(n^2u^2 + 2n^2u + n^2)} = n$$

since $n(u^2 - n) + n - u^2 > 0$ as $n > 1$. Thus v is also an upper bound for A. Therefore the least upper bound for A is not in $\mathbb{Q}$. But since A is nonempty and bounded, a least upper bound exists in $\mathbb{R}$. $\square$

Problem 5. *Suppose that A and B are bounded sets in $\mathbb{R}$. Prove or disprove the following:*
1) The $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.
2) If $A + B = \{a + b \mid a \in A, b \in B\}$, *then* $\sup(A + B) = \sup(A) + \sup(B)$.
3) If the elements of A and B are positive and $A \cdot B = \{ab \mid a \in A, b \in B\}$, then
$\sup(A \cdot B) = \sup(A)\sup(B)$.
4) Formulate the analogous problems for the greatest lower bound.

Proof. Note the the statements only make sense if A and B are nonempty. Otherwise the $\sup A$ and $\sup B$ do not exist. Hence, assume that A and B are nonempty, bounded subsets of $\mathbb{R}$ and let $a = \sup A$ and $b = \sup B$.

1) Let $x \in A$. Then $x \le a \le \max\{a, b\}$. Let $y \in B$. Then $y \le b \le \max\{a, b\}$. Thus every element in A or in B is less than or equal to $\max\{a, b\}$. Therefore $\max\{a, b\}$ is an upper bound for $A \cup B$. Suppose there exists $c < \max\{a, b\}$ such that c is an upper bound for $A \cup B$. Then c is an upper bound for A and an upper bound for B. Since $c < \max\{a, b\}$, $c < a$ or $c < b$. Without loss of generality, assume that $c < a$. Then c is an upper bound for A which is less than $\sup A$. This is a contradiction and so there exists no upper bounds for $A \cup B$ which are less than $\max\{a, b\}$. Therefore $\sup(A \cup B) = \max\{a, b\}$.

2) Let $k \in A + B$. Then $k = x + y$ where $x \in A$ and $y \in B$. Since $x \le a$ and $y \le b$ we know that $k = x + y \le a + b$. Thus $a + b$ is an upper bound for $A + B$. Suppose there exists $c < a + b$ such that c is an upper bound for $A + B$. Consider the value $r = (a + b - c)/2 > 0$. Since a is the least upper bound for A, it must be the case that there exists some element $p \in A$ such that $a - r < p \le a$, otherwise $a - r$ would be an upper bound for A which is less than a. Likewise, there exists a $q \in B$ such that $b - r < q \le b$. Then $p + q \in A + B$ and

$$c = a + b - (a + b - c) = (a - r) + (b - r) < p + q \le a + b.$$

Thus there exists an element of $A + B$ which is greater than c and so $\sup(A + B) = a + b$.

3) Let $k \in A \cdot B$. Then $k = xy$ where $x \in A$ and $y \in B$. Since $0 < x \le a$ and $0 < y \le b$ we have $k = xy \le ab$. Thus ab is an upper bound for $A \cdot B$. Now suppose there exists $c < ab$ such that c is an upper bound for $A \cdot B$. Let $r = ab - c$. Then there exists $x, y \in \mathbb{R}$ such that $xy = ab - r/2$. Let $p = a - x$ and $q = b - y$. Then there exists $u \in A$ such that $u > a - p/2 > x$ and there exists $v \in B$ such that

$v > b - q/2 > y$. But then $uv \in A \cdot B$, but $uv > xy = ab - r/2 > c$.

4) The analogous problems for the greatest lower bound are:

1) $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$.

2) If $A + B = \{a + b \mid a \in A, b \in B\}$ then $\inf(A + B) = \inf(A) + \inf(B)$.

3) If the elements of A and B are positive and $A \cdot B = \{ab \mid a \in A, b \in B\}$ then $\inf(A \cdot B) = \inf(A)\inf(B)$. $\square$

Problem 6. *Let F be an Archimedean ordered field. Show that F is order isomorphic to a subfield of $\mathbb{R}$.*

Proof. Note that since F is an ordered field, the rationals exist as a subfield which we will refer to as $\mathbb{Q}$. Define the function $f : \mathbb{Q} \to \mathbb{R}$ where $f(x) = \{p \in \mathbb{Q} \mid p < x\}$. Using the Archimedean property, we know that for all $x \in F$, $f(x) \neq \emptyset$. From here it's easy to see that for all $x \in F$, $f(x)$ is a Dedekind cut. Using the definitions of addition and multiplication from Problem 3 for $p, q \in F$ we see that $f(p + q) = f(p) + f(q)$ and $f(pq) = f(p)f(q)$. Also, since the ordering of $\mathbb{Q}$ holds in $\mathbb{R}$, f preserves the ordering of F. Finally, we can show that f is injective because if $p \neq q$ in $\mathbb{Q}$, there exists some number $r \in \mathbb{Q}$ such that $p < r < q$ because F is Archimedean. Therefore $f(p) \neq f(q)$. $\square$