Sheet 31: Taylor Series

Definition 1 *A function of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

is called a power series centered at a.

Theorem 2 *Suppose that the series*

$$\sum_{n=0}^{\infty} a_n x_0^n$$

converges and let $0 < a < |x_0|$. Then on $B(0,a)$ the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and

$$g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

uniformly and absolutely converge. Also f is differentiable and

$$f'(x) = g(x)$$

for all $x \in B(0,a)$.

Proof. Note that for $x \in B(0,a)$ we have $|x/x_0| < 1$ and so

$$\sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n$$

is convergent since it's a geometric series. Then by the Comparison Criterion we have

$$\sum_{n=0}^{\infty} |a_n| \left| \frac{x}{x_0} \right|^n = \sum_{n=0}^{\infty} \left| a_n \frac{x^n}{x_0^n} \right|$$

is convergent and so

$$\sum_{n=0}^{\infty} |a_n x^n|$$

is convergent. A similar proof holds to show that $g(x)$ is absolutely convergent using the fact that $1/n$ converges to 0. Also we have $a_n x^n$ is bounded by $|a_n a^n|$ on $B(0,a)$ and $n a_n x^{n-1}$ is bounded by $|n a_n a^{n-1}|$ on $B(0,a)$ and since the series absolutely converge, we can use the Weierstrass M-test to show that f and g are uniformly convergent (30.10). Finally since $n a_n x^{n-1}$ is integrable on $[a;b]$, $n a_n x^{n-1}$ uniformly converges and $n a_n x^{n-1}$ is continuous so g is continuous, we have $f'(x) = g(x)$ for all $x \in B(0,a)$ (30.9). $\square$

Theorem 3 *For a power series $\sum_{n=0}^{\infty} a_n x^n$ let*

$$A = \left\{ x \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}$$

be the set of converge for the power series. Then either A is everything or there exists a such that

$$B(0,a) \subseteq A \subseteq \overline{B(0,a)}.$$

This a is called the radius of convergence of the power series.

Proof. Suppose that A is not everything. Then there exists $b \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} a_n b^n$ diverges. Note then that for all $x \in \mathbb{R}$ with $x \geq b$ we have $\sum_{n=1}^{\infty} a_n x^n$ diverges. Note also that $\sum_{n=1}^{\infty} a_n (0)^n$ converges. Then note that b is an upper bound for A and A is nonempty so let $a = \sup A$. Then we have $B(0,a) \subseteq A$. If we have $c > a$ then $\sum_{n=1}^{\infty} a_n c^n$ diverges so it must also be the case that $A \subseteq \overline{B(0,a)}$. ☐

Exercise 4 *Find real power series centered at* 0 *with sets of convergence* 0, $\mathbb{R}$, $(-1;1)$, $[-1;1)$ *and* $[-1;1]$.

0:

$$\sum_{n=0}^{\infty} n! x^n.$$

$\mathbb{R}$:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$(-1;1)$:

$$\sum_{n=0}^{\infty} -x^{2n}.$$

$[-1;1)$:

$$\sum_{n=0}^{\infty} x^n.$$

$[-1;1]$:

$$\sum_{n=1}^{\infty} (-1)^n x^{2n}.$$

Theorem 5 *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge absolutely and* (c_n) *is a sequence containing the products* $a_i b_j$ *for each pair* (i,j) *then*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

Proof. Note that

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, we can rearrange the terms and they will still converge to the same thing. Then the partial sums of $\sum_{n=0}^{\infty} b_n$ can be rearranged in the same way as c_n so that the partials sums of $\sum_{n=0}^{\infty} c_n$ are just the product of the partial sums of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Then since the product of limits is the limit of products we have the desired relation. ☐

Theorem 6 (Cauchy Product) *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *be the power series with radius of convergence at least* a. *Let*

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

Then the power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

has radius of convergence of at least a *and for* $x \in B(0,a)$ *we have*

$$h(x) = f(x)g(x).$$

Proof. We know that $f(x)$ and $g(x)$ are absolutely convergent on $B(0,a)$ (31.2). Also we know that $h(x)$ is uniformly and absolutely convergent on $B(0,a)$ because $f(x)$ and $g(x)$ are (31.2, 31.5). Also using Theorem 5 we know that for $x \in B(0,a)$ we have $h(x) = f(x)g(x)$. $\square$

Definition 7 *Let f be a real function such that $f^{(n)}(a)$ exists for all n. Then the Taylor series of f at a is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem 8 *For all real x we have*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Proof. Consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^n x^{2n}}{(2n)!}$$

and note that

$$f'(x) = \sum_{n=0}^{\infty} -\frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^n x^{2n}}{(2n)!}$$

and

$$f''(x) = \sum_{n=0}^{\infty} -\frac{(-1)^n x^{2n+1}}{(2n+1)!} - \frac{(-1)^n x^{2n}}{(2n)!}.$$

Then we can easily verify $f + f'' = 0$, $f(0) = 1$ and $f'(0) = 1$. Then we must have $f = \cos + \sin$ (27.14). Then since $\sin' = \cos$ it must be the case that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Also we have $(e^x)' = e^x$ and $e^0 = 1$ so the Taylor series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

But note then that for all n, the remainder terms in the Taylor polynomial will converge to zero because of the $n!$ factor. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$\square$

Theorem 9 *For $x \in (-1;1)$ we have*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Proof. We have $1/(1-x)$ is a geometric series (15.6). Also, using the Taylor polynomial definition we have the Taylor series for log is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

Note that for $x < 1$ we know this series converges so the remainder terms must go to zero. Thus

$$\log x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

□

Theorem 10 *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ be a convergent sequence in $B(a,r)$ for some $r > 0$. Then the Taylor series of $f(x)$ at a equals $\sum_{n=0}^{\infty} a_n(x-a)^n$.*

Proof. Note that since

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

we have

$$f'(x) = f(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1}$$

and in general

$$f^{(j)}(x) = \sum n = 0^{\infty} \frac{n!}{(n-j)!} a_n (x-a)^{n-j}$$

using Theorem 2 (31.2). But then each term in $f^{(j)}(a)$ is zero unless $n = j$ in which case we have

$$f^{(j)}(a) = \frac{j!}{(j-j)!} a_j (a-a)^{j-j} = j! a_j (0)^0 = j! a_j$$

Thus $f^{(n)}(a) = n! a_n$. Using this in the Taylor Series definition we have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{n! a_n}{n!}(x-a)^n = \sum_{n=0}^{\infty} a_n (x-a)^n = f(x).$$

□