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Homework 1

Theorem 9 *Let (f_n) be a sequence of continuous functions on $[a;b]$ that uniformly converges to f on $[a;b]$. Then f is continuous on $[a;b]$.*

Proof. Let $\varepsilon > 0$ and consider $\varepsilon/3$. We know (f_n) uniformly converges to f so there exists N such that for all $n > N$ and for all $x, y \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon/3$ and $|f(y) - f_n(y)| < \varepsilon/3$. Also f_n is continuous for all n so for all $n > N$ and for all $x \in [a;b]$ there exists $\delta_n > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta_n$ we have $|f_n(x) - f_n(y)| < \varepsilon/3$. Consider δ_{N+1}. Then for all $x \in [a;b]$ there exists $\delta_{N+1} > 0$, which may depend on x, such that for all $y \in [a;b]$ with $|x - y| < \delta_{N+1}$ we have $|f_{N+1}(x) + f_{N+1}(y)| < \varepsilon/3$. By the triangle inequality we have $|f(x) - f_{N+1}(y)| \leq |f_{N+1}(x) - f_{N+1}(y)| + |f(x) - f_{N+1}(x)| < 2\varepsilon/3$ and then $|f(x) - f(y)| < |f(x) - f_{N+1}(y)| + |f(y) - f_{N+1}(y)| < \varepsilon$. Thus for all $x \in [a;b]$ there exists some $\delta > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Therefore f is continuous on $[a;b]$. $\square$

Theorem 3 (Division Remainder) *Let $a, b \in \mathbb{R}[x]$ be polynomials with $b \neq 0$. Then there exists unique $q, r \in \mathbb{R}[x]$ such that*

$$a = bq + r$$

and

$$\deg r < \deg b.$$

Proof. To show existence consider the set $S = \{a - bc \mid c \in \mathbb{R}[x]\}$. Suppose that for all $r \in S$, $\deg(r) \geq \deg(b)$. Choose $p \in S$ such that $\deg(p)$ is the minimum degree of all elements of S using the Well Ordering Principle. Note that $p = a - bc$ for some $c \in \mathbb{R}[x]$. Now let $q = p - bd$ for some $d \in \mathbb{R}[x]$. Then $q = a - bc - bd = a - b(c + d)$ and so $q \in S$. Thus $\deg(q) \geq \deg(p)$. But then if $p(x) = \sum_{i=0}^{n} a_i x^i$ and $b(x) = \sum_{i=0}^{m} b_i x^i$ then consider $d = a_n/b_m x^{(n-m)}$. Then $\deg(bd) = n$ and so $\deg(q) < \deg(p)$ since $q = p - bd$. This is a contradiction and so there exists $r \in S$ such that $\deg(r) < \deg(b)$. For uniqueness suppose that there exists q, q', r, r' with $q \neq q'$ and $r \neq r'$ such that $a = bq + r$, $a = bq' + r'$, $\deg(r) < b$ and $\deg(r') < b$. Then $bq + r = bq' + r'$ and $b(q - q') = r' - r$. Note that since $q \neq q'$ and $r \neq r'$, $\deg(q - q') \geq 0$ and $\deg(r - r') \geq 0$. But then using Theorem 2 we have $\deg(r - r') < b$ and $\deg(b(q - q')) = \deg(b) + \deg(q - q') \geq \deg(b)$. This is a contradiction and so $q = q'$ and $r = r'$ which means q and r are unique. $\square$