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Homework 9

Problem 1. *Let (A, M) a Noetherian local ring of dimension n. Show that $\dim_{A/M}(M/M^2) \geq n$. (Here $\dim_{A/M}(M/M^2)$ denotes the dimension of the A/M-vector space M/M^2).*

Proof. Let $x_1, \ldots, x_m$ be a basis of M/M^2 over A/M. Since A is Noetherian, M is a finitely generated A-module. Since A is local, M is contained in the Jacobson radical of A. So we may apply Nakayama's Lemma to conclude that m preimages of the x_i generate M. But since A is local, ht $M = \dim(A) = n$ and by the principal ideal theorem, $n = \text{ht } M \leq m$. $\square$

A Noetherian local ring (A, M) of dimension n is called a regular local ring if $\dim_{A/M}(M/M^2) = n$.

Problem 2. *Show that a Noetherian local ring (A, M) is regular if and only if M is generated by $n = \dim(A)$ elements.*

Proof. If M is generated by $x_1, \ldots, x_n$, then M/M^2 is generated by $x_1 + M^2, \ldots, x_n + M^2$ by simply taking an element $y + M^2$ and writing y as a linear combination of the x_i. On the other hand, if $x_1 + M^2, \ldots, x_n + M^2$ is a basis for M/M^2 over A/M, then take $N = (x_1, \ldots, x_n) \subseteq M$. Then $M \subseteq N + M^2$, so $M/N \subseteq M(M/N)$. Now apply Nakayama's Lemma to get $M/N = 0$ and $M = N$. $\square$

Problem 3. *Let (A, M) be a regular local ring of dimension $n \geq 1$. Show that there exists an $x \in M$, $x \notin M^2$ and x not in any minimal prime ideal P with $\dim(A/P) = n$. Further show that for any such choice of x, $A/Ax = \overline{A}$ is a regular local ring of dimension $n - 1$.*

Proof. Since $\dim(A) = \text{ht } M \geq 1$, we know $M \neq 0$. Thus $M \neq M^2$ by Nakayama's Lemma. Suppose M is contained in the union of M^2 and the minimal primes of A. Since A is Noetherian, this is a finite union only (possibly) one ideal of which is not prime. By the prime avoidance lemma, M is contained in one of these ideals. But $M \nsubseteq M^2$, and M cannot be minimal since ht $M = 1$. Thus we can find $x \in M$ which is not in M^2 and not in any minimal prime of A. Now note that $\overline{M} = M/Ax$ is a maximal ideal in $\overline{A}$ which must contain any other ideal, since M contains all ideals in A. Thus $\overline{A}$ is a local ring.

Since $x \notin M^2$, we can pick it to be in a basis for the space M/M^2 over A/M. Then M/M^2 has basis $(\overline{x}, \overline{x_2}, \ldots, \overline{x_n})$. Using Nakayama's Lemma, the preimages $(x, x_1, \ldots, x_n)$ generate M in A. But then $(x_2 + Ax, \ldots, x_n + Ax)$ generate $\overline{M}$. Therefore ht $\overline{M} = \dim(\overline{A}) \leq n - 1$ by the principal ideal theorem.

This shows that $\overline{A}$ is a regular ring since $\overline{M}$ is generated by $n - 1 = \dim(\overline{A})$ elements. Problem 2 now shows that $\overline{A}$ is regular. Now note that since x is not in any minimal prime, we can pick a minimal prime P and consider A/P. This is a local integral domain with $\overline{x} = x + P$ not in the maximal ideal. By a previous problem, and since P is minimal, we know $\dim(A/Ax) = \dim((A/P)/(A/P)\overline{x}) = \dim(A/P) - 1 = n - 1$. $\square$

Problem 4. *Show that regular local ring (A, M) is an integral domain.*

Proof. Induct on n. If $n = 0$ then ht $M = 0$ so M is minimal and A is a field, thus an integral domain. Suppose $n \geq n$. Using Problem 3, we can find $x \in M$ with $\dim(A/Ax) = n - 1$, so by the inductive hypothesis, this is an integral domain. Therefore Ax is a prime ideal. Since x is not in any minimal prime of A, Ax cannot be a minimal prime. Therefore Ax strictly contains some minimal prime $P \in \text{Spec}(A)$. Pick $y \in P$, and since $P \subsetneq Ax$, $y = ax$ for some $a \in A$. Since $x \notin P$, we have $a \in P$, so $P = Px$. By Nakayama's Lemma, $P = 0$. Since 0 is a prime ideal in A, A must be an integral domain. $\square$

Problem 5. *(a) Let A be a UFD. Show that A is an integrally closed integral domain.*
(b) Let A be a UFD with $2 \in A^*$. Let $d \in A$, $d = p_1 \cdots p_r$, $r > 0$, p_i prime elements of A such that $Ap_i \neq Ap_j$ if $i \neq j$. Let B be a ring containing A. Suppose $B = A[\alpha]$, $\alpha^2 = d$. Let K, L be the fields of fractions of A and B respectively. Show that L/K is an algebraic extension of degree 2. Further, show that B is an integrally closed domain. (We recall that an integral domain A with field of fractions K is an integrally closed domain *if $x \in K$, integral over A implies $x \in A$).*

Proof. (a) Let K be the field of fractions of A and let $a_n(r/s)^n+\cdots+a_0=0$ be a polynomial with coefficients in A where we take $a_n=1$. Assume that the factorizations of $r,s\in A$ into irreducibles are distinct. Then multiply by s^n to get $r^n+\cdots+a_0s^n=0$ and subtract to get $r^n=-s(-a_{n-1}r^{n-1}+\cdots+a_0s^{n-1})$. Since s and r share no irreducible factors, we must have $s\in A^*$, which means $r/s=rs^{-1}\in A$. Thus A is integrally closed.

(b) Note that x^2-d is an irreducible polynomial since $d=p_1\cdots p_r$ is square free (since $Ap_i\neq Ap_j$ if $i\neq j$). Then $K[x]/(x^2-d)$ is a field, and since $B=A[\alpha]$ and A is an integral domain, we have $L=K[x]/(x^2-d)$ using the map $x\mapsto\alpha$. Then L is an algebraic extension of $\deg(x^2-d)=2$.

To show that B is integrally closed, we can take any element $a+b\alpha\in L$ with $a,b\in K$ as an arbitrary element of L which is integral over B. Let $(a+b\alpha)^n+c_{n-1}(a+b\alpha)^{n-1}+\cdots+c_0=0$ be a monic polynomial that $a+b\alpha$ satisfies with $c_i\in B$. We can write $c_i=a_i+b_i\alpha$ with $a_i,b_i\in A$. Separate this polynomial into it's α and non α parts so we have

$$(a+b\alpha)^n+a_{n-1}(a+b\alpha)^{n-1}+\cdots+a_0=b_{n-1}\alpha(a+b\alpha)^{n-1}+\cdots+b_0\alpha.$$

Now square both sides and note that since the left side is purely α, it becomes purely in A since $\alpha^2\in A$. Moving the terms back to the right gives a polynomial in $A[x]$ which $a+b\alpha$ satisfies. Thus $a+b\alpha$ is integral over A.

Note that α is a degree 2 algebraic element over K, so there exists some order 2 element $\sigma\in\mathrm{Gal}(K(\alpha)/K)$ with $\sigma:\alpha\to-\alpha$. But then if $a+b\alpha$ is integral over A, then apply σ to any monic polynomial that $a+b\alpha$ satisfies, and $a-b\alpha$ will satisfy the same polynomial. So $a-b\alpha$ is integral over A as well.

Note that A is integrally closed since A is a UFD. Then note that $a+b\alpha+a-b\alpha=2a$ is in A, since the sum of two integral elements is integral. Since $2\in A^*$, we have $a\in A$. Similarly, $(a+b\alpha)(a-b\alpha)=a^2-(b\alpha)^2=a^2-b^2d$ is in A since it's the product of two integral elements. Since $a\in A$, $b^2d\in A$. Since $b\in K$, we can write $b^2=b_1^2/b_2^2$ for $b_1,b_2\in A$ with b_1 and b_2 are relatively prime. But then since $b^2d\in A$, we must have $b_2^2\mid d$, but d is squarefree, as noted earlier. Thus b_2^2 is a unit, and $b^2=ub_1^2$ for some $u\in A^*$. Because A is a UFD, $b=b_1$ is in A. Therefore $a,b\in A$, so $a+b\alpha\in B$ and B is integrally closed. ☐

Let B be a ring and $A\subseteq B$ a subring. Recall that $A'=\{b\in B\mid b\text{ is integral over A}\}$ is a subring of B called the *integral closure* of A in B.

Problem 6. *With A, B and A' as above, let $S\subseteq A$ be a multiplicatively closed subset. Show that $S^{-1}A'$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Proof. Since localization preserves integrality, and A' is integral over A, we know $S^{-1}A'$ is integral over $S^{-1}A$. On the other hand, if $b/s\in S^{-1}B$ is integral over $S^{-1}A$ then there exists a polynomial $(b/s)^n+(a_{n-1}/s_{n-1})(b/s)^{n-1}+\cdots+a_0/s_0=0$ with coefficients in $S^{-1}A$. Let $t=s_0\cdots s_{n-1}$ and multiply by $(st)^n$. We now have an polynomial of which bt is a root. Therefore $bt\in A'$ and $b/s=bt/st$ is in $S^{-1}A'$. This gives the reverse inclusion. ☐

Problem 7. *Let B be integral over its subring A.*
(a) Let $P\in\mathrm{Spec}(A)$. Show that $PB\cap A=P$.
(b) Let $I\subseteq A$ be an ideal. Show that $IB\cap A\subseteq\sqrt{I}$. Thus deduce that if $I=\sqrt{I}$, then $IB\cap A=I$.
(c) $I\subsetneq A$ an ideal implies $IB\neq B$.
(d) $B^*\cap A=A^*$.

Proof. (a) This follows immediately from part (b) since for a prime ideal $\sqrt{P}=P$.

(b) Take $x\in IB\cap A$. Since $x\in IB$, $x=ab$ for $a\in I$ and $b\in B$. Since B is integral over A, b satisfies $b^n+a_{n-1}b^{n-1}+\cdots+a_0=0$ with $a_i\in A$. Now multiply both sides by a^n to get $(ab)^n+aa_{n-1}(ab)^{n-1}+\cdots+a^na_0=0$. Note that $a^ia_{n-i}\in I$ since $a\in I$. In particular, $a^na_0\in I$, so we must also have $(ab)^n+aa_{n-1}(ab)^{n-1}+\cdots+a^{n-1}a_1(ab)\in I$ (since this is $-a^na_0\in I$). Now note that since $ab\in IB\cap A$, $ab\in A$ so $a^{n-1}a_1(ab)\in I$ as well. Thus the sum of the first $n-1$ terms is in I. Inductively, we see that $(ab)^n\in I$, so $ab\in\sqrt{I}$. Thus $IB\cap A\subseteq\sqrt{I}$. If $\sqrt{I}=I$, then $IB\cap A\subseteq I$. But note that $I\subseteq A$ and $I\subseteq IB$, so $I\subseteq IB\cap A$ and $IB\cap A=I$.

(c) Suppose that $IB = B$. Then by part (b) we have $A = B \cap A = IB \cap A \subseteq \sqrt{I}$. But clearly $\sqrt{I} \subseteq A$, so then $A = I$.

(d) We definitely have $A^* \subseteq B^*$ so $A^* \subseteq B^* \cap A$. Conversely, let $u \in B^* \cap A$. Then $u^{-1} \in B$ so we have $(u^{-1})^n + a_{n-1}(u^{-1})^{n-1} + \cdots + a_0 = 0$. Multiply both sides by u^{n-1} to get $u^{-1} = -(a_{n-1} + \cdots + a_0 u^{n-1})$. Since $u \in A$ and $a_i \in A$, we have $u^{-1} \in A$. Then we also have $u^{-1} \in A^*$. $\square$

Problem 8. *Let K be a field and $B = K[X,Y]/(Y)(Y+1,X) = K[x,y]$, x, y images of X, Y in B. Let $A = K[x] \subseteq B$. Show that B is integral over A. Let $Q = Bx + B(y+1)$. Compute* ht Q *and* ht $Q \cap A$ *and show that* ht $Q <$ ht $Q \cap A$.

Proof. Note that $A = K[x] \subseteq K[x,y] = B$ so $B = A[y]$. Note also that $y = \overline{Y} = Y + (Y)(Y+1,X)$. Then $y^2 + y = \overline{Y}^2 + \overline{Y} = (Y^2+Y) + (Y)(Y+1,X) = 0$ since $Y^2 + Y \in (Y)(Y+1,X)$. Therefore y is integral over A since B is generated by A and y and all of these elements are integral over A, we must have B is integral over A.

Suppose that $P \subsetneq Q$ is a prime contained in Q. Then $0 \in P$, and $0 = xy = y(y+1)$, so either $x \in P$ or $y \in P$. But $y \notin Q$ since $y+1 \in Q$ and then 1 would be in Q. So $x \in P$ and likewise $y+1 \in P$. Therefore $P \supseteq Q$ and $P = Q$. So Q is minimal. Note that Q is prime since $B/Q \cong K$. We can see this by noting that any term with x goes to zero in the quotient and any term with y goes to some constant in K. Therefore Q is a minimal prime and ht $Q = 0$. Note now that $Q \cap A = Bx \cap A = Ax$. But in A, 0 is prime so $0 \subseteq Ax$ is a chain of length one, and by the principal ideal theorem we know ht $Q \cap A \leq 1$. Thus ht $Q = 0 < 1 =$ ht $Q \cap A$. $\square$

Problem 9. *Let B be integral over its subring A. Let $P \in \operatorname{Spec}(A)$. Show that $B_P = S^{-1}B$, ($S = A \backslash P$) is a local ring if and only if there exists exactly one prime ideal $Q \subseteq B$ such that $Q \cap A = P$. Is this true if B is* not *integral over A. Justify your answer.*

Proof. Suppose there exists exactly one prime $Q \subseteq B$ with $Q \cap A = P$. Note that $\operatorname{Spec}(B_P) = \{S^{-1}R \mid R \in \operatorname{Spec}(B), R \cap S = \emptyset\}$. But note for a prime $R \in \operatorname{Spec}(B)$, $R \cap S = R \cap (A \backslash P)$ so if $R \cap S = \emptyset$ then $R \cap A \subseteq P$. We have $Q \cap A = P$, so Q is maximal with respect to these primes. This follows because if we consider any $S^{-1}R$, we must have $S^{-1}R \subseteq S^{-1}Q$. Then since Q is unique in this respect, B_P must be a local ring.

Conversely, suppose B_P is a local ring. Then there exists $Q \in \operatorname{Spec}(B)$ such that $Q \cap S = \emptyset$ and Q is maximal with respect to $R \in \operatorname{Spec}(B)$ with $R \cap A \subseteq P$. Using the going up theorem, we know there exists a prime $R \in \operatorname{Spec}(B)$ with $R \cap A = P$. Then we must have $R = Q$ so that $Q \cap A = P$, as desired.

The statement is not true if B is not integral over A. Take $B = \mathbb{R}$ and $A = \mathbb{Z}$. Take $P = 2\mathbb{Z}$. Then $B_P = B$ since B already contains the field of fractions for A. So B_P is a field and thus a local ring. But since the only prime of B is 0, there are no primes $Q \subseteq B$ with $Q \cap A = P$. $\square$

Problem 10. *Let R be a Noetherian ring with $R \neq 0$. Let $A = R[x_{ij}]$, $1 \leq i \leq m$, $1 \leq j \leq n$ be a polynomial ring in mn variables x_{ij} over R. Let $\alpha = (x_{ij})$ be the $m \times n$ matrix with entries x_{ij}. Let $P \in \operatorname{Spec}(A)$ be a minimal prime ideal of $V(I_r(\alpha))$, where $r \leq \min(m,n)$ and $I_r(\alpha)$ is the ideal of A generated by all $r \times r$ minors of α. Show that* ht $P = (m-r+1)(n-r+1)$. *Thus* ht $I_r(\alpha) = (m-r+1)(n-r+1)$.

Proof. We use induction on r. In the case $r = 1$ we have $I_1(\alpha) = (x_{11}, \ldots, x_{mn})$. By the principal ideal theorem, we know ht $I \leq mn$, and thus ht $P \leq mn$. To construct a chain of mn primes in P, we use induction on mn. In the base case, we have $R[x]$ and a prime $P \subseteq R[x]$. There are two possibilities for P. Either $P \cap R = Q[x]$, $Q \in \operatorname{Spec}(R)$, or $P \cap R \neq Q[x]$, $Q \in \operatorname{Spec}(R)$. The first case is impossible since $x \in P$. In the second case we know that ht $P =$ ht $Q + 1 \geq 1$. The general case follows by letting $R = R[x_{ij}]$, $1 \leq i \leq n$, $1 \leq j \leq m-1$ and $x = x_{mn}$. Thus ht $P = mn$ and ht $I_1(\alpha) = mn$ as well. So assume $r > 1$ and that the statement is true for $r - 1$.

We know already that ht $P \leq (m-r+1)(n-r+1)$, so it just remains to show ht $P \geq (m-r+1)(n-r+1)$. Since $r > 1$, $m - r + 1 < m$ and $n - r + 1 < n$ so ht $P \leq (m-r+1)(n-r+1) < mn$. Thus at least one $x_{ij} \notin P$, otherwise the ideal $(x_{11}, \ldots, x_{mn}) \subseteq P$, in which case ht $P = mn$ by the base case. Note that interchanging rows and columns of α will only affect the minors by multiplication by a unit, so we may assume $x_{ij} = x_{11}$, with $x_{11} \notin P$.

Now define $A' = A[x_{11}^{-1}]$ and $P' = PA' = S^{-1}P$ with $S = \{x_{11}^n \mid n \geq 0\}$ and $A' = S^{-1}A$. Let $I'_r(\alpha)$ be the ideal generated by $I_r(\alpha)$ in A', that is, $I'_r(\alpha) = S^{-1}I_r(\alpha)$. Now perform a single row reduction as follows. Subtract $x_{11}^{-1}x_{ij}$ times the first column of α from the j^{th} column for $j \geq 2$. Multiply the first row by x_{11}^{-1} to get $(1, 0, \ldots, 0)$ in the first row. Subtract x_{i1} times the first row from each each i^{th} row for $i \geq 2$. This now transforms α into

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \alpha' & \\ 0 & & & \end{pmatrix}$$

where $\alpha' = (x'_{ij}) = (x_{ij} - x_{11}^{-1}x_{ij}x_{11})$, $2 \leq i \leq m$ and $2 \leq j \leq n$. Since addition of a scaled row to another doesn't change the determinant, we now have $I'_r(\alpha)I'_{r-1}(\alpha')$, the ideal generated by all $(r-1)\times(r-1)$ minors of α'. Define $R' = R[x_{11}, x'_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{31}, \ldots, x_{n1}]$. Then we see that $A' = A[x_{11}^{-1}] = R'[x_{ij}]$, $2 \leq i \leq m$, $2 \leq j \leq n$. This is the same (through an invertible transformation of variables) as $R'[x'_{ij}]$, $2 \leq i \leq m$, $2 \leq j \leq n$. This ring is a polynomial ring in $(m-1)(n-1)$ variables over R'. We can now use the induction hypothesis to note that

$$\text{ht } I'_r(\alpha) = \text{ht } I'_{r-1}(\alpha') = ((m-1)-(r-1)+1)((n-1)-(r-1)+1) = (m-r+1)(n-r+1).$$

Since $x_{11} \notin P$, $x_{11} \notin I_r(\alpha)$, so P' contains $I'_r(\alpha)$ and is minimal over $I'_r(\alpha)$. Therefore ht $P' = (m-r+1)(n-r+1) =$ ht P since $P \cap S = \emptyset$. Thus ht $I_r(\alpha) = (m-r+1)(n-r+1)$ and so we're done by induction. □

Problem 11. *Let K be a field and $A = K[x_1, \ldots, x_n]$. Let $I \subsetneq A$ be an ideal. Show that* ht $I = 1$ *if and only if $I = Jf$ for some $f \in A$, $f \notin K$ and J some ideal of A.*

Proof. Suppose $I = Jf$ for $f \in A\backslash K$ and J an ideal of A. Then $I \subseteq Af$ which has height less than or equal to 1 by the principal ideal theorem. The I has height at most one, and since $f \neq 0$, ht $I = 1$. Conversely, suppose ht $I = 1$. Then I is contained in some height one prime P. Since A is a UFD, we know P is a principal ideal, so $P = Af$ for some prime f. Therefore $x \in I$ means $x = af$ for some $a \in A$. So define $J = \{a \in A \mid af \in I\}$. Then $I = Jf$, so if we can show J is an ideal, then we're done. Take $a, b \in J$ so $af, bf \in I$. Then $af + bf = (a+b)f$ is in I, so $a + b \in J$. Similarly, $afbf = (abf)f$ is in I, so $abf \in J$. Note that $abf \in I$ since $bf \in I$, so $ab \in J$. Finally, take $c \in A$, then $caf = c(af)$ is in I since $af \in I$. Thus $ca \in J$ and J is an ideal. □

Problem 12. *Let $A = K[x, y, z, t]$. Let $I \subseteq A$ be the ideal generated by $xt - yz$, $y^2 - xz$, $z^2 - yt$. Compute the height of I.*

Proof. Consider the matrix

$$\alpha = \begin{pmatrix} t & z & y \\ z & y & x \end{pmatrix}.$$

The minors of this matrix are $ty - z^2$, $xz - y^2$ and $xt - yz$. Note that up to multiplication by a unit, these are precisely the generators of I, so $I = I_2(\alpha)$. Then we know ht $I \leq (2-2+1)(3-2+1) = 2$. Now note that there is no $f \in A$ with f dividing each generator of I. Then by Problem 11, we know ht $I \neq 1$, so ht $I = 2$ (since $I \neq 0$). □

Problem 13. *Let A be a finitely generated algebra over a field K. Suppose A is a finite dimensional K-vector space of dimension n. Show that* $|\text{Max}(A)| \leq n$.

Proof. We induct on n. In the case $n = 1$ we have $A = K$, $\text{Max}(A) = 0$ and so $|\text{Max}(A)| \leq 1$. Suppose the inequality holds for $1 \leq j \leq n-1$. Every ideal of A is a K-submodule and therefore a vector space over K with finite dimension. Let I be the proper nonzero ideal of A with the least dimension (assuming $\text{Max}(A) \neq \emptyset$, such an ideal exists). Then I must be minimal over 0. Let $\dim(I) = k < n$.

Now note that I is maximal in $K+I$, so $K+I/I$ is a field and A/I is a finitely dimensional vector space over $K+I/I$ with dimension $n-k \geq 1$. By the inductive hypothesis there exist at most $j \leq n-k$ maximal ideals $M_1, \ldots, M_j$ of A/I, so there are at most j ideals in A containing I. Pick $M, N \in \mathrm{Max}(A) \backslash \{M_1, \ldots, M_j\}$. Then $I \nsubseteq M$ so $I \cap M \subsetneq I$. But I is minimal so $I \cap M = 0$. Similarly $I \cap N = 0$. Thus $I + M \supsetneq M$ so $I + M = A$. Now we have $N = AN = (M+I)N = MN + IN = MN$ since $IN \subseteq I \cap N = 0$. Thus $N = MN \subseteq M \cap N \subseteq M$. If we interchange M and N we get $M \subseteq N$ so $M = N$. Thus there are at most $j + 1 \leq n - k + 1 \leq n$ maximal ideals of A (since $0 < k < n$). $\square$