Kris Harper
MATH 25700
Homework 7 December 2, 2009

Problem 1 (6.1.1)**.** *Prove that $Z_i(G)$ is a characteristic subgroup of G for all i.*

Proof. We proceed by induction on i. In the base case we know that the trivial subgroup is preserved by any automorphism of G, so Z_0 char G. Assume that $Z_i(G)$ char G and let $\varphi \in \operatorname{Aut}(G)$. This naturally induces a function $\varphi' : G/Z_i(G) \to G/Z_i(G)$ defined by $\varphi'(xZ_i(G)) = \varphi(x)Z_i(G)$. Note that this function is a homomorphism because

$$\begin{aligned}
\varphi'(xZ_i(G)yZ_i(G)) &= \varphi'(xyZ_i(G)) \\
&= \varphi(xy)Z_i(G) \\
&= \varphi(x)\varphi(y)Z_i(G) \\
&= \varphi(x)Z_i(G)\varphi(y)Z_i(G) \\
&= \varphi'(xZ_i(G))\varphi'(yZ_i(G)).
\end{aligned}$$

It's also clearly surjective since φ is an automorphism of G. Now suppose $\varphi'(aZ_i(G)) = \varphi'(bZ_i(G))$. Then we have $\varphi(a)Z_i(G) = \varphi(b)Z_i(G)$ and $\varphi(b^{-1}a) \in Z_i(G)$. Thus $b^{-1}a \in \varphi^{-1}(Z_i(G))$ But by our inductive hypothesis, $\varphi^{-1}(Z_i(G)) = Z_i(G)$ and so $b^{-1}a \in Z_i(G)$. Therefore $aZ_i(G) = bZ_i(G)$ and so φ' must be injective. Now note that $Z(G/Z_i(G))$ is characteristic and so $\varphi'(Z_{i+1}(G)/Z_i(G)) = Z_{i+1}(G)/Z_i(G)$. Thus, if $x \in Z_{i+1}$ then $\varphi(x)Z_i = \varphi'(xZ_i) = yZ_i$ for some $y \in Z_{i+1}$. Hence $\varphi(x) \in Z_{i+1}$ and Z_{i+1} char G. □

Problem 2 (6.1.3)**.** *If G is finite prove that G is nilpotent if and only if it has a normal subgroup of each order dividing $|G|$, and is cyclic if and only if it has a unique subgroup of each order dividing $|G|$.*

Proof. Suppose G has a normal subgroup of each order dividing $|G|$. Then every Sylow p-subgroup of G is normal in G and G is nilpotent. Now suppose that G is nilpotent and has Sylow p-subgroups P_i for $1 \le i \le s$ so that $|G| = p_1^{a_1} \dots p_s^{a_s}$. Then $G \cong P_1 \times \dots \times P_s$. But each P_i has a normal subgroup of order p_i^b for each $1 \le b \le a_i$. Now let $k \mid |G|$ such that $k = p_1^{b_1} \dots p_s^{b_s}$. We simply take $N = N_1 \times \dots \times N_s$ where $N_i \trianglelefteq P_i$ and $|N_i| = p_i^{b_i}$. Clearly N has the appropriate order, but we also have $N \trianglelefteq G$ since multiplication is performed coordinate-wise and each $N_i \trianglelefteq P_i$.

Now suppose that G is cyclic. Then we know that G has a unique subgroup of each order dividing the order of G. On the other hand, if G has a unique subgroup of each order n dividing the order of G, then each of these subgroups must contain all elements of G such that $x^n = 1$. Otherwise $\langle x \rangle$ would form another subgroup of order n. Thus for each n dividing $|G|$ there are at most n elements with $x^n = 1$. Therefore G is cyclic. □

Problem 3 (6.1.6)**.** *Show that if $G/Z(G)$ is nilpotent then G is nilpotent.*

Proof. We first show that $[G/Z(G), G^n/Z(G)] = [G, G^n]/Z(G)$. Let $\varphi : [G, G^n] \to [G/Z(G), G^n/Z(G)]$ be defined so that $\varphi(x^{-1}y^{-1}xy) = (x^{-1}y^{-1}xy)Z(G)$. Then φ is a homomorphism because if $z_1 = x_1^{-1}y_1^{-1}x_1y_1$ and $z_2 = x_2^{-1}y_2^{-1}x_2y_2$ then $\varphi(z_1z_2) = z_1z_2Z(G) = z_1Z(G)z_2Z(G) = \varphi(z_1)\varphi(z_2)$. Furthermore, if $\varphi(x^{-1}y^{-1}xy) = Z(G)$, then $x^{-1}y^{-1}xy \in Z(G)$ and conversely if $x^{-1}y^{-1}xy \in Z(G)$ then $\varphi(x^{-1}y^{-1}xy) = x^{-1}y^{-1}xyZ(G) = Z(G)$. Thus $\ker\varphi = Z(G)$. Finally, since elements of the form $x^{-1}y^{-1}xy$ are generators of $[G, G^n]$ and $x^{-1}y^{-1}xyZ(G)$ are generators of $[G/Z(G), G^n/Z(G)]$, we see that $\varphi([G, G^n]) = [G/Z(G), G^n/Z(G)]$. Thus by the First Isomorphism Theorem we have $[G/Z(G), G^n/Z(G)] = [G, G^n]/Z(G)$.

Now we proceed by induction on n to show that $G^n/Z(G) = (G/Z(G))^n$. If $n = 0$ the result is clearly true as $G^0 = G$ and $(G/Z(G))^0 = G/Z(G)$. Suppose the result is true for some n. Now using this assumption and the above result we have

$$(G/Z(G))^{n+1} = [G/Z(G), (G/Z(G))^n] = [G/Z(G), G^n/Z(G)] = [G, G^n]/Z(G) = G^{n+1}/Z(G).$$

Since $G/Z(G)$ is nilpotent, we know it's lower central series terminates in $(G/Z(G))^n = 1$ for some n. But now this is the same as saying $G^n/Z(G) = 1$, or alternatively, $G = Z(G)$ and G is abelian. Thus G is nilpotent as well. □

Kris Harper
MATH 25700
Homework 7 December 2, 2009

Problem 4 (6.1.8)**.** *Prove that p is a prime and P is a non-abelian group of order p^3 then $|Z(P)| = p$ and $P/Z(P) \cong Z_p \times Z_p$.*

Proof. Note that $|Z(P)| \neq p^3$ because P is nonabelian, $|Z(P)| \neq 1$ by Cauchy's Theorem and $|Z(P)| \neq p^2$ because othewise $|P/Z(P)| = p$ and P would be abelian. Thus $|Z(P)| = p$. Also note that if $P/Z(P) \cong Z_{p^2}$ then once again P would be abelian as Z_{p^2} is cyclic. Thus $P/Z(P) \cong Z_p \times Z_p$. □

Problem 5 (6.1.12)**.** *Find the upper and lower central series for A_4 and S_4.*

Proof. We have $Z_0(A_4) = 1$ and $Z_1(A_4) = 1$ which means $Z_n(A_4) = 1$ for all $0 \leq n$. On the other hand $A_4^0 = A_4$ and $A_4^1 = [A_4, A_4] = \langle (12)(34), (13)(24) \rangle$. Now $A_4^2 = [A_4, A_4^1]$. Since $A_4^1 \trianglelefteq A_4$ we know that $[A_4, A_4^1] \leq A_4^1$. But $(12)(34)(132)(12)(34)(123) = (14)(23)$ which shows that we can write every element of A_4^1 as a commutator in $[A_4, A_4^1]$. Therefore $A_4^n = \langle (12)(34), (13)(24) \rangle$ for $n \geq 1$.

Now $Z_0(S_4) = 1$ and $Z_1(S_4) = 1$ which means $Z_n(S_4) = 1$ for all $0 \leq n$. On the other hand $S_4^0 = S_4$ and $S_4^1 = [S_4, S_4] = A_4$. Now $S_4^2 = [S_4, A_4] = A_4$ since $(ab)(abc)(ab)(acb) = (abc)$ shows that any 3-cycle can be written as a commutator. Since 3-cycles generate A_4, we have $S_4^n = A_4$ for all $n \geq 1$. □

Problem 6 (6.1.21)**.** *Prove that $\Phi(G)$ is a characteristic subgroup of G.*

Proof. Let $M < G$ be a maximal subgroup and let $\varphi \in \mathrm{Aut}(G)$. Then $\varphi(M)$ is also a proper subgroup of G. Suppose there exists a proper subgroup $H < G$ such that $\varphi(M) < H$. Since φ is an automorphism we know $\varphi^{-1}(H)$ is a subgroup. Furthermore, if $m \in M$ then $\varphi(m) \in H$ and so $\varphi^{-1}(\varphi(m)) \in \varphi^{-1}(H)$. Thus $m \in \varphi^{-1}(H)$. Once again, since φ is an automorphism all inclusions are proper. But this contradicts the maximality of M since now $M < \varphi^{-1}(H) < G$. Thus $\varphi(M)$ is also a maximal subgroup of G. Now let the maximal subgroups of G be indexed M_i. Note that since φ is injective, we have

$$\varphi(\Phi(G)) = \varphi\left(\bigcap_i M_i\right) = \bigcap_i \varphi(M_i).$$

And by the above argument for each i $\varphi(M_i) = M_j$ for some j. Thus since φ is an automorphism we may write $\bigcap_i \varphi(M_i) = \bigcap_i M_i$. Thus $\varphi(\Phi(G)) = \Phi(G)$ and this subgroup is characteristic. □

Problem 7 (6.1.26a)**.** *Let p be a prime, let P be a finite p-group and let $\overline{P} = P/\Phi(P)$.*
(a) Prove that $\overline{P}$ is an elementary abelian p-group.

Proof. First let $M < P$ be a maximal subgroup of P. We know that $|P : M| = p$ and $M \trianglelefteq P$. Thus $P/M \cong Z_p$ and is thus abelian. Therefore $P' \leq M$. Since this is true of every maximal subgroup we must have $P' \leq \Phi(P)$. Now choose some element $x \notin M$. Then note that $x \notin \langle M, x^p \rangle$ since p is prime. Thus $\langle M, x^p \rangle$ is proper subgroup of G and so it must be the case that $x^p \in M$. In the case that $x \in M$ we clearly have $x^p \in M$. Thus x^p is in every maximal subgroup for every element x and therefore $x^p \in \Phi(P)$. Now we know $\overline{P}$ is abelian (since $P' \leq \Phi(P)$ and $\Phi(P) \trianglelefteq P$ by Problem 6) and every element of $\overline{P}$ has order p (since $x^p \in \Phi(P)$ for all x so $|x\Phi(P)| = p$). Thus $\overline{P}$ must be an elementary abelian p-group. □

Problem 8 (6.2.2)**.** *In the group $S_3 \times S_3$ exhibit a pair of Sylow 2-subgroups that intersect in the identity and exhibit another pair that intersect in a group of order 2.*

Proof. Take the pair $\langle (12) \rangle \times \langle (23) \rangle$ and $\langle (23) \rangle \times \langle (12) \rangle$. These groups each have order 4 and so they are Sylow 2-subgroups of $S_3 \times S_3$. But they must intersect in the identity since each of the coordinates intersect in the identity.

Now consider $\langle (12) \rangle \times \langle (12) \rangle$ and $\langle (12) \rangle \times \langle (23) \rangle$. These groups each have order 4 and so they are Sylow 2-subgroups of $S_3 \times S_3$. But they must intersect in a group of order 2 since the second coordinate intersects trivially. Thus, all that's left in the intersection is $\langle (12) \rangle \times 1$. □

Problem 9 (6.2.6)**.** *Prove that there are no simple groups of order* 2205*,* 4125*,* 5103*,* 6545*,* 6435*.*

Proof. We prove the case where a group G has order 5103. This has factorization $3^6 \cdot 7$. Possibilities for n_3 are 1 and 7 and possibilities for n_7 are 1 and 729. Note that the smallest integer k for which $|G| \mid k!$ is 12. Thus, if G is to be simple there are no subgroups of G with index less than 12, otherwise we would obtain a permutation representation with a nontrivial kernel, which would then be normal in G. But note that if $n_3 = 7$ we would have $|G : N_G(P)| = 7$ for some Sylow 3-subgroup P. Therefore for G to be simple we must have $n_3 = 1$, which immediately produces a normal Sylow 3-subgroup proving that G is in fact not simple. □

Problem 10 (6.2.10)**.** *Prove that there are no simple groups of order* 4095*,* 4389*,* 5113 *or* 6669*.*

Proof. We prove the case where a group G has order 5313. This has factorization $3 \cdot 7 \cdot 11 \cdot 23$. Possibilities for n_7 are 1 or 253 and possibilities for n_11 are 1 or 23. Now let $Q \in Syl_11(G)$. Supposing that G is not simple, we have $n_11 = 23$ which means $|N_G(Q)| = 231$. Since $7 \mid 231$ we have a subgroup $P \leq N_G(Q)$ with $|P| = 7$. This shows that PQ is a group with order $7 \cdot 11 = 77$. Since $7 \nmid 11$ we know that PQ is cyclic and therefore abelian. This means $PQ \leq N_G(P)$ and $11 \mid |N_G(P)|$. But if G is to be simple, $n_7 = 253$ which means $|N_G(P)| = 21$. But $11 \nmid 21$ which is a contradiction. Therefore G cannot be simple. □

Problem 11 (6.2.13)**.** *Let G be a group with more than one Sylow p-subgroup. Over all pairs of distinct Sylow p-subgroups let P and Q be chosen so that $|P \cap Q|$ is maximal. Show that $N_G(P \cap Q)$ has more than one Sylow p-subgroup and that any two distinct Sylow p-subgroups of $N_G(P \cap Q)$ intersect in the subgroup $P \cap Q$. (Thus $|N_G(P \cap Q)|$ is divisible by $p \times |P \cap Q|$ and by some prime other than p. Note that Sylow p-subgroups of $N_G(P \cap Q)$ need not be Sylow in G.)*

Proof. Let $N = N_G(P \cap Q)$. Note that since P is a p-group, $P \cap Q < N_P(P \cap Q) \leq N$ so it can't be the case that $P \cap Q \in Syl_p(N)$. We claim that $P \cap N \in Syl_p(N)$. Clearly $P \cap N$ is a p-group since it's order must divide $|P|$. Suppose that $P \cap N < R$ where $R \in Syl_p(N)$. Then R is a p-group and so $R \leq R'$ where $R' \in Syl_p(G)$. Thus $P \cap Q < P \cap R \leq P \cap R'$. But then $|P \cap Q| < |P \cap R'|$ which contradicts the choice of P and Q. Thus $P \cap N$, and similarly $Q \cap N$, are Sylow p-subgroups of N. Suppose now that $P \cap N = Q \cap N$. Then $(P \cap N) \cap Q = (Q \cap N) \cap Q = Q \cap N$ and $Q \cap N = P \cap Q = Q \cap N$. But we've already shown that $P \cap Q$ isn't a Sylow p-subgroup of N while $P \cap N$ is. Thus it can't be the case that $P \cap N = Q \cap N$, which shows that there are two distinct Sylow p-subgroups of N, namely $P \cap N$ and $Q \cap N$.

We know $P \cap Q$ is a p-group and thus if $R \in Syl_p(N)$, $x(P \cap Q)x^{-1} \leq R$ for some $x \in N$. That is, for every Sylow p-subgroup $R \leq N$, $P \cap Q$ is a subgroup of some conjugate of R, or more helpfully, some conjugate of $P \cap Q$ is a subgroup of R. But since $P \cap Q$ is clearly in N, we simply have $x(P \cap Q)x^{-1} = P \cap Q$ so $P \cap Q \leq R$ for all $R \in Syl_p(N)$. Therefore if $R, S \in Syl_p(N)$ we have $P \cap Q \leq R \cap S$.

Now take $R, S \in Syl_p(N)$ distinct. There exists $x \in N$ such that $P \cap N = xRx^{-1}$ and so let $S' = xSx^{-1}$. Since $x(P \cap Q)x^{-1} = P \cap Q$, without loss of generality we can simply show $(P \cap N) \cap S' \leq P \cap Q$ to finish the proof. Suppose this is not the case. Then $P \cap Q < (P \cap N) \cap S'$ by the above inclusion. Once again, we know there exists a Sylow p-subgroup of G, S'', such that $S' \leq S''$. Note that $S'' \neq P$ because otherwise $S' \leq P$ and $P \cap N \leq P$. But then $P \cap Q < (P \cap N) \cap S' \leq P \cap S''$ which means $|P \cap Q| < |P \cap S''|$, contradicting the maximality of $|P \cap Q|$. Thus we must have $(P \cap N) \cap S' \leq P \cap Q$. Since both inclusions have been shown, we have $R \cap S = P \cap Q$ for any two Sylow p-subgroups R and S of N. □

Problem 12 (6.3.2)**.** *Prove that if $|S| > 1$ then $F(S)$ is non-abelian.*

Proof. Let $\{a, b\} \subseteq S$. We show that $a^{-1}b^{-1}ab \neq 1$ by showing that $a^{-1}b^{-1}ab$ is in reduced form. Note that $a \neq b^{-1}$ and $b \neq a^{-1}$. Thus $a^{-1}b^{-1}ab$ is in reduced form and $F(S)$ is nonabelian. □