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CMSC 27500
April 29, 2009

Homework 4

Problem 1. *Let G be a connected graph on at least three vertices, and let $e = uv$ be a cut edge of G. Show that either u or v is a cut vertex of G.*

Proof. Note that since e is a cut edge of G, then e belongs to no cycle. Since there is clearly a path from u to v through e, there can thus be no other path from u to v. Because G has at least three vertices, either u or v must have one other neighbor, w. Without loss of generality, suppose u has neighbor w. Then $G - u$ results in a graph in which there is no path from w to v, otherwise that path adjoined with wu and uv would make a cycle containing e. Since $G - u$ is now disconnected we have $c(G - u) > c(G)$ and so u is cut vertex of G. □

Problem 2. *Let G be a nonseparable graph, and let e be an edge of G. Show that the graph obtained from G by subdividing e is nonseparable.*

Proof. Since G is nonseparable, it is connected and has no separating vertices. It's clear that subdividing e will keep G connected, and so it remains to be shown that no separating vertices are introduced. In the case that G consists of one vertex and a loop, subdividing e creates the graph on two vertices with two parallel edges which is nonseparable. Assume that G has more than one vertex, and so G has no loops as it's nonseparable. Let $e = uv$ and subdivide e so that the path uwv is created. Call this new graph G'. Suppose that w is a separating vertex. Then G' consists of two connected components joined only at w. But the only neighbors of w are u and v and so G/uw is also separable with a separating vertex u. But this graph is G, which is nonseparable. Therefore w is not a separating vertex. Since this is the only vertex introduced in the process, we know that G' is nonseparable. □

Problem 3. *Let G be a graph, and let e be an edge of G. Show that:*
1) If $G \backslash e$ is nonseparable and e is not a loop of G, then G is nonseparable.

Proof. If $G \backslash e$ is nonseparable then it is connected and the graph G is clearly connected as well as adding edges will not destroy any paths. We must show that if $e = uv$ then neither u nor v is a separating vertex in G. Suppose u is a separating vertex of G. Then G can be decomposed into two connected subgraphs which have only u in common. Note that one of these subgraphs must not contain e. Suppose we remove e from the other subgraph. Since $G \backslash e$ is nonseparable, it is connected and so there exists a path connecting each point in this subgraph to u. But then this subdivides $G \backslash e$ into two connected subgraphs which only intersect at u. This is a contradiction so u is not a separating vertex and a similar case holds for v. Since e is not a loop, we can state that G is nonseparable. □

2) If G/e is nonseparable and e is neither a loop nor a cut edge of G, then G is nonseparable.

Proof. Note that since e is not a cut edge nor a loop, it must belong to a cycle, C, of G. Then since G/e is nonseparable, an edge of C and any other edge of G lie in a common cycle. Now take e' in G. We know that e' lies on common cycle with some edge, e'' in C. Then we can form a cycle containing e and e' by combining the path connecting one end of e' to e'', the path connecting the two ends of e'', which contains e, and the path connecting the other end of e'' to e'. This path combined with e' creates a cycle containing e and e'. Thus, any two edges of G are contained in a common cycle and G is nonseparable. □

Problem 4. *1) Let B be a block of a graph G, and let P be a path in G connecting two vertices of B. Show that P is contained in B.*

Proof. Suppose that P is not entirely contained in B. Since B is nonseparable, it is connected and so there exists a path entirely in B connecting the ends of P. But then this path, combined with P creates a cycle which is not contained in a single block. This is a contradiction and so P must be entirely contained in B. □

2) Deduce that a spanning subgraph T of a connected graph G is a spanning tree of G if and only if $T \cap B$ is a spanning tree of B for every block B of G.

Proof. Suppose T is a spanning tree of G and let B be a block of G. Consider two vertices u and v of B. Since T is a tree there exists a path in T between u and v, and by Part 1) this path lies entirely in B. Thus $T \cap B$ is a spanning tree of B. Conversely, suppose that T is a spanning subgraph of G such that $T \cap B$ is a spanning tree of B for each block B of G. Consider two vertices u and v of G such that $u \in B_1$ and $v \in B_n$. Since G is connected, it has a block tree $B(G)$. Then there's a path in $B(G)$ which connects B_1 and B_n. This path alternates blocks $B_1, B_2, \ldots B_n$ with separating vertices, $s_1, s_2, \ldots, s_{n-1}$. Now take the path in $T \cap B_1$ which connects u with s_1, then adjoin it with the path in $T \cap B_2$ which connects s_1 with s_2. Continue in the way until we adjoin the path in $T \cap B_n$ which connects s_{n-1} with v. This shows that T is a spanning tree. □

Problem 5. *Let G be a nonseparable graph and v a vertex of G of degree at least 4 with at least two distinct neighbors. Also let f be an edge of G not incident to v. Let $H = G \backslash f$ be a separable graph whose block tree is a path. Suppose that v is a separating vertex of H such that there are two edges, $e_1 = vv_1$ and $e_2 = vv_2$, which are incident to v and lie in distinct blocks of H. Let G' be the graph obtained by splitting off e_1 and e_2 with an edge e. Show that:*
1) G' is connected.

Proof. We need only consider paths in G which contain v. Note that any such path must contain at least one of e_1 or e_2 since the block tree of H is a path. Any path which contains e_1 and e_2 in succession remains intact in G' since the connection $e_1 e_2$ is simply replaced by e. Consider a path with endpoints p_1 and p_2 in G which contains e_1 and is not followed by e_2. Since G is nonseparable and v has degree at least 4, we know that e_1 lies on a cycle in G, and that this cycle must be contained in the block containing e_1. Then any path which contains e_1 can be replaced by the remainder of this cycle and likewise for e_2. This shows that G' is connected. □

2) Each edge of G' lies in a cycle.

Proof. Consider an edge, $e' \neq e$, in G'. Suppose first that e' is completely contained in a block, B, of H and does not have vertices which are separating vertices of H. We know that G is nonseparable and so each pair of edges lies on a common cycle. Thus e' and some other edge in B lie on a common cycle of G' and this cycle must lie entirely in B. If e' contains a separating vertex of H, then it lies in part of the cycle formed by the path connecting all the separating vertices of H, and f. Finally, if $e' = e$, then e lies in this cycle as well. Therefore every edge of G' lies in a cycle. □

Problem 6. *Let G be a nonseparable graph, and let e be an edge of G such that $G \backslash e$ is separable. Show that the block tree of $G \backslash e$ is a path.*

Proof. Suppose that $B(G)$ is not a path and consider some vertex, v, which has degree at least 3. Suppose that this vertex corresponds to a separating vertex, s, in $G \backslash e$. Then there are at least three blocks, B_1, B_2 and B_3 which intersect at s. Consider two edges $e_1 \in B_1$ and $e_2 \in B_2$. Note that since G is nonseparable, there exists a cycle, C, containing e_1 and e_2. Since $G \backslash e$ is separable and e_1 and e_2 are in different blocks, this cycle must contain e. Moreover, e must join B_1 and B_2 directly. Likewise, for two edges $e_2 \in B_2$ and $e_3 \in B_3$, there exists a cycle C' which contains e, and e joins B_2 and B_3 directly. But in a similar way, e must join B_1 and B_3 directly, which is impossible.

Now suppose that v corresponds to a block, B. Then there are at least three separating vertices which are part of B. Note that a leaf of $B(G)$ must correspond to a block, and so each of these separating vertices belongs to exactly one other block. There are then at least three blocks which are connected to B and the above proof holds to show that this situation is impossible. Therefore each vertex of $B(G)$ has degree less than 3. □

Problem 7. *Let F be a nonseparable proper subgraph of a graph G, and let P be an ear of F. Then $F \cup P$ is nonseparable.*

Proof. We know any two edges of F lie on a common cycle in $F \cup P$. Consider the ends of P, p_1 and p_2. Since F is nonseparable there exists a path in F which connects p_1 and p_2. Then this path together with P creates a cycle in $F \cup P$. Thus any two edges in P lie in a cycle in $F \cup P$. Now consider some edge $e_1 = uv$ in F. There exist paths P_1 from u to p_1 and P_2 from v to p_2. Then for any edge e_2 in P the cycle $P_1PP_2e_1$ contains both e_1 and e_2. Thus any two edges of $F \cup P$ lie on a common cycle. □

Problem 8. *Show that every edge of a nonseparable graph is either deletable or contractible.*

Proof. In the case that $G = K_2$, the edge can be contracted to form K_1 which is also nonseparable. Assume that G has three or more vertices. Let e be an edge of a nonseparable graph G and suppose that $G \backslash e$ is separable. Note that since G is nonseparable, there are no cut vertices and so any two distinct vertices are connected by internally disjoint paths. Note that e cannot be a loop since G is nonseparable. Thus, $G \backslash e$ must have at least one cut vertex since it has a separating vertex. Therefore there exist two vertices, u and v which are not connected by internally disjoint paths in $G \backslash e$, but are in G. Therefore one of the paths contain e and identifying the two ends of e will necessarily preserve this connection. Then there are no cut vertices in G/e and G/e is thus nonseparable.

Now suppose that G/e is separable. Since no connections are broken when contracting e, no paths are disconnected and so there are no cut vertices in G/e except at the point where e was contracted. Suppose that e is contracted to the vertex v. Then v must be a separating vertex of G/e. Note the a connected subgraph containing v could not have only contained one vertex of e in G, since then G would have been separable. Thus, some connected subgraph contained both vertices of e and thus made a cycle with e. Then in $G \backslash e$ this cycle is broken, but any paths which contained e can be replaced with the remaining part of the cycle which joins the two ends of e. Thus, $G \backslash e$ is nonseparable. □