Kris Harper
MATH 20800
March 2, 2009

Homework 9

**** Problem 1.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Then*

$$f((x_1, x_2, \ldots, x_n)) = (f_1((x_1, x_2, \ldots, x_n)), f_2((x_1, x_2, \ldots, x_n)), \ldots, f_m((x_1, x_2, \ldots, x_n))).$$

If $f_k : \mathbb{R}^n \to \mathbb{R}$ is differentiable for all $1 \le k \le m$, then f is differentiable.

Proof. Since f_k is differentiable for all $1 \le k \le m$, there exists a linear transformation L_k such that for all $x \in \mathbb{R}^n$ we have

$$\lim_{h\to 0} \frac{|f_k(x+h) - f_k(x) - L_k h|}{|h|} = \lim_{h\to 0} \frac{|f_k((x_1+h_1, x_2+h_2, \ldots, x_n+h_n)) - f_k((x_1, x_2, \ldots, x_n)) - L_k h|}{|h|} = 0$$

But then we must have

$$\begin{aligned}
0 &= \lim_{h\to 0} \frac{|(f_1((x_1+h_1,\ldots,x_n+h_n)) - f_1((x_1,\ldots,x_n)),\ldots,f_m((x_1+h_1,\ldots,x_n+h_n)) - f_m((x_1,\ldots,x_n)))|}{|h|} \\
&= \lim_{h\to 0} \frac{|(f_1((x_1+h_1,\ldots,x_n+h_n)),\ldots,x_n+h_n)),\ldots,f_m((x_1+h_1,\ldots,x_n+h_n)))|}{|h|} \\
&- \frac{(f_1((x_1,\ldots,x_n)),\ldots,f_m((x_1,\ldots,x_n))) - Lh|}{|h|} \\
&= \lim_{h\to 0} \frac{|f(x+h) - f(x) - Lh|}{|h|}
\end{aligned}$$

□

**** Problem 2.** *Show that if $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$ is differentiable at $x \in U$ then $D_v f(x) = \nabla f(x) \cdot v$.*

Proof. We have

$$\nabla f(x) \cdot v = \sum_{i=1}^{n} D_i f(x) v_i = \sum_{i=1}^{n} \lim_{t\to 0} \frac{f(x+te_i) - f(x)}{t} v_i = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} = D_v f(x).$$

□

**** Problem 3.** *Relative to the standard basis, we can represent $Df(a)$ by the $m \times n$ matrix $[D_j f_i(a)]$ where $j = 1, \ldots, n$ and $i = 1, \ldots, m$.*

Proof. We already know that in one variable, $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$Df(x) = (D_1 f(x), \ldots, D_n f(x)).$$

This immediately extends to m dimensions if $f : \mathbb{R}^n \to \mathbb{R}^m$. We know each component function, $f_i : \mathbb{R}^n \to \mathbb{R}$ with $i = 1, \ldots m$, is differentiable. Then the ith row of $f'(x)$ is $f_i'(x)$. □

**** Problem 4.** *Let U be an open set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ such that f is differentiable on U. Suppose $x, y \in U$ and the line segment*

$$L = \{(1-t)x + ty \mid 0 \le t \le 1\} \subseteq U.$$

Then there exists $z \in L$ such that $f(y) - f(x) = Df(z)(y-x)$.

Proof. Let $F(t) = f((1-t)x + ty)$ for $0 \le t \le 1$. Then by the Mean Value Theorem there exists $s \in [0,1]$ such that $F'(s) = f(x) - f(y)$. Then by the Chain Rule note that $F'(s) = f'((1-s)x + sy)(x-y)$. Taking $z = (1-s)x + sy$ gives the desired result. $\square$

** **Problem 5.** *Let*

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{x^3y+xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Are $D_1 f(x,y)$ and $D_2 f(x,y)$ continuous at $(0,0)$?

Yes.

Proof. We have

$$D_1 f(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}.$$

Since the power on the numerator always exceeds that of the denominator, we must have

$$\lim_{(x,y)\to(0,0)} \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} = 0.$$

A similar proof holds for $D_2 f(x,y)$. $\square$

** **Problem 6.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined in ** Problem 5. Is $D_2(D_1 f)(0,0) = D_1(D_2 f)(0,0)$?*

No.

Proof. We have

$$D_1 f(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4}{(x^2+y^2)^2}$$

and

$$D_2 f(x,y) = \frac{x(x^4 - 4x^2y^2 - y^4}{(x^2+y^2)^2}.$$

Note that $D_2 f(x,0) = x$ for all x and $D_1 f(0,y) = -y$ for all y. Now $D_2(D_1 f)(0,0) = D(D_1 f(0,y)) = D(-y) = -1$ and $D_1(D_2 f)(0,0) = D(D_2 f(x,0)) = D(x) = 1$. $\square$

** **Problem 7.** *Take $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^n$ such that f is differentiable and $D_j(D_i f)(x)$ exists for all $x \in U$. If $D_i(D_j f)$ is continuous for all i, j, then $D_i(D_j f) = D_j(D_i f)$.*

Proof. Let $x \in U$. Consider the function

$$\begin{aligned} F_{ij}(h) = {} & (f(x_1, \dots, x_i + h, \dots, x_j + h, \dots, x_n) - f(x_1, \dots, x_i + h, \dots, x_j, \dots, x_n)) \\ & - (f(x_1, \dots, x_i, \dots, x_j + h, \dots, x_n) + f(x_1, \dots, x_n)) \end{aligned}$$

and let

$$g(y) = f(x_1, \dots, y, \dots, x_j + h, \dots, x_n) - f(x_1, \dots, y, \dots, x_j, \dots, x_n)$$

then

$$F_{ij}(h) = g(x_i + h) - g(x_i).$$

By the Mean Value Theorem there exists $c \in [x_i, x_i + h]$ such that

$$g(x_i + h) - g(x_i) = g'(c)h = h(D_i f(x_1, \dots, c, \dots, x_j + h, \dots, x_n) - D_1 f(x_1, \dots, c, \dots, x_j, \dots, x_n)).$$

Now use the Mean Value Theorem again on $D_i f$ so that there exists $d \in [x_j, x_j + h]$ such that

$$D_i f(x_1, \dots, c, \dots, x_j + h, \dots, x_n) - D_i f(x_1, \dots, c, \dots, x_j, \dots, x_n) = D_{ij}(x_1, \dots, c, \dots, d, \dots, x_n)h.$$

Now we have

$$F_{ij}(h) = h^2 D_{ij}(x_1, \ldots, c, \ldots, d, \ldots, x_n).$$

Note that as $h \to 0$ we have $c \to x_i$ and $d \to x_j$, so by the continuity of $D_{ij}f$ we have

$$\lim_{h\to 0} \frac{F_{ij}(h)}{h^2} = \lim_{c,d\to 0,0} D_{ij}f(x_1, \ldots, c, \ldots, d, \ldots, x_n) = D_{ij}f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n).$$

But then it's clear that $F_{ij} = F_{ji}$ which results in $D_{ij}f = D_{ji}f$. $\square$

Problem 1. *Find f' for the following:*
1) $f(x, y, z) = x^y$
2) $f(x, y, z) = (x^y, z)$
3) $f(x, y) = \sin(x \sin y)$
4) $f(x, y, z) = \sin(x \sin(y \sin z))$
5) $f(x, y, z) = x^{y^z}$
6) $f(x, y, z) = x^{y+z}$
7) $f(x, y, z) = (x + y)^z$
8) $f(x, y) = \sin(xy)$
9) $f(x, y) = (\sin xy)^{\cos 3}$
10) $f(x, y) = (\sin xy, \sin(x \sin y), x^y)$.

Proof. 1)

$$\begin{pmatrix} yx^{y-1} & e^{y \ln x} \ln x & 0 \end{pmatrix}$$

2)

$$\begin{pmatrix} yx^{y-1} & e^{y \ln x} \ln x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3)

$$\begin{pmatrix} \cos(x \sin y) \sin y & \cos(x \sin y) x \cos y \end{pmatrix}$$

4)

$$\begin{pmatrix} \cos(x \sin(y \sin z)) \sin(y \sin z)) & \cos(x \sin(y \sin z)) \cos(y \sin z) \sin z & \cos(x \sin(y \sin z)) \cos(y \sin z) y \cos z \end{pmatrix}$$

5)

$$\begin{pmatrix} y^z x^{y^z - 1} & e^{y^z \ln x} z y^{z-1} \ln x & e^{e^{z \ln y} \ln x} \left(\frac{1}{x} e^{z \ln y} + \ln x e^{z \ln y} \ln y\right) \end{pmatrix}$$

6)

$$\begin{pmatrix} (y+z) x^{y+z-1} & e^{(y+z) \ln x} \ln x & e^{(y+z) \ln x} \ln x \end{pmatrix}$$

7)

$$\begin{pmatrix} z(x+y)^{z-1} & z(x+y)^{z-1} & e^{z \ln(x+y)} \ln(x+y) \end{pmatrix}$$

8)

$$\begin{pmatrix} \cos(xy) y & \cos(xy) x \end{pmatrix}$$

9)

$$\begin{pmatrix} \cos(3) \sin(xy)^{\cos(3)-1} y \cos(xy) & \cos(3) \sin(xy)^{\cos(3)-1} x \cos(xy) \end{pmatrix}$$

10)

$$\begin{pmatrix} \cos(xy) y & \cos(xy) x \\ \cos(x \sin y) \sin y & \cos(x \sin y) x \cos y \\ yx^{y-1} & e^{y \ln x} \ln x \end{pmatrix}$$

$\square$

Problem 2. *Find f' for the following where $g : \mathbb{R} \to \mathbb{R}$ is continuous:*
1) $f(x, y) = \int_a^{x+y} g$
2) $f(x, y) = \int_a^{xy} g$
3) $f(x, y, z) = \int_{xy}^{\sin(x \sin(y \sin z))} g$.

Proof. 1)

$$\begin{pmatrix} g(x+y) & g(x+y) \end{pmatrix}$$

2)

$$\begin{pmatrix} g(xy)y & g(xy)x \end{pmatrix}$$

3)

$$g(\sin(x \sin(y \sin z)))Dh_1(x, y, z) - g(x^y)Dh_2(x, y, z)$$

where $h_1 = \sin(x \sin(y \sin z))$ and $h_2 = x^y$ have solutions in Parts 1) and 4) of Problem 1. □

Problem 3. *A function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$ is bilinear if for $x, x_1, x_2 \in \mathbb{R}^n$, $y, y_1, y_2 \in \mathbb{R}^m$ and $a \in \mathbb{R}$ we have*

$$f(ax, y) = af(x, y) = f(x, ay),$$
$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y),$$
$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2).$$

1) Prove that if f is bilinear, then

$$\lim_{(h,k)\to 0} \frac{|f(h, k)|}{|(h, k)|} = 0.$$

2) Prove that $Df(a, b)(x, y) = f(a, y) + f(x, b)$.
3) Show that $Dp(a, b)(x, y) = bx + ay$ where $p : \mathbb{R}^2 \to \mathbb{R}$ is defined by $p(x, y) = xy$ is a special case of Part 2).

Proof. Note that

$$f(h, k) = \sum_{i=1}^{n} \sum_{j=1}^{m} h_i k_j f(e_i, e_j)$$

and so this function is linear. Thus there exists some $M > 0$ such that

$$|f(h, k)| \leq M \max(|h_i|) \max(|k_j|) \leq M|h||k|.$$

Since $|(h, k)| = \sqrt{|h|^2 + |k|^2}$, we need only show the result is true when $n = m = 1$ and f is simply p, mentioned in Part 3). But this has already been shown to be true.

2) We have

$$\lim_{(h,k)\to 0} \frac{|f(a + h, b + k) - f(a, b) - f(a, k) - f(h, b)|}{|(h, k)|} = \lim_{(h,k)\to 0} \frac{|f(h, k)|}{|(h, k)|} = 0$$

by bilinearity and Part 1).

3) Taking $n = m = p = 1$ we have $f : \mathbb{R}^2 \to \mathbb{R}$. If $f(x, y) = xy$ then from Part 2) we have $Df(a, b)(x, y) = f(a, y) + f(b, x) = bx + ay$. □

Problem 4. *Define $IP : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $IP(x, y) = \langle x, y\rangle$.*
1) Find $D(IP)(a, b)$ and $(IP)'(a, b)$.
2) If $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable and $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = \langle f(t), g(t)\rangle$, show that

$$h'(a) = \langle f'(a)^T, g(a)\rangle + \langle f(a), g'(a)^T\rangle.$$

3) If $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable and $|f(t)| = 1$ for all t, show that $\langle f'(t)^T, f(t)\rangle = 0$.
4) Exhibit a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that the function $|f|$ defined by $|f|(t) = |f(t)|$ is not differentiable.

Proof. 1) Since IP is bilinear, we have $D(IP)(a,b)(x,y) = \langle b, x\rangle + \langle a, y\rangle$. Then $(IP)'(a,b) = (a,b)$

2) Note that $h(t) = (IP) \circ (f, g)$. Now we simply use the Chain Rule and Part 1) to obtain the result.

3) This is just Part 2) applied to $\langle f(t), f(t)\rangle = 1$. Differentiating both sides gives the desired result.

4) Take $f(t) = t$. Then $|f(t)|$ is not differentiable at 0. $\square$

Problem 5. *Let E_i with $i = 1, \dots k$ be Euclidean spaces of various dimensions. A function $f : E_1 \times \cdots \times E_k \to \mathbb{R}^p$ is called multilinear if for each choice of $x_j \in E_j$, $j \neq i$ the function $g : E_i \to \mathbb{R}^p$ defined by $g(x) = f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k)$ is a linear transformation.*
1) If f is multilinear and $i \neq j$, show that for $h = (h_1, \dots h_k)$, with $h_l \in E_l$ we have

$$\lim_{h \to 0} \frac{|f(a_1, \dots, h_i, \dots, h_j, \dots, a_k)|}{|h|} = 0.$$

2) Prove that

$$Df(a_1, \dots a_k)(x_1, \dots x_k) = \sum_{i=1}^{k} f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_k).$$

Proof. 1) Since $f(a_1, \dots, h_i, \dots, h_j, \dots, a_k)$ is bilinear, this is an immediate result of Part 2) of Problem 3.

2) This is a similar case to Part 3) of Problem 3. Using the definition of a derivative, we can expand the numerator in a similar fashion as in Part 3) of Problem 3. Then using Part 1) we obtain a similar result, with more terms. This limit finally goes to 0 for the same reasons as in Part 3) of Problem 3. $\square$

Problem 6. *Regard and $n \times n$ matrix as a point in the n-fold product $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ by considering each row as a member of $\mathbb{R}^n$.*
1) Prove that $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ *is differentiable and*

$$D(\det)(a_1, \dots, a_n)(x_1, \dots, x_n) = \sum_{i=1}^{n} \det(a_1, \dots, x_i, \dots, a_n)^T.$$

2) If $a_{ij} : \mathbb{R} \to \mathbb{R}$ are differentiable and $f(t) = \det(a_{ij}(t))$, show that

$$f'(t) = \sum_{j=1}^{n} \det \begin{pmatrix} a_{11}(t), & \dots & , a_{1n}(t) \\ \vdots & & \vdots \\ a'_{j1}(t), & \dots & , a'_{jn}(t) \\ \vdots & & \vdots \\ a_{n1}(t), & \dots & , a_{nn}(t) \end{pmatrix}.$$

3) If $\det(a_{ij}(t)) \neq 0$ for all t and $b_1, \dots, b_n : \mathbb{R} \to \mathbb{R}$ are differentiable, let $s_1, \dots, s_n : \mathbb{R} \to \mathbb{R}$ be the functions such that $s_1(t), \dots, s_n(t)$ are the solutions of the equations

$$\sum_{j=1}^{n} a_{ji}(t) s_j(t) = b_i(t)$$

for $i = 1, \dots, n$. Show that s_i is differentiable and find $s_i'(t)$.

Proof. 1) Since det is multilinear, this follows immediately from Problem 5, Part 2).

2) This is a direct consequence of Part 1) and the Chain Rule.

3) Using Cramer's Rule, we can write $s_i = \det(B_i)/\det(A)$ where $A = [a_{ij}(t)]$ and B_i is the matrix obtained by replacing the ith column of A with $(b_1(t), \ldots, b_n(t))^T$. Taking the transpose of these matrices doesn't change the determinant, which allows us to use Part 2) and the quotient rule to find

$$s_i'(t) = \frac{\det(B_i)\det'(A) - \det(A)\det'(B_i)}{\det^2(B_i)}.$$

□

Problem 7. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and has a differentiable inverse $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. Show that $(f^{-1})'(a) = (f'(f^{-1}(a)))^{-1}$.*

Proof. Note that $f \circ f^{-1}(x) = x$. Differentiating both sides we have $f'(f^{-1}(x))(f^{-1})'(x) = 1$. Dividing gives the result. □

Problem 8. *A function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_0 \in \mathbb{C}$ if*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. A function f is analytic on an open set $U \subseteq \mathbb{C}$ if f is differentiable at each point of U. Write $f(z) = u(x, y) + iv(x, y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$, and $z = x + iy$.
1) Suppose f is analytic on an open set $U \subseteq \mathbb{C}$. Show that u and v are differentiable on U considered as a subset of $\mathbb{R}^2$.
2) Suppose f is analytic on an open set $U \subseteq \mathbb{C}$. Show that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These are the Cauchy-Riemann Equations.
3) If $U \subseteq \mathbb{C}$ is an open set and u and v are in $C^1(U)$ and satisfy the Cauchy-Riemann Equations, show that $f(z) = u(x, y) + iv(x, y)$ is analytic on U.
4) Find an example of a function $f : \mathbb{C} \to \mathbb{C}$ that is differentiable at one point, but not in a neighborhood of that point.

Proof. 1) Let $z_0 = x_0 + iy_0 \in U$ and consider

$$\begin{aligned}
f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\
&= \lim_{x+iy \to x_0+iy_0} \frac{u(x, y) + iv(x, y) - u(x_0, y_0) + iv(x_0, y_0)}{x + iy - x_0 + iy_0} \\
&= \lim_{x+iy \to x_0+iy_0} \frac{u(x, y) - u(x_0, y_0)}{x + iy - x_0 + iy_0} + i \lim_{x+iy \to x_0+iy_0} \frac{v(x, y) - v(x_0, y_0)}{x + iy - x_0 + iy_0}.
\end{aligned}$$

In $\mathbb{R}^2$ these last two terms correspond to

$$\lim_{(x,y) \to (x_0,y_0)} \frac{u(x, y) - u(x_0, y_0)}{(x, y) - (x_0, y_0)}$$

and

$$\lim_{(x,y) \to (x_0,y_0)} \frac{v(x, y) - v(x_0, y_0)}{(x, y) - (x_0, y_0)}.$$

Since these two limits exist, u and v are differentiable functions in $\mathbb{R}^2$.

2) We have

$$Df = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

Substituting for $f(x+iy) = u(x,y) + iv(x,y)$ we have

$$Df = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)\right).$$

Along the real axis $\partial f/\partial y = 0$, thus

$$Df = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right).$$

Along the imaginary axis $\partial f/\partial x = 0$, thus

$$Df = \frac{1}{2}\left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right).$$

The value of the derivative must be the same in so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

3) Given that u and v satisfy the Cauchy-Riemann equations, then we must have

$$\frac{1}{2}\left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{df}{d\overline{z}}.$$

But then this directly implies the differentiability of f since the conjugate function is continuous.

4) Define $f(z) = x^2 + y^2 + ixy$ for $z = x + iy$. Then the Cauchy-Riemann equations are satisfied only at the origin. Thus, f is differentiable at the origin, but not in any neighborhood of it. □

Problem 9. *Define $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$ as follows: $f(z) = f(x+iy) = e^x \cos y + ie^x \sin y$. Show that f is analytic on $\mathbb{C}$.*

Proof. We define $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Note that

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

By Part 3) of Problem 8 we see that f is analytic on $\mathbb{C}$. □

Problem 10. *Let $z_0 \in \mathbb{C}$ and define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $a_n \in \mathbb{C}$ for all n. Let $r > 0$ be the radius of convergence of this power series.*
1) Show that $f(z)$ is analytic on $B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$.
2) Show that the radius of convergence of the power series for $f'(z)$ is equal to r.

Proof. 1) Within the radius of convergence we can write

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$$

which represents the term by term differentiation of $f(z)$.

2) We have $r = 1/\limsup n \to \infty |a_n|^{1/n}$. The series $\sum_{n=0}^{\infty} na_n x^{n-1}$ will converge when the series $\sum_{n=0}^{\infty} na_n x^n$ converges. Now consider
$\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} n^{1/n}|a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}$. Thus the radius of convergence of this series is the same as that of $f(z)$. $\square$

Problem 11. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sqrt{|x| + |y|}$. Find those points in $\mathbb{R}^2$ at which f is differentiable.*

Proof. We have f is differentiable at all points such that $x \neq 0$ and $y \neq 0$. Suppose that $x = 0$. Then we have

$$f'(x, y) = \lim_{h\to 0} \frac{|\sqrt{|y + h_2|} - \sqrt{|y|}}{\sqrt{h_1^2 + h_2^2}}.$$

Based on the powers of the numerator and the denominator, we see that this limit doesn't exist. A similar case holds for $y = 0$. $\square$

Problem 12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq ||x||^{\alpha}$ for some $\alpha > 1$. Show that f is differentiable at 0.*

Proof. For $x = 0$ we have

$$f'(x) = \lim_{h\to 0} \frac{|f(x + h) - f(x)|}{|h|} \leq \lim_{h\to 0} \frac{||h||^{\alpha}}{||h||}.$$

Since α is strictly greater than 0, this limit goes to 0 and so f is differentiable at 0. $\square$

Problem 13. *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x, y) = x \cdot y$.*
1) Show that f is differentiable on $\mathbb{R}^n \times \mathbb{R}^n$.
2) Show that $Df(a, b)(x, y) = ay + bx$.

Proof. Both parts follow from Problems 3 and 4. $\square$