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Homework 1

Problem 1. *Let x, y and z be loops in X based at $x_0 \in X$. Based on the above picture, write down an explicit homotopy $F(s,t)$ between $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$.*

Proof. Note that if $f = (x \cdot y) \cdot z$ then $f \circ g = x \cdot (y \cdot z)$ where

$$g = \begin{cases} \frac{1}{2}s & 0 \le s \le \frac{1}{2} \\ s - \frac{1}{4} & \frac{1}{2} \le s \le \frac{3}{4} \\ 2s - 1 & \frac{3}{4} \le s \le 1 \end{cases}.$$

So now the homotopy $F : I^2 \to X$ defined by $F(s,t) = f((1-t)s + tg(s))$ gives a homotopy from $(x \cdot y) \cdot z$ to $x \cdot (y \cdot z)$. Expanding this out we get the homotopy

$$F(s,t) = \begin{cases} x\left(\frac{4}{1+t}s\right) & 0 \le s \le \frac{1+t}{4} \\ y(4s - (1+t)) & \frac{1+t}{4} \le s \le \frac{2+t}{4} \\ z(2(1+t)s - (1+2t)) & \frac{2+t}{4} \le s \le 1 \end{cases}.$$

Then $F(s,0) = (x \cdot y) \cdot z$, $F(s,1) = x \cdot (y \cdot z)$ for all s and $F(0,t) = F(1,t) = x(0) = z(1) = x_0$ for all t. $\square$

Problem 2. *For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.*

Proof. Suppose all β_h are independent of paths. Let $[f], [g] \in \pi_1(X, x_0)$. We wish to show that the composed loop $f \cdot g$ is homotopic to $g \cdot f$. Note that f is homotopic to a loop $h\overline{h'}$ where h and h' are paths from x_0 to x_1. To see this, let y be a point on f and let f' be a path from y to x_1. Then let h be the path along f to y composed with f' and let $\overline{h'}$ be $\overline{f'}$ composed with the path from y to x_0 along f. Now we've assumed that $\pi_1(X, x_0) \approx \pi_1(X, x_1)$ with the associated maps $\overline{h}gh$ and $\overline{h'}gh'$ the same. This relation can be rewritten as $h'\overline{h}g \simeq gh'\overline{h}$ and we've just shown that this is the same as $fg \simeq gf$ so $[fg] = [gf]$ and $\pi_1(X)$ is abelian.

Conversely, suppose $\pi_1(X)$ is abelian and let h and h' be a paths in X from x_0 to x_1 and $f \mapsto \overline{h}fh$ and $f \mapsto \overline{h'}fh'$ their associated homomorphisms. We know $h'\overline{h}$ is a loop in X and is thus an element of $\pi_1(X, x_0)$. Thus for any loop f we have $f(h'\overline{h}) \simeq (h'\overline{h})f$ which can be rewritten as $\overline{h}fh \simeq \overline{h'}fh'$ so the maps must be equal. $\square$

Problem 3. *Show that for a space X, the following three conditions are equivalent:*
(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Proof. Note that $\pi_1(X, x_0)$ is the set of maps $I \to X$ with x_0 the image of 0 and 1. But this is the same as the set of maps from $S^1 \to X$ with a fixed point s_0 mapping to x_0. Thus, every map $S^1 \to X$ being nullhomotopic is precisely the same as $\pi_1(X, x_0) = 0$. Therefore (a) and (c) are equivalent.

To show that (b) implies (a), let $f : S^1 \to X$ be a map and let f' be it's extension from D^2 to X. Note that in D^2, S^1 is nullhomotopic so there exists a homotopy f_t taking S_1 to s_0 where $f(s_0) = x_0$. But then $f'f_t$ is a homotopy taking f to x_0. Note that $f'f_0 = f$ and $f'f_1 = f'(s_0) = x_0$ so that this is indeed the homotopy we're after. Thus f is nullhomotopic.

Finally, suppose every map $S^1 \to X$ is nullhomotopic and let $f : S^1 \to X$ be a map. Then there exists a homotopy f_t taking f to x_0. Now for each $t \in [0, 1]$ define $f'(t, \theta) = f_t(\theta)$. Since t takes on all values in $[0, 1]$ and for each t f_t takes on all values in S^1, we see that f' is a map $D^2 \to X$. Moreover, f' is continuous since each f_t is continuous in θ and f_t varies continuously with t since it's a homotopy. This gives an extension of f to D^2 and proves that (a) implies (b). $\square$

Problem 4. *We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and set of conjugacy classes in $\pi_1(X)$, when X is path connected.*

Proof. Suppose X is path connected and let $[f] \in [S^1, X]$. Let $g \in [f]$ with basepoint x_1 and let y be a point on g. Since X is path connected there exists a path p from y to x_0. Then the path which goes along g from x_1 to y, then along p from y to x_0, then along $\overline{p}$ and finally along g from y to x_1 is a loop which is homotopic to g and includes x_0. With an appropriate shift, this path is homotopic to a loop with basepoint x_0, so $[f]$ is the image of some element of $\pi_1(X, x_0)$ and Φ is surjective.

Now suppose $\Phi([f]) = \Phi([g])$ for some elements $[f], [g] \in \pi_1(X, x_0)$. Then there exists a homotopy $F : I^2 \to X$ such that $F(0, t) = F(1, t)$ for all t and $F(s, 0) = f(s)$ and $F(s, 1) = g(s)$ for all s. Then Let $h : I \to X$ be defined by $h(t) = F(0, t)$. Note that $h(0) = F(0, 0) = f(0) = g(0) = F(0, 1) = h(1)$ so

$h \in \pi_1(X, x_0)$. Now note that

$$f \simeq \begin{cases} h(3s) & s = 0 \\ F(s,0) & 0 \le s \le 1 \\ \overline{h}(3s-2) & s = 1 \end{cases}$$

and

$$hg\overline{h} \simeq \begin{cases} h(3s) & 0 \le s \le \frac{1}{3} \\ F\left(3\left(s - \frac{1}{3}\right), 1\right) & \frac{1}{3} \le s \le \frac{2}{3} \\ \overline{h}(3s-2) & \frac{2}{3} \le s \le 1 \end{cases}.$$

So now we can create the homotopy $F' : I^2 \to X$ which takes f to $hg\overline{h}$ as

$$F'(s,t) = \begin{cases} h(3s) & 0 \le s \le \frac{t}{3} \\ F\left((1+2t)\left(s - \frac{t}{3}\right), s\right) & \frac{t}{3} \le s \le 1 - \frac{t}{3} \\ \overline{h}(3s-2) & 1 - \frac{t}{3} \le s \le 1 \end{cases}.$$

We see that $F'(s,0) = f(t)$, $F'(s,1) = hg\overline{h}$ and $F'(0,t) = F'(1,t) = h(0) = x_0$, so f and g are conjugate through h. Thus Φ is injective. □

Problem 5. *Suppose you have a sandwich consisting of bread, ham and cheese (each a compact set in $\mathbb{R}^3$). Then the sandwich can be bisected with a single cut, i.e., a plane, such that each half contains the same amount of bread, ham and cheese.*

Proof. Call the three sets A, B and C and suppose that A is open, connected and bounded instead of compact. Draw a sphere S big enough to encompass A, B and C. Let $\mathbf{x}$ denote the vector pointing from 0 to $x \in S$. Note that there is a unique plane containing x and normal to $\mathbf{x}$. Define $f : S \times [-1,1] \to \mathbb{R}$ by $f(x,t)$ is the measure of A lying on the side of the plane corresponding to $t\mathbf{x}$ in the direction that $\mathbf{x}$ points. This means that $f(x,t) + f(-x,-t) = \mu(A)$. Note that f is a continuous function since small changes in

x and t amount to small changes in the corresponding plane and thus small changes in the measure of A on either side of said plane. Note also that for each $x \in S$ we have $f(x,1) = 0$ and $f(x,-1) = m(A) \ge 0$ and for a fixed x, f is monotonically decreasing. Thus, using the intermediate value theorem there is some point $g(x) \in [-1,1]$ such that $f(x, g(x)) = \mu(A)/2$. Note that $g(x)$ is a unique point because A is open and connected so the plane corresponding to $g(x)\mathbf{x}$ necessarily intersects A, and A is open so we could draw a ball with nonzero measure intersecting two potential planes dividing A in half. The fact that g is continuous follows from the fact that f is continuous. Note that $g(x) = -g(-x)$.

Now define f_B and f_C in the same way we defined f, but for the sets B and C. Let $h : S \to \mathbb{R}^2$ be defined by $h(x) = (f_B(x, g(x)), f_C(x, g(x)))$. By the Borsuk-Ulam theorem there exists a pair of antipodal points x and $-x$ such that $h(x) = h(-x)$. This means that $f_B(x, g(x)) = f_B(-x, g(-x)) = f_B(-x, -g(x))$ and likewise $f_C(x, g(x)) = f_C(-x, -g(x))$. But this precisely says that the measure of B on one side of a plane normal to x ($f_B(x, g(x))$) is the same as the measure of B on the other side ($f_B(-x, -g(x))$). Thus, this plane must bisect the set B. Likewise, the same plane must bisect C. Then from how we defined g we see that this plane also bisects A so we're done. $\square$

Problem 6. *Suppose X is path-connected. Define $\pi_1(X, x_0, x_1)$, and show that this is a left $\pi_1(X, x_0)$-torsor.*

Proof. Define $\pi_1(X, x_0, x_1)$ as the set of homotopy classes of paths in X from x_0 to x_1. Define an action of $\pi_1(X, x_0)$ on $\pi_1(X, x_0, x_1)$ as $[f] \circ [h] = [f] \cdot [h] = [f \cdot h]$. That is, a loop in $\pi_1(X, x_0)$ acting on a path in $\pi_1(X, x_0, x_1)$ is just the composed path going around the loop and then following the path. This clearly satisfies the axioms of a group action since path composition is associative and composing with the identity loop in $\pi_1(X, x_0)$ will leave any element of $\pi_1(X, x_0, x_1)$ unaffected.

Now suppose we have h and h' paths in X from x_0 to x_1. Let f be the loop which traverses h' from x_0 to x_1, then traverses $\overline{h}$ from x_1 to x_0. So $f \in \pi_1(X, x_0)$ and $f \simeq h'\overline{h}$. But then $[f] \circ [h] = [f \cdot h] = [(h'\overline{h})h] = [h']$. Thus for any two paths h and h' there exists an element of $\pi_1(X, x_0)$ taking h to h'. Suppose that

$g \in \pi_1(X, x_0)$ such that $[g] \circ [h] = [h']$. Since X is path connected, g is homotopic to a loop which contains x_1 using a similar argument as that in Problem 4. Thus $[gh]$ can be decomposed as a path from x_0 to x_1, followed by a path from x_1 to x_0 and then h, a path from x_0 to x_1. Since $[gh] = [h']$ it follows that the first of these paths is homotopic to h', and the second is homotopic to $\overline{h}$. Thus $g \simeq f$ and $[f]$ is the unique

element of $\pi_1(X, x_0)$ taking $[h]$ to $[h']$. The fact that $\pi_1(X, x_0, x_1)$ is a right $\pi_1(X, x_1)$-torsor follows by a similar argument where we switch the order of all the functions involved. $\square$