Sheet 7: Return of the Continuum

Definition 1 (Open Cover) *Let $X \subseteq C$ be a set and let $\mathcal{A}$ be a set of subsets of C. We say that $\mathcal{A}$ is an open cover for X if for all $A \in \mathcal{A}$ the set A is open and*

$$X \subseteq \bigcup_{A \in \mathcal{A}} A.$$

Exercise 2 *Let $p \in C$ be a point and let*

$$\mathcal{A} = \{\text{ext}(a; b) \mid p \in (a; b)\}.$$

Show that $\mathcal{A}$ is an open cover for $C \backslash p$.

Proof. Let $x \in C \backslash p$. Then $x \in C$ and $x \neq p$ and so $x < p$ or $p < x$. Suppose $x < p$. Since regions are nonempty there exists $a \in C$ such that $x < a < p$ (5.8). And because C has no last point there exists $b \in C$ such that $p < b$ (A2.3). But then $p \in (a; b)$ and since $x < a$, $x \in \text{ext}(a; b)$. Because this is true for some region $(a; b)$, we see $x \in \bigcup_{A \in \mathcal{A}} A$. Therefore, $C \backslash p \subseteq \bigcup_{A \in \mathcal{A}} A$. From Exercise 12 we see that $\text{ext}(a; b)$ is open and so $\mathcal{A}$ is an open cover for $C \backslash p$ (7.12). A similar argument holds if $p < x$ because C has no first point (A2.3). Note that Exercise 12 does not depend on this exercise. □

Definition 3 (Subcover) *Let $\mathcal{A}$ be an open cover for X. A subset $\mathcal{B} \subseteq \mathcal{A}$ is a subcover if*

$$X \subseteq \bigcup_{B \in \mathcal{B}} B.$$

Exercise 4 *Show that the set*

$$A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}.$$

is closed.

Proof. Let $p \in C$ be point such that $p \notin A$. Then there are three cases.

Case 1: Let $p < 0$. Then since C has no first point there exists a point $x \in C$ such that $x < p$ and so the region $(x; 0)$ contains p but no points in A (A2.3).

Case 2: Let $p > 1$. Then since C has no last point there exists a point $y \in C$ such that $p < y$ and so the region $(1; y)$ contains p but no points in A (A2.3).

Case 3: Let $p \in (0; 1)$. Then $p = \frac{a}{b}$ for some $a, b \in \mathbb{N}$ and since $0 < \frac{a}{b} < 1$, we have $a < b$. Since $0 < \frac{b}{a}$, by the Archimedean Property there exists a natural number k such that $\frac{b}{a} < k$ (4.20). But since $k \in \mathbb{N}$, by the Well Ordering Principle there exists a least such element n. Since $p \notin A$, $a \neq 1$ and so $\frac{b}{a} \notin \mathbb{N}$. But then $n - 1 < \frac{b}{a} < n$ and so $\frac{1}{n} < p < \frac{1}{n-1}$. Therefore $p \in \left(\frac{1}{n}; \frac{1}{n-1}\right)$ which doesn't contain any elements of A.

In all three cases there exists a region containing p which contains no elements of A and so p cannot be a limit point of A. Therefore if A has any limit points, they must be in A. Since A contains all its limit points, it is closed. □

Exercise 5 *Prove that every open cover of A has a finite subcover.*

Proof. Let $\mathcal{A}$ be a cover of A. Then for every element of A, there exists an open set in $\mathcal{A}$ which contains that element. But then there exists an open set B in $\mathcal{A}$ containing 0. And so there exists a region $(a;b) \subseteq B$ such that $0 \in (a;b)$ by the open condition (3.17). There are three cases.

Case 1: Let $1 < b$. Then $A \subseteq B$ and so the set containing B is a finite subcover of $\mathcal{A}$.

Case 2: Let $b = 1$. Then the region $(a;b)$ contains all the elements of A except for 1. Thus the set containing B and a set from $\mathcal{A}$ containing 1 is a finite subcover of A.

Case 2: Let $b < 1$. Then $b = \frac{p}{q}$ for some $p, q \in \mathbb{N}$ and since $0 < \frac{p}{q} < 1$, we have $p < q$. Since $0 < \frac{q}{p}$, by the Archimedean Property there exists a natural number k such that $\frac{q}{p} < k$ (4.20). But since $k \in \mathbb{N}$, by the Well Ordering Principle there exists a least such element n. There are a finite number of natural numbers less than n and since every element of A is a reciprocal of a natural number, there are a finite number of elements x of A such that $\frac{1}{n} < x$. All the other elements of A are less than b so they are contained in $(a;b)$. For each element of A greater than $\frac{1}{n}$ there exists a set in $\mathcal{A}$ containing that element. There are finitely many of these elements so there exist finitely many sets of $\mathcal{A}$ containing them. So those sets and B form a finite subcover of $\mathcal{A}$. $\square$

Definition 6 (Compact Set) *A set X is compact if every open cover of X has a finite subcover.*

Exercise 7 *Let $\mathcal{A}$ be the set of all regions. Show that no finite subset of $\mathcal{A}$ covers C.*

Proof. Let $\mathcal{B}$ be a finite subset of $\mathcal{A}$. If $\mathcal{B} = \emptyset$ then it is clear that it is not an open cover for C. Then $\mathcal{B} = \{(a_1;b_1), (a_2;b_2), \ldots, (a_n;b_n)\}$. But since there are a finite number of lower boundary points a_i for regions in $\mathcal{B}$, we can order them so that x is a lower boundary point and $x \leq a_i$ for all regions in $\mathcal{B}$. Then x is less than every point in every region in $\mathcal{B}$. But since C has no first point there exists a point $p \in C$ such that $p < x$ and so $C \nsubseteq \bigcup_{(a;b)\in\mathcal{B}}(a;b)$ (A2.3). $\square$

Exercise 8 *Let $p \in C$ be a point and let $\mathcal{A} = \{\text{ext}(a;b) \mid p \in (a;b)\}$. Show that no finite subset of $\mathcal{A}$ covers $C\backslash p$.*

Proof. Let $\mathcal{B}$ be a finite subset of $\mathcal{A}$. Clearly if $\mathcal{B} = \emptyset$ then it is not an open cover for $C\backslash p$. Then $\mathcal{B} = \{\text{ext}(a_1;b_1), \text{ext}(a_2;b_2), \ldots, \text{ext}(a_n;b_n)\}$ such that $p \in (a;b)$ for all $\text{ext}(a;b) \in \mathcal{B}$. Consider the finite set of values of a_i for exteriors in $\mathcal{B}$. Since this set is finite there exists a last point x so that $x \geq a_i$ for all exteriors in $\mathcal{B}$ (2.2). Since regions are nonempty there exists a point $y \in C$ such that $x < y < p$ and so $y \notin \text{ext}(a_i;b_i)$ for any exterior in T (5.8). But then $C\backslash p \nsubseteq \bigcup_{B\in\mathcal{B}} B$. $\square$

Theorem 9 (Compact Sets Are Bounded) *If $X \subseteq C$ is not bounded, then X is not compact.*

Proof. Let $X \subseteq C$ be a set which is not bounded below and let $\mathcal{A}$ be the set of all regions. Consider a finite subset of $\mathcal{A}$, $\mathcal{B}$. Since $\emptyset$ is bounded below, $X \neq \emptyset$. So in the case where $\mathcal{B} = \emptyset$ we see that $\mathcal{B}$ is not an open cover for X. Then $\mathcal{B} = \{(a_1;b_1), (a_2;b_2), \ldots, (a_n;b_n)\}$. But since there are a finite number of lower boundary points a_i for regions in $\mathcal{B}$, we can order them so that x is a lower boundary point and $x \leq a_i$ for all regions in $\mathcal{B}$ (2.2). Then x is less than every point in every region in $\mathcal{B}$. But since X has no lower bound, for all $p \in C$ there exists $q \in X$ such that $q < p$. Therefore there exists a $q \in X$ such that $q < x$ and so $X \nsubseteq \bigcup_{B\in\mathcal{B}} B$. A similar proof holds if X is a set which is not bounded above. $\square$

Theorem 10 (Compact Sets Are Closed) *If $X \subseteq C$ is not closed, then X is not compact.*

Proof. Let $X \subseteq C$ be a set which is not closed and $p \notin X$ be a limit point of X. Let $\mathcal{A} = \{\text{ext}(a;b) \mid p \in (a;b)\}$. Since $p \notin X$ we see that $\mathcal{A}$ covers X. Suppose that $\mathcal{B}$ is a finite subset of $\mathcal{A}$. We see that $X \neq \emptyset$ because $\emptyset$ is closed (3.13). So in the case where $\mathcal{B} = \emptyset$ we see that $\mathcal{B}$ does not cover X. Then $\mathcal{B} = \{\text{ext}(a_1;b_1), \text{ext}(a_2;b_2), \ldots, \text{ext}(a_n;b_n)\}$. But then the set of lower boundary points a_i and the set of upper boundary points b_i for exteriors in $\mathcal{B}$ are finite. Thus there exists a last point x such that x is a lower boundary point of some exterior in $\mathcal{B}$ and $x \geq a_i$ for all exteriors in $\mathcal{B}$. Likewise there exists a smallest upper boundary point y for exteriors in $\mathcal{B}$. Note that x and y need not define the same exterior in $\mathcal{B}$. But then the region $(x; y)$ must contain p because all lower boundary points are less than p and all upper boundary points are greater than p. Since p is a limit point of X, then $(x; y)$ also contains a point in X. But $(x; y)$ is defined so that $(x; y) \nsubseteq \bigcup_{B \in \mathcal{B}} B$. Therefore $X \nsubseteq \bigcup_{B \in \mathcal{B}} B$ and so $\mathcal{B}$ is not a finite subcover for $\mathcal{A}$ and X is not compact. □

Definition 11 *For $a < b$ let the closed interval $[a; b]$ be defined as*

$$[a; b] = (a; b) \cup \{a\} \cup \{b\}.$$

Exercise 12 *Closed intervals are closed*

Proof. Let $a, b, p \in C$ be points such that $a < b$ and $p \notin [a; b]$. Then $p < a$ or $p > b$. Let $p < a$. Since C has no first point there exists a point $x \in C$ such that $x < p$ (A2.3). But then the region $(x; a)$ contains x but no points in $[a; b]$. A similar argument holds for $b < p$ and so p cannot be a limit point of $[a; b](A2.3)$. But then any limit points of $[a; b]$ must be in $[a; b]$ and so $[a; b]$ is closed. □

Definition 13 (Chain of Regions) *Let $a < b$. A chain of regions going from a to b is defined as a finite sequence $R_1, R_2, \ldots, R_n$ of regions such that $a \in R_1$, $b \in R_n$ and for $1 \leq i \leq n-1$ we have $R_i \cap R_{i+1} \neq \emptyset$.*

Exercise 14 *A chain of regions from a to b covers the the closed interval $[a; b]$.*

Proof. Let $R_1, R_2, \ldots R_n$ be a chain of regions going from a to b such that $R_i = (p_i; q_i)$. Let $x \in [a; b]$. Then x is greater than a finite number of upper boundary points q_i. Consider the set of indexes for these points. If the set is empty then $x \in R_1$. If the set is not empty then we can take the last point of the set k (2.2). By definition $R_k \cap R_{k+1} \neq \emptyset$ and so $p_{k+1} < q_k$. But $q_k < x$ and $x < q_{k+1}$ and so $x \in (p_{k+1}; q_{k+1}) = R_{k+1}$. Therefore, if $x \in [a; b]$ then x is in one of the regions $R_1, R_2, \ldots, R_n$. Thus $[a; b] \subseteq R_1 \cup R_2 \cup \cdots \cup R_n$. Since all regions are open, the chain of regions covers $[a; b]$ (3.16). □

Theorem 15 *Let $a < b$ and let $\mathcal{A}$ be a set of regions that covers $[a; b]$. Let $X = \{x \in [a; b] \mid$ there is a chain of regions $R_1, R_2, \ldots, R_n \in S$ going from a to $x\}$. Then $\sup X = b$. Moreover $b \in X$.*

Proof. Since $X \subseteq [a; b]$ we see that X is bounded above by b. Additionally, $X \neq \emptyset$ because there exists a region $R_1 \in \mathcal{A}$ which contains a and so there is a finite chain of regions going from a to all points in R_1 greater than or equal to a. Therefore $\sup X$ exists (6.11). Let $u = \sup X$. If $u > b$ then we have $b \geq x$ for all $x \in [a; b]$ and thus $b \geq x$ for all $x \in X$. Therefore b is an upper bound of X which is less than u. This is a contradiction and so $u \leq b$. So we have $a < u$ and $u \leq b$ so $u \in [a; b]$. Since $\mathcal{A}$ is an open cover of $[a; b]$ there exists a region $R_i \in \mathcal{A}$ such that $u \in R_i$. Suppose to the contrary that all the points in R_i which are between a and u are not in X and consider one of these points p. We see that there are no elements of X between p and u and because $\sup X = u$, p is an upper bound of X. But this is a contradiction because $p < u$. Therefore there exists a point $c \in X$ such that $c \in R_i$ and $a < c < u$ (5.8). Thus there exists a chain of regions from $\mathcal{A}$ which goes from a to c which ends in some region S. Since $c \in R_i$ and $c \in S$ we have $R_i \cap S \neq \emptyset$. Then there exists a finite chain of regions from $\mathcal{A}$ going from a to u and $u \in X$. Now assume to the contrary that $u < b$. Then there exists another point $u' \in R_i$ such that $u < u' < b$ (5.8). But then $u' \in X$ and $u < u'$. This is a contradiction since $u = \sup X$. Therefore $\sup X = b$ and $b \in X$. □

Theorem 16 (Closed Intervals Are Compact With Respect To Regions) *Let $a < b$. Then any set of regions that covers $[a; b]$ has a finite subcover.*

Proof. This follows from Theorem 15 and Exercise 14. Because $b \in X$ we see that there exists a finite chain of regions going from a to b (7.15). Since regions are open sets, this chain forms a finite subcover for $[a; b]$ (7.14). $\square$

Theorem 17 *Let $a < b$ be points in C and let $\mathcal{A}$ be an open cover for $[a; b]$. Let*

$$S = \{(c; d) \mid c < d, \text{ there exists } A \in \mathcal{A} \text{ with } (c; d) \subseteq A\}.$$

We have

$$[a; b] \subseteq \bigcup_{(c;d) \in S} (c; d).$$

Proof. We know that $\mathcal{A}$ is an open cover for $[a; b]$. Thus, for all $x \in [a; b]$ there exists $A \in \mathcal{A}$ such that A is open and $x \in A$. But by the open condition there exists a region $(c; d) \subseteq A$ such that $x \in (c; d)$ (3.17). Then $x \in \bigcup_{(c;d) \in S}(c; d)$ because $x \in \bigcup_{A \in \mathcal{A}} A$ and $(c; d) \subseteq A$ for all $A \in \mathcal{A}$. Therefore $[a; b] \subseteq \bigcup_{(c;d) \in S}(c; d)$. $\square$

Corollary 18 *For $(c; d) \in S$ let $A_{(c;d)} \in \mathcal{A}$ such that $(c; d) \subseteq A_{(c;d)}$. We have*

$$[a; b] \subseteq \bigcup_{(c;d) \in S} A_{(c;d)}.$$

Proof. From Theorem 17 we have $[a; b] \subseteq \bigcup_{(c;d) \in S}(c; d)$ (7.17). For all $(c; d) \in S$ we have $(c; d) \subseteq A_{(c;d)}$. Therefore $\bigcup_{(c;d) \in S}(c; d) \subseteq \bigcup_{(c;d) \in S} A_{(c;d)}$. And so $[a; b] \subseteq \bigcup_{(c;d) \in S} A_{(c;d)}$. $\square$

Theorem 19 (Closed Intervals Are Compact) *For $a < b$ the closed interval $[a; b]$ is compact*

Proof. Let $\mathcal{A}$ be an open cover for $[a; b]$ for $a, b \in C$. Define

$$S = \{(c; d) \mid c < d, \text{ there exists } A \in \mathcal{A} \text{ with } (c; d) \subseteq A\}.$$

From Theorem 17 we know that S is a cover for $[a; b]$ (7.17). Since S is composed entirely of regions, by Theorem 16 there exists a finite subcover of S for $[a; b]$. So there exists finitely many regions from S which will form an open cover of $[a; b]$. Call this set T. Then for $(c; d) \in T$ let $B_{(c;d)} \in \mathcal{A}$ such that $(c; d) \subseteq B_{(c;d)}$. From Corollary 18 we know that the set of all $A_{(c;d)}$ for $(c; d) \in S$ is an open cover for $[a; b]$ (7.18). But $T \subseteq S$ and so the set of all $B_{(c;d)}$ is a subset of the set of all $A_{(c;d)}$. And because $(c; d) \subseteq B_{(c;d)}$ for all $(c; d) \in T$, and T is an open cover for $[a; b]$ we have $[a; b] \subseteq \bigcup_{(c;d) \in T} B_{(c;d)} \subseteq \bigcup_{(c;d) \in S} A_{(c;d)}$. So the set of all $B_{(c;d)}$ is a finite open subcover for $[a; b]$ because T is finite and $B_{(c;d)} \in \mathcal{A}$ for all $(c; d) \in T$. $\square$

Theorem 20 *Let $X \subseteq C$ be a closed set and let $\mathcal{A}$ be an open cover of X. Then $\mathcal{A} \cup \{C \backslash X\}$ is an open cover of C.*

Proof. We know that $X = C \backslash (C \backslash X)$ is closed and so $C \backslash X$ is open. Then let $p \in C$. Then $p \in X$ or $p \notin X$. If $p \in X$ then $p \in \bigcup_{A \in \mathcal{A}} A$. If $p \notin X$ then $p \in C \backslash X$. Therefore $p \in \bigcup_{A \in \mathcal{A}} \cup (C \backslash X)$. Thus $C \subseteq \bigcup_{A \in \mathcal{A}} A \cup (C \backslash X)$. Since all the sets in $\mathcal{A} \cup \{C \backslash X\}$ are open, $\mathcal{A} \cup \{C \backslash X\}$ is an open cover for C. $\square$

Theorem 21 *Let $X \subseteq C$ be a set and let $\mathcal{B}$ be an open cover of X such that $C \backslash X \in \mathcal{B}$. Then $\mathcal{B} \backslash \{C \backslash X\}$ is an open cover of X.*

Proof. There are no points of X which are in $C \backslash X$. Therefore, since $X \subseteq \bigcup_{B \in \mathcal{B}} B$, we also have $X \subseteq \bigcup_{B \in \mathcal{B}, B \neq (C \backslash X)} B$. And so $\mathcal{B} \backslash \{C \backslash X\}$ is an open cover for X. $\square$

Theorem 22 (Bounded Closed Sets Are Compact) *Let $X \subseteq C$ be a bounded closed set. Then X is compact.*

Proof. Let $\mathcal{A}$ be an open cover of X. Then from Theorem 20 we have $\mathcal{A} \cup \{C \backslash X\}$ is an open cover of C (7.20). Since X is bounded we see that $\inf X$ and $\sup X$ exist (6.11, 6.12). But then $[\inf X; \sup X] \subseteq C$ and so $\mathcal{A} \cup \{C \backslash X\}$ is an open cover for $[\inf X; \sup X]$. But from Theorem 19 we know that $[\inf X; \sup X]$ is compact and so we let $\mathcal{B} \subseteq \mathcal{A} \cup \{C \backslash X\}$ be a finite subset which covers $[\inf X; \sup X]$ (7.19). Then $X \subseteq [\inf X; \sup X]$ by definition and so $\mathcal{B}$ is an open cover for X. But then we know that $\mathcal{B} \subseteq \mathcal{A} \cup \{C \backslash X\}$ and so $\mathcal{B} \backslash \{C \backslash X\} \subseteq \mathcal{A}$. From Theorem 21 we know that $\mathcal{B} \backslash \{C \backslash X\}$ is an open cover for X because $\mathcal{B}$ is an open cover for X (7.21). Since $\mathcal{B} \backslash \{C \backslash A\} \subseteq \mathcal{A}$ is finite we now have a finite open subset of $\mathcal{A}$ which covers X so X is compact. $\square$