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CMSC 27500
April 22, 2009

Homework 3

Problem 1. *Let G be a connected graph and e a link of G.*
1) Describe a one-to-one correspondence between the set of spanning trees of G that contain e and the set of spanning trees of G/e.

Proof. Let T be a spanning tree of G such that T contains e and consider the graph T/e. Since T is a tree, T/e is a tree as well, and a subgraph of G/e, thus it's a spanning tree of G/e. Now consider A and B distinct spanning trees of G/e and let A' and B' be the resulting graphs with e added back in. Note that A and B must differ in some edge and this edge cannot be e since they are subgraphs of G/e. Therefore A' and B' are distinct spanning trees of G. This shows that edge contraction of e is an injective map from the set of spanning trees of G that contain e and the set of spanning trees of G/e. □

2) Show $t(G) = t(G\backslash e) + t(G/e)$.

Proof. Note that the map defined in Part 1) is surjective, since given a spanning tree of G/e we can add e back in to find a spanning tree of G which contains e. We can break $t(G)$ into the number of spanning trees containing e and the number not containing e. We know there's an injection between spanning trees of G which contain e and spanning trees of G/e. Then this shows $t(G) - t(G/e) = t(G\backslash e)$. □

Problem 2. *Show that the incidence matrix of a graph is totally unimodular if and only if the graph is bipartite.*

Proof. Let A be a matrix whose rows can be partitioned into two disjoint sets B and C such that every column of A contains at most two nonzero entries, every entry is either -1, 0 or $+1$, if two 1s appear in a column then one entry is in a row from B and the other from C, and if two elements of different sign appear in a column then they are both in B or both in C. The A is totally unimodular. The proof of this follows from the proof that an incidence matrix of a digraph is totally unimodular.

Now consider the incidence matrix, A, of a bipartite graph. Call the two sets of vertices B and C. Since there are no edges between two vertices in B, there are no loops in B, and similarly for C. Thus, every element in A is a 1 or a 0. Note also that one edge has exactly one head and one tail and so there are precisely two 1s in each column of A. Finally, if a 1 appears in a row in B, then the other 1 in the column must be in C since the corresponding edge goes from B to C. Since it fulfills all the conditions, A is totally unimodular.

Conversely, suppose the incidence matrix, A, of a graph is totally unimodular. Suppose that the graph is not bipartite. There there exists some odd cycle with vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{e_1, e_2, \ldots, e_n\}$. Take the submatrix of A with these rows and columns and order both the rows and columns by their indices. Note that this creates 1s on the main diagonal and 1s on the first lower diagonal as well as one 1 in the upper right-hand corner. But this matrix will have determinant 2 and so A has a submatrix with determinant not equal to -1, 0 or 1. This is a contradiction and so the graph is bipartite. □

Problem 3. *Show that a digraph contains a directed odd cycle if and only if some strong component is not bipartite.*

Proof. Let D be a digraph which contains a directed odd cycle. Considering the underlying graph G of D, we know G contains an odd cycle, which means G, and thus D is not bipartite. Conversely, assume some strong component of D is not bipartite. Then the underlying graph of this component contains an odd cycle. But then the component contains a directed odd cycle. □

Problem 4. *1) Show that a digraph D has a spanning x-branching if and only if $\partial^+(X) \neq \emptyset$ for every proper subset X of V that includes x.*

Proof. Create the spanning x branching as follows. First let $X_1 = \{x\}$. Then since $\partial^+(X_1) \neq \emptyset$, there are vertices $U_1 = \{v_{1_1}, v_{1_2}, \dots, v_{1_n}\}$ such that there are arcs which join x to all elements in U_1. Now let $X_2 = X_1 \cup U_1$. If $X_2 = V$ then we're done. Otherwise, we have $\partial^+(X_2) \neq \emptyset$ and so there are vertices $U_2 = \{v_{2_1}, v_{2_2}, \dots, v_{2_m}\}$ such that there are arcs which join elements of U_1 to elements of U_2. Note that x is not joined to any elements of U_2 since all of those connections are made to elements of U_1. Now let $X_3 = X_2 \cup U_2$. Proceed in this way until $X_k = V$. At this point we have an x-branching which covers every vertex in V and is thus spanning.

Conversely, suppose that D has a spanning x-branching and consider some proper subset $X \subseteq V$ such that $x \in X$. Let $v \notin X$ be a vertex of D. Note that since D has a spanning x-branching, there exists a directed path from x to v, and since $v \notin X$ we must have $\partial^+(X) \neq \emptyset$. □

2) Deduce that a digraph is strongly connected if and only if it has a spanning v-branching for every vertex v.

Proof. If a directed graph D has a spanning v-branching for every vertex v, then we have that for every proper subset $X \subseteq V$ we have $\partial^+(X) \neq \emptyset$. But this is the definition of being strongly connected. Conversely, if D is strongly connected then there's a directed connection between every vertex v and every other vertex, which implies that there exists a spanning v-branching for every vertex v. □

Problem 5. *Let G be a connected graph, let T_1 and T_2 be the edges sets of two spanning trees of G, and let $e \in T_1 \backslash T_2$. Show that:*
1) There exists $f \in T_2 \backslash T_1$ such that $(T_1 \backslash \{e\}) \cup \{f\}$ is a spanning tree of G.

Proof. Let a and b be the ends of e. We must find $f \in T_2 \backslash T_1$ such that there is a path from a to b, which doesn't go through e. Since T_2 is a tree, there exists a path, P, from a to b which lies in T_2. Note that all of P cannot also lie in T_1 because then it would form a cycle with e. Thus there exists some edge f with ends c and d which lies only in T_2. But since T_1 is spanning, there are paths from a to c and from b to d. These paths together with f form a path connecting a to b without using e. □

2) There exists $f \in T_2 \backslash T_1$ such that $(T_2 \backslash \{f\}) \cup \{e\}$ is a spanning tree of G.

Proof. Use the edge f from Part 1). Then there are paths from c to a and d to b which lie in T_2 and these paths together with e form a path from c to d without using f. □

Problem 6. *Let T be a spanning tree of a connected graph G. Show the following:*
1) The fundamental cycles of G with respect to T form a basis of its cycle space.

Proof. Let C be an even subgraph of G and let $S = C \cap \overline{T}$. We know that $C = \Delta\{C_e \mid e \in S\}$ and that this expresses C uniquely. Then every even subgraph can be generated through symmetric differences of fundamental cycles and so these form a basis of the cycle space. □

2) The fundamental bonds of G with respect to T form a basis of its bond space.

Proof. A similar proof as in Part 1) shows that every edge cut can be expressed uniquely as a symmetric difference of fundamental bonds. This shows that these form a basis of the bond space. □

3) Determine the dimensions of these two spaces.

Proof. The dimension of a finite dimensional vector space is equal to the number of vectors in its basis. In this case, for every edge e in $\overline{T}$ there is a unique path through T which connects the ends. This shows that the number of fundamental cycles is number of edges in $\overline{T}$ and so $|\overline{T}|$ is the dimension of the cycle space. Each fundamental bond is created from one edge of T, and so the dimension of the bond space is $|T|$. □