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Homework 5

Problem 1. *Construct a Δ-complex structure on $\mathbb{R}P^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$.*

Proof. To make the described Δ-complex structure on S^n we first take the $2n$ vertices that are distance 1 away from the origin in $\mathbb{R}^n$ on the coordinate axes. For each of these vertices, attach a 1-simplex between this vertex and the $2n-2$ vertices not lying on the same axis. Now attach 2-simplexes between any three vertices not lying in a plane determined by two coordinate axes. Next attach 3-simplexes between any four vertices not lying in a 3-dimensional subspace determined by the coordinate axes. Continue in this way until all points $(x_1, \ldots, x_{n+1})$ with $\sum_i x_i = 1$ are included in our complex. This space, which is composed of 2^{n+1} n-simplexes, is then homeomorphic to S^n. To get $\mathbb{R}P^n$ we need to identify antipodal points. For each n-simplex in our structure, there is another one reflected about the origin. Identify these two simplexes (after doing a reflection) so that antipodal points of our space are identified. Then this is a Δ-complex structure on $\mathbb{R}P^n$. $\square$

Problem 2. *Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.*

Proof. We can view the Klein bottle as a union of two 2-simplexes U and L with sides a, b and c and a vertex v as follows. Then our chain complex $\Delta_2 \to \Delta_1 \to \Delta_0 \to 0$ is $\mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$. We also have $\partial_2 : U \mapsto b-c+a$ and $\partial_2 : L \mapsto a-b+c$ while $\partial_1 : a \mapsto v-v$, $\partial_1 : b \mapsto v-v$ and $\partial_1 : c \mapsto v-v$. Since the images of U and L are distinct under ∂_2, it must be injective and we see that $H_2(X) = \ker \partial_2 / \operatorname{im} \partial_3 = 0$. Since ∂_1 maps all elements to 0 we have $\ker \partial_1 = \mathbb{Z}^3\{a, b, c\} = \mathbb{Z}^3\{a, a+b-c, c\}$. Also $\operatorname{im} \partial_2 = \mathbb{Z}^2\{a+b-c, a-b+c\} = \mathbb{Z}^2\{a+b-c, 2a\}$. Thus $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Finally, $\ker \partial_0 = \mathbb{Z}\{v\}$ and $\operatorname{im} \partial_1 = 0$ so $H_0(X) = \mathbb{Z}$. $\square$

Problem 3. *Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \ldots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}$, $\mathbb{Z}_n$, 0, $\mathbb{Z}$ respectively.*

Proof. Note that all the outer vertices are identified with each other in the first step, and the middle vertices are identified in the second step, so there are only two 0-simplexes, v_0 and v_1. Call the outer vertex v_0 and the inner vertex v_1. Each tetrahedron has 6 edges, but the outer edges are identified in the first step as is the middle edge, so there are 4 left for each T_i. Two of these get paired off in the first step, and two more get paired off when the bottom faces are identified with the top faces. This leaves n edges plus the outer and middle edges for a total of $n+2$ 1-simplexes. Each tetrahedron has four faces, but these are paired off in the first step and then again in the second step so we're left with $2n$ 2-simplexes. There are n 3-simplexes. We thus have the following chain complex

$$0 \xrightarrow{\partial_4} \mathbb{Z}^n \xrightarrow{\partial_3} \mathbb{Z}^{2n} \xrightarrow{\partial_2} \mathbb{Z}^{n+2} \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0\ .$$

Note that each 1-simplex either connects v_0 to v_1 or connects v_0 or v_1 to itself. Thus ∂_1 takes each 1 cell either to 0 or to $v_1 - v_0$ so $\operatorname{im}\partial_1(X) = \mathbb{Z}\{v_1 - v_0\}$ and $H_0(X) = \ker\partial_0/\operatorname{im}\partial_1 = \mathbb{Z}^2/\mathbb{Z} \approx \mathbb{Z}$.

Number the T_i in a counterclockwise fashion and label the face on the bottom of T_i f_i and the face on the right side of T_i f_{n+i} so that the bottom and top faces are labeled f_1 through f_n and the vertical faces are labeled f_{n+1} through f_{2n}. Label the outer edge d_1 and the inner edge d_2. Label the bottom edge of f_{n+i} e_i. Using the labeling in the diagram we see that for $1 \le i \le n$ we have $\partial_2(f_i) = e_i - e_{i-1} + d_1$ and $\partial_2(f_{n+i}) = d_2 - e_{i-1} + e_i$ where $e_0 = e_n$. Order the edges as $d_1, e_1, e_2, \ldots, e_n, d_2$. We can take the coefficients from the images of ∂_2 and express them as the rows in the following $2n \times (n+2)$ matrix

$$\begin{pmatrix}
1 & 1 & 0 & 0 & \ldots & 0 & -1 & 0 \\
1 & -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\
 & & & & \vdots & & & \\
1 & 0 & 0 & 0 & \ldots & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & -1 & 1 \\
0 & -1 & 1 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & \ldots & 0 & 0 & 1 \\
 & & & & \vdots & & & \\
0 & 0 & 0 & 0 & \ldots & -1 & 1 & 1
\end{pmatrix}.$$

For $1 \le i \le n$ we can subtract row i from row $n+1$ (namely, replace a generator in the image with that generator plus another) and note that the last n rows are the same. This leaves the following $(n+1)\times(n+2)$ matrix

$$\begin{pmatrix}
1 & 1 & 0 & 0 & \ldots & 0 & -1 & 0 \\
1 & -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\
 & & & & \vdots & & & \\
1 & 0 & 0 & 0 & \ldots & -1 & 1 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}.$$

Now for $1 \le i \le n$ we can add the first $i-1$ rows to the i^{th} row as

$$\begin{pmatrix}
1 & 1 & 0 & 0 & \ldots & 0 & -1 & 0 \\
2 & 0 & 1 & 0 & \ldots & 0 & -1 & 0 \\
3 & 0 & 0 & 1 & \ldots & 0 & -1 & 0 \\
 & & & & \vdots & & & \\
n & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}.$$

This means that the image is generated by the $n+1$ elements $id_1+e_i-e_n$, nd_1 and d_2-d_1 for $1 \le i \le n-1$. On the other hand, ∂_1 takes d_1 and d_2 to 0 while it takes e_i to v_1-v_0. Thus $\ker\partial_1$ is generated by d_1, d_2 and all the differences $e_i - e_j$ for $1 \le i < j \le n$. We can express these last generators as $e_i - e_n$ for $1 \le i \le n-1$. Now if we add d_1 to each generator a particular number of times we get the set of generators $\{d_1, d_2 - d_1, id_1 + e_i - e_n\}$. Comparing this to the generators for $\operatorname{im} \partial_2$ we see that $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}/n\mathbb{Z}$.

Let $a_1 f_1 + \cdots + a_{2n} f_{2n} \in \ker \partial_2$. From the first matrix above representing the image of ∂_2 we see that we must have $a_1 + \cdots + a_n = a_{n+1} + \cdots + a_{2n} = 0$ and for $1 < i \le n$ we have $a_i + a_{n+i} = a_{i-1} + a_{n+i-1}$ and $a_1 + a_{n+1} = a_n + a_{2n}$. Then $a_1 + a_{n+1} = a_2 + a_{n+2} = \cdots = a_n + a_{2n}$. Since the sum of these terms is 0 we must have $a_i = -a_{n+i}$ for $1 \le i \le n$. Now consider

$$\begin{aligned} \partial_3(b_1 T_1 + \cdots + b_n T_n) &= b_1(f_{n+1} - f_{2n} + f_n - f_1) + \cdots + b_n(f_{2n} - f_{2n-1} + f_{n-1} - f_n) \\ &= (b_2 - b_1) f_1 + (b_3 - b_2) f_2 + \cdots + (b_1 - b_n) f_n \\ &\quad + (b_1 - b_2) f_{n+1} + (b_2 - b_3) f_{n+2} + \cdots + (b_n - b_1) f_{2n}. \end{aligned}$$

Fix b_1 as any integer. Pick b_2 such that $b_2 - b_1 = a_1$. This determines b_2 and in a similar fashion we can determine b_i by picking it such that $b_i - b_{i-1} = a_{i-1}$. We need to make sure that $b_n - b_1 = a_n$. Note $b_n - b_1 = -((b_2 - b_1) + (b_3 - b_2) + \cdots + (b_n - b_{n-1})) = -(a_1 + \cdots + a_{n-1}) = a_n$. Thus any element of $\ker \partial_2$ is also in $\operatorname{im} \partial_3$, $\ker \partial_2 = \operatorname{im} \partial_3$ and $H_2(X) = 0$.

For $1 < i \le n$ we have $\partial_3(T_i) = f_{n+i} - f_{n+i-1} + f_{i-1} - f_i$ and $\partial_3(T_1) = f_{n+1} - f_{2n} + f_n - f_1$. Any 2-simplex f_k appearing in the image of ∂_3 belongs to two neighboring 3-simplexes, T_i and T_{i+1}. But the coefficient of f_k in $\partial_3(T_i)$ and $\partial_3(T_{i+1})$ have opposite sign so they cancel out. Therefore if $a_1 T_1 + \cdots + a_n T_n \in \ker \partial_3$ then $a_1 = a_2 = \cdots = a_n$ so $\ker \partial_3 = \mathbb{Z}$ and $H_3(X) = \mathbb{Z}$. $\square$

Problem 4. *Show that a chain homotopy of chain maps is an equivalence relation.*

Proof. Let (C_*, ∂_*) and (D_*, ∂'_*) be chain complexes. If $f_*, g_* : C_* \to D_*$ are chain maps we will write $f \sim g$ if f and g are chain homotopic, that is, if there are maps $P_n : C_n \to D_{n+1}$ with $\partial_{n+1} P_n + P_{n-1} \partial'_n = g_n - f_n$. If P is the 0 map then $f \sim f$. If P is such a map that $f \sim g$ then $-P$ is a map giving $g \sim f$ since $f_n - g_n = -(g_n - f_n) = -(\partial_{n+1} P_n + P_{n-1} \partial'_n) = \partial_{n+1}(-P_n) + (-P_{n-1}) \partial'_n$. Finally if $f \sim g$ using P and $g \sim h$ using Q then

$$h_n - f_n = (h_n - g_n) + (g_n - f_n) = \partial_{n+1} Q_n + Q_{n-1} \partial'_n + \partial_{n+1} P_n + P_{n-1} \partial'_n = \partial_{n+1}(Q_n + P_n) + (Q_{n-1} + P_{n-1}) \partial'_n$$

and $f \sim h$ using $Q + P$. Thus a chain homotopy is an equivalence relation. $\square$

Problem 5. *Determine whether there exists a short exact sequence* $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$*. More generally, determine which abelian groups A fit into a short exact sequence* $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ *with p prime. What about the case of short exact sequences* $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$*?*

Proof. Suppose $0 \longrightarrow \mathbb{Z}_{p^m} \xrightarrow{\varphi} A \xrightarrow{\psi} \mathbb{Z}_{p^n} \longrightarrow 0$ is an exact sequence. Suppose A is infinite. Then ψ must map infinitely many elements to the identity in $\mathbb{Z}_{p^n}$. Since $\ker \psi = \operatorname{im} \varphi$, it follows that $\operatorname{im} \varphi$ is infinite, but this is a contradiction since $\mathbb{Z}_{p^m}$ is finite. Thus A must be finite, and since A is abelian we know $A = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ where $n_1 \mid n_2 \mid \cdots \mid n_k$. Furthermore, by Lagrange's Theorem we know $|A| = |\mathbb{Z}_{p^n}||\mathbb{Z}_{p^m}| = p^{n+m}$. Thus $n_i \mid p^{n+m}$ and so $n_i = p^{\alpha_i}$. We know φ maps a generator of $\mathbb{Z}_{p^m}$ to an element of order p^m in A and this element and its powers make up the entire kernel of ψ. Then $n_j \ge m$ for some j and the other n_i add up to n. But then if we have more than two summands the only possible element of A with order p^n is in the kernel of ψ, thus ψ cannot be surjective. This is a contradiction so we must only have two components. Hence $A = \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}}$ where $m \le \alpha_1$ and $\alpha_1 + \alpha_2 = m + n$. The map φ takes a generator of $\mathbb{Z}_{p^m}$ to the element $(p^{\alpha_1 - m}, 1)$. In particular, $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_{\not\geq} \to \mathbb{Z}_4 \to 0$ is a short exact sequence given by the map which takes a generator of $\mathbb{Z}_4$ to $(2,1)$ and then mapping $(1,1)$ to a generator of $\mathbb{Z}_4$.

Now suppose we have the exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\varphi} A \xrightarrow{\psi} \mathbb{Z}_n \longrightarrow 0$. Note that A must have a $\mathbb{Z}^r$ component with $r > 0$ so that φ maps $\mathbb{Z}$ into one component of this. But also, if $r > 1$ then more than one copy of $\mathbb{Z}$ will be mapped by ψ to the identity in $\mathbb{Z}_n$. Since there is no injection $\mathbb{Z} \to \mathbb{Z}^r$ for $r > 1$, we must have $r = 1$. By a similar argument as above, $A = \mathbb{Z} \oplus \mathbb{Z}_m$ for some $m \geq n$. Since ψ is a homomorphism from $\mathbb{Z}_m \to \mathbb{Z}_n$ it must be the case that $n \mid m$. In this case $\varphi : 1 \mapsto (1, n)$ so that the element $(1, 1)$ has order n under ψ. $\square$

Problem 6. *Suppose we have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "p"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r, "j"] & Y \arrow[r, "q"] & Z \arrow[r] & 0
\end{tikzcd}$$

of abelian groups where the horizontal sequences are exact. Show that we get a long exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

Proof. Let's label the maps in question as follows

$$0 \longrightarrow \ker(f) \xrightarrow{\alpha} \ker(g) \xrightarrow{\beta} \ker(h) \xrightarrow{\gamma} \operatorname{coker}(f) \xrightarrow{\delta} \operatorname{coker}(g) \xrightarrow{\varepsilon} \operatorname{coker}(h) \longrightarrow 0.$$

Pick $a \in \ker(f)$ so that $f(a) = 0$ in X. Then $jf(a) = 0 = gi(a)$. Thus $i(a) \in \ker(g)$ so we get a map $\alpha : \ker(f) \to \ker(g)$ given by $\alpha(a) = i(a)$. Now let $b \in \ker(g)$ so that $g(b) = 0$ in Y. Then $qg(b) = 0 = hp(b)$ so $p(b) \in \ker(h)$. We get a map $\beta : \ker(g) \to \ker(h)$ by $\beta(b) = p(b)$. Let $x + f(A) \in \operatorname{coker}(f)$ and apply j to get $j(x) + jf(A) = j(x) + gi(A) \in \operatorname{coker}(g)$. Thus we get a map $\delta : \operatorname{coker}(f) \to \operatorname{coker}(g)$ as $\delta(x + f(A)) = j(x + f(A))$. Finally let $y + g(B) \in \operatorname{coker}(g)$ and apply q to get an element of $\operatorname{coker}(h)$. This gives the map $\varepsilon : \operatorname{coker}(g) \to \operatorname{coker}(h)$ given by $\varepsilon(y + g(B)) = q(y + g(B))$.

Pick $c \in \ker(h)$. Since p is surjective, there exists $b \in B$ with $p(b) = c$. Then $q(g(b)) = h(p(b)) = h(c) = 0$ since $c \in \ker(h)$. Thus $g(b) \in \ker(q)$ and $\ker(q) = \operatorname{im}(j)$. Find $x \in X$ such that $j(x) = g(b)$ and note that x is unique since j is injective. Now define $\gamma(c) = x + f(A)$. Suppose now we chose some $b' \in B$ also with $p(b') = c$. We would then get some other element x' such that $j(x') = g(b')$. Note though that $p(b - b') = c - c = 0$ so $b - b' \in \ker(p)$ and $\ker(p) = \operatorname{im}(i)$. Pick $a \in A$ so that $i(a) = b - b'$. Then $j(f(a)) = g(i(a)) = g(b - b') = g(b) - g(b') = j(x) - j(x') = j(x - x')$. Since j is injective $f(a) = x - x'$ so $x - x' \in f(A)$. This shows that we still get the same element $x + f(A) = \gamma(c)$ so that γ is well-defined.

Suppose now we pick another element $c' \in \ker(h)$ so that $\gamma(c') = x' + f(A)$. Suppose that $b, b' \in B$ with $p(b) = c$ and $p(b') = c'$. From the definition of γ we know $j(x) = g(b)$ and $j(x') = g(b')$. Then $j(x+x') = g(b)+g(b') = g(b+b')$ and also $p(b+b') = p(b)+p(b') = c+c'$. This means $\gamma(c+c') = (x+x')+f(A)$. Then $\gamma(c) + \gamma(c') = (x + f(A)) + (x' + f(A)) = (x + x') + f(A) = \gamma(c + c')$ so γ is a homomorphism.

Now let $a \in \ker(\alpha)$. Then $\alpha(a) = i(a) = 0$. Since i is injective, $a = 0$. Take $b \in \operatorname{im}(\alpha)$ so $b = i(a)$ for some $a \in \ker(f)$. Then $\beta(b) = \beta(i(a)) = p(i(a))$ and since the top sequence is exact, we get $p(i(a)) = \beta(b) = 0$ and $b \in \ker(\beta)$. Thus $\operatorname{im}(\alpha) = \ker(\beta)$. This shows the long sequence is exact at $\ker(f)$ and $\ker(g)$.

Let $c \in \operatorname{im}(\beta)$ so $c = p(b)$ for some $b \in \ker(g)$. From above we know $g(b) \in \ker(q)$ and $\ker(q) = \operatorname{im}(j)$ so we can find $x \in X$ such that $j(x) = g(b) = 0$. But then $x = 0$ since j is injective so $\gamma(c) = 0 + f(A) = 0$. Conversely, let $c \in \ker(\gamma)$ so that $\gamma(c) \in f(A)$. Then we can find $a \in A$ such that $f(a) = \gamma(c)$. Note that $\gamma(c) = x$ with $j(x) = g(b)$ for $b \in B$ with $p(b) = c$. Since $f(a) = x$ we have $gi(a) = jf(a) = j(x) = g(b)$ so $b = i(a)$. Applying p we see that $c \in \operatorname{im}(\beta)$ so that $\operatorname{im}(\beta) = \ker(\gamma)$ and the sequence is exact at $\ker(h)$.

Now let $x + f(A) \in \operatorname{im}(\gamma)$ so that $\gamma(c) = x + f(A)$. Note that $x \in X$ such that $j(x) = g(b)$ for some $b \in B$ with $p(b) = c$. Then $j(x + f(A)) = j(x) + jf(A) = g(b) + jf(A) = g(b) + gi(A) = 0$. So $x + f(A) \in \ker(\delta)$. Conversely, suppose $x + f(A) \in \ker(\delta)$ so that $j(x) + jf(A) = j(x) + gi(A) = 0$. This is the same as saying $j(x) = g(b)$ for some $b \in B$. Since p is surjective, $p(b) = c$ and it follows that $\gamma(c) = x + f(A)$. Thus $x + f(A) \in \operatorname{im}(\gamma)$ and $\operatorname{im}(\gamma) = \ker(\delta)$. This shows that the sequence is exact at $\operatorname{coker}(f)$.

Let $y + g(B) \in \operatorname{im}(\delta)$ so $y + g(B) = j(x + f(A))$ for some $x + f(A) \in \operatorname{coker}(f)$. Then $\varepsilon(y + g(B)) = q(y + g(B)) = q(j(x + f(A)) = 0$ since the bottom sequence is exact. Thus $y + g(B) \in \ker(\varepsilon)$ and $\operatorname{im}(\delta) = \ker(\varepsilon)$. Finally pick some element $z + h(C) \in \operatorname{coker}(h)$. Note that since q is surjective, $z + h(C) = q(y + g(B))$ for some $y + g(B) \in \operatorname{coker}(g)$. Since $\varepsilon(y + g(B)) = q(y + g(B))$ we see that ε is surjective. This finally shows that the sequence is exact at $\operatorname{coker}(g)$ and $\operatorname{coker}(h)$. $\square$