Kris Harper
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Homework 7

Problem 1. *Find the periodic solutions of the system*

$$\dot{x}_1 = -x_2 + x_1 f(r), \quad \dot{x}_2 = x_1 + x_2 f(r)$$

where $r^2 = x_1^2 + x_2^2$ *and* $f(r) = -r(1-r^2)(4-r^2)$.

We can switch to polar coordinates labeling $x_1 = x$ and $x_2 = y$. Then we're need to find $\dot{r}$ and $\dot{\theta}$ where $r = x^2 + y^2$ and $\tan(\theta) = x/y$. Differentiating the equation for r, we have $r\dot{r} = x\dot{x} + y\dot{y}$. Putting in the given equations for $\dot{x} = \dot{x_1}$ and $\dot{y} = \dot{x_2}$, we have $\dot{r} = f(r) = -r(1-r^2)(4-r^2)$. Similarly, differentiating the equation for θ gives $\sec^2(\theta)\dot{\theta} = (x^2+y^2)/x^2 = 1 + \tan^2(\theta)$. Multiplying by $\sec^2(\theta)$ gives $\dot{\theta} = \cos^2(\theta) + \sin^2(\theta) = 1$.

Since $\dot{\theta}$ is never 0, the only fixed point of this system can be at 0, and it's easy to see that $x_1 = x_2 = 0$ gives a fixed point. The periodic orbits will then be points where $\dot{r} = f(r) = 0$. Looking at $f(r)$, we have $r = 2$ and $r = 1$ give $f(r) = 0$, so the periodic orbits of the system are the circles centered at the origin with radii 1 and 2.

Problem 2. *Consider the* nonautonomous*, periodic system*

$$\dot{x} = f(x,t), \quad f(x, t+T) = f(x,t).$$

Let $x(t)$ *be a solution such that, at some time* t_1, $x(t_1) = x(t_1 + T)$. *Show that this solution is periodic with period* T.

Proof. Define $\widetilde{x}(t) = x(t+T)$. Then $\dot{\widetilde{x}} = \dot{x} = f(x,t)$ from the chain rule. Further $\widetilde{x}(t_1) = x(t_1+T) = x(t_1)$. So $\widetilde{x}(t)$ is a solution that satisfies the initial condition at t_1. By uniqueness, $x(t) = \widetilde{x}(t) = x(t+T)$. So $x(t)$ is periodic. □

Problem 3. *Consider the planar autonomous system*

$$\frac{dx}{dt} = f(x), \quad x \in \Omega,\ \Omega \subseteq \mathbb{R}^2$$

and suppose

$$\text{div } f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

has one sign in Ω. *Show that this system can have no periodic orbits other than equilibrium points.*

Proof. Suppose $x(t)$ is a periodic solution. If $x(t)$ is not an equilibrium point, then $x(t)$ traces out a simple closed curve, C in the plane. Note that f is defined as the component-wise derivative of x, so f will always point tangential to $x(t)$. If $\hat{\mathbf{n}}$ is the normal vector to $x(t)$ at any time t, then $f \cdot \hat{\mathbf{n}} = 0$ since these two vectors are perpendicular. This means that

$$\oint_C f \cdot \hat{\mathbf{n}} ds = 0.$$

But note that by the divergence theorem

$$\oint_C f \cdot \hat{\mathbf{n}} ds = \iint_D (\text{div } f) dA$$

where D is the area enclosed by C. But this integral cannot be nonzero if div f has only one sign on D. Thus $x(t)$ cannot be a periodic solution. □

Problem 4. *Consider the gradient system*

$$\frac{dx}{dt} = \nabla\phi, \quad x \in \Omega,\, \Omega \subseteq \mathbb{R}^n,$$

where $\phi(x)$ is a smooth, single-valued function. Draw the same conclusion as in the preceding problem.

Proof. Suppose $x(t)$ is a periodic solution. If $x(t)$ is not an equilibrium point then $x(t)$ traces out a simple closed curve C in the plane. Let $\mathbf{v} = \nabla\phi$. By definition, $\mathbf{v}$ is a conservative vector field, so we must have

$$\oint_C \mathbf{v} \cdot \hat{\mathbf{n}} ds = 0.$$

From the divergence theorem we know that

$$\oint_C \mathbf{v} \cdot \hat{\mathbf{n}} ds = \iint_D (\nabla \cdot \mathbf{v}) dA = 0.$$

But note that C could be any curve here, so we must have $\nabla \cdot \mathbf{v} = \nabla^2\phi = 0$. Thus ϕ is a harmonic function, which means that on any compact set, ϕ takes its maximum and minimum on the boundary. Consider the compact set D given by the C unioned with its interior. Note that $\nabla\phi$ is defined as the component-wise derivative of $x(t)$, so $\nabla\phi$ always points tangentially to $C = \partial D$. But this means that ϕ must be constantly 0 on C and therefore constantly 0 on D. Thus $x(t)$ is an equilibrium point, a contradiction. □

Problem 5. *Consider the system*

$$\dot{x} = xf(x, y), \quad \dot{y} = yg(x, y)$$

where f, g are arbitrary, smooth functions defined in $\mathbb{R}^2$. Show that the lines $x = 0$ and $y = 0$ are invariant curves for this system. Infer that each of the four quadrants of the xy-plane is an invariant region for this system.

Proof. Let $p_0 = (x_0, 0)$ be some point on the line $y = 0$. Then note that $\dot{y} = 0$ at p_0. Therefore, the orbit $\gamma(p_0)$ for times $t > 0$ is entirely governed by $\dot{x}$. Since the derivative of the y-coordinate is 0 at p_0, the only path the orbit can take is in the x-direction. But then $\gamma(p_0)$ must stay on the line $y = 0$; any deviation would imply a nonzero y-derivative. The same can be said for times $t < 0$, so $\gamma(p_0)$ is entirely contained on the curve $y = 0$. A similar argument holds for the line $x = 0$ since $\dot{x} = 0$ there.

Now suppose we have an orbit containing a point p in one of the four quadrants. If this orbit is to leave this quadrant, then it must pass through some point on the lines $x = 0$ or $y = 0$. But we've already seen that these are invariant, so p must have not been in one of the quadrants to start with, a contradiction. Thus $\gamma(p)$ is entirely contained in the quadrant containing p and so each quadrant is an invariant set. □

Problem 6. *Show that the nonwandering set is closed and positively invariant.*

Proof. Let W be the nonwandering set and take a convergent sequence (p_n) in W with limit p. Let U be a neighborhood of p and let $T > 0$. Since p is the limit of (p_n), infinitely many points of (p_n) lie within U. Take $p_k \in U$ and note that since p_k is nonwandering, $\phi(t, x) \in U$ for some $x \in U$ and $t \geq T$. Therefore p is nonwandering. Thus every convergent sequence has a limit in W, so W must be closed.

Now take an orbit $\gamma(p)$ for some point $p \in W$. Take some time $t_0 > 0$ and let $q = \phi(t_0, p)$ be another point point $\gamma(p)$. Let U be a neighborhood of q and take $T > 0$. Take a neighborhood V and find some point $\phi(t_1, x) \in V$ for some $t_1 \geq T$. Now consider the point $\phi(t_1 + t_0, x)$. Because ϕ is continuous, this point must be close to q. In particular, if U is contained inside a ball of radius ε, then we can enclose V in a ball of radius δ such that $||p - \phi(t_1, x)|| < \delta$ implies that $||q - \phi(t_1 + t_0, x)|| < \varepsilon$. Thus $\phi(t_1 + t_0, x) \in U$ and q must be in W as well. □