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MATH 16300
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April 15, 2008

Sheet 15: Series

Definition 1 *A series of real numbers is an expression* $\sum_{n=1}^{\infty} a_n$*, where* (a_n) *is a real sequence.*

Definition 2 (Convergent Series) *Let* $\sum_{n=1}^{\infty} a_n$ *be a series. The sequence of partial sums is defined as*

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ *converges to* s *(or* $\sum_{n=1}^{\infty} a_n = s$*) if* $\lim_{n\to\infty} s_n = s$*. If such an* s *exists, we say that* $\sum_{n=1}^{\infty} a_n$ *is convergent, otherwise it is divergent.*

Exercise 3 *Reformulate convergence using the Cauchy property.*

We say a series $\sum_{n=1}^{\infty} a_n$ is convergent if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have $|s_n - s_m| < \varepsilon$.

Lemma 4 *If* $\sum_{n=1}^{\infty} a_n$ *is a convergent series, the the sequence* (a_n) *converges to* 0*.*

Proof. Let $\sum_{n=1}^{\infty} a_n = s$. Then the sequence of partial sums (s_n) converges to s and (s_n) is a Cauchy sequence. Thus for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have $|s_n - s_m| < \varepsilon$. But note that $s_{n+1} - s_n = a_n$ so for $n > N + 1$ we have $|a_n| < \varepsilon$ which means $\lim_{n\to\infty} a_n = 0$. □

Lemma 5 *Let* $\sum_{n=1}^{\infty} a_n$ *be convergent with a partial sum sequence* (s_n)*. Let* $n_0 = 0$ *and* $n_1 < n_2 < \dots$ *be a sequence of natural numbers. For* $k \in \mathbb{N}$ *let*

$$b_k = a_{n_{k-1}+1} + \cdots + a_{n_k} = \sum_{i=n_{k-1}+1}^{n_k} a_i.$$

Then

$$\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n.$$

Proof. Let $s_{b_k} = \sum_{i=1}^{k} b_i$ and $s_{a_n} = \sum_{i=1}^{n} a_i$. Then note that

$$s_{b_k} = \sum_{i=1}^{k} b_i = \sum_{i=1}^{n_1} a_i + \sum_{i=n_1+1}^{n_2} a_i + \cdots + \sum_{i=n_{k-1}+1}^{n_k} a_i = s_{a_{n_k}}.$$

We know $\sum_{n=1}^{\infty} a_n$ is convergent so (s_{a_n}) converges. Also $(s_{a_{n_k}})$ is a subsequence of (s_{a_n}) so it converges as well (13.12). But $(s_{b_k}) = (s_{a_{n_k}})$ so $\lim_{n\to\infty} s_{b_k} = \lim_{n\to\infty} s_{a_{n_k}}$ which implies

$$\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n.$$

□

Theorem 6 (Geometric Series) *For all $t < |1|$, we have*

$$\sum_{n=0}^{\infty} t^n \frac{1}{1-t}.$$

Proof. Consider a partial sum of $\sum_{n=0}^{\infty} t^n$,

$$s_k = \sum_{n=0}^{\infty} t^n = 1 + t + \cdots + t^k = \frac{1-t^{k+1}}{1-t} = \frac{1}{1-t} - \frac{t^k}{1-t}.$$

But since $t < |1|$ we have $\lim_{k\to\infty} t^k/(1-t) = 0$. So then $\lim_{k\to\infty} s_k = 1/(1-t) + 0$ which means

$$\sum_{n=0}^{\infty} t^n \frac{1}{1-t}.$$

$\square$

Theorem 7 *The series $\sum_{n=1}^{\infty} 1/n$ is not convergent.*

Proof. Suppose that $\sum_{n=1}^{\infty} 1/n$ is convergent. Create a sequence (b_k) as in Lemma 5 such that

$$b_k = \sum_{i=n_{k-1}+1}^{n_k} \frac{1}{n}$$

where $n_k = 2^{k-1}$ for $k \in \mathbb{N}$ and $n_0 = 0$. Note that for $k \geq 2$, b_k has $2^{k-1} - 2^{k-2} = 2^{k-2}$ terms, the smallest of which is $1/2^{k-1}$. Thus, for all $k \geq 2$, $b_k \geq 2^{k-2}/2^{k-1} = 1/2$. Also $b_1 = \sum_{n=1}^{1} 1/n = 1$. So for all $k \in \mathbb{N}$ we have $b_k \geq 1/2$. But then there are infinitely many $k \in \mathbb{N}$ such that $b_k \notin (-1/2; 1/2)$ so $\lim_{k\to\infty} b_k \neq 0$. Thus, $\sum_{k=1}^{\infty} k_n$ is not convergent (15.4). But we know that $\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n$ which is a contradiction (15.5). Thus $\sum_{n=1}^{\infty} 1/n$ is not convergent. $\square$

Theorem 8 (Alternating Sign Series) *Let $\sum_{n=1}^{\infty} a_n$ be a series with the following properties: 1) a_n is positive if n is odd and negative if n is even; 2) $|a_{n+1}| < |a_n|$ for all n; 3) $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty} a_n$ is convergent.*

Proof. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n| < \varepsilon$. Let $n \in \mathbb{N}$ such that $n > N$ and n is even. Then $a_{n+1} > 0$. We have $s_{n+1} = s_n + a_{n+1} > s_n$. Also $a_{n+2} < 0$ and $|a_{n+2}| < |a_{n+1}|$ so $a_{n+1} + a_{n+2} > 0$. Then $s_{n+1} > s_{n+1} + a_{n+2} = s_{n+2} = s_n + a_{n+1} + a_{n+2} > s_n$. So for $n > N$ even we have $s_n \leq s_{n+2} \leq s_{n+1}$ and a similar proof shows that for $n > N$ odd we have $s_n \geq s_{n+2} \geq s_{n+1}$. Use strong induction on n to show that for $k + N$ even $s_N \leq s_{k+N} \leq s_{N+1}$. We see that for $k = 1$ we have $s_N \leq s_{N+1} \leq s_{N+1}$ which is true since a_{N+1} is positive. We've also shown the case for $k = 2$. Assume that for $n + N$ even we have $s_N \leq s_{N+n} \leq s_{N+1}$. Consider s_{N+n+2}. We know $s_{N+n} \leq s_{N+n+2} \leq s_{N+n+1}$ and $s_{N+n-1} \leq s_{N+n+1} \leq s_{N+n}$. Combining these three inequalities we have $s_N \leq s_{N+n+2} \leq s_{N+1}$. Thus for all even $N + n$ we have $s_N \leq s_{N+n} \leq s_{N+1}$. A similar proof holds to show that for odd $N + n$ we have $s_N \leq s_{N+n} \leq s_{N+1}$. Since this is true for any N given ε, for any region $(s_N; s_{N+1})$ there are finitely many n with s_n not in the region. Thus $\sum_{n=1}^{\infty} a_n$ is convergent. $\square$

Exercise 9 *The series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is convergent.

Proof. Note that for n odd we have $a_n = (-1)^{n+1}/n$ and since $n+1$ is even and $n > 0$ we have $a_n = 1/n > 0$. For n even $n+1$ is odd so $a_n = (-1)^{n+1}/n = -1/n < 0$. Also $|a_{n+1}| = 1/(n+1) < 1/n = |a_n|$. Finally we know that $\lim_{n\to\infty} a_n = 0$ (13.4). Since this series fulfills the requirements of Theorem 8, it must be convergent. $\square$

Definition 10 *A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

Lemma 11 *$\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if there exists $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$, $\sum_{n=1}^{N} |a_n| \leq C$.*

Proof. Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Let $s_k = \sum_{n=1}^{k} |a_n|$. Then (s_n) is convergent and therefore bounded (13.15). Thus there exists $C \in \mathbb{R}$ such that for all N we have $s_N = \sum n = 1^N |a_n| \leq C$.

Now suppose there exists $C \in \mathbb{R}$ such that $s_N \leq C$ for all N. Thus (s_n) is bounded. Note that $s_n = s_{n-1} + |a_n|$ and since $|a_n| \geq 0$ for all n we have (s_n) is an increasing sequence. Since (s_n) is bounded and increasing we know it is convergent (13.18). Thus $\sum_{n=1}^{\infty} |a_n|$ is convergent and so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. $\square$

Theorem 12 (Comparison Criterion) *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series. Suppose there is some N such that for all $n \geq N$ we have $|a_n| \leq |b_n|$. Then if $\sum_{n=1}^{\infty} b_n$ is absolutely convergent so is $\sum_{n=1}^{\infty} a_n$.*

Proof. For all $M \geq N$ note that

$$\sum_{n=N}^{M} |a_n| \leq \sum_{n=N}^{M} |b_n| \leq \sum_{n=1}^{M} \leq C$$

for some $C \in \mathbb{R}$ because every term in $(|b_n|)$ is greater than or equal to zero (15.11). Also note that

$$\sum_{n=1}^{M} |a_n| \leq C + \sum_{n=1}^{N-1} |a_n| \leq C'$$

for some $C' \in \mathbb{R}$ because every term of $(|a_n|)$ is greater than or equal to zero. Also note that for $M' < N \leq M$ we have

$$\sum_{n=1}^{M'} |a_n| \leq \sum_{n=1}^{M} \leq C'$$

so that for all M we have $\sum_{n=1}^{M} |a_n| \leq C'$. By Lemma 11 $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (15.11). $\square$

Corollary 13 (Quotient Criterion) *Let $\sum_{n=1}^{\infty} a_n$ be a series. Suppose that there is an $N \in \mathbb{N}$ and $0 < r < 1$, such that $|a_{n+1}/a_n| \leq r$ for all $n \geq N$. Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*

Proof. Use induction on n to show that $|a_{N+n}| \leq |a_N| r^n$. For the base case, $n = 1$ we have $|a_{N+1}| \leq |a_N| r$ by assumption. Assume that for all $n \in \mathbb{N}$ we have $|a_{N+n}| \leq |a_N| r^n$ so $|a_{N+n}| r \leq |a_N| r^{n+1}$. Then note that $|a_{N+n+1}| \leq |a_{N+n}| r \leq |a_N| r^{n+1}$ as desired. Thus for $n \geq N$ we have $|a_n| \leq |a_N| r^{n-N}$. Let $b_n = |a_N| r^{n-N}$. Then for $n > N$ we have $|a_n| \leq |a_N| r^{n-N} = |a_N r^{n-N}| = |b_n|$ since $r > 0$. But also

$$\sum_{n=1}^{\infty} |a_N r^{n-N}| = \sum_{n=0}^{\infty} |a_N| r^{n-N+1} = |a_N| r^{-N+1} \sum_{n=0}^{\infty} r^n$$

and so $\sum_{n=1}^{\infty} b_n$ is absolutely convergent by Theorem 6, because $r > 0$ and because $|a_N| r^{-N+1}$ is a constant value (15.6). Thus, by Theorem 12 we have $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. $\square$

Definition 14 *Let $\sum_{n=1}^{\infty} a_n$ be a series. A reordering of $\sum_{n=1}^{\infty} a_n$ is a series of the form $\sum_{n=1}^{\infty} b_n$, where $b_n = a_{f(n)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$.*

Lemma 15 *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, and let $\sum_{n=1}^{\infty} b_n$ be a reordering of it. Then for every $k \in \mathbb{N}$ there exists $L \in \mathbb{N}$ such that for all $l \geq L$,*

$$\left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{l} b_n\right| \leq \sum_{n=k+1}^{\infty} |a_n|.$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(x) = |x|$. We know that since g is continuous, for a sequence (a_n), if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} |a_n| = |a|$ (13.7). We have $\sum_{n=1}^{\infty} a_n$ is absolutely convergent so $|\sum_{n=1}^{\infty} a_n| = \lim_{n\to\infty} |s_n|$. Then use induction on n to show that $|s_n| \leq \sum_{k=1}^{n} |a_k|$. For $n = 1$ we have $|s_1| = |a_1| = \sum_{k=1}^{1} |a_1|$. Assume that for $n \in \mathbb{N}$, $\sum_{k=1}^{n} |a_k| \geq |s_n|$. Then

$$\sum_{k=1}^{n+1} |a_k| = \sum_{k=1}^{n} |a_k| + |a_{n+1}| \geq |s_n| + |a_{n+1}| \geq |s_n + a_{n+1}| = |s_{n+1}|$$

by the triangle inequality and our inductive hypothesis (9.36). Therefore we have

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Let $k \in \mathbb{N}$ and consider the sets $A = \{a_n \mid n \leq k\}$ and $S = \{f(n) \mid n \leq k\}$ Let $L = \sup S$. Consider $l \geq L$ and let $B = \{b_n \mid n \leq l\}$ and $T = \{n \mid b_n \in B\}$. Note that every element of B is in A because $L \geq k$. Finally let $C = \{a_n \mid n \notin T\}$. Make a new sequence c_n where n is the nth element of C. Note that by definition, $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{l} b_n$. Then

$$\left|\sum_{n=1}^{\infty} c_n\right| = \left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{l} b_n\right| \leq \sum_{n=1}^{\infty} |c_n| \leq \sum_{k+1}^{\infty} |a_n|.$$

The last inequality holds because (c_n) is the sequence (a_n), but with at least k terms missing. □

Theorem 16 (Abel Resummation Theorem) *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, and let $\sum_{n=1}^{\infty} b_n$ be a reordering of it. Then $\sum_{n=1}^{\infty} b_n$ absolutely convergent and*

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

Proof. Let $k \in \mathbb{N}$ and consider the sets $A = \{a_n \mid n \leq k\}$ and $S = \{f(n) \mid n \leq k\}$ Let $L = \sup S$ and let $B = \{b_n \mid n \leq L\}$. Note that every element of B is in A because $L \geq k$. But then

$$\sum_{n=1}^{L} |b_n| = \sum_{n=1}^{L} |a_n| \leq C$$

for some $C \in \mathbb{R}$ (15.11). Since f is a bijection, L can be any value of $\mathbb{N}$, so every partial sum of $\sum_{n=1}^{\infty} |b_n|$ is bounded and thus $\sum_{n=1}^{\infty} b_n$ is absolutely convergent (15.11). Now consider $\sum_{n=k+1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |a_n| - \sum_{n=1}^{k} |a_n|$ (15.5). Take the limit as k goes to infinity. We have

$$\lim_{k\to\infty} \sum_{n=k+1}^{\infty} |a_n| = \lim_{k\to\infty} \left(\sum_{n=1}^{\infty} |a_n| - \sum_{n=1}^{k} |a_n|\right) = \sum_{n=1}^{\infty} |a_n| - \lim_{k\to\infty} s_k = 0.$$

But then we have

$$\lim_{l\to\infty}\left|\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{l}b_n\right|=\left|\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n\right|\le\lim_{n\to\infty}\sum_{n=k+1}^{\infty}|a_n|=0\ (15.15).$$

Thus,

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}a_n.$$

□

Theorem 17 *Let $\sum_{n=1}^{\infty}a_n$ be a convergent, but not absolutely convergent series. Then for all $c\in\mathbb{R}$ there exists a reordering $\sum_{n=1}^{\infty}b_n$ of $\sum_{n=1}^{\infty}a_n$ such that*

$$\sum_{n=1}^{\infty}b_n=c.$$

Proof. Let $A=\{a_n \mid n\in\mathbb{N}\}$. Then A is nonempty and bounded, so $\sup A$ exists (6. 11, 13.15). Suppose that for any positive term of (a_n) there are infinitely many terms greater than or equal to it. Consider some term $a_k>0$ and the region $(-a_k;a_k)$. Then there are infinitely many terms of (a_n) which are not in $(-a_k;a_k)$. But then (a_n) does not converge to zero which means $\sum_{n=1}^{\infty}a_n$ is not convergent (13.4). This is a contradiction and so for all positive terms of (a_n) there are finitely many terms greater than or equal to it. A similar proof holds to show that for a negative term of (a_n), there are finitely many terms less than or equal to it.

We have $\sum_{n=1}^{\infty}a_n=a$ for some $a\in\mathbb{R}$. Assume that $a_n=0$ for finitely many n. We can order the positive elements of (a_n) in decreasing order and the negative elements of (a_n) in increasing order because there are finitely many positive or negative terms of (a_n) greater than or less than any given term respectively. Define (x_k) where x_k is the kth positive element of (a_n) and (y_k) where y_k is the kth negative element of (a_n). Then for all $k\in\mathbb{N}$ we have $y_k<0\le x_k$. Suppose there are finitely many negative terms of (a_n). Then there exists a largest element, j, of N so that

$$\sum_{k=1}^{j}y_k=q\text{ and }\sum_{k=1}^{j}|y_k|=-q$$

for some $q\in\mathbb{R}$ because $y_k<0$ for all k. Then we have

$$\sum_{n=1}^{\infty}a_n=\sum_{k=1}^{\infty}x_k+\sum_{k=1}^{j}y_k\text{ and so }\sum_{k=1}^{\infty}x_n=\sum_{k=1}^{\infty}|x_k|=a-q.$$

This follows from Lemma 5. But then

$$(a-q)+q=\sum_{k=p_1}^{\infty}|x_k|+\sum_{k=n_1}^{n_j}|y_k|=\sum_{n=1}^{\infty}|a_n|$$

which means $\sum_{n=1}^{\infty}a_n$ is absolutely convergent which is a contradiction. Thus there are infinitely many terms of (y_k) and a similar proof shows there are infinitely many terms of (x_k).

Let $c\in\mathbb{R}$. Now suppose that for all $j\in\mathbb{N}$ we have $\sum_{k=1}^{j}x_k\le c$. Since $x_k>0$ for all k, we have the partial sums of $\sum_{k=1}^{\infty}x_k$ are bounded and increasing so it must converge to x for some $x\in\mathbb{R}$ (13.18). Suppose that $\sum_{k=1}^{\infty}|y_k|=y$ for some $y\in\mathbb{R}$. Then $\sum_{n=1}^{\infty}|a_n|=\sum_{k=1}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=x+y$ which is a contradiction (15.16). Thus $\sum_{k=1}^{\infty} y_k$ is not absolutely convergent so there exists $l \in \mathbb{N}$ such that $\sum_{k=1}^{l} |y_k| > c$ (15.11). But since $y_k < 0$ for all k we have $-c < \sum_{k=1}^{l} y_k$.

Now consider the sequence (a'_n) where $a'_n = a_n$ if $a_n < 0$ and 0 if $a_n \geq 0$. Then a partial sum of

$$\sum_{n=1}^{\infty} a'_n \text{ is } s_{a'_n} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n'} x_k$$

supposing there are n' positive terms in the first n terms of (a_n). Then if we consider $\lim_{n\to\infty} s_{a'_n}$ we simply have $a - x$ since n' will go to ∞ as n does. Hence

$$\sum_{n=1}^{\infty} a'_n = \sum_{k=1}^{\infty} y_k + 0 = a - x.$$

Thus $\sum_{k=1}^{\infty} y_k$ is convergent, but we just showed that the partial sums of this series are unbounded which is a contradiction (13.15). Thus, for $c \in \mathbb{R}$ there exists $j \in \mathbb{N}$ such that $\sum_{k=1}^{j} x_k > c$. A similar proof shows that for $c \in \mathbb{R}$ there exists $j \in \mathbb{N}$ such that $\sum_{k=1}^{j} y_k < c$

Define a reordering of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ where the first n_1 terms of b_n are the least number of terms of (x_k) such that $\sum_{k=1}^{n_1} x_k > c$. Then let the next n_2 terms be the least number of terms of (y_k) such that $\sum_{k=1}^{n_1} x_k + \sum_{k=1}^{n_2} y_k < c$. Note that we can always do this because the partial sums of

$$\sum_{k=1}^{\infty} x_k \text{ and } \sum_{k=1}^{\infty} y_k$$

are unbounded. Then for odd $i \in \mathbb{N}$, n_i is the least number of terms of (x_k) such that

$$\sum_{n=1}^{n_i} b_n = \sum_{k=1}^{n_i} x_k + \sum_{k=1}^{n_{i-1}} y_k > c$$

and for even i, n_i is the least number of terms of (y_k) such that

$$\sum_{n=1}^{n_i} b_n = \sum_{k=1}^{n_{i-1}} + \sum_{k=1}^{n_i} y_k < c.$$

Let $s_k = \sum_{n=1}^{k} b_n$. Note that s_k for k between n_i and n_{i+1} for $i \in \mathbb{N}$ is between s_{n_i} and $s_{n_{i+1}}$ because the terms of b_n change sign at n_i. Consider some region $(p; q)$ such that $c \in (p; q)$. Since the least number of elements of (y_k) are added to $s_{n_{i-1}}$ so that $s_{n_i} < c$, we have $|c - s_{n_i}|$ is always less than or equal to the absolute value of some element of (y_k). Suppose that $p > s_{n_i}$ for an infinite number of odd i. Then $|c - p|$ is less than or equal to an infinite number of absolute values of terms of (y_k). But then if we consider some $|y_k| > |c - p|$ there are an infinite number of n such that $|y_n| > |y_k|$. This is a contradiction and so $p > s_{n_i}$ for finitely many odd i. But also for all s_{n_i} with odd i there are finitely many s_k such that $i < k < i + 1$ because the positive and negative partial sums are unbounded. Thus there are finitely many n such that $s_n < p$. A similar proof shows that there are finitely many n with $s_n > q$ so there are finitely many n with $s_n \notin (p; q)$. Therefore $\lim_{n\to\infty} s_n = c$ and so

$$\sum_{n=1}^{\infty} b_n = c.$$

If there are infinitely many n such that $a_n = 0$ the change b_n so that a zero term is added to each n_ith partial sum. This will not change the resulting series convergence. $\square$