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Homework 7

Problem 1. *Let G and H be any compact topological groups. Let V be an irreducible (continuous) G-representation and let W be an irreducible H-representation.*
(a) Prove that $V \otimes W$ is an irreducible representation of $G \times H$.
(b) Prove that every irreducible representation of $G \times H$ is a tensor product of the above form.

Proof. (a) We have

$$\begin{aligned}
\langle \chi_{V\otimes W}, \chi_{V\otimes W} \rangle &= \int_{G\times H} \chi_{V\otimes W}((g,h))\overline{\chi_{V\otimes W}}((g,h))d(g,h) \\
&= \int_{G\times H} \chi_V(g)\chi_W(h)\overline{\chi_V}(g)\overline{\chi_W}(h)dgdh \\
&= \int_G \chi_V(g)\overline{\chi_V}(g)dg \int_H \chi_W(h)\overline{\chi_W}(h)dh \\
&= 1.
\end{aligned}$$

(b) Let U be a representation of $G \times H$. Consider the homomorphism of H-representations

$$\varphi : \bigoplus_j \mathrm{Hom}_H(W_j, U) \otimes W_j \to U$$

defined as $\varphi(f \otimes w) = f(w)$. Note that this is the isotypic decomposition of U. Thus we know that φ is an isomorphism by Shur's Lemma.

We have an action of G on $\mathrm{Hom}_H(W_j, U)$ given by $(gf)(w) = gf(w)$ where $gf(w)$ is defined as $(g,1)f(w)$. Note that $(g,1)(1,h)f(w) = (g,h)f(w) = (1,h)(g,1)f(w)$ so $gf \in \mathrm{Hom}_H(W_j, U)$. Thus $\mathrm{Hom}_H(W_j, U)$ has some decomposition into irreducible G-representations as $\mathrm{Hom}_H(W_j, U) \cong \bigoplus_i a_{ij}V_{ij}$ for each j. Since φ is an isomorphism and tensor products and direct sums commute, we now have

$$U \cong \bigoplus_{i,j} a_{ij}V_{ij} \otimes W_j.$$

where V_{ij} is an irreducible G-representation and W_j is an irreducible H-representation. □

Problem 2. *Let G be a topological group. A* continuous family of representations *of G is a continuous map*

$$F : G \times [0,1] \to GL(n, \mathbb{C})$$

with the property that, for each $t \in [0,1]$ the map $F_t : G \to GL(n, \mathbb{C})$ given by $g \to F(g,t)$ is a (continuous of course) representation.
(a) For a continuous family of representations of a compact *topological group, the representations F_0 and F_1 are isomorphic.*
(b) Give an example to show that this does not necessarily hold if G is not compact.

Proof. (a) Note that since trace is continuous, by composition we immediately get a homotopy of characters $\chi_t : G \to \mathbb{C}$ which is continuous in t. Now consider the function $\varphi : t \mapsto \langle \chi_0, \chi_t \rangle$. Note that this is an integer-valued continuos function since taking the inner product is continuous. But now note that $\varphi(t) = \varphi(0)$ for all t because φ is both continuous and integer-valued so it's impossible for $\varphi(t)$ to move to a different value than $\varphi(0)$. Thus F_0 is isomorphic to F_t for each $t \in [0,1]$.

(b) Consider the representation of $\mathbb{R}^*$ given by

$$r \mapsto \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}.$$

We can find a homotopy from the trivial representation to this one as

$$(r,t) \mapsto \begin{pmatrix} r^t & 0 \\ 0 & r^{-t} \end{pmatrix}.$$

This is continuous in t since the exponential map is continuous, but $(r,0)$ and $(r,1)$ are not isomorphic because they have different traces. $\square$

Problem 3. *Let V be an irreducible representation of a compact topological group G. Prove that*

$$\chi_V(x)\chi_V(y) = \dim(V) \int \chi_V(gxg^{-1}y)dg.$$

Proof. Let $\rho : G \to GL(V)$ be a representation of G. Define

$$A = \int_G \rho(gxg^{-1})dg.$$

Note that A represents a G-action as

$$Av = \int_G \rho(gxg^{-1})vdg.$$

Then using left-invariance we have

$$\begin{aligned}
A(hv) &= \int_G \rho(gxg^{-1})\rho(h)vdg \\
&= \int_G \rho(h)\rho(h^{-1})\rho(gxg^{-1})\rho(h)vdg \\
&= \rho(h)\int_G \rho(h)\rho(g)\rho(x)\rho(g^{-1})\rho(h^{-1})vdg \\
&= \rho(h)\int_G \rho(gh)\rho(x)\rho((gh)^{-1})vdg \\
&= h(Av).
\end{aligned}$$

Thus A respects the G-action on V so by Shur's Lemma we know $A = \lambda \mathrm{id}_V$. Also since $\rho(y)$ is independent of g, we have $\rho(y)A = A\rho(y)$.

Note that since trace is linear it commutes with integration so we have

$$\mathrm{tr}(A) = \mathrm{tr}\left(\int_G \rho(gxg^{-1})dg\right) = \int_G \mathrm{tr}(\rho(g)\rho(x)\rho(g^{-1}))dg = \int_G \chi_\rho(x)dg = \chi_\rho(x)\int_G 1dg = \chi_\rho(x).$$

Then from the above we know $\chi_\rho(x) = \mathrm{tr}(A) = \mathrm{tr}(\lambda \mathrm{id}_V) = \lambda \dim(V)$ so $\lambda = (\dim V)^{-1}\chi_\rho(x)$. Now we have the following using the above and the left-invariance of the Haar measure

$$\int_G \rho(gxg^{-1}y)dg = \left(\int_G \rho(gxg^{-1})dg\right)\rho(y) = \lambda \mathrm{id}_V \rho(y) = (\dim V)^{-1}\chi_\rho(x)\rho(y).$$

Now take the trace of both sides so we have

$$\begin{aligned}
\int_G \chi(gxg^{-1}y)dg &= \int_G \operatorname{tr}(\rho(gxg^{-1}y))dg \\
&= \operatorname{tr}\left(\int_G \rho(gxg^{-1}y)dg\right) \\
&= \operatorname{tr}((\dim V)^{-1}\chi_\rho(x)\rho(y)) \\
&= (\dim V)^{-1}\chi\rho(x)\operatorname{tr}(\rho(y)) \\
&= (\dim V)^{-1}\chi_\rho(x)\chi_\rho(y).
\end{aligned}$$

$\square$

Problem 4. *Let $\{V_n\}$ be the irreducible representations of $SU(2)$, as discussed in class. The* Clebsch-Gordan Formula *gives a direct sum decomposition of $V_k \otimes V_\ell$ as follows: Let $q = \min\{k, \ell\}$. Then*

$$V_k \otimes V_\ell = \bigoplus_{j=0}^{q} V_{k+\ell-2j}.$$

(b) Decompose the following representations $V_3 \otimes V_4$, $V_1^{\otimes n}$ and $\wedge^2 V_3$.

Proof. (b) Using the formula

$$V_3 \otimes V_4 = \bigoplus_{j=0}^{3} V_{7-2j} = V_7 \oplus V_5 \oplus V_3 \oplus V_1.$$

To decompse $V_1^{\otimes n}$ denote

$$V_1^{\otimes n} = \bigoplus_{k=0}^{n} a_k V_k.$$

We will show by induction that

$$a_k = \begin{cases} \frac{(k+1)n!}{(\frac{n-k}{2})!(\frac{n+k}{2}+1)!} & \text{if } n+k \equiv 0 \pmod 2 \\ 0 & \text{if } n+k \equiv 1 \pmod 2). \end{cases}$$

For $n = 1$ we have $a_0 = 0$ and

$$a_1 = \frac{(1+1)(1!)}{\left(\frac{1-1}{2}\right)!\left(\frac{1+1}{2}+1\right)!} = \frac{2}{2} = 1$$

as desired. Now assume the formula holds for n. Then using the Clebsch-Gordon Formula we have

$$V_1^{\otimes(n+1)} = V_1 \otimes \left(\bigoplus_{k=0}^{n} a_k V_k\right) = \bigoplus_{k=0}^{n} a_k(V_1 \otimes V_k) = \bigoplus_{k=0}^{n} a_k(V_{k+1} \oplus V_{k-1}) = \bigoplus_{k=0}^{n+1}(a_{k-2} + a_k)V_{k-1}$$

where we define $a_k = 0$ if $k < 0$ and $V_{-1} = 0$. Now note that for $k \neq 1$ we have

$$\begin{aligned}
a_{k-2} + a_k &= \frac{(k-1)n!}{\left(\frac{n-k+2}{2}\right)!\left(\frac{n+k-2}{2}+1\right)!} + \frac{(k+1)n!}{\left(\frac{n-k}{2}\right)!\left(\frac{n+k}{2}+1\right)!} \\
&= \frac{\left(\frac{n+k}{2}+1\right)(k-1)n! + \left(\frac{n-k+2}{2}\right)(k+2)n!}{\left(\frac{n-k+2}{2}\right)!\left(\frac{n+k}{2}+1\right)!} \\
&= \frac{\left(\left(\frac{(n+k)(k-1)}{2}+k-1\right)+\left(\frac{nk-k^2+k}{2}\right)\right)n!}{\left(\frac{(n+1)-k+1}{2}\right)!\left(\frac{(n+1)+k-1}{2}+1\right)!} \\
&= \frac{k(n+1)n!}{\left(\frac{(n+1)-k+1}{2}\right)!\left(\frac{(n+1)+k-1}{2}+1\right)!} \\
&= \frac{k(n+1)!}{\left(\frac{(n+1)-k+1}{2}\right)!\left(\frac{(n+1)+k-1}{2}+1\right)!}
\end{aligned}$$

which is the claimed a_{k-1} for $V^{\otimes(n+1)}$. In the case $k = 1$ we have

$$a_1 = \frac{2n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}+1\right)!} = \frac{\left(\frac{n+1}{2}\right)2n!}{\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}+1\right)!} = \frac{(n+1)!}{\left(\frac{n+1}{2}\right)!\left(\frac{n+1}{2}+1\right)!} = a_0$$

which is the coefficient of V_0 for $V^{\otimes(n+1)}$.

We also note that a_k can be expressed as the difference of two binomial coefficients as

$$a_k = \binom{n-1}{\frac{n+k}{2}-1} - \binom{n-1}{\frac{n+k}{2}+1}.$$

Finally, to find $\wedge^2 V_3$ we note that this sits as a subspace inside $V_3 \otimes V_3$ which by the Clebsch-Gordon Formula is $V_0 \oplus V_2 \oplus V_4 \oplus V_6$. Since the dimension of $\wedge^2 V_3 = \binom{4}{2} = 6$, counting dimensions leaves the only possibility as $\wedge^2 V_3 = V_0 \oplus V_4$. $\square$

Problem 5. *Consider the 9-dimensional complex representation of $SU(2)$ on 3×3 complex matrices given by $A \in SU(2)$ acting on M via $M \mapsto A_1 M A_1^{-1}$ where A_1 is the 3×3 block matrix with A in the upper left and 1 in the lower right. Decompose this representation as a direct sum of irreducibles.*

Proof. Let M_i be the 3×3 matrix $[m_{jk}]$ where $m_{jk} = 1$ if $j + k = i$ and 0 otherwise. Then note the 9 M_i matrices form a basis for the space of 3×3 complex matrices. Let $A \in SU(2)$ have the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

We have the following computations

$$A_1M_1A_1^{-1} = \begin{pmatrix} |a|^2 & -ab & 0 \\ -\overline{ab} & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1M_2A_1^{-1} = \begin{pmatrix} a\overline{b} & a^2 & 0 \\ -\overline{b}^2 & -a\overline{b} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1M_3A_1^{-1} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & -\overline{b} \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1M_4A_1^{-1} = \begin{pmatrix} \overline{a}b & -b^2 & 0 \\ \overline{a}^2 & -\overline{a}b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1M_5A_1^{-1} = \begin{pmatrix} |b|^2 & ab & 0 \\ \overline{ab} & |a|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1M_6A_1^{-1} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & \overline{a} \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_1M_7A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overline{a} & -b & 0 \end{pmatrix}$$

$$A_1M_8A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overline{b} & a & 0 \end{pmatrix}$$

$$A_1M_9A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now let T be the 9×9 transformation matrix representing this action. Note that the entires of $A_1M_iA_1^{-1}$, read from left to right, top to bottom, form the i^{th} column of T. This T_{ii} is the i^{th} entry from $A_1M_iA_1^{-1}$. Reading these off we see that

$$\operatorname{tr}(T) = \sum_{i=1}^{9} T_{ii} = |a|^2 + a^2 + a + \overline{a}^2 + |a|^2 + \overline{a} + \overline{a} + a + 1 = 2(|a|^2 + a + \overline{a}) + a^2 + \overline{a}^2 + 1.$$

Now note that the matrix representation of A acting on V_1 is simply A itself since $(x, y)A = (ax - \overline{b}y, bx + \overline{a}y)$. Thus

$$\chi_{V_1}(A) = \operatorname{tr}(A) = a + \overline{a}.$$

Using the Clebsch-Gordon Formula we know $V_1 \otimes V_1 = V_2 \oplus V_0$. Thus we must have

$$\chi_{V_2}(A) = \chi_{V_1}(A)^2 - \chi_{V_0}(A) = (a + \overline{a})^2 - 1 = a^2 + 2|a|^2 + \overline{a}^2 - 1.$$

Then note that

$$\chi_{V_2} + 2\chi_{V_1} + 2\chi_{V_0} = a^2 + 2|a|^2 + \overline{a}^2 - 1 + 2a + 2\overline{a} + 2 = 2(|a|^2 + a + \overline{a}) + a^2 + \overline{a}^2 + 1 = \operatorname{tr}(T)$$

so this representation decomposes as $V_2 \oplus 2V_1 \oplus 2V_0$. $\square$