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Homework 5

Exercise 1 *Show that if* $\lim_{n\to\infty} a_n = 0$ *and* (b_n) *is bounded, then*

$$\lim_{n\to\infty} (a_n b_n) = 0.$$

Proof. We have (a_n) converges to 0 and (b_n) is bounded. Then there exists $l, u \in \mathbb{R}$ such that $l \le b_n \le u$ for all $n \in \mathbb{N}$. For $\varepsilon > 0$ we have there exists $N \in \mathbb{N}$ such that for all $n > N$, $a_n \in (-\varepsilon; \varepsilon)$ and $b_n \in (l; u)$. Then using Lemma 12.10 we have $a_n b_n \in (\min(-\varepsilon l, \varepsilon l, \varepsilon u, \varepsilon u); \max(-\varepsilon l, \varepsilon l, \varepsilon u, \varepsilon u))$. But then let $\varepsilon = \max(-\varepsilon l, \varepsilon l, \varepsilon u, \varepsilon u)$ so that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n| < \varepsilon$. □

Exercise 2 *Prove that the sequence* $a_n = 2^{-n}$ *converges to* 0*.*

Proof. Note that for all $n \in \mathbb{N}$ we have $a_n \in (0; 1)$. Let $(p; q)$ be a region such that $0 \in (p; q)$. If $1/2 \le q$ then there are no elements of $(a_n) \in (p; q)$ and so there are finitely many $n \in \mathbb{N}$ with $a_n \notin (p; q)$. Suppose that $q \in (0; 1/2)$. Then $1/q > 1$. By the Archimedean Property exists $n \in \mathbb{N}$ with $1/q < n$. But since $1/q > 1$, we have $1/q < 2^n$. Then 2^n for $n \in \mathbb{N}$ is a subset of $\mathbb{N}$ so there exists a least $k \in \mathbb{N}$ such that $2^k > 1/q$. Then $q < 2^{-k}$. But there are a finite number of elements of $\mathbb{N}$ which are less than or equal to k and so there are a finite number of $n \in \mathbb{N}$ with $a_n > q$. Thus $\lim_{n\to\infty} a_n = 0$. □

Exercise 3 *Show that if* $s > 1$ *and* $n \in \mathbb{N}$*, we have*

$$s^n \ge 1 + n(s-1)$$

Proof. Use induction on n. We see that for $s > 1$ with $n = 1$ we have $s = 1 + s - 1 = 1 + n(s-1)$. Now assume that for $k \in \mathbb{N}$ we have $s^k \ge 1 + k(s-1)$. Consider

$$s^{k+1} \ge s + sn(s-1) = s + s^2 n - sn.$$

At this point notice that

$$0 \le (s-1)^2 = s^2 - 2s + 1.$$

Thus $s^2 - s \ge s - 1$ so $ns^2 - ns \ge ns - n$. Note that this implies

$$s + s^2 n - sn \ge ns - n + s = 1 + (n+1)(s-1)$$

as desired. □

Exercise 4 *Prove that for each real number* t *with* $0 < t < 1$*, the sequence* $x_n = t^n$ *converges to* 0*.*

Proof. Let $s = t^{-1}$ for $0 < t < 1$ so that we have $s > 1$. Then $s^n \geq 1 + n(s-1)$ so $(t^{-1})^n \geq 1 + n(t^{-1} - 1)$ which means

$$\begin{aligned} t^n &\leq \frac{1}{1 + n(t^{-1} - 1)} \\ &= \frac{1}{1 + n(\frac{1-t}{t})} \\ &= \frac{1}{\frac{t+n(1-t)}{t}} \\ &= \frac{t}{t + n(1-t)} \\ &= \frac{\frac{t}{n}}{\frac{t}{n} + (1-t)}. \end{aligned}$$

Using sum and product rules from Theorem 13.5 and convergent functions from Exercise 13.4 we have this expression converges to 0. Using comparative limits from Theorem 13.5 and since $t^n > 0$ for all $n \in \mathbb{N}$ we know that (a_n) must converge to 0 as well. $\square$

Exercise 5 *Show that for each real number $|t| < 1$, the sequence*

$$a_n = 1 + t + t^2 + \cdots + t^{n-1}$$

satisfies

$$\lim_{n\to\infty} a_n = \frac{1}{1-t}.$$

Proof. Note that $a_n = 1 + t + t^2 + \cdots + t^{n-1} = \frac{1-t^n}{1-t} = \frac{1}{1-t} - \frac{t^n}{1-t}$. From Exercise 4 we have that t^n converges to 0 and using the product rule we have $\frac{t^n}{1-t}$ converges to 0. Then using the addition rule we have $\lim_{n\to\infty} a_n = \frac{1}{1-t}$. $\square$

Exercise 6 *Prove that for any real number a, the sequence $x_n = a^n/(n!)$ converges to 0.*

Proof. If $a = 0$ the theorem is trivial. Let $a > 0$. For all $n > a$ we have $a_{n-1} \leq \frac{a_n}{a_{n-1}}$. But then $\frac{a_n}{a_{n-1}} = \frac{a}{n}$ which converges to 0 from Exercise 13.4. Using the sequence comparisons from Theorem 13.5 we know that $\lim_{n\to\infty} a_n \leq 0$. In the case where $a < 0$ every other element of (a_n) is less than 0. In the case where $a > 0$ we concluded that for every region R with $0 \in R$ there were finitely many n such that $a_n \notin R$. If we consider $|a_n|$ for all n then this condition still holds which means that if half of the elements of (a_n) are less than 0 then there are still finitely many $n \in \mathbb{N}$ with $a_n \notin R$. $\square$

Exercise 7 *Let*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

be polynomials. What is

$$\lim_{n\to\infty} \frac{p(n)}{q(n)}?$$

Proof. We have

$$p(x) = x^n(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n})$$

and

$$q(x) = x^m(b_m + \frac{b_{m-1}}{x} + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}).$$

Then

$$\frac{p(x)}{q(x)} = x^{n-m}\left(\frac{a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}{b_m + \frac{b_{m-1}}{x} + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}}\right).$$

When we take the limit as x approaches ∞, we have $\frac{a_k}{x^{n-k}}$ converges to zero because $\frac{1}{x}$ converges to 0 from Exercise 13.4 and Theorem 13.5. Then if $m = n$ we have $\lim_{x\to\infty} p(x)/q(x) = a_n/b_m$. If $n > m$ then the sequence diverges since $n - m > 0$ and it is no longer bounded. If $n < m$ then $n - m < 0$ and the sequence converges to 0 by the same reasoning as above (i.e. $1/x$ converges to 0 and we can make use of the product of two sequence). □

Exercise 8 *What is*

$$\lim_{n\to\infty} \sqrt[n]{n}?$$

Proof. Let $\varepsilon > 0$ and consider $(1+\varepsilon)^k$. Let $k = 1/\varepsilon^2$. Then we have $\varepsilon^2(1+\varepsilon)^{1/\varepsilon^2} = \varepsilon^{2\varepsilon^2}(1+\varepsilon) > 1$. By the Archimedean Property there exists some $n \in \mathbb{N}$ such that $n > 1/\varepsilon^2$. Then $(1+\varepsilon)^n > n$ and $\sqrt[n]{n} > 0$ so $|\sqrt[n]{n} - 1| < \varepsilon$. Thus for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|\sqrt[n]{n} - 1| < \varepsilon$. Therefore $\lim_{n\to\infty} \sqrt[n]{n} = 1$. □