Sheet 21: Derivatives

Definition 1 *A function f is differentiable at a if*

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

exists.

Definition 2 *The function f', called the derivative of f, is defined as the function whose domain is all a such that f is differentiable at a and whose value at a is*

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

The function $f''=(f')'$ is the second derivative of f. Similarly $f'''=(f'')'$. We denote $f^{(n)}$ as the nth derivative of f for $n\geq 4$.

Theorem 3 *If f is differentiable at a, then f is continuous at a.*

Proof. We have

$$\lim_{h\to 0} f(a+h)-f(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}h=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\lim_{h\to 0}h=f'(a)\cdot 0=0.$$

Thus $\lim_{h\to 0} f(a+h)=f(a)$ which means that f is continuous at a. $\square$

Exercise 4 *Give and prove an example of a function that is continuous but not differentiable.*

Proof. Let $f(x)=|x|$ and consider $x=0$. Let $\varepsilon>0$ and let $\delta=\varepsilon$. Then if we have $|x|<\delta=\varepsilon$ we have $|f(x)|=||x||=|x|<\varepsilon$. Thus f is continuous at $x=0$. Then consider

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0^+}\frac{|h|}{h}=1$$

and

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0^-}\frac{|h|}{h}=-1$$

because $|h|\geq 0$. Since the left and right hand limits are not the same the limit does not exist and f is not differentiable at 0. $\square$

Exercise 5 *If $g(x)=f(x+c)$ then $g'(x)=f'(x+c)$. Also if $g(x)=f(cx)$ then $g'(x)=cf'(cx)$.*

Proof. Both of these can be proved with the Chain rule. Let $h(x)=x+c$. Then $f(h(x))'=f'(h(x))h'(x)=f'(x+c)$ (21.16). If $h(x)=cx$. Then $f(h(x))'=f'(h(x))h'(x)=cf'(cx)$ (21.16). $\square$

Exercise 6 *Let f be a function such that $|f(x)|\leq x^2$ for all x. Show that f is differentiable at 0.*

Proof. Note that $f(0)=0$ because $0\leq|f(0)|\leq 0^2=0$. We have $|f(h)/h|\leq|h^2/h|\leq|h|$ which means that $\lim_{h\to 0} f(h)/h=0$. Thus $f'(0)=0$. $\square$

Theorem 7 *If $f(x)=c$ then $f'(x)=0$.*

Proof. We have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}\lim_{h\to 0}\frac{0}{h}=0.$$

$\square$

Theorem 8 *If* $f(x) = ax + b$ *then* $f'(x) = a$.

Proof. We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(a(x+h)+b) - (ax+b)}{h} \lim_{h\to 0} \frac{ax + ah + b - ax - b}{h} = \lim_{h\to 0} \frac{ah}{h} = \lim_{h\to 0} a = a.$$

□

Theorem 9 *If* f *and* g *are differentiable at* a *then* $f+g$ *is also differentiable at* a *and*

$$(f+g)'(a) = f'(a) + g'(a).$$

Proof. Since f and g are both differentiable at a we know

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

and

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$$

both exist. Then

$$\begin{aligned} f'(a) + g'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a) + g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \qquad = (f+g)'(a). \end{aligned}$$

We know this limit exists because the sum of the limits of two functions is the limit of their sum. □

Theorem 10 *If* f *and* g *are differentiable at* a *then*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof. Since f and g are both differentiable at a we know

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

and

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$$

both exist. Then $f(a)$ and $g(a)$ are both constants so

$$\begin{aligned} f'(a)g(a) + f(a)g'(a) &= \lim_{h\to 0} g(a) \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} f(a+h) \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a)g(a)}{h} + \lim_{h\to 0} \frac{g(a+h)f(a+h) - g(a)f(a+h)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a+h)g(a) + g(a+h)f(a+h) - g(a)f(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\ &= (fg)'(a). \end{aligned}$$

□

Theorem 11 *If $g(x) = cf(x)$ and f is differentiable at a then g is differentiable at a and*

$$g'(a) = cf'(a).$$

Proof. We have f is differentiable at a so

$$\begin{aligned}
cf'(a) &= c \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{cf(a+h) - cf(a)}{h} \\
&= \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \\
&= g'(a).
\end{aligned}$$

We know this limit exists because the limit of the product of two functions is the product of their limits. $\square$

Theorem 12 *If $f(x) = x^n$ for some $n \in \mathbb{N}$ then*

$$f'(a) = na^{n-1}.$$

Proof. Note that for $n = 1$ we have $f'(a) = 1 \cdot a^0 = 1$ by Theorem 8 (21.8). Use induction on n and suppose that if $f(x) = x^n$ for $n \in \mathbb{N}$ we have $f'(a) = na^{n-1}$. Consider a function $f(x) = x^{n+1} = x \cdot x^n$. Then from Theorem 10 we have

$$f'(a) = x^n + x \cdot (nx^{n-1}) = x^n + nx^n = (n+1)x^n$$

as desired. $\square$

Theorem 13 *If f is differentiable at a and $f(a) \neq 0$ then $1/f$ is differentiable at a and*

$$\left(\frac{1}{f}\right)'(a) = \frac{-f'(a)}{(f(a))^2}.$$

Proof. We have

$$\begin{aligned}
\left(\frac{1}{f}\right)'(a) &= \lim_{h \to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} \\
&= \lim_{h \to 0} \frac{\frac{f(a) - f(a+h)}{f(a+h)f(a)}}{h} \\
&= \lim_{h \to 0} \frac{1}{f(a+h)f(a)} \frac{f(a) - f(a+h)}{h} \\
&= \lim_{h \to 0} \frac{1}{f(a+h)f(a)} \lim_{h \to 0} \frac{f(a) - f(a+h)}{h} \\
&= \frac{1}{(f(a))^2} \left(- \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\right) \\
&= \frac{-f'(a)}{(f(a))^2}.
\end{aligned}$$

Note that $1/f$ is differentiable at a because of the product rules for limits and $f'(a)$ exists. $\square$

Corollary 14 *If f and g are differentiable at a and $g(a) \neq 0$ then f/g is differentiable at a and*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof. We have

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)'(a) \\ &= \frac{f'(a)}{g(a)} + \frac{-g'(a)f(a)}{(g(a))^2} \\ &= \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}.\end{aligned}$$

using Theorems 10 and 13 (21.10, 21.13). □

Lemma 15 *Let g be continuous at a and let f be differentiable at $g(a)$. Let*

$$\phi(h) = \begin{cases} \frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{if } g(a+h) - g(a) = 0. \end{cases}$$

Then $\phi(x)$ is continuous at 0.

Proof. Since $f'(g(a))$ exists we have

$$\lim_{k \to 0} \frac{f(g(a)+k) - f(g(a))}{k} = f'(g(a))$$

which means that for all $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if $0 < |m| < \delta_1$ we have

$$\left| \frac{f(g(a)+m) - f(g(a))}{k} - f'(g(a)) \right| < \varepsilon.$$

Since $g'(a)$ exists then g is continuous at a (21.3). Thus for all $\delta_1 > 0$ there exists $\delta_2 > 0$ such that for all h if $|h| < \delta_2$ we have $|g(a+h) - g(a)| < \delta_1$. Now let$|h| < \delta_2$. If $k = g(a+h) - g(a) \neq 0$ then we have

$$\phi(h) = \frac{f(g(a+h) - f(g(a))}{g(a+h) - g(a)} = \frac{f(g(a)+k) - f(g(a))}{k}.$$

We know from our second continuity statement that $|k| < \delta_1$ and from our first continuity statement that $|\phi(h) - f'(g(a))| < \varepsilon$. If $g(a+h) - g(a) = 0$ then $\phi(h) = f'(g(a)$ and so we have $0 = |\phi(h) - f'(g(a)| < \varepsilon$. Thus

$$\lim_{h \to 0} \phi(h) = f'(g(a))$$

which means ϕ is continuous at 0. □

Theorem 16 (Chain Rule) *If g is differentiable at a and f is differentiable at $g(a)$ then $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. Use the function from Lemma 15 and note that if $h \neq 0$ we have

$$\frac{f(g(a+h)) - f(g(a))}{h} = \phi(h)\frac{g(a+h)-g(a)}{h}.$$

Then

$$(f \circ g)'(a) = \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h \to 0} \phi(h) \lim_{h \to 0} \frac{g(a+h)-g(a)}{h} = f'(g(a))g'(a)$$

which exists because $g'(a)$ exists and because of the product rules for limits. □

Exercise 17 *Differentiate*

$$f(x) = \sin\left(\frac{x^3}{\cos(x^3)}\right).$$

Proof. Using the chain rule we have

$$f'(x) = \cos\left((x^3)(\cos x^3)^{-1}\right)\left(3(x^5)(\cos x^3)^{-2}(\sin x^3) + 3(x^2)(\cos x^3)^{-1}\right).$$

□

Exercise 18 *Let a be a double root of the polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Show that a is a double root of f if and only if a is a root of both f and f'.*

Proof. Let a be a double root of f. Then $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Then $f(a) = (a-a)^2 g(a) = (0)g(x) = 0$ so a is a root of f. Also using the product and chain rules we have $f'(x) = (x-a)^2 g'(x) + 2(x-a)g(x) = (x-a)((x-a)g'(x) + 2g(x))$. Then $f'(a) = (a-a)((a-a)g'(a) + 2g(a)) = 0$ so a is a root of f'. Conversely assume that a is a root of both f and f'. Then $f(a) = f'(a) = 0$. Thus $f(x) = (x-a)g(x)$ for some polynomial function $g(x)$ and $f'(x) = (x-a)g'(x) + g(x)$. But since $f'(a) = 0$ we have $g(a) = 0$. Thus $g(a) = (x-a)h(x)$ for some polynomial function h. But then $f(x) = (x-a)^2 h(x)$. Therefore a is a double root of f. □

Definition 19 *Let f be a function and A a set of numbers contained in the domain of f. A point $x \in A$ is a maximum point for f on A if $f(x) \geq f(y)$ for all $y \in A$. The number $f(x)$ itself is called the maximum value of f on A and we say that f has its maximum value on A at x.*

Theorem 20 *Let f be a function defined on $(a;b)$. If x is a maximum or minimum point for f on $(a;b)$ and f is differentiable at x then $f'(x) = 0$.*

Proof. Consider h such that $x + h \in (a;b)$. Then $f(x+h) - f(x) \leq 0$. If $h > 0$ then we have

$$\frac{f(x+h) - f(x)}{h} \leq 0$$

which means

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \leq 0.$$

If $h < 0$ then we have

$$\frac{f(x+h) - f(x)}{h} \geq 0$$

which means

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \geq 0.$$

Since f is differentiable at x these two limits must be equal to $f'(x)$ which means $0 \leq f'(x) \leq 0$ and so $f'(x) = 0$. If x is a minimum point for f on $(a;b)$ then consider $-f$ and we end up with the equality $0 \leq -f'(x) \leq 0$ as well. □

Definition 21 *Let f be a function and A a set of numbers contained in the domain of f. A point x in A is a local maximum or minimum point for f on A if there is some $\delta > 0$ such that x is a maximum or minimum point for f on $A \cap (x - \delta; x + \delta)$.*

Theorem 22 *Let f be a function defined on $(a; b)$. If x is a local maximum or local minimum point for f on $(a; b)$ and f is differentiable at x then $f'(x) = 0$.*

Proof. Let x be a local maximum or minimum for f on $(a; b)$ then there exists $\delta > 0$ such that x is a maximum or minimum for f on $(a; b) \cap (x - \delta; x + \delta)$. But this set is a subset of the domain of f and so $f'(x) = 0$ (21.20). □

Definition 23 *A critical point of a function f is a number x such that $f'(x) = 0$. The number $f(x)$ itself is called a critical value of f.*

Exercise 24 *Prove that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has at most $n - 1$ critical points.*

Proof. Taking the derivative of f we have $f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$. This is a polynomial of degree $n - 1$ and so it must have at most $n - 1$ roots which means that $f'(x) = 0$ at at most $n - 1$ points (19.9). Thus f has at most $n - 1$ critical points. □

Theorem 25 (Rolle's Theorem) *If f is continuous on $[a; b]$, differentiable on $(a; b)$ and $f(a) = f(b)$ then there is some $x \in (a; b)$ such that $f'(x) = 0$.*

Proof. Since f is continuous on $[a; b]$ there exists $x_1, x_2 \in [a; b]$ such that $f(x_1) \geq f(x)$ and $f(x_2) \leq f(x)$ for all $x \in [a; b]$ (10.9). If $x_1 \in (a; b)$ or $x_2 \in (a; b)$ then we have a maximum or minimum point for f on $(a; b)$ in $(a; b)$. Thus $f'(x_1) = 0$ or $f'(x_2) = 0$ and we're done. If $x_1, x_2 \notin (a; b)$ then x_1 and x_2 are the values a and b, not necessarily respectively. Then since $f(a) = f(b)$ the maximum and minimum values of f are the same so f must be constant on $[a; b]$. Then $f'(x) = 0$ for all $x \in [a; b]$. □

Corollary 26 (Mean Value Theorem) *If f is continuous on $[a; b]$ and differentiable on $(a; b)$ then there exists some $x \in (a; b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $g(x)$ is continuous on $[a; b]$ and differentiable on $(a; b)$ and we have $g(a) = f(a)$, $g(b) = f(a) = g(a)$. Then we know that there exists some $x \in (a; b)$ such that

$$0 = g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

from Rolle's Theorem (21.25). Thus we have

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

□

Exercise 27 *If f is defined on an interval and $f'(x) = 0$ for all x in the interval then f is constant on the interval.*

Proof. Consider two points a and b in the interval with $a \neq b$. We know that there exists $x \in (a;b)$ such that

$$0 = f'(x) = \frac{f(b) - f(a)}{b - a}$$

which means that $f(a) = f(b)$ (21.26). So for any two points in the interval the value of f is the same which means f is constant on the interval. □

Exercise 28 *If f and g are defined on the same interval and $f'(x) = g'(x)$ for all x in the interval then there is some number c such that $f = g + c$.*

Proof. For all x in the interval we have $f'(x) - g'(x) = (f - g)'(x) = 0$. Then we must have $(f - g)(x) = c$ for some constant c (21.27). Thus $f = g + c$. □

Definition 29 *A function is increasing on an interval if $f(a) < f(b)$ for all a and b in the interval with $a < b$. The function f is decreasing on an interval if $f(a) > f(b)$ for all a and b in the interval with $a < b$.*

Exercise 30 *If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval. If $f'(x) < 0$ for all x in the interval then f is decreasing on the interval.*

Proof. Let $f'(x) > 0$ for all x in the interval and let a and b be two points in the interval with $a < b$. Then there exists $x \in (a;b)$ such that

$$0 < f'(x) = \frac{f(b) - f(a)}{b - a}$$

and so $f(b) - f(a) > 0$ (21.26). But then $f(b) > f(a)$ and so f is increasing on the interval. A similar proof holds for decreasing f. □

Theorem 31 *Suppose $f'(a) = 0$. If $f''(a) > 0$ then f has a local minimum at a. If $f''(a) < 0$ then f has a local maximum at a.*

Proof. Suppose that $f''(a) > 0$. Since $f'(a) = 0$ we have

$$f''(a) = \lim_{h \to 0} \frac{f'(a + h)}{h} > 0.$$

Then $f'(a + h)/h > 0$ for small enough values of h. Thus for small values of $h > 0$ we have $f'(a + h) > 0$ which means f is increasing on an interval to the right of a. Similarly f is decreasing on an interval to the left of a. Then f must have a minimum at a. A similar proof holds for $f''(a) < 0$. □

Exercise 32 *Let $a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} = 0$. Show that the polynomial $p(x) = a + bx + cx^2 + dx^3 + ex^4$ has at least one real zero.*

Proof. Let $P(x) = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4 + \frac{e}{5}x^5$ and note that $P'(x) = p(x)$. Also note that $P(0) = P(1) = 0$. Then we know there exists some $x \in (0;1)$ such that

$$p(x) = P'(x) = \frac{P(1) - P(0)}{1 - 0} = 0$$

from the Mean Value Theorem (21.26). □

Theorem 33 *Suppose that f is continuous at a and that $f'(a)$ exists for all x in some interval containing a, except perhaps for $x = a$. Suppose, moreover, that $\lim_{x \to a} f'(x)$ exists. Then $f'(a)$ also exists and*

$$f'(a) = \lim_{x \to a} f'(x).$$

Proof. Note that if $h > 0$ is small enough then f is continuous on $[a; a+h]$ and differentiable on $(a; a+h)$. We know there exists some value y such that

$$f'(y) = \frac{f(a+h) - f(a)}{h}$$

by the Mean Value Theorem (21.26). Note that y goes to a as h goes to 0 because $y \in (a; a+h)$. Then

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^+} f'(y) = \lim_{x \to a^+} f'(x).$$

If $h < 0$ is small enough then f is continuous on $[a+h; a]$ and differentiable on $(a+h; a)$. We know there exists some value y such that

$$f'(y) = \frac{f(a) - f(a+h)}{-h} = \frac{f(a+h) - f(a)}{h}$$

by the Mean Value Theorem (21.26). Note that y goes to a as h goes to 0 because $y \in (a; a+h)$. Then

$$f'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^-} f'(y) = \lim_{x \to a^-} f'(x).$$

Since the left and right hand limits are the same we must have

$$f'(a) = \lim_{x \to a} f'(x).$$

□

Theorem 34 (Cauchy Mean Value Theorem) *If f and g are continuous on $[a; b]$ and differentiable on $(a; b)$ then there exists $x \in (a; b)$ such that*

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

If $g(b) \neq g(a)$ and $g'(x) \neq 0$ this equation can be written

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x}{g'(x)}.$$

Proof. Let

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

Then h is continuous on $[a; b]$, differentiable on $(a; b)$ and $h(a) = h(b)$. Then $h'(x) = 0$ for some $x \in (a; b)$ (21.25). Thus

$$0 = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a))$$

giving the desired equality. □

Theorem 35 (L'Hôpital's Rule) *Suppose that*

$$\lim_{x \to a} f(x) = 0,$$

$$\lim_{x \to a} g(x) = 0$$

and $\lim_{x \to a} f'(x)/g'(x)$ exists. Then $\lim_{x \to a} f(x)/g(x)$ exists and

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x).$$

Proof. Note that $f(a)$ and $g(a)$ need not necessarily defined so let $f(a) = g(a) = 0$. Then f and g are continuous on $[a; x]$ and differentiable on $(a; x)$. Then there exists some $y \in (a; x)$ such that

$$(f(x) - f(a))g'(y) = (g(x) - g(a))f'(y)$$

which means

$$\frac{f(x)}{g(x)} = \frac{f'(y)}{g'(y)}$$

after using the Cauchy Mean Value Theorem on f and g (21.34). But then y goes to a as x goes to a because $y \in (a; x)$. Then we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(y)}{g'(y)} = \lim_{z \to a} \frac{f'(z)}{g'(z)}.$$

$\square$