Sheet 14: Cauchy Sequences

Definition 1 (Cauchy Sequence) *We say that a sequence (a_n) is a Cauchy sequence if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then $|a_n - a_m| < \varepsilon$.*

Lemma 2 *Every convergent sequence has the Cauchy property.*

Proof. Let (a_n) converge to a and let $\varepsilon > 0$. Consider $\varepsilon/2$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n \in (a - \varepsilon/2; a + \varepsilon/2)$. But then also for all $m, n > N$ we have $a_m, a_n \in (a - \varepsilon/2; a + \varepsilon/2)$. Then the distance between a_m and a_n is no more than $\varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus, there exists $N \in \mathbb{N}$ such that for all $m, n > N$ we have $|a_m - a_n| < \varepsilon$. □

Lemma 3 *Let (a_n) be a Cauchy sequence and let $(b_k = a_{n_k})$ be a subsequence. If (b_k) converges then so does (a_n).*

Proof. Let $(b_k = a_{n_k})$ be a subsequence of (a_n) which converges to a and let $\varepsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$ we have $|a - b_k| < \varepsilon/2$. But also (a_n) is a Cauchy sequence and so there exists some $N_2 \in \mathbb{N}$ such that for all $n, m > N_2$ we have $|a_m - a_n| < \varepsilon/2$. Let $N = \max(N_1, N_2)$. Then for all $n, m > N$ we have $|a - b_n| < \varepsilon/2$ and $|a_m - a_n| < \varepsilon/2$. Now choose $n > N$ such that $a_n = b_n$. Then for all $m > n > N$ we have $|a - a_n| < \varepsilon/2$ and $|a_m - a_n| < \varepsilon/2$. Thus by the triangle inequality for all $m > n > N$ we have $|a - a_m| \leq |a - a_n| + |a_n - a_m| < \varepsilon$ and so (a_n) converges to a. □

Lemma 4 *Every Cauchy sequence is bounded.*

Proof. Let (a_n) be a Cauchy sequence and let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_N - a_n| < \varepsilon$. Then there are finitely many $n \in \mathbb{N}$ such that $a_n \notin (-\varepsilon + a_N; \varepsilon + a_N)$. Then the largest of these a_n is greater than or equal to every other term of (a_n). Note that if there are no terms of (a_n) greater than $a_N + \varepsilon$, then we can choose a smaller epsilon so that such a term exists. A similar argument shows that there is a lower bound of (a_n). □

Theorem 5 *A sequence is convergent if and only if it is Cauchy.*

Proof. Let (a_n) be a Cauchy sequence. Then by Lemma 4 we know (a_n) is bounded and therefore there exists a convergent subsequence of (a_n) (13.16, 14.4). But then by Lemma 3 we know (a_n) converges (14.3). Conversely if a sequence is convergent then it Cauchy by Lemma 2 (14.2). □

Definition 6 *Let (a_n) be a bounded sequence and A be the set of its accumulation points. We define its limes inferior, $\liminf_{n\to\infty} a_n$, to be the first point of A and the limes superior, $\limsup_{n\to\infty} a_n$, to be the last point of A.*

Corollary 7 *Let (a_n) be a bounded sequence. Then $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ and equality holds if and only if the sequence is convergent.*

Proof. Let A be the set of accumulation points for (a_n). Since $\liminf_{n\to\infty} a_n$ is the first point of A, we have $\liminf_{n\to\infty} a_n \leq a$ for all $a \in A$. But since $\limsup_{n\to\infty} a_n \in A$ we have $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$. Suppose now that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. Then the first and last points of A are equal and so A only has one accumulation point. But then since (a_n) is bounded we have (a_n) is convergent (13.17). Conversely assume that (a_n) is convergent. Then (a_n) only has one accumulation point and so A contains one point (13.17). But then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. □

Theorem 8 *Let (a_n) be a bounded sequence. Then*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf\{a_k \mid k > n\})$$

and

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_k \mid k > n\}).$$

Proof. Consider the sequence (b_n) where $b_n = \inf\{a_k \mid k > n\}$. Then (b_n) is bounded because (a_n) is bounded and it's increasing because each infimum will either be less than or equal to the previous one. Thus $\lim_{n\to\infty} b_n = \sup\{b_n \mid n \in \mathbb{N}\} = s$ (13.18). Now consider some region $(p; q)$ with $s \in (p; q)$. Note that $p < \inf\{a_k \mid k > n\} = r$ for some n, otherwise there would exist some point in $(p; s)$ which would be an upper bound for $\{b_n \mid n \in \mathbb{N}\}$. Note that there are finitely many n such that $a_n < r$ because of how r is defined. Thus there are finitely many n with $a_n < p$. But also there must be finitely many n with $a_n > q$ because if there were infinitely many then there would exist $a_k > q$ such that k is greater than every index of $a_n \leq q$. But this contradicts how s is defined. Thus there are infinitely many n with $a_n \in (p; q)$ and so s is an accumulation point of (a_n). But there can't be an accumulation point of (a_n) less than s because for each term or (b_n) there are finitely many n with a_n less that it and an accumulation point would imply infinitely many such n. Thus $s = \liminf_{n\to\infty} a_n$. A similar proof holds to show $\limsup_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_k \mid k > n\})$. □

Theorem 9 *Let (a_n) be a bounded sequence. Then*

$$\liminf_{n\to\infty} a_n = \sup\{x \mid \text{ there are finitely many } n \text{ with } a_n \in (-\infty; x)\}$$

and

$$\limsup_{n\to\infty} a_n = \inf\{x \mid \text{ there are finitely many } n \text{ with } a_n \in (x; \infty)\}$$

Proof. Let $S = \{x \mid \text{ there are finitely many } n \text{ with } a_n \in (-\infty; x)\}$. Note that S is nonempty because (a_n) is bounded. Thus a lower bound for (a_n) shows that S is nonempty and an upper bound for (a_n) shows that S is bounded. Thus $\sup S = t$ exists. Let (b_n) be defined such that $b_n = \inf\{a_k \mid k > n\}$ and let $s = \lim_{n\to\infty} b_n = \sup\{b_n \mid n \in \mathbb{N}\}$ (13.18, 14,8). First suppose that $t > s$. Then there exists $x \in (s; t)$ such that there are finitely many n with $a_n < x$. But then if we take the largest index, i, of all such a_n we have $\inf\{a_k \mid k > i\} > s$ which is a contradiction. So $t \leq s$. Suppose that $t < s$. Then for all $x \in (t; s)$ there are infinitely many n with $a_n < x$. But this implies that there are infinitely many n with $a_n \in (t; s)$ because there exists $x < t$ such that there are finitely many n with $a_n < x$. But then there exists some element of b_n which is less than s, but greater than infinitely many terms of (a_n). This cannot happen and so $s = t$. But then using Theorem 8 we have $t = \liminf_{n\to\infty} a_n$ (14.8). □