Kris Harper
MATH 26300
April 20, 2010

Homework 3

Problem 1. *Consider two arcs α and β embedded in $D^2 \times I$ as shown in the figure. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.*

Proof. We will preform a series of homeomorphisms to the space. First we move the endpoints of both α and β toward the center of the cylinder. These maps will necessarily move γ to the position indicated in the figure. Now we transform the cylinder to a 2-sphere so that the arcs α and β are chords in the sphere.

Now we can draw a plane intersecting the sphere separating α and β into two hemispheres. We deformation retract this space by taking the plane to a point, which we'll call x_0. The resulting space is a wedge sum of two 2-spheres with diameters (namely the arcs α and β) removed. Note that this retraction will take γ to a composition of four loops based at x_0. Now we can deformation retract each of the two 2-spheres into

a disk which will take α and β each to a point removed from these disks. These two spaces can then be deformation retracted to a copy of S^1 so the resulting space is a wedge sum $S^1 \vee S^1$ with fundamental group $\mathbb{Z} * \mathbb{Z}$. While keeping track of γ through this process we now see that γ is the commutator $aba^{-1}b^{-1}$ so it

cannot be nullhomotopic. $\square$

Problem 2. *Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0-cells, four 1-cells, three 2-cells, and one three cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order 8.*

Proof. Label the cube I^3 with vertices a, b, c, d, e, f, g and h as shown in the figure. Making the identifications described we have the following identifications of points. (a, h), (a, f), (a, c), (c, f), (c, h) and (f, h) as well as (b, e), (e, g), (b, g), (d, g), (b, d), (d, e). From these pairings we see that a, c, f and h get identified to one vertex u and b, d, e and g get identified to another vertex v. So we have two distinct 0-cells, u and v.

We can make similar identification pairs with the edges so we have (ad, cg), (ad, ef), (ef, ch) as well as (ae, bc), (ae, gh), (bc, gh) as well as (ab, fg), (ab, dh), (dh, fg) and finally (bf, eh), (cd, eh), (bf, cd). Thus we have four distinct 1-cells. Note that in each edge is identified with either u or v so our 1-complex looks like a graph with two vertices and four double edges. For convenience relabel the 1-cells as a, b, c and d. Now the

six faces of the cube are identified into three pairs so we have three 2-cells. The 2-cells are attached via the maps $abcd$, $d^{-1}a^{-1}cb$ and $ac^{-1}db$. Finally we attach a 3-cell appropriately to form the middle of the cube.

We can contract the edge d to the point u in the 1-skeleton and retain the same fundamental group overall. We now have a fundamental group consisting of $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ quotiented out by the relations given by the attaching maps for the 2-cells. With d contracted to a point the relations are $abc = 1$, $a^{-1}cb = 1$ and $ac^{-1}b = 1$. From the first relation we see $a^{-1} = bc$ so $(a^{-1})^2 = a^{-1}bc$. Also from the first relation we have $b = a^{-1}c^{-1}$ and from the third relation we have $b = ca^{-1}$. Thus $b^2 = a^{-1}c^{-1}ca^{-1} = (a^{-1})^2 = a^{-1}bc$. Finally from the first relation we have $c = b^{-1}a^{-1}$ and from the second relation we have $c = ab^{-1}$ thus $c^2 = b^{-1}a^{-1}ab^{-1} = (b^{-1})^2 = a^{-1}bc$. Making the identifications $a^{-1} = i$, $b = j$, $c = k$ and $-1 = a^{-1}bc = ijk$ we have the following group presentation $\langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$ which is the quaternion group Q_8. □

Problem 3. *Given a space X with basepoint $x_0 \in X$, we may construct a CW complex $L(X)$ having a single 0-cell, a 1-cell e^1_γ for each loop γ in X based at x_0, and a 2-cell e^2_τ for each map τ of a standard triangle PQR into X taking the three vertices P, Q and R of the triangle to x_0. The 2-cell e^2_τ is attached to the three 1-cells that are the loops obtained by restricting τ to the three oriented edges PQ, PR, and QR. Show that the natural map $L(X) \to X$ induces an isomorphism $\pi_1(L(X)) \approx \pi_1(X, x_0)$.*

Proof. Let f be any loop in $\pi_1(X, x_0)$. Then f is a 1-cell in $L(X)$ and since $L(X)$ has only one 0-cell, this is a loop in $\pi_1(L(X))$. Thus the map from $\pi_1(L(X)) \to \pi_1(X, x_0)$ is surjective. Now suppose f is a loop in $\pi_1(L(X))$ which gets mapped to a nullhomotopic loop g in $\pi_1(X, x_0)$. Since g is nullhomotopic we can use this homotopy to make a map of a triangle PQR into X with P, Q and R mapping to x_0. Restricting this map to the edges PQ, PR and QR we have the boundary of a 2-cell in $L(X)$. But then the boundary of this 2-cell is precisely the loop $f \in \pi_1(L(X))$ and so f is nullhomotopic. Thus our map has trivial kernel and is injective. □

Problem 4. *For a covering space $p : \widetilde{X} \to X$ and a subspace $A \subseteq X$, let $\widetilde{A} = p^{-1}(A)$. Show that the restriction $p : \widetilde{A} \to A$ is a covering space.*

Proof. Since $p : \widetilde{X} \to X$ is a covering space there exists an open cover $\{U_\alpha\}$ such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$ each of which is mapped homeomorphically onto U_α. Note that $\{A \cap U_\alpha\}$

is an open cover of A and that $p^{-1}(A \cap U_\alpha) = p^{-1}(A) \cap p^{-1}(U_\alpha) = \widetilde{A} \cap p^{-1}(U_\alpha)$. Since each $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$ it follows that $p^{-1}(A \cap U_\alpha) = \widetilde{A} \cap p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{A}$. Moreover, for each alpha there is a homeomorphism from $p^{-1}(U_\alpha)$ onto U_α and so the restriction of these maps to $\widetilde{A}$ gives homeomorphisms from $\widetilde{A} \cap p^{-1}(U_\alpha)$ to $p(\widetilde{A} \cap p^{-1}(U_\alpha)) = A \cap U_\alpha$. Thus $p : \widetilde{A} \to A$ is a covering space. □

Problem 5. *Show that if $p_1 : \widetilde{X}_1 \to X_1$ and $p_2 : \widetilde{X}_2 \to X_2$ are covering space, so is their product $p_1 \times p_2 : \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$.*

Proof. Let $\{U_\alpha\}$ and $\{V_\beta\}$ be the open covers of X_1 and X_1 corresponding to p_1 and p_2 respectively. Then $\{U_\alpha \times V_\beta\}$ is an open cover of $X_1 \times X_2$. Now note that $(p_1 \times p_2)^{-1}(U_\alpha \times V_\beta) = p_1^{-1}(U_\alpha) \times p_2^{-1}(V_\beta)$. Since $p_1^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}_1$ and $p_2^{-1}(V_\beta)$ is a disjoint union of open sets in $\widetilde{X}_2$ we see that $(p_1 \times p_2)^{-1}(U_\alpha \times V_\beta)$ is a disjoint union of open sets in $\widetilde{X}_1 \times \widetilde{X}_2$. Moreover, for each α and β there exists homeomorphisms from $p_1^{-1}(U_\alpha)$ to U_α and from $p_2^{-1}(V_\beta)$ to V_β. Taking the product of these homeomorphisms produces a homeomorphism from $p_1^{-1}(U_\alpha) \times p_2^{-1}(V_\beta) = (p_1 \times p_2)^{-1}(U_\alpha \times V_\beta)$ to $U_\alpha \times V_\beta$. Thus $p_1 \times p_2 : \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ is a covering space as well. □

Problem 6. *Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \mathbb{1}$ and $hf \simeq \mathbb{1}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.*

Proof. Composing the first homotopy with h and the second homotopy with g we have $hfg \simeq h$ and $hfg \simeq g$. Since homotopy is transitive we have that $g \simeq h$ so that we must have $gf \simeq \mathbb{1}$ as well and f is a homotopy equivalence.

If fg and hf are homotopy equivalences then there exist maps $k : X \to Y$ and $k' : Y \to X$ such that $k'fg \simeq \mathbb{1}$, $fgk \simeq \mathbb{1}$, $khf \simeq \mathbb{1}$ and $hfk' \simeq \mathbb{1}$. Then we have $k \simeq k'fgk \simeq k'$ so that k and k' are homotopic. Thus f must be a homotopy equivalence. □

Problem 7. *Let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected space spaces X and Y. Show that if $X \simeq Y$ then $\widetilde{X} \simeq \widetilde{Y}$.*

Proof. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be the covering spaces for X and Y in question. We know there exists a map $f : Y \to X$ and a map $g : X \to Y$ such that there is a homotopy $f_t : Y \to X$ taking $fg = f_0$ to $\mathbb{1} = f_1$. Furthermore $f_*(\pi_1(Y)) \subseteq p_*(\pi_1(\widetilde{X}))$. Thus there exists a lift $\widetilde{f} : Y \to \widetilde{X}$ of f and similarly a lift $\widetilde{g} : X \to \widetilde{Y}$ of g. Now $\widetilde{g}p : \widetilde{X} \to \widetilde{Y}$ and $\widetilde{f}q : \widetilde{Y} \to \widetilde{X}$. Furthermore $\widetilde{g}p\widetilde{f}q = \widetilde{g}fq \simeq \mathbb{1}$ using the homotopy f_t. Thus $\widetilde{g}p$ is a homotopy equivalence of $\widetilde{X}$ and $\widetilde{Y}$. □

Problem 8. *Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.*

Proof. Let $f : X \to S^1$ be a map so that we have the inclusion $f_*(\pi_1(X, x_0)) \subseteq \pi_1(S^1) = \mathbb{Z}$. Let $p : \mathbb{R} \to S^1$ be a covering space. Since $\pi_1(X, x_0)$ is finite we must have $f_*(\pi_1(X, x_0))$ is trivial since the only trivial subgroups of $\mathbb{Z}$ are trivial. This implies that $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(\mathbb{R}))$. Now we have a lift $\widetilde{f} : (X, x_0) \to \mathbb{R}$ so there exists a homotopy f_t taking $\widetilde{f}$ to a constant map into $\mathbb{R}$. But then pf_t is a homotopy of f to the constant map. □