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Homework 8

Problem 1. *(a) Let A be a Noetherian UFD. Show that a prime ideal P is of height one if and only if $P = Aa$, with $a \neq 0$ and $a \notin A^*$.*
(b) Let A be a Noetherian integral domain. Show that every $a \in A$, $a \neq 0$, $a \notin A^*$ is a product of irreducible elements. (Recall that $a \in A$, $a \neq 0$, $a \notin A^*$ is irreducible if $a = bc$ implies $b \in A^*$ or $c \in A^*$.)
(c) A Noetherian integral domain is a UFD if and only if every irreducible element is a prime element.

Proof. (a) An integral domain is a UFD if and only every nonzero prime ideal contains a prime element. So suppose P has height one. Then we have a maximal length chain $0 \subsetneq P$. Take $p \in P$ and note that $0 \subsetneq (p) \subseteq P$. Since p is prime, (p) is prime so $P = (p)$. Take $a = p$.

Conversely, suppose $P = Aa$. Let $0 \subsetneq Q \subsetneq P$ be a prime ideal. Then there is some prime $q \in Q$. Since $q \in P$, $q = ab$ for some $b \in A$. Since a and q both generate prime ideals, they're both prime and thus both irreducible. Since $q = ab$, either a or b must be a unit. But $a \notin A^*$ by assumption, so $b \in A^*$ and $Q \subseteq P = Aa = Aq \subseteq Q$. Thus P is minimal over 0 and so ht $P = 1$.

(b) Suppose false and let

$$T = \{Aa \mid a \neq 0, a \notin A^*,\ a \text{ is not a product of irreducibles }\}.$$

Then T is nonempty by assumption, so let $I = Aa$ be a maximal element of T. We can write $a = bc$, $b, c \notin A^*$. Note that $b \notin I$ since if it were then we could write $b = bcx$ and since A is an integral domain, $cx = 1$ and $c \in A^*$, a contradiction. Similarly, $c \notin I$. Thus $I \subsetneq Ab$ and $I \subsetneq Ac$. Thus b and c are both products of irreducibles since I is maximal, which means $a = bc$ is as well. This is a contradiction, so all elements can be written as a product of irreducibles.

(c) Let A be a Noetherian integral domain and suppose A is a UFD. Let p be an irreducible and suppose $p \mid ab$ in A. Then $pc = ab$ for some $c \in A$. Since A is a UFD, we can write a and b as products of irreducibles as $a = up_1 \cdots p_n$, $b = vq_1 \cdots q_m$, $u, v \in A^*$. Because p is irreducible and the product of the irreducibles in ab is unique, we must have p appearing as some p_i or q_i. Without loss of generality, we may assume $a = upp_2 \cdots p_n$, so that $p \mid a$. Thus p is prime.

Conversely, suppose every irreducible element of A is prime. Let P be any nonzero prime ideal and pick $a \in P$. By part (b) we can write $a = p_1 \cdots p_n$ where the p_i are irreducible. Since P is prime, P must contain some p_i. By assumption, p_i is prime. So every nonzero prime ideal contains a prime element, which is equivalent to being a UFD. ☐

Problem 2. *(a) Let A be a Noetherian domain. Show that $a \in A \backslash (\{0\} \cup A^*)$ is contained in a prime ideal of height one.*
(b) Show that a Noetherian integral domain is a UFD if and only if every prime ideal of height one is a principal ideal.

Proof. (a) Suppose false and consider

$$T = \{Aa \mid a \in A\backslash(\{0\} \cup A^*),\ P \in \operatorname{Spec}(A) \text{ with } a \in P \text{ implies ht } P > 1\ \}.$$

Then since A is Noetherian we know T has a maximal ideal $I = Aa$. By part (b) of Problem **??** we know $a = p_1 \cdots p_n$ where each p_i is irreducible. Note that if $p_i \in I$, then $p_i = p_1 \cdots p_n b$ for some $b \in A$. Since A is an integral domain, $1 = p_1 \cdots p_{i-1} p_{i+1} \cdots p_n b$ so p_1 is a unit. But p_1 is irreducible, so this is a contradiction and $p_i \in I$. Thus $I \subsetneq Ap_i$. But since I is maximal with respect to the above property, and $a \in Ap_i$, there must be a prime P_i with $a \in P_i$ and ht $P \leq 1$. Since 0 is prime in A and $a \neq 0$, ht $P = 1$. This is a contradiction so T must be empty and every $a \in A\backslash(\{0\} \cup A^*)$ is contained in a prime ideal of height one.

(b) By part (a) of Problem **??**, if A is a UFD then all height one primes are principal. Conversely, assume all height one primes are principal and take $a \in A$ irreducible. Then a is nonzero and not a unit, so by part (a) it's contained in a prime ideal P of height one. By assumption, $P = Ap$ for some prime $p \in A$. Then $Aa \subseteq Ap$ so $a = pb$ for some $b \in A$. But a is irreducible and p is not a unit, so $b \in A^*$. Thus $Aa = Abp = Ap$ so a is prime. Thus every irreducible element of A is prime, so A is a UFD by part (c) of Problem **??**. $\square$

Problem 3. *Let K be a field and $A = K[x_1, x_2, x_3, x_4, x_5]$. Determine the heights of the following ideals.*
(a) $\sum_{i=1}^4 Ax_i$
(b) $\sum_{i=1}^4 Ax_ix_5$
(c) $Ax_1x_3 + Ax_2x_3 + Ax_1x_4 + Ax_2x_4$
(d) $(Ax_1 + Ax_2) \cap (Ax_3x_5 + Ax_4x_5)$

Proof. (a) By the principal ideal theorem we know the height is less than or equal to 4. But we also have a chain $0 \subseteq Ax_1 \subseteq Ax_2 \subseteq Ax_3 \subseteq Ax_4 \subseteq A$. Thus the height is equal to 4.

(b) Note that each generator is contained in Ax_5, which is of height 1. Since the ideal is not 0, we must have the height equal to 1.

(c) Let $I = Ax_1x_3 + Ax_2x_3 + Ax_1x_4 + Ax_2x_4$. Note that $I \subseteq (Ax_1 + Ax_2) \cap (Ax_3 + Ax_4)$. Each of these has height 2, so $\operatorname{ht} I \leq 2$. Note that since A is a Noetherian UFD, we have every ideal of height 1 is a principal ideal. So if $I \subseteq Aa$ for some $a \in A$, then a is prime and we have $a \mid x_1x_3$ and $a \mid x_2x_4$. So without loss of generality, $a \mid x_1$ and $a \mid x_2$, a contradiction. Thus $\operatorname{ht} I = 2$.

(d) Note that $(Ax_1 + Ax_2) \cap (Ax_3x_4 + Ax_4x_5) = (Ax_1 + Ax_2) \cap (Ax_3 + Ax_4) \cap Ax_5$. So Ax_5 contains this ideal, and since $\operatorname{ht} Ax_5 = 1$ and the ideal is nonzero, it must have height one as well. $\square$

Problem 4. *Let A be a Noetherian ring with* $\dim(A) \geq 2$. *Show that there exist infinitely many prime ideals of height one.*

Proof. If A is not an integral domain, then 0 is not a prime ideal, so there exists some minimal prime ideal $P \neq 0$. Then any prime $\overline{Q} \subseteq A/P$ contains 0 in A/P and no other primes. Then $Q \subseteq A$ contains P and no other primes since P is minimal. So if we can show there are infinitely many prime ideals of height one in A/P, then we're done. So we may assume A is an integral domain.

Take any prime $P \in \operatorname{Spec}(A)$. By part (a) of Problem **??**, we know $a \in P$ is contained in some prime ideal P_a with $\operatorname{ht} P_a = 1$. Therefore $P \subseteq \bigcup_{a \in P} P_a$, a union of height one primes. Now since $\dim(A) \geq 2$, we can take P with $\operatorname{ht} P = 2$. Then since P is contained in a union of height one primes, if this union were finite, then by prime avoidance, P would be contained in some P_a, $a \in P$. But $\operatorname{ht} P = 2 > 1 = \operatorname{ht} P_a$, a contradiction. Thus the union must be infinite, so there are infinitely many height one primes. $\square$

Problem 5. *Let A be a Noetherian ring of* $\dim(A) = n \geq 2$. *Show that for all i, with $1 \leq i \leq n-1$, there exist infinitely many prime ideals P such that* $\operatorname{ht} P = i$ *and* $\dim(A/P) = n - i$.

Proof. Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subsetneq A$, $P_i \in \operatorname{Spec}(A)$ be a chain of primes in A of maximal length $n = \dim(A)$. Note that $\operatorname{ht} P_i = i$ since we've demonstrated a series of length i, and any longer series could be added to $P_{i+1} \subsetneq \cdots \subsetneq P_n$ to make a series with length greater than n. Also, note that $P_i \subsetneq \cdots \subsetneq P_n$ corresponds to a series of primes $0 = \overline{P_i} \subsetneq \cdots \subsetneq \overline{P_n} \subsetneq A/P_i$. Thus $\dim(A/P_i) \geq n - i$. But if there were a longer series then the preimage of each term under the natural projection would contain P_i, so we could add these terms to the series $P_0 \subsetneq \cdots \subsetneq P_i$ and show that $\dim(A) > n$. Thus $\dim(A/P_i) = n - i$.

Now note that if $1 \leq i \leq n-1$ we have $\dim(A/P_{i-1}) = n - i + 1 \geq 2$. Further, consider $\overline{P_{i+1}} \supseteq \overline{P_{i-1}} = 0$ and localize at this prime. Then we have the ring $\overline{A}_{\overline{P_{i+1}}}$ which has dimension $\operatorname{ht} \overline{P_{i+1}} = 2$. By Problem **??**, there are infinitely many prime ideals $\overline{P}_{\overline{P_{i+1}}}$ of height one in this ring. Note that their height being one means that they're strictly contained in the maximal ideal $\overline{P_{i+1}}_{\overline{P_{i+1}}}$ and they strictly contain the 0 ideal $\overline{P_{i-1}}_{\overline{P_{i+1}}}$, since every ideal is contained in and contains these two ideals respectively. Now note that each $\overline{P}_{\overline{P_{i+1}}}$ corresponds to an ideal $\overline{P} \subseteq \overline{A} = A/P_{i-1}$, and $\overline{P} \subsetneq \overline{P_{i+1}} \subsetneq \cdots \subsetneq \overline{P_n} \subsetneq \overline{A}$. Further, $\overline{P} \supsetneq \overline{P_{i-1}} = 0$, so we also have $P_0 \subsetneq \cdots \subsetneq P_{i-1} \subsetneq P$. Taking the preimage under the natural projection of the first chain and adding it to this chain shows gives a chain $P_0 \subsetneq \cdots \subsetneq P_{i-1} \subsetneq P \subsetneq P_{i+1} \subsetneq \cdots \subsetneq P_n \subsetneq A$, for infinitely many primes P. This shows that there are infinitely many primes of height i which are also contained in a chain of length n. This gives $\dim(A/P) = n - i$ by the same reasoning as the P_i case above. $\square$

Problem 6. *Let A be a Noetherian local integral domain with maximal ideal M. Let $x \in M$, $x \neq 0$. Show that* $\dim(A/Ax) = \dim(A) - 1$.

Proof. Take a maximal length chain $0 = P_0 \subsetneq \cdots \subsetneq P_n = M \subsetneq A$ of prime ideals in A. Note that since $x \in M$ and A is local, $x \in P_i$ for some $1 \leq i \leq n$. Let P_{i+1} be the smallest prime in the series such that $x \in P_{i+1}$. That is, $x \notin P_i$ and $x \notin P_{i-1}$. Now quotient by this prime to get $0 = \overline{P_{i-1}} \subsetneq \cdots \subsetneq \overline{P_n} \subsetneq \overline{A} = A/P_{i-1}$. Note that $x \notin P_{i-1}$ so $\overline{x} = x + P_{i-1} \neq 0$. We can now localize at the prime $\overline{P_{i+1}}$ to get $0 = \overline{P_{i-1}}_{\overline{P_{i+1}}} \subsetneq \overline{P_i}_{\overline{P_{i+1}}} \subsetneq \overline{P_{i+1}}_{\overline{P_{i+1}}} \subseteq \overline{A}_{\overline{P_{i+1}}}$. This ring has dimension ht $\overline{P_{i+1}} = 2$ and $\overline{x}$ is still nonzero. By part (a) of Problem **??**, we know $\overline{x} \in \overline{P}_{\overline{P_{i+1}}}$, a height one prime. Note that this prime strictly contains the 0 ideal $\overline{P_{i-1}}_{\overline{P_{i+1}}}$ and is strictly contained in the maximal ideal $\overline{P_{i+1}}_{\overline{P_{i+1}}}$, since it is of height one and all ideals contain and are contained in these two ideals respectively.

Note that $\overline{P}_{\overline{P_{i+1}}}$ corresponds to an ideal $\overline{P} \subseteq \overline{A}$ with $0 = \overline{P_{i-1}} \subsetneq \overline{P} \subsetneq \overline{P_{i+1}} \subsetneq \cdots \subsetneq \overline{P_n} \subsetneq \overline{A}$ and that $\overline{x} \in \overline{P}$. Taking the preimage under the quotient map gives a chain $P_{i-1} \subsetneq P \subsetneq P_{i+1} \subsetneq \cdots \subsetneq P_n \subsetneq A$, with $x \in P$. Further, $P_{i-1} \supsetneq \cdots \supsetneq P_0 = 0$, so we have a chain $0 = P_0 \subsetneq \cdots \subsetneq P_{i-1} \subsetneq P \subsetneq P_{i+1} \subsetneq \cdots \subsetneq P_n \subsetneq A$. This shows that ht $P = i$, since it contains a chain of length i and any longer chain could be added to $P_{i+1} \subsetneq \cdots \subsetneq P_n$ to get a chain longer than $\dim(A) = n$. But note that $x \in P$, so we we've decreased the lowest height of a prime containing x by one. Inductively, we can then find a height one prime P containing x which is part of a maximal length chain. Then taking the quotient by Ax gives a length $\dim(A) - 1$ chain in A/Ax by looking at the quotients of all the primes in this chain. So $\dim(A/Ax) \geq \dim(A) - 1$. But note that we must also have $\dim(A/Ax) \leq \dim(A) - 1$ since ht $Ax = 1$ by the principal ideal theorem so any longer chain in A/Ax could be made into a chain of length greater than $\dim(A)$ by taking the preimages and adding 0. We must then have $\dim(A/Ax) = \dim(A/P) = \dim(A) - 1$. $\square$

Problem 7. *Show that every maximal ideal of $K[x_1, \ldots, x_n]$, K a field (respectively $\mathbb{Z}[x_1, \ldots, x_{n-1}]$) is of height n. Deduce that for all $f \in K[x_1, \ldots, x_n]$, $f \notin K$ (for all $f \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$, $f \neq 0$, $f \neq \pm 1$),* $\dim(K[x_1, \ldots, x_n]/(f)) = n - 1$ *(*$\dim(\mathbb{Z}[x_1, \ldots, x_{n-1}]/(f)) = n - 1$*).*

Proof. For $n = 0$ the only maximal ideal of K is 0, so ht $M = 0$ for all $M \in \operatorname{Max}(K)$. Similarly, if $n = 1$ then the maximal ideals of $\mathbb{Z}$ are $p\mathbb{Z}$ for p prime, and $0 \subseteq p\mathbb{Z}$ shows ht $p\mathbb{Z} = 1$. Now we induct on n. Assume the statement is true for some $n - 1$ and take $M \in \operatorname{Max}(K[x_1, \ldots, x_n])$. Now localize at M and note that ht $M = \dim((K[x_1, \ldots, x_n])_M)$. Now quotient by $(Ax_n)_M$ and use Problem **??** to get

$$\begin{aligned}
\text{ht } M &= \dim((K[x_1, \ldots, x_n])_M) \\
&= 1 + \dim((K[x_1, \ldots, x_n])_M/(Ax_n)_M) \\
&= 1 + \dim((K[x_1, \ldots, x_n]/Ax_n)_M) \\
&= 1 + \dim((K[x_1, \ldots, x_{n-1}])_M) \\
&= 1 + n - 1 \\
&= n
\end{aligned}$$

where we've used the induction hypothesis to note that $\dim((K[x_1, \ldots, x_{n-1})_M) = \text{ht } M = n - 1$. Similarly, if we assume the statement for $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ is true for $n - 2$, then since $\mathbb{Z}[x_1, \ldots, x_{n-1}]$ is a Noetherian integral domain, we can apply Problem **??** to get

$$\begin{aligned}
\text{ht } M &= \dim((\mathbb{Z}[x_1, \ldots, x_{n-1}])_M) \\
&= 1 + \dim((\mathbb{Z}[x_1, \ldots, x_{n-1}])_M/(Ax_{n-1})_M) \\
&= 1 + \dim((\mathbb{Z}[x_1, \ldots, x_{n-2})_M) \\
&= 1 + n - 1 \\
&= n.
\end{aligned}$$

Now since $f \notin K$, $f \in M$ for some $M \in \operatorname{Max}(A)$. Then localize $K[x_1, \dots, x_n]/(f)$ at M and apply Problem **??** to get

$$\begin{aligned}\dim(K[x_1, \dots, x_n]/(f)) &= \text{ht } M \\ &= \dim((K[x_1, \dots, x_n]/(f))_M) \\ &= \dim((K[x_1, \dots, x_n])_M) - 1 \\ &= \text{ht } M - 1 \\ &= n - 1.\end{aligned}$$

The same proof works for $\mathbb{Z}[x_1, \dots, x_{n-1}]$ since this ring satisfies the conditions of Problem **??**. □

Problem 8. *Let $f : A \to B$ be a ring homomorphism of Noetherian local rings with maximal ideals $M_A \subseteq A$ and $M_B \subseteq B$. Suppose $f(M_A) \subseteq M_B$. Show that $\dim(B) \leq \dim(A) + \dim(B/M_A B)$. (Here $M_A B = f(M_A)B$).*

Proof. Suppose $\dim(A) = m$ and $\dim(B/M_A B) = n$. Since A is local, ht $M_A = n$ and we can find an ideal generated by n elements $x_1, \dots, x_n \in M_A$ such that M_A is minimal over $(x_1, \dots, x_n)$. Then $A/(x_1, \dots, x_n)$ has only one prime ideal, which must be the nilradical of this ring. Therefore there is some $p > 0$ such that $M_A^p \subsetneq (x_1, \dots, x_n)$. Similarly, we can find elements $y_1, \dots, y_m \in M_B$ such that $M_B^q \subseteq (y_1, \dots, y_m) + M_A B$ for some $q > 0$. Then we have

$$M_B^{pq} \subseteq ((y_1, \dots, y_m) + M_A B)^p \subseteq M_A^P B + (y_1, \dots, y_m) \subseteq (x_1, \dots, x_n, y_1, \dots, y_m)B \subseteq M_B.$$

Here we've identified $M_A B = f(M_A)B$ so x_i is identified with $f(x_i)$. Then M_B must be minimal over $(x_1, \dots, x_n, y_1, \dots, y_m)$ since M_B is in the radical of this ideal. Since B is local $\dim(B) = \text{ht } M_B$ and by the principal ideal theorem, ht $M_B \leq m + n = \dim(A) + \dim(B/M_A B)$. □

Let A be a commutative ring and E an A-module. We define *dimension of* E ($\dim(E)$) as the dimension of the subspace $\operatorname{Supp}(E)$ of $\operatorname{Spec}(A)$. Thus if E is a finite A-module, $\dim(E) = \dim(V(\operatorname{ann}(E))) = \dim(A/\operatorname{ann}(E))$.

Problem 9. *Let A be a Noetherian local ring with maximal ideal M. Let E be a finite A-module. Then $\dim(E) \leq \infty$. Show that*
(a) $x_1, \dots, x_r$ elements of M with $\ell(E/(x_1 E + \cdots + x_r E)) < \infty$ implies $\dim(E) \leq r$.
(b) $\dim(E)$ is the least integer n such that there exists $x_i \in M$, $1 \leq i \leq n$ with $\ell(E/(x_1 E + \cdots + x_n E)) < \infty$.

Proof. (a) Note that $\ell(E/(x_1 E + \cdots + x_r E))$ implies that $\operatorname{Supp}(E/(x_1 E + \cdots + x_r E)) \subseteq \{M\}$. Note that if $\operatorname{Supp}(E/(x_1 E + \cdots + x_r E)) = \emptyset$, then $E = (x_1 A + \cdots + x_r A)E$. Since $x_1 A + \cdots + x_r A \subseteq M$, is in the Jacobson radical of A and E is finitely generated, then $E = 0$, so $\dim(E) \leq r$. Otherwise, $\operatorname{Supp}(E/(x_1 E + \cdots + x_r E)) = \operatorname{Supp}(E) \cap V(x_1 A + \cdots + x_r A) = \{M\}$. Now note that $\operatorname{Supp}(E) \cap V(x_1 A + \cdots + x_r A) = V(\operatorname{ann}(E)) \cap V(x_1 A + \cdots + x_r A) = V(\operatorname{ann}(E) \cup (x_1 A + \cdots + x_r A))$. So M is minimal over $\operatorname{ann}(E) \cup (x_1 A + \cdots + x_r A)$, which means $M/\operatorname{ann}(E)$ is minimal over $x_1 A + \cdots + x_r A$, which means ht $M/\operatorname{ann}(E) \leq r$ by the principal ideal theorem. But note that $\dim(E) = \dim(A/\operatorname{ann}(E)) = \text{ht } M/\operatorname{ann}(E)$, so $\dim(E) \leq r$.

(b) If $\dim(E) = r$ then ht $M/\operatorname{ann}(E) = \dim(A/\operatorname{ann}(E)) = \dim(E) = r$. Then we can find an ideal $I/\operatorname{ann}(E) \subseteq A/\operatorname{ann}(E)$ generated by r elements such that $M/\operatorname{ann}(E)$ is minimal over $I/\operatorname{ann}(E)$. Then M is minimal over $\operatorname{ann}(E) \cup I$, so $\{M\} = V(\operatorname{ann}(E) \cup I) = V(\operatorname{ann}(E)) \cap V(I) = \operatorname{Supp}(E) \cap V(I) = \operatorname{Supp}(E/IE)$. Thus $\ell(E/IE) < \infty$. Note that I is generated by elements in M since A is local and so none of the generators can be units. □

Problem 10. *Let A be a commutative ring and E a finite A-module. Show that $\dim(E_P) + \dim(E/PE) \leq \dim(E)$ for $P \in \operatorname{Spec}(A)$. (We put $\dim(0) = -\infty$).*

Proof. If $P \notin \mathrm{Supp}(E)$ then $E_P = 0$ so $\dim(E_P) + \dim(E/PE) = \dim(E_P) = -\infty \leq \dim(E)$. Also if $\dim(E) = \infty$, then we're done. So assume $P \in \mathrm{Supp}(E)$ and $\dim(E) < \infty$. Take a maximal length series in A_P, $(\mathrm{ann}(E))_P \subseteq (P_0)_P \subsetneq \cdots \subsetneq (P_n)_P = P_P$. Note that $(P_0)_P \supseteq \mathrm{ann}(E_P) \supseteq (\mathrm{ann}(E))_P$ and that P is maximal in A_P, so $(P_n)_P = P_P$. Now note that $\dim(E/PE) = \dim(\mathrm{Supp}(E) \cap V(P)) = \dim(V(\mathrm{ann}(E)) \cap V(P)) = \dim(V(P))$. Take a maximal length series for this space as $P = Q_0 \subsetneq \cdots \subsetneq Q_m$. Since each of the primes $(P_i)_P$ correspond to primes $P_i \subseteq P$ with $P_i \supseteq \mathrm{ann}(E)$, we have a chain

$$\mathrm{ann}(E) \subseteq P_0 \subsetneq \cdots \subsetneq P_n = P = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_m.$$

This shows that $\dim(E) \geq n + m$. □

Problem 11. *Let A be a nonzero ring and $R = A[x]$. Let P be a prime ideal of A. Show that the maximum cardinality of any totally ordered subset of $\{Q \in \mathrm{Spec}(R) \mid Q \cap A = P\}$ is 2. In other words, any maximal chain of prime ideals lying over a single prime ideal P of A is of length 2.*

Proof. Since we're only concerned with primes lying over P, we need not consider any primes contained in P. In particular, we can consider the quotient ring A/P and take the zero ideal. With that in mind, replace A by an integral domain and take $P = 0$. Suppose we have a chain of length 2, $0 \subsetneq P_1 \subsetneq P_2 \subsetneq R$ such that $P_1 \cap A = P$ and $P_2 \cap A = P$. Now localize at P. Since $P = 0$, A_P is the field of fraction for A and $A_P[x] = R_P$ is a polynomial ring over a field. By Problem **??** we know $\dim(R_P) = 1$, so the longest possible series of primes is $0 \subsetneq Q_P \subsetneq R_P$ for some $Q \in \mathrm{Spec}(R)$ with $Q \cap (A \backslash P) = \emptyset$. But since $P_1 \cap A = 0$ and $P_2 \cap A = 0$, we also have $P_1 \cap (A\backslash P) = \emptyset$ and $P_2 \cap (A \backslash P) = \emptyset$, so we can form the chain $0 \subsetneq (P_1)_P \subsetneq (P_2)_P \subsetneq R_P$, a contradiction. Thus the longest possible chain in R is $0 \subsetneq P_1 \subsetneq R$ with $P_1 \cap A = P$ and $0 \cap A = P$. □

Problem 12. *Let A be a commutative ring. Show that* $\dim(A[x]) \leq 2\dim(A) + 1$.

Proof. Set $\dim(A) = n$. Let $Q_0 \subsetneq \cdots \subsetneq Q_n$ be a chain of prime ideals in $A[x]$. Note that such a chain exists for if we take a maximal length chain $P_0 \subsetneq \cdots \subsetneq P_n \subsetneq A$, then just set $Q_i = P_i[x]$. Now consider the series $Q_0 \cap A \subseteq \cdots \subseteq Q_n \cap A$. From Problem **??**, we know that for each $0 \leq i \leq n$ there can be at most one prime ideal Q_i' with either $Q_i' \subsetneq Q_i$ or $Q_i \subsetneq Q_i'$. Without loss of generality, assume the later for each i. Then the longest possible chain we could make using the Q_i is $Q_0 \subsetneq Q_0' \subsetneq \cdots \subsetneq Q_n \subsetneq Q_n' \subsetneq R$. Note that we started with any arbitrary series of length n. A shorter series would clearly produce a new series less than or equal in length. A longer series would produce $Q_i \cap A = Q_j \cap A$ for some $i \neq j$. Thus, this series is maximal in length for R and it is of length $2n + 1 = 2\dim(A) + 1$. □

Problem 13. *Let A be a Noetherian ring and $R = A[x]$. Show that*
(a) $P \in \mathrm{Spec}(A)$, $P \in \mathrm{Ass}(A)$ if and only if $P[x] \in \mathrm{Ass}(R)$.
(b) Let $Q \in \mathrm{Ass}(R)$. Show that $Q = P[x]$ with some $P \in \mathrm{Ass}(R)$. Thus $\mathrm{Ass}(R) = \{P[x] \mid P \in \mathrm{Ass}(A)\}$.

Proof. (a) Let $P \in \mathrm{Ass}(A)$ so that $P = \mathrm{ann}(y) = \{a \in A \mid ay = 0\}$ for some $y \in A$. Then we certainly have $P[x] \subseteq \mathrm{ann}(y)$ when y is considered as an element of R since any polynomial with coefficients in P will annihilate y on a term by term basis. On the other hand, if we pick any polynomial $a_n x^n + \cdots + a_0 \in R$ with $y(a_n x^n + \cdots + a_0) = y a_n x^n + \cdots + y a_0 = 0$, then we must have $y a_i = 0$ for each $0 \leq i \leq n$. Therefore $a_i \in P$ for each $0 \leq i \leq n$ and this polynomial is in $P[x]$.

Conversely, suppose $P[x] \in \mathrm{Ass}(R)$ so that $P[x] = \mathrm{ann}(p) = \{a \in R \mid ap = 0\}$ where $p = a_n x^n + \cdots + a_0$ is an element of R. Now consider $I = \bigcap_{i=0}^n \mathrm{ann}(a_i) \subseteq A$. If $a \in I$ then $a a_i = 0$ for $0 \leq i \leq n$ so $ap = 0$ and $a \in P[x]$. On the other hand if $a \in P$ then $0 = ap = a a_n x^n + \cdots + a a_0$ so $a a_i = 0$ for $0 \leq i \leq n$ and $a \in I$. Thus $P = I$. But note that if a prime ideal contains an intersection of ideals, it contains their product, and thus contains one of them, by primality. Then since $P \subseteq \bigcap_{i=0}^n \mathrm{ann}(a_i)$, we know $P \subseteq \mathrm{ann}(x_i)$ for each $0 \leq i \leq n$. Therefore $P = \mathrm{ann}(a_i)$ for some i and $P \in \mathrm{Ass}(A)$.

(b) Since A is Noetherian, we know there exists a series of ideals $0 = I_n \subseteq \cdots \subseteq I_0 = A$ with $I_i/I_{i+1} \cong A/P_i$ for $0 \leq i \leq n-1$ where $\mathrm{Ass}(A) \subseteq \{P_0, \ldots, P_{n-1}\}$. Now take the series $0 = I_n[x] \subseteq \cdots \subseteq I_0[x] = R$. Note that

$$I_i[x]/I_{i+1}[x] \cong (I_i/I_{i+1})[x] \cong (A/P_i)[x] \cong A[x]/P_i[x] = R/P_i[x].$$

But now we know that all the associated primes of R must appear among the $P_i[x]$, so $Q = P_i[x]$ for some i. Using the inclusion of part (a), we now have $\operatorname{Ass}(R) = \{P[x] \mid P \in \operatorname{Ass}(A)\}$. ☐

Problem 14. *Let A be a Noetherian ring and $M \in \operatorname{Max}(A)$. Suppose each $x \in M$ is a zero divisor. Show that $M \in \operatorname{Ass}(A)$. Give an example a prime ideal P which consists of zero divisors, but $P \notin \operatorname{Ass}(A)$.*

Proof. Since A is Noetherian, the union of all primes in $\operatorname{Ass}(A)$ is the set of zero divisors of A. Since M is contained in this union of primes and M is an ideal, we know $M \subseteq P$, with $P \in \operatorname{Ass}(A)$ by the prime avoidance lemma. Since M is maximal we must have $M = P$.

Let $A = \mathbb{Z}[x, y, z]/(Axy + Ayz)$ and let $P = \overline{Ax}$. Then P is prime because the preimage is prime in $\mathbb{Z}[x, y, z]$. Note that P consists of zero divisors since $\overline{xy} = \overline{xz} = 0$ and x, y and z are generators. But if $P = \operatorname{ann}(\overline{a})$ then we must have $y \mid a$ and so $\overline{za} = 0$, but $\overline{z} \notin P$. ☐