Sheet 27: Sine and Cosine

Definition 1 *Let*

$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}dx.$$

Definition 2 *For $-1 \leq x \leq 1$ let*

$$A(x) = x\sqrt{1-x^2} + 2\int_{x}^{1} \sqrt{1-t^2}dt.$$

Theorem 3 *For $-1 < x < 1$ the function $A(x)$ is differentiable at x and*

$$A'(x) = \frac{-1}{\sqrt{1-x^2}}.$$

Proof. Let $f(x) = \sqrt{x} = x^{1/2}$. Then

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to\infty} \frac{1}{2\sqrt{x}}$$

which shows that $f(x)$ is differntiable. Note that x is differentiable on $[-1;1]$ and so using products of differentiable functions and the Fundamental Theorem of Calculus we have $A(x)$ is differentiable on $(-1;1)$ (21.10, 22.17). Also we have

$$\begin{aligned} A'(x) &= \frac{1}{2}x\left(1-x^2\right)^{-\frac{1}{2}}(-2x) + \sqrt{1-x^2} - 2\sqrt{1-x^2} \\ &= \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} - 2\sqrt{1-x^2} \\ &= \frac{-x^2+1-x^2-2(1-x^2)}{\sqrt{1-x^2}} \\ &= \frac{-1}{\sqrt{1-x^2}} \end{aligned}$$

from the Chain Rule and the Fundamental Theorem of Calculus (21.16, 22.17). □

Theorem 4 *$A(-1) = \pi$, $A(1) = 0$ and A is decreasing between -1 and 1.*

Proof. We have

$$A(-1) = (-1)\sqrt{1-(-1)^2} + 2\int_{-1}^{1} \sqrt{1-t^2}dt = 0 + \pi = \pi,$$

and

$$A(1) = \sqrt{1-1^2} + 2\int_{1}^{1} \sqrt{1-t^2}dt = 0.$$

Note that for $a \in (-1,1)$ we have $0 \leq a^2 < 1$. Thus

$$A'(a) = \frac{-1}{\sqrt{1-a^2}} < 0$$

which means that A is decreasing between -1 and 1 because its derivative is negative there. □

Definition 5 *For* $0 \leq x \leq \pi$ *let* $\cos x$ *be the unique number such that*

$$A(\cos x) = x.$$

Also let

$$\sin x = \sqrt{1 - (\cos x)^2}.$$

Theorem 6 *For* $0 < x < \pi$ *the following hold:*

$$\cos'(x) = -\sin x$$

$$\sin'(x) = \cos x.$$

Proof. We have

$$A'(\cos x)\cos' x = 1$$

using the inverse function identity from Theorem 3 (27.3). Then $\sin x = \sqrt{1-(\cos x)^2}$ and thus

$$\sin' x = \frac{1}{2}\frac{1}{\sqrt{1-(\cos x)^2}}(-2\cos x)\cos' x = \cos x\left(\frac{-1}{\sqrt{1-(\cos x)^2}}\right)\cos' x = \cos x A'(\cos x)\cos' x = \cos x$$

using the Chain Rule and the above identity (22.16) Also $\cos x = \sqrt{1-(\sin x)^2}$ and thus

$$\cos' x = \frac{1}{2}\frac{1}{\sqrt{1-(\sin x)^2}}(-2\sin x)\sin' x = -\sin x\frac{\sin' x}{\cos x} = -\sin x$$

using the Chain Rule and the fact that $\sin' x = \cos x$ (21.16). □

Exercise 7 *Analyze* cos *and* sin *on* $[0;\pi]$ *(extremal places, monotonicity, convexity etc.)*

Proof. We have cos and sin on $[0;\pi]$ are both functions which map to $[-1;1]$. Then note that $A(-1) = \pi$ so $\cos\pi = -1$. Likewise $A(1) = 0$ and so $\cos 0 = 1$. We know that $A(x)$ and $\cos x$ are inverse functions on $[0,\pi]$ so these values will only be taken on once. Also, $\sin x = \sqrt{1-(\cos x)^2}$ and letting $\sin x = 1$ we have $\cos x = 0$. Then

$$A(0) = 2\int_0^1 \sqrt{1-t^2}dt = \int_{-1}^1 \sqrt{1-x^2}dt = \frac{\pi}{2}$$

because t^2 takes on the same values on $[-1;0]$ as on $[0;1]$. Thus $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$. Note that cos will only take on 0 once on $[0;\pi]$ and so sin takes on 1 only once. Note also that $\sin x$ is defined to be always positive on $[0;\pi]$. Thus the lowest value it could take on is 0. Letting $\sin x = 0$ we have $\cos x = \pm 1$. Thus $\sin 0 = \sin\pi = 0$. Hence cos has a maximum at 0 and a minimum at π and sin has a maximum at $\pi/2$ and a minimum at 0 and π.

We already determined that $\sin x > 0$ on $[0;\pi]$ and so $\cos' x = -\sin x < 0$ on $[0;\pi]$. Thus cos is decreasing on $[0;\pi]$. We also know that $\cos 0 = 1$, $\cos(\pi/2) = 0$ and $\cos\pi = -1$ and since cos is decreasing on $[0;\pi]$, it must be the case that $\cos x > 0$ for $x \in [0;\pi/2]$ and $\cos x < 0$ for $x \in [\pi/2;\pi]$. Thus, since $\sin' x = \cos x$ we have sin is increasing on $[0;\pi/2]$ and decreasing on $[\pi/2;\pi]$.

Finally, we have $\sin'' x = -\sin x$ and since $-\sin x < 0$ for $x \in [0;\pi]$, we have sin is concave down on $[0;\pi]$. Additionally we have $\cos'' x = -\cos x$ and so we have cos is concave down on $[0;\pi/2]$ and concave up on $[\pi/2;\pi]$ based on where cos is positive or negative. □

Definition 8 *For $\pi \leq x \leq 2\pi$ let*

$$\sin x = -\sin(2\pi - x)$$

$$\cos x = \cos(2\pi - x).$$

For $0 \leq x \leq 2\pi$ and a nonzero integer k let

$$\sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x.$$

Definition 9 *For $x \neq k\pi + \pi/2$ let*

$$\sec x = \frac{1}{\cos x}$$

$$\tan x = \frac{\sin x}{\cos x}.$$

For $x \neq k\pi$ let

$$\csc x = \frac{1}{\sin x}$$

$$\cot x = \frac{\cos x}{\sin x}.$$

Exercise 10 *Compute the derivatives of the above functions.*

Proof. We have

$$\sec' x = \left(\frac{1}{\cos x}\right)' = \frac{-\cos' x}{(\cos x)^2} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x,$$

$$\tan' x = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \sin' x - \sin x \cos' x}{(\cos x)^2} = \frac{(\sin x)^2 + (\cos x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \sec^2 x,$$

$$\csc' x = \left(\frac{1}{\sin}\right)' = \frac{-\sin x}{(\sin x)^2} = \frac{1}{\sin x}\frac{-\cos x}{\sin x} = -\csc x \cot x,$$

and

$$\cot' x = \left(\frac{\cos x}{\sin x}\right)' = \frac{\sin x \cos' x - \cos x \sin' x}{(\sin x)^2} = \frac{-((\sin x)^2 + (\cos x)^2)}{(\sin x)^2} = \frac{-1}{(\sin x)^2} = -\csc^2 x$$

using the rules of differentiation (21.13, 21.14). □

Definition 11 *Let* arcsin *be the inverse of* sin *restricted to* $[-\pi/2; \pi/2]$. *Let* arccos *be the inverse of* cos *restricted to* $[0; \pi]$. *Let* arctan *be the inverse of* tan *restricted to* $[-\pi/2; \pi/2]$.

Theorem 12 *For $-1 < x < 1$ we have*

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\arccos'(x) = \frac{-1}{\sqrt{1 - x^2}}$$

and for all x we have

$$\arctan'(x) = \frac{1}{1 + x^2}.$$

Proof. We have

$$\arcsin' x = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-(\sin(\arcsin x))^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\arccos' x = \frac{1}{\cos'(\arccos x)} = \frac{-1}{\sin(\arccos x)} = \frac{-1}{\sqrt{1-(\cos(\arccos))^2}} = \frac{-1}{\sqrt{1-x^2}}$$

and

$$\begin{aligned}
\arctan' x &= \frac{1}{\tan'(\arctan x)} \\
&= \frac{1}{(\sec(\arctan x))^2} \\
&= \frac{1}{\frac{1}{(\cos(\arctan x))^2}} \\
&= \frac{1}{\frac{\sin^2 x+\cos^2 x}{(\cos(\arctan x))^2}} \\
&= \frac{1}{1+\left(\frac{\sin(\arctan x)}{\cos(\arctan x)}\right)^2} \\
&= \frac{1}{1+(\tan(\arctan x))^2} \\
&= \frac{1}{1+x^2}
\end{aligned}$$

from the identity in Theorem 3 (27.3). □

Theorem 13 *Suppose that f has a second derivative everywhere and that*

$$f + f'' = 0$$

$$f(0) = 0$$

$$f'(0) = 0.$$

Then $f = 0$.

Proof. We have $ff' + f'f'' = 0$. Then consider

$$\int_0^x ff' + \int_0^x f'f'' = \int_0^x 0$$

$$\frac{1}{2}f^2(x) - \frac{1}{2}f^2(0) + \frac{1}{2}f'^2(x) - \frac{1}{2}f'^2(0) = 0$$

$$\frac{1}{2}f^2 + \frac{1}{2}f'^2 = 0$$

and since f^2 and f'^2 are both greater than or equal to 0, they must both be 0. Then $f = 0$. □

Theorem 14 *Suppose that f has a second derivative everywhere and that*

$$f + f'' = 0$$

$$f(0) = a$$

$$f'(0) = b.$$

Then $f = b\sin + a\cos$.

Proof. Let $g = f - b\sin - a\cos$. Then $g(0) = a - 0 - a = 0$, $g' = f' - b\cos + a\sin$, $g'(0) = b - b + 0 = 0$ and $g'' = f'' + b\sin + a\cos$ (27.6). Then $g + g'' = f - b\sin - a\cos + f'' + b\sin + a\sin = f + f'' = 0$. Then $g = 0$ and so $f = b\sin + a\cos$ (27.13). $\square$

Theorem 15 *For all x, y we have*

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Proof. Let $f(x) = \sin(x+y)$ for some $y \in \mathbb{R}$. Then $f'(x) = \cos(x+y)$, $f''(x) = -\sin(x+y)$ and $f + f'' = 0$. Also $f(0) = \sin y$ and $f'(0) = \cos y$. Then we have $f(x) = \sin x \cos y + \cos x \sin y$ (27.14). Letting $f = \cos(x+y)$ gives the second identity. $\square$