Kris Harper
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Miklós Abért
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Homework 3

Problem 9.15 *Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.*

Proof. Note that $f(0) = 0$ because $0 \leq |f(0)| \leq 0^2 = 0$. We have $|f(h)/h| \leq |h^2/h| \leq |h|$ which means that $\lim_{h\to 0} f(h)/h = 0$. Thus $f'(0) = 0$. □

This result can be generalized if x^2 is replaced by $|g(x)|$ where g has what property?

We need the fact that $|f(x)| \leq |g(x)|$ and $g(0) = 0$.

Problem 10.2 *Let*

$$f(x) = \sin\left((x^3)(\cos x^3)^{-1}\right)$$

and find $f'(x)$.

Using the chain rule we have

$$f'(x) = \cos\left((x^3)(\cos x^3)^{-1}\right)\left(3(x^5)(\cos x^3)^{-2}(\sin x^3) + 3(x^2)(\cos x^3)^{-1}\right)$$

Problem 10.22 *Let a be a double root of the polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Show that a is a double root of f if and only if a is a root of both f and f'.*

Proof. Let a be a double root of f. Then $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Then $f(a) = (a-a)^2 g(a) = (0)g(x) = 0$ so a is a root of f. Also using the product and chain rules we have $f'(x) = (x-a)^2 g'(x) + 2(x-a)g(x) = (x-a)((x-a)g'(x) + 2g(x))$. Then $f'(a) = (a-a)((a-a)g'(a) + 2g(a)) = 0$ so a is a root of f'. Conversely assume that a is a root of both f and f'. Then $f(a) = f'(a) = 0$. Thus $f(x) = (x-a)g(x)$ for some polynomial function $g(x)$ and $f'(x) = (x-a)g'(x) + g(x)$. But since $f'(a) = 0$ we have $g(a) = 0$. Thus $g(a) = (x-a)h(x)$ for some polynomial function h. But then $f(x) = (x-a)^2 h(x)$. Therefore a is a double root of f. □

Problem 10.29 *Show that it is impossible to have $x = f(x)g(x)$ where f and g are differentiable and $f(0) = g(0) = 0$.*

Proof. Suppose we have the above equality. Then $f(x)g(x) - x = 0$ and we have $f(x)g'(x) + f'(x)g(x) - 1 = 0$. But then for $x = 0$ we have $f(0)g'(0) + f'(0)g(0) - 1 = 0 - 1 = -1 = 0$. This is a contradiction and so the equality must be false. □

Problem 11.11 *A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.*

Proof. Let the two legs of the triangle be h and r. Then the volume of the cone is $V(h,r) = \pi/3r^2h$. But $r^2 = a^2 - h^2$ so $V(h) = \pi/3(ha^2 - h^3)$. Then $V'(h) = \pi/3(a^2 - 3h^2)$ and V' has a zero at $h = a/\sqrt{3}$. Thus the maximum volume is then $V(a/\sqrt{3}) = \pi/3(a^3/\sqrt{3} - a^3/(3\sqrt{3}))$. □

Problem 11.52 *Suppose that* $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ *and* $\lim_{x\to\infty} f'(x)/g'(x) = l$. *For every* $\varepsilon > 0$ *there exists* a *such that*

$$\left|\frac{f'(x)}{g'(x)} - l\right| < \varepsilon$$

for $x > a$. *Show that*

$$\left|\frac{f(x) - f(a)}{g(x) - g(a)} - l\right| < \varepsilon$$

for $x > a$.

Proof. From our assumption we know that $g'(x) \neq 0$ for $x > a$. But then $g(x) - g(a) \neq 0$ for $x > a$ by Rolle's Theorem. Then use the Cauchy Mean Value Theorem to state that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(y)}{g'(y)}$$

for some $y \in (a; x)$. But since $y > a$ we have our desired inequality. □

Conclude that

$$\left|\frac{f(x)}{g(x)} - l\right| < 2\varepsilon$$

for sufficiently large x.

Proof. Note that

$$\frac{f(x)}{g(x)} = \frac{(f(x) - f(a)}{g(x) - g(a)} \frac{f(x)}{f(x) - f(a)} \frac{g(x) - g(a)}{g(x)}$$

and $f(x) - f(a) \neq 0$ and $g(x) - g(a) \neq 0$ for large x because $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. But then we have

$$\lim_{x\to\infty} \frac{f(x)}{f(x) - f(a)} = \lim_{x\to\infty} \frac{g(x)}{g(x) - g(a)} = 1.$$

Then we can make $|f(x)/g(x) - (f(x) - f(a))/(g(x) - g(a)) < \varepsilon$ for large enough x. Using this with the previous inequality we have

$$\left|\frac{f(x)}{g(x)} - l\right| < 2\varepsilon$$

for sufficiently large x. □

Problem 11.56 *If* $|f|$ *is differentiable at* a *and* f *is continuous at* a *then* f *is also differentiable at* a.

Proof. If $f(a) \neq 0$ then by continuity we know that $f = |f|$ or $f = -|f|$ for some region around a which means that f is differentiable at a. Consider a such that $f(a) = 0$. Then a is a minimum point for $|f|$ which means that

$$0 = |f|'(a) = \lim_{h\to 0} \frac{|f(a + h)| - |f(a)|}{h} = \lim_{h\to 0} \frac{|f(a + h)|}{h}$$

which implies that $f'(a) = 0$. □

Problem 11.59 *Show that if* f' *is increasing then every tangent line intersects* f *only once.*

Proof. Let a be in the domain of f. Then the tangent line to f at $(a, f(a))$ is $g(x) = f'(a)(x - a) + f(a)$. Suppose there exists $b \neq a$ such that $g(b) = f(b)$. Then there must exist $x \in (a; b)$ or $x \in (b; a)$ such that $g'(x) = f'(x)$ which means $f'(a) = f'(x)$. But this can't happen since f is increasing. □