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MATH 26700
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Homework 6

Problem 1. *(a) Prove that $SO(n)$ is connected.*
(b) Prove that $O(n)$ is not connected. Do this by first proving that $\det : GL(n, \mathbb{R}) \to \mathbb{R}^*$ *is continuous.*

Proof. (a) Note that $SO(n)$ is the group of rotations in $\mathbb{R}^n$. Any rotation can be specified by picking a point on $(x_1, \ldots, x_n) \in S^{n-1}$ and forming the isometry where $(1, 0, \ldots, 0)$ moves to $(x_1, \ldots, x_n)$. We need $n-1$ angles to specify such a point, so we get a map

$$\varphi : [0, 2\pi]^{n-2} \times [0, \pi] \to SO(n).$$

This map takes an $(n-1)$-tuple of angles and uses sine and cosine to specify a point on S^{n-1}, and then this point corresponds to some rotation of $SO(n)$ which is a matrix with entires sin and cos of the angles from the $(n-1)$-tuple. But note then that since sine and cosine are both continuous, we must have φ is continuous. Since the domain of φ is connected, its image must also be connected. But we've already shown that φ is onto, so $SO(n)$ is connected.

(b) The determinant of $A = [a_{ij}]$ is a continuous function simply because it's a polynomial in the n^2 variables $a_{11}, \ldots, a_{21}, \ldots, a_{nn}$, given by

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i\sigma(j)}.$$

Since polynomials are continuous, the determinant must be. Now note that R^* is not connected since $(-\infty, 0)$ and $(0, \infty)$ provide a separation. The determinant map is surjective since for $a \in \mathbb{R}$ we can form the matrix $a_{11} = a$, $a_{ii} = 1$ for $i \neq 1$ and $a_{ij} = 0$ for $i \neq j$. Then if $O(n)$ were connected, we would have a continuous map from a connected space into a disconnected space, which is a contradiction. $\square$

Problem 2. *Let G be a topological group. Prove that a representation $\rho : G \to GL(n, \mathbb{C})$ is a continuous (by the definition given in class) if and only if ρ is a continuous map, where $GL(n, \mathbb{C})$ is given the standard topology.*

Proof. Suppose ρ is continuous as a representation. Then the map $\varphi : G \times V \to V$ given by $\varphi : (g, v) \mapsto gv$ is continuous. Now, V is n-dimensional, so if we fix a basis for V as $e_1, \ldots, e_n$, then $(g, e_i) \mapsto ge_i$ is a continuous map $G \to V$ for each $1 \leq i \leq n$. The product of theses maps is still continuous. Now fix some g for each component in the product. Then we have a continuous map which takes $g \in G$ to an n-tuple of basis vectors under the image of g. But this is precisely the matrix representation of g, so this is the map $\rho : G \to GL(n, \mathbb{C})$. Thus ρ is continuous as a topological map.

Now suppose ρ is continuous as a topological map. We have n projection maps $\pi_i : GL(n, \mathbb{C}) \to V$ which give the i^{th} column of an element of $GL(n, \mathbb{C})$. Then the maps $\pi_i \rho : G \to V$ are each continuous maps which give the image under g of the i^{th} basis vector of V. Now if we have some vector $v = \sum_{i=1}^{n} a_i e_i$ then we can form the map $(g, v) \mapsto \sum_{i=1}^{n} a_i(\pi_i \rho(g)) = gv$. This is a sum of scaled continuous maps, so it's continuous. Thus ρ is continuous as a representation. $\square$

Problem 3. *Let $\psi : \mathbb{R} \to \mathbb{C}^*$ be a continuous map satisfying for all $s, t \in \mathbb{R}$:*
(a) $\psi(s + t) = \psi(s)\psi(t)$.
(b) $\psi(t) = 1$ for all $t = 2\pi n$, $n \in \mathbb{Z}$.
Prove that there exists $c \in \mathbb{C}^*$ and $\zeta \in \mathbb{C}$ so that $\psi(t) = ce^{t\zeta}$ for all t.

Proof. Define ζ to be the real number such that $\psi(1) = ce^{i\zeta}$ for some $c \in \mathbb{R}$ (since $\psi(1)$ is some complex number it has this form). Then by property (a) we know for any integer n we have $\psi(n) = \psi(1)^n = ce^{n\zeta}$. In particular, $\psi(0) = ce^0 = c = 1$. Now if we have $1/n \in \mathbb{Q}$ then $\psi(1) = \psi(1/n + \cdots + 1/n) = \psi(1/n)^n$ so $\psi(1/n) = \psi(1)^{1/n} = e^{i\zeta/n}$. Then if we have $p/q \in \mathbb{Q}$ we must have $\psi(p/q) = \psi(1/q)^p = \psi(1)^{p/q} = e^{ip\zeta/q}$. Now let $t \in \mathbb{R}$ and pick a sequence of rationals (x_n) converging to t. Then $\lim_{n\to\infty} x_n = t$ and since ψ is continuous $\psi(t) = \lim_{n\to\infty} \psi(x_n) = \lim_{n\to\infty} e^{ix_n\zeta} = e^{it\zeta}$. □

Problem 4. *Let $V_{m,n}$ denote the vector space of the homogenous complex polynomials of degree m in n variables (under addition).*
(a) Extend the case $m = 3$ in class to define a continuous representation

$$\pi_{m,n} : SO(n) \to GL(V_{m,n}).$$

Prove this is indeed a continuous representation.
(b) What is the degree of $\pi_{m,n}$, i.e. what is the dimension of $V_{m,n}$?
(c) For which $\pi_{m,n}$ does there exists an $SO(n)$-invariant vector?

Proof. (a) To show continuity we need to show that the map $(g, p) \mapsto gp$ is continuous for all $g \in G$ and polynomials $p \in V_{m,n}$. Since g acts linearly and $V_{m,n}$ is a space under addition, it's enough to show continuity for monomials p. Note that if $p(x_1, \ldots, x_n) = c\prod_{j=1}^n x_j^{m_j}$ with $\sum_{j=1}^n m_j = m$, and $g^{-1} = [a_{ij}]$ then we have

$$gp = p(g^{-1}(x_1, \ldots, x_n)) = p\left(\sum_{i=1}^n a_{i1}x_1, \ldots, \sum_{i=1}^n a_{in}x_n\right) = c\prod_{j=1}^n \left(\sum_{i=1}^n a_{ij}x_j\right)^{m_j}.$$

To check continuity we need to make sure that small changes in the entires of g and small changes c will result in a small change in gp. Clearly if c changes by δ, then gp will change by a corresponding amount. If the entries g_{ij} in g change by some δ_{ij} then the entires a_{ij} will also change by some small δ'_{ij} since the inverse map is continuous. Then we note that the terms in gp are just polynomial functions in a_{ij}, that is, just summing them and raising them to m_j. This is a continuous function, so the change in gp is small if the δ'_{ij} are small enough. Thus $(g, p) \mapsto gp$ is continuous.

(b) The dimension is given by the number of n-variable monomials of degree m. To count these consider an arbitrary monomial $x_1^{m_1} \cdots x_n^{m_n}$ and take the corresponding multiset of terms

$$\{x_1, x_1, \ldots, x_2, x_2, \ldots, x_n, x_n\}$$

so that the cardinality of this multiset is m. We can group these terms as

$$\{x_1, \ldots, x_1 \mid x_2, \ldots, x_2 \mid x_3, \ldots \mid \ldots, x_{n-1} \mid x_n, \ldots, x_n\}.$$

Now the number of such multisets (and thus, of such monomials) is the number of ways to arrange the $n-1$ vertical bars. But this is just the number of ways to choose an $(n-1)$-sized subset out of an $(n+m-1)$-sized set. Thus the dimension is

$$\binom{n+m-1}{n-1} = \binom{n+m-1}{m}.$$

(c) Note that by definition for $g \in SO(n)$ and $x \in \mathbb{R}^n$ we have

$$\langle gx, gx \rangle = \langle x, x \rangle = x_1^2 + \cdots + x_n^2.$$

So if $p(x_1, \ldots, x_n) = \langle x, x \rangle^k$ for some k, then this vector is fixed under g by definition since g acts by first acting with g^{-1} on $(x_1, \ldots, x_n)$ and then evaluating at $p(x_1, \ldots, x_n)$. Thus, for all even m we have an $SO(n)$-invariant vector $\langle x, x \rangle^{m/2}$. □

Problem 5. *Let $g \in SU(2)$.*
(a) Prove that g must have the form

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.
(b) For $\alpha, \theta \in [0, 2\pi]$ define:

$$s(\alpha) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \quad \text{and} \quad r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Prove that each $g \in SU(2)$ can be decomposed as a product

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = s(-\alpha/2)r(-\beta/2)s(-\gamma/2)$$

for some $\alpha, \gamma \in [0, 2\pi]$ and $\beta \in [0, \pi]$. As a corollary note that $SU(2)$ is generated by all matrices of the form $s(\alpha)$, $r(\beta)$.
(c) Check that the measure

$$dg = (1/8\pi^2)\sin\beta d\alpha d\beta d\gamma$$

is left invariant (it is actually bi-invariant) and has total mass 1, so it is the Haar measure on $SU(2)$.

Proof. (a) Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We know

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \bar{g}^T = g^{-1}\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and $ad - bc = 1$ since $\det(g) = 1$. Then we immediately have $d = \bar{a}$ and $c = -\bar{b}$.

(b) Note that

$$\begin{aligned}
s(-\alpha/2)r(-\beta/2)s(-\gamma/2) &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} \cos(-\beta/2) & \sin(-\beta/2) \\ -\sin(-\beta/2) & \cos(-\beta/2) \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2}\cos(-\beta/2) & e^{i\gamma/2}\sin(-\beta/2) \\ -e^{-i\gamma/2\sin(-\beta/2)} & e^{i\gamma/2}\cos(-\beta/2) \end{pmatrix} \\
&= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(-\beta/2) & e^{i(\gamma-\alpha)/2}\sin(-\beta/2) \\ -e^{-i(\gamma-\alpha)/2}\sin(-\beta/2) & e^{i(\alpha+\gamma)/2}\cos(-\beta/2) \end{pmatrix}.
\end{aligned}$$

Now let $g \in SU(2)$ and from part (a) we know g has the form

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} ce^{i\theta} & de^{i\psi} \\ -de^{i\psi} & ce^{-i\theta} \end{pmatrix}$$

where $\theta, \psi \in [0, 2\pi]$ and $|c|^2 + |d|^2 = 1$. Then (c, d) lies on the unit circle so we can find $\beta \in [0, \pi]$ such that $\cos(-\beta/2) = c$ and $\sin(-\beta/2) = d$. Also, if we let $\alpha = -\theta - \psi$ and $\gamma = \psi - \theta$ then g has the exact form above.

(c) To check that the total mass is 1 we simply find

$$\begin{aligned}
\int_G dg &= \frac{1}{8\pi^2}\int\int\int \sin\beta d\alpha d\beta d\gamma \\
&= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} \sin\beta d\alpha d\beta d\gamma \\
&= \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\gamma \\
&= \frac{1}{8\pi^2}(2\pi)(2)(2\pi) \\
&= 1.
\end{aligned}$$

To show that the measure is left invariant, we note that by part (a) $SU(2)$ is the set of complex numbers $x_1 + ix_2, x_3 + ix_4 \in \mathbb{C}$ such that

$$1 = |x_1 + ix_2|^2 + |x_3 + ix_4|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

But this is exactly the subset $S^3 \subseteq \mathbb{R}^4$. The standard measure on $\mathbb{R}^4$, $dx_1dx_2dx_3dx_4$ is left invariant, so if we can change variables to $r^3 \sin\beta d\alpha d\beta d\gamma dr$, we will show that dg is left invariant.

Now, to find equations for x_1, x_2, x_3 and x_4 in terms of α, β, γ and r we can use part (b) and note that $x_1 + ix_2 = e^{-i(\alpha+\gamma)/2}\cos(-\beta/2)$ and $x_3 + ix_4 = e^{i(\gamma-\alpha)/2 \sin(-\beta/2)}\sin(-\beta/2)$. Thus

$$x_1 = r\cos\left(-\frac{\beta}{2}\right)\cos\left(-\frac{\alpha+\gamma}{2}\right)$$

$$x_2 = r\cos\left(-\frac{\beta}{2}\right)\sin\left(-\frac{\alpha+\gamma}{2}\right)$$

$$x_3 = r\sin\left(-\frac{\beta}{2}\right)\cos\left(\frac{\gamma-\alpha}{2}\right)$$

$$x_4 = r\sin\left(-\frac{\beta}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right)$$

This gives (after some simplification) the sixteen partial derivatives

$$\frac{\partial x_1}{\partial \alpha} = -\frac{r}{2}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right)$$

$$\frac{\partial x_1}{\partial \beta} = -\frac{r}{2}\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)$$

$$\frac{\partial x_1}{\partial \gamma} = -\frac{r}{2}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right)$$

$$\frac{\partial x_1}{\partial r} = \cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)$$

$$\frac{\partial x_2}{\partial \alpha} = -\frac{r}{2}\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)$$

$$\frac{\partial x_2}{\partial \beta} = \frac{r}{2}\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right)$$

$$\frac{\partial x_2}{\partial \alpha} = -\frac{r}{2}\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)$$

$$
\begin{aligned}
\frac{\partial x_2}{\partial r} &= -\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right) \\
\frac{\partial x_3}{\partial \alpha} &= \frac{r}{2}\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right) \\
\frac{\partial x_3}{\partial \beta} &= -\frac{r}{2}\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha-\gamma}{2}\right) \\
\frac{\partial x_3}{\partial \gamma} &= -\frac{r}{2}\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right) \\
\frac{\partial x_3}{\partial r} &= -\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha-\gamma}{2}\right) \\
\frac{\partial x_4}{\partial \alpha} &= \frac{r}{2}\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha-\gamma}{2}\right) \\
\frac{\partial x_4}{\partial \beta} &= \frac{r}{2}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right) \\
\frac{\partial x_4}{\partial \gamma} &= -\frac{r}{2}\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha-\gamma}{2}\right) \\
\frac{\partial x_4}{\partial r} &= \sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\alpha-\gamma}{2}\right).
\end{aligned}
$$

We now transform the differentials using the change of variables formula so $dx_1dx_2dx_3dx_4 = \det(D)d\alpha d\beta d\gamma dr$ where D is the Jacobian of the transformation. Thus

$$
\det(D) = \left|\begin{pmatrix} \frac{\partial x_1}{\partial \alpha} & \frac{\partial x_1}{\partial \beta} & \frac{\partial x_1}{\partial \gamma} & \frac{\partial x_1}{\partial r} \\ \frac{\partial x_2}{\partial \alpha} & \frac{\partial x_2}{\partial \beta} & \frac{\partial x_2}{\partial \gamma} & \frac{\partial x_2}{\partial r} \\ \frac{\partial x_3}{\partial \alpha} & \frac{\partial x_3}{\partial \beta} & \frac{\partial x_3}{\partial \gamma} & \frac{\partial x_3}{\partial r} \\ \frac{\partial x_4}{\partial \alpha} & \frac{\partial x_4}{\partial \beta} & \frac{\partial x_4}{\partial \gamma} & \frac{\partial x_4}{\partial r} \end{pmatrix}\right| = \frac{1}{8}r^3\sin(\beta).
$$

We're fixing $r = 1$ in this case, so this leaves us with $dx_1dx_2dx_3dx_4 = (1/8)\sin(\beta)d\alpha d\beta d\gamma$. Thus, up to a constant, this is the same as dg so it must be left invariant. $\square$