Sheet 28: Primes

Lemma 1 *Let $N > 2$ be an integer. We have*

$$\sum_{i=1}^{N} \frac{1}{i} > \log(N).$$

Proof. Let $P = \{1, 2, 3, \dots, N\}$ be a partition of $[1; N]$ and $f = 1/x$. Note that

$$\log(N) = \int_1^N \frac{1}{t} dt = \inf\{U(f, P) \mid P \text{ is a partition of } [1; N]\} < U(f, P) = \sum_{i=1}^{N} \frac{1}{i}.$$

□

Lemma 2 *If $n > 1$ then*

$$\sum_{i=0}^{N} \frac{1}{n^i} < 1 + \frac{1}{n-1}.$$

Proof. Note that

$$\begin{aligned}
\sum_{i=0}^{N} \frac{1}{n^i} &= \frac{\sum_{i=0}^{N} n^i}{n^N} \\
&= \frac{(n-1)\sum_{i=0}^{N} n^i}{(n-1)n^N} \\
&= \frac{n^{N+1} - 1}{(n-1)n^N} \\
&< \frac{n^{N+1}}{(n-1)n^N} \\
&= \frac{n}{n-1} \\
&= 1 + \frac{1}{n-1}.
\end{aligned}$$

□

Lemma 3 *If $n \geq 2$ then*

$$\frac{1}{n-1} \leq \frac{2}{n}.$$

Proof. Note that since $n \geq 2$ we have $n \leq 2n - 2$ which gives

$$\frac{1}{n-1} \leq \frac{2}{n}.$$

□

Lemma 4 *If $x > 0$ then*

$$\log(1 + x) < x.$$

Proof. This follows from Theorem 16 on Sheet 26 (26.16). $\square$

Lemma 5 *Let $p_1, \ldots, p_k$ be the positive primes less than or equal to N. We have*

$$\prod_{i=1}^{k} \sum_{j=0}^{N} \frac{1}{p_i^j} = \left(1 + \frac{1}{p_1} + \cdots + \frac{1}{p_1^N}\right) \ldots \left(1 + \frac{1}{p_k} + \cdots + \frac{1}{p_k^N}\right) > \sum_{i=i}^{N} \frac{1}{i}.$$

Proof. Note that for each $n \leq N$ there exists a unique prime factorization

$$n = p_{n_1}^{a_1} p_{n_2}^{a_2} \ldots p_{n_j}^{a_j}$$

where $0 \leq n_i \leq k$ and $0 \leq a_i \leq N$ for all i. But then we know that $1/n$ will be in the product

$$\prod_{i=1}^{k} \sum_{j=0}^{N} \frac{1}{p_i^j}$$

since this will contain the reciprocals of all possible combinations of products of primes less than or equal to N raised to powers less than or equal to N. Note also that since $N > 2$ there must be a term in the product whose reciprocal is greater than N. Thus we have the strict inequality

$$\prod_{i=1}^{k} \sum_{j=0}^{N} \frac{1}{p_i^j} > \sum_{i=i}^{N} \frac{1}{i}.$$

$\square$

Theorem 6 *We have*

$$\sum_{i=1}^{k} \frac{1}{p_i} > \frac{1}{2} \log(\log(N)).$$

Proof. We have

$$\begin{aligned}
\frac{1}{2} \log(\log(N)) &< \frac{1}{2} \log\left(\sum_{i=1}^{N} \frac{1}{i}\right) \\
&< \frac{1}{2} \log\left(\prod_{i=1}^{k} \sum_{j=0}^{N} \frac{1}{p_i^j}\right) \\
&< \frac{1}{2} \log\left(\prod_{i=1}^{k} \left(1 + \frac{1}{p_i - 1}\right)\right) \\
&= \frac{1}{2} \sum_{i=1}^{k} \log\left(1 + \frac{1}{p_i - 1}\right) \\
&< \frac{1}{2} \sum_{i=1}^{k} \frac{1}{p_i - 1} \\
&\leq \frac{1}{2} \sum_{i=1}^{k} \frac{2}{p_i} \\
&= \sum_{i=1}^{k} \frac{1}{p_i}
\end{aligned}$$

from Lemmas 1, 2, 3, 4 and 5 (28.1, 28.2, 28.3, 28.4, 28.5). $\square$

Corollary 7 *We have*

$$\sum_{p \text{ is a prime}}^{\infty} \frac{1}{p}$$

is divergent.

Proof. Note that from Theorem 6 we have the partial sums of

$$\sum_{p \text{ is a prime}}^{\infty} \frac{1}{p}$$

are unbounded. Thus

$$\sum_{p \text{ is a prime}}^{\infty} \frac{1}{p}$$

is divergent (13.15). □