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Homework 2

As before A denotes a commutative ring.

Problem 1. *Let A_1, A_2, B be commutative rings and $f_i : A_i \to B$, $i = 1, 2$, be surjective ring homomorphisms. Let $A_1 \times_B A_2$ (called the fiber product of A_1 and A_2 over B) denote the subring of the direct product $A_1 \times A_2$:*

$$A_1 \times_B A_2 = \{(a_1, a_2) \mid f_1(a_1) = f_2(a_2)\}.$$

Suppose A_1 and A_2 are Noetherian rings. Show that $A_1 \times_B A_2$ is also a Noetherian ring.

Proof. Let $\pi : A_1 \times_B A_2 \to A_1$ be the projection map $\pi : (a_1, a_2) \mapsto a_1$. Then $\ker \pi$ is the subset of $A_1 \times_B A_2$ consisting of pairs (a_1, a_2) with $a_1 = 0$ and $a_2 \in A_2$. For each pair we must have $f_2(a_2) = f_1(a_1) = f_1(0) = 0$. Thus, this set is isomorphic to $\ker f_2$. Now consider the exact sequence

$$0 \longrightarrow \ker f_2 \xrightarrow{\iota} A_2 \xrightarrow{f_2} B \longrightarrow 0.$$

Since A_2 is Noetherian, we must have $\ker f_2$ is a Noetherian A_2-module, and thus $\ker \pi$ is a $A_1 \times_B A_2$-module. Now consider $\operatorname{im} \pi$. Since f_2 is surjective, this is just A_1, which is Noetherian by assumption.

Now consider the exact sequence

$$0 \longrightarrow \ker \pi \xrightarrow{\iota} A_1 \times_B A_2 \xrightarrow{\pi} \operatorname{im} \pi \longrightarrow 0.$$

Since the outer two terms are Noetherian $A_1 \times_B A_2$-modules, the middle term must be as well. □

Problem 2. *Let $f : A \to B$ be a ring homomorphism and E a finite B-module. Suppose B is a finite A-module (via f). Show that E is a finite A-module. Deduce that if A is a Noetherian ring, then B is a Noetherian ring and E a Noetherian B-module. E is also a Noetherian A-module.*

Proof. Since E is a finite B-module, we can write $E = Bx_1 + \cdots + Bx_n$ with $x_i \in E$. Furthermore, since B is a finite A-module, we can write $B = Ab_1 + \cdots + Ab_m$ where $b_i \in B$ and $ab_i = f(a)b_i$ for each $a \in A$. Then

$$E = (Ab_1 + \cdots + Ab_m)x_1 + \cdots + (Ab_1 + \cdots + Ab_m)x_n = Ab_1x_1 + \cdots + Ab_mx_1 + \cdots + Ab_1x_n + \ldots Ab_mx_n$$

where $b_ix_j \in E$ and $ab_ix_j = (f(a)b_i)x_j$ for each $a \in A$.

If A is Noetherian, then since B is a finite A-module, B is a Noetherian A-module and thus a Noetherian B-module as well. Since B is Noetherian, and E is a finite B-module, E is a Noetherian B-module as well. We now know E is a finite A-module, so it's a Noetherian A-module as well. □

Problem 3. *Let B be a commutative ring and A a Noetherian subring of B. Suppose B is a finite A-module. Let R be a subring of B with $A \subseteq R$. Show that R is a Noetherian ring.*

Proof. Since B is a finite A-module we have $B = Ax_1 + \cdots + Ax_n$, $x_i \in B$. In particular, B contains all polynomials in the elements x_i with coefficients in A. Thus $B = A[x_1, \ldots, x_n]$. Furthermore, since R contains every element of A and is still a subring of B, we also have $B = Rx_1 + \cdots + Rx_n$. Thus, we are in the situation $A \subseteq R \subseteq B$ are rings, A is Noetherian, B is a finite A-algebra and B is a finite R module. We can then conclude that R is a finitely generated A-algebra and thus Noetherian. □

Problem 4. *Let $A \subseteq B$ be commutative rings. Assume that B is a finitely generated A-algebra, say $B = A[u_1, \ldots, u_m]$, $u_i \in B$. Let $f_i \in A[x]$ be unitary polynomials, $1 \le i \le m$ such that $f_i(u_i) = 0$, $1 \le i \le m$. Show that B is a finite A-module.*

Proof. Let g be an arbitrary polynomial in $A[x]$. Since each f_i is unitary, there exist unique $q, r \in A[x]$ such that $g = qf_i + r$ where $\deg r < \deg f_i$. This is true for each i, so for any polynomial $g_i(u_i)$ in the variables u_i, we have $g_i(u_i) = q(u_i)f_i(u_i) + r(u_i) = q(u_i) \cdot 0 + r(u_i) = r(u_i)$. Since $\deg r < \deg f_i$, we know the highest possible exponent of u_i in $g_i(u_i)$ is $\deg f_i - 1$. Now note that every element of B can be written as a polynomial in the variables u_i. But each polynomial in these variables reduces to one with degree no higher than $\deg f_i - 1$. Therefore B is generated as a finite A-module by the elements $u_1^{i_1}, \ldots, u_m^{i_m}$ where $1 \leq i_j < \deg f_j$. □

Problem 5. *As in Problem 4, let $A \subseteq B = A[u_1, \ldots, u_m]$. Suppose that A is a Noetherian ring and C a subring of B with $A \subseteq C \subseteq B$. Suppose there exist unitary polynomials $f_i \in A[x]$ with $f_i(u_i) \in C$, $1 \leq i \leq m$. Show that C is a finitely generated A-algebra (i.e. there exist $t_1, \ldots, t_r \in C$ such that $C = A[t_1, \ldots t_r]$). In particular, C is a Noetherian ring.*

Proof. Let $R = A[f_1(u_1), \ldots, f_m(u_m)]$ be a finitely generated A-algebra. Note that $f(u_i) \in C$ so $R \subseteq C$. Using a similar argument to Problem 4, we see that C is a finite R-module with the generators $u_i^{i_j}$ where $1 \leq i_j < \deg f_j$. Since C is a finite R module and R is a finite A-algebra, it follows that C is a finite A-algebra. □

Problem 6 (Theorem of Emmy Noether)**.** *Let A be a commutative ring and K a subfield of A. Suppose $A = K[u_1, \ldots, u_m]$ is a finitely generated K-algebra. Let G be a finite group of K-automorphisms of the ring A (i.e. $\sigma(a) = a$, for all $a \in K$). Let $A^G = \{x \in A \mid \sigma(x) = x$ for all $\sigma \in G\}$. (A^G is called the* ring of G-invariants*). Show that A^G is a finitely generated K-algebra. In particular A^G is a Noetherian ring.*

Proof. For each $1 \leq i \leq m$ define $f_i(x) = \prod_{\sigma \in G}(x - \sigma(u_i))$. Now let R be the finite K-algebra generated by all the coefficients of the polynomials f_i. Then for each i, $f_i \in R[x]$ and $f_i(u_i) = 0$ which is in A^G. Also $R \subseteq A^G$ because all the $f_i(x)$ are G-invariant. Using Problem 5 we are able to conclude that A^G is a finite R-algebra. But since R is a finite K-algebra, we also have A^G is a finite K-algebra. □

Problem 7. *Let K be a field and $A = K[x]$. Let R be a subring of A, with $K \subsetneq R$. Show that R is a finitely generated K-algebra. In particular R is Noetherian.*

Proof. Pick any nonconstant monic polynomial $f \in R$ (which we can do since $K \subsetneq R$). Then $f \in K[x]$ and $f(x) \in R$. We can now apply Problem 5 where K, A and R take the place of A, B and C respectively. Thus, R is a finitely generated K-algebra and is Noetherian since K is Noetherian. □

Problem 8. *Let A be a Noetherian ring. Let G be a finite group of automorphisms of the ring A of order n. Suppose $n \cdot 1 \in A^*$. Show that*

$$A^G = \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in G\}$$

is a Noetherian ring.

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of A^G ideals. Then $AI_1 \subseteq AI_2 \subseteq \cdots$ is an ascending chain of A ideals. Since A is Noetherian, we know $AI_n = AI_{n+1}$ for some n. Take $x \in I_{n+1} \backslash I_n$. Then $x \in AI_n$ which is finitely generated so $x = \sum_{i=1}^{m} a_i y_i$, $a_i \in A$, $y_i \in I_n$. Since $x \in A^G$, we know $\sigma(x) = x$ for all $\sigma \in G$. Thus

$$x = \frac{1}{n} \sum_{\sigma \in G} \sigma(x) = \frac{1}{n} \sum_{\sigma \in G} \sum_{i=1}^{m} \sigma(a_i y_i) = \sum_{i=1}^{m} \left(\frac{1}{n} \sum_{\sigma \in G} \sigma(a_j) \right) y_i.$$

Since $\frac{1}{n} \sum_{\sigma \in G} \sigma(a) \in A^G$ for each $a \in A$ and $y_i \in I_n$ for each i, we see that $x \in A^G I_n$ so $x \in I_n$. Thus $I_{n+1} = I_n$. □

Problem 9. *Let $R = A[[x]]$. Then for any $f \in R$ $(1 - fx) \in R^*$. Thus $x \in$* J-rad *A.*

Proof. Let $f \in R$ and consider $g = \sum_{i=0}^{\infty} (fx)^i$. Then

$$g(1 - fx) = g - gfx = \sum_{i=0}^{\infty} (fx)^i - \sum_{i=0}^{\infty} (fx)^{i+1} = 1.$$

Thus $(1 - fx) \in R^*$ and $x \in \text{J-rad}\, A$. $\square$

Problem 10. *Let K be a field and m, n positive integers such that $\gcd(m, n) = 1$. Let $C \subseteq K^2$ be a subset defined by*

$$C = \{(a^m, a^n) \mid a \in K\}.$$

Show that C is an affine algebraic set in K^2.

Proof. Clearly $C \subseteq V(\{x^n - y^m\})$. Now pick $(a, b) \in K^2$ with $a^n - b^m = 0$. Since $\gcd(m, n) = 1$ there exist $p, q \in \mathbb{Z}$ such that $mp + nq = 1$. Then $a^{nq} = b^{mq}$ and $a^{np} = b^{mp}$. Now $a^{1-mp} = b^{mq}$ and $a^{np} = b^{1-nq}$. Multiply the first equation by a^{mp} and the second by b^{nq} to obtain $a = a^{mp} b^{mq} = (a^p b^q)^m$ and $b = a^{np} b^{nq} = (a^p b^q)^n$. Thus $(a, b) = ((a^p b^q)^m, (a^p b^q)^n)$ is in C so $V(\{x^n - y^m\}) \subseteq C$. $\square$

Problem 11. *Let K be a field and $H_i \subseteq K$, $1 \leq i \leq n$ infinite subsets of K. Let $f \in K[x_1, \dots, x_n]$. Suppose $f(a) = 0$ for all $a \in H_1 \times \cdots \times H_n \subseteq K^n$. Show that $f = 0$.*

Proof. We induct on n. In the base case we have a polynomial $f(x) \in K[x]$ which is 0 on an infinite set. It's well known that a nonconstant polynomial f of one variable can have at most $\deg f$ zeros. Thus f must be constantly 0.

Suppose the statement is true for $n-1$ and consider $f \in K[x_1, \dots, x_n]$ which is 0 for all $a \in H_1 \times \cdots \times H_n$. Suppose that f is nonconstant, so there is some variable, say x_n, such that we can write $f = g_0 + g_1 x_n + \cdots + g_m x_n^m$ where $g_i \in K[x_1, \dots, x_{n-1}]$, $m > 0$ and $g_m \neq 0$. Now for $a \in H_1 \times \cdots \times H_n$ we have $f(a) = g_0(a) + g_1(a)a + \cdots + g_m(a)a^m = 0$. Since the a^i terms are linearly independent we must have $g_0(a) = \cdots = g_m(a) = 0$. But by the inductive hypothesis, this means each g_i is constantly 0. Thus f is constantly 0. $\square$

Problem 12. *Let K be an algebraically closed field and $A = K[x_1, \dots, x_n]$, $n \geq 2$. Let $f \in A$, $f \notin K$. Show that $V(f)$ is an infinite set.*

Proof. Since $n \geq 2$ and f is nonconstant, we can write $f = g_0 + g_1 x_n + \cdots + g_m x_n^m$ where each $g_i \in K[x_1, \dots, x_{n-1}]$, $m > 0$ and $g_m \neq 0$. Suppose there are only finitely many points

$$(a_{1,1}, \dots, a_{n-1,1}), \dots, (a_{1,k}, \dots, a_{n-1,k})$$

for which g_m is nonzero. Then let $H_i = K \backslash \{a_{i,1}, \dots, a_{i,k}\}$. Note that each H_i is an infinite subset of K and g_m is zero on $H_1 \times \cdots \times H_n$. By Problem 11 we know $g_m = 0$, but this is a contradiction. Thus there are infinitely many points $(a_1, \dots, a_{n-1}) \in K^{n-1}$ with $g_m(a_1, \dots, a_{n-1}) \neq 0$.

Then, evaluating each g_i at $(a_1, \dots, a_{n-1})$ makes f a polynomial with coefficients in K. Since K is algebraically closed, for each of these points there is some $a_n \in K$ such that $f(a_1, \dots, a_n) = 0$. $\square$

Problem 13. *(a) Let V be an affine algebraic set in $\mathbb{R}^n$. Show that there exists an $f \in \mathbb{R}[x_1, \dots, x_n]$ such that $V(f) = V$.*
(b) Let K be an algebraically closed field and $A = K[x_1, \dots, x_n]$. Show that there does not exist any $f \in A$ such that $V(f) = V(\{x_1, \dots, x_n\}) = \{0\}$.

Proof. (a) Let $V = V(\{f_1, \dots, f_n\})$. Then consider $f = f_1^2 + \cdots + f_n^2$. Since $f_i^2(x) \geq 0$, for all $x \in \mathbb{R}$, we must have $f(x) = 0$ if and only if $f_i(x) = 0$ for all $1 \leq i \leq n$. Thus $V(f) = V$.

(b) The problem is not true for $n = 1$ since $V(\{x_1\}) = \{0\}$. If $n \geq 2$ then apply Problem 12 to see that $V(f)$ is an infinite set and thus not $\{0\}$. $\square$

Problem 14. *Let I be an ideal in a commutative ring A. Show that $\sqrt{I}$ radical I is the intersection of all prime ideals containing I.*

Proof. First we show that the nilradical of a commutative ring A is the intersection of all prime ideals $J = \bigcap_{P \subseteq A} P$. First, note that if $a \in A$ is nilpotent, then $a^n = 0$. Since $0 \in J$, we know $a^n \in P$ for each prime ideal P and thus $a \in P$ for each P.

Conversely, suppose a is not nilpotent. Consider $\Sigma = \{I \subseteq A \mid a^k \notin I, k \in \mathbb{N}\}$. Note that Σ can be partially ordered by inclusion and it's nonempty since it contains the 0 ideal. For any totally ordered chain in Σ, we can take the union of those ideals to obtain an upper bound (it's clear that this ideal will not contain a^k for any $k \geq 1$). Apply Zorn's Lemma and let P be a maximal element for Σ. To show P is prime, pick $xy \in P$ with $x, y \in A$ and suppose $x, y \notin P$. Then $P \subsetneq xA$ and $P \subsetneq yA$ so $a^k \in P + xA$ and $a^j \in P + yA$ for some $k, j \in \mathbb{N}$. Then $x^{j+k} \in P + xyA$, and so $P + xyA \notin \Sigma$, but $P + xyA = P$ since $xy \in P$, which is a contradiction. Thus P is prime. Since $a \notin P$, we must have $a \notin J$. This shows the second inclusion, so the nilradical of any ring is the intersection of all prime ideals in that ring.

Now let I be an arbitrary ideal in a commutative ring A. Note that $\sqrt{I}$ is the preimage of the nilradical of A/I under the natural projection. By the above, we know the nilradical of A/I the intersection of all prime ideals of A/I. By the fourth isomorphism theorem, each prime ideal of A/I is of the form P/I where P is a prime ideal of A containing I. Then, the preimage of this intersection is just the intersection of all ideals $P \subseteq A$ which contain I, as desired. $\square$