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Homework 7

Problem 1. *(a) Let A be a Noetherian ring and E an A-module. Show that the map $\eta : E \to \prod_{P \in \mathrm{Ass}(E)} E_P$, $\eta(x) = (i_p(x))_{P \in \mathrm{Ass}(E)}$ ($i_P(x) = x/1$ in E_P) is injective.*
(b) Let A be a Noetherian ring and I, J ideals in A. Suppose $I_P \subseteq J_P$, for all $P \in \mathrm{Ass}(A/J)$. Show that $I \subseteq J$.

Proof. (a) Let $Q \in \mathrm{Ass}(\ker \eta)$ with $Q = \mathrm{ann}(x)$ so $x \in \ker \eta$. Then $x/1 = 0/s$ in E_P for each $P \in \mathrm{Ass}(E)$. So for each $P \in \mathrm{Ass}(E)$, there exists $t \notin P$ such that $stx = 0$. But since $x \in E$, $Q \in \mathrm{Ass}(E)$ as well which means there exists $s, t \notin Q$ with $stx = 0$, a contradiction since $Q = \mathrm{ann}(x)$. Therefore $\mathrm{Ass}(\ker \eta) = \emptyset$, but this is impossible if $\ker \eta \neq 0$, since A is Noetherian. Thus $\ker \eta = 0$ and η is injective.

(b) By part (a) we have an injective map $\eta : A/J \to \prod_{P \in \mathrm{Ass}(A/J)} (A/J)_P$. Note also that

$$\prod_{P \in \mathrm{Ass}(A/J)} (A/J)_P \cong \prod_{P \in \mathrm{Ass}(A/J)} A_P/J_P$$

since $(A/J)_P \cong A_P/J_P$ using the exact sequence $0 \to J \to A \to A/J \to 0$. Now pick $x \in I$ and let $\overline{x} = x + J$ in A/J. Then $\eta(\overline{x}) = (\overline{x}/1)_{P \in \mathrm{Ass}(A/J)} = (x/1 + J_P)_{P \in \mathrm{Ass}(A/J)}$ in the second product above. But note that since $x \in I$, $x/1 \in I_P$ for all $P \in \mathrm{Ass}(A/J)$. By assumption then, $x/1 \in J_P$ for all $P \in \mathrm{Ass}(A/J)$. But this is 0 in each A_P/J_P, so $\eta(\overline{x}) = 0$. Since η is injective, we know $\overline{x} = 0$, which means $x \in J$. Thus $I \subseteq J$. □

Problem 2. *Let A be a Noetherian local ring with maximal ideal M. Let $P \in \mathrm{Spec}(A)$ with $0 \neq P \neq M$. Compute $\mathrm{Ass}(A/PM)$.*

Proof. Note that A is a local ring, so M is the only maximal ideal. Since all ideals are contained in some maximal ideal, then we have $P \subseteq M$. Thus $P + M = M$ is a maximal ideal of A. Furthermore, $P \cap M = P$. Suppose that $PM = P \cap M = P$. Then since A is Noetherian, P is finitely generated and since M is inside the Jacobson radical of A, we know $P = 0$, a contradiction. Thus $PM \neq P \cap M$. Since we also have $P \neq M$ we can use a previous problem to conclude that $\mathrm{Ass}(A/PM) = \{P, M, P + M\} = \{P, M\}$. □

Problem 3. *Let $f : A \to B$ be a ring homomorphism and E a B-module. Let $\mathrm{Ass}_B(E)$ denote the set of all associated prime ideals of E considered as a B module and $\mathrm{Ass}_A(E) \subseteq \mathrm{Spec}(A)$ denote the set of all associated prime ideals of E considered as an A-module via f. Let $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $f^*(A) = f^{-1}(Q)$.*
(a) Show that $f^*(\mathrm{Ass}_B(E)) \subseteq \mathrm{Ass}_A(E)$.
(b) Suppose further that B is a Noetherian ring. Show that $f^*(\mathrm{Ass}_B(E)) = \mathrm{Ass}_A(E)$.

Proof. (a) Let $P \in \mathrm{Ass}_B(E)$ so that $P = \mathrm{ann}(x)$, $x \in E$. Then $P = \{a \in B \mid ax = 0\}$. Now consider

$$f^*(P) = f^{-1}(P) = \{a \in A \mid f(a) \in \mathrm{ann}(x)\} = \{a \in A \mid f(a)x = 0\} = \mathrm{ann}(x).$$

So $f^*(P)$ is the annihilator of x in A since the action of $a \in A$ on $x \in E$ is $f(a)x$. Thus $P \in \mathrm{Ass}_A(E)$.

(b) Let $P \in \mathrm{Ass}_A(E)$. Now we pass to the localization at P, so we wish to show $P \in f^*(\mathrm{Ass}_{B_P}(E_P))$ where $B_P = S^{-1}B$ and $E_P = S^{-1}E$ with $S = f(A \backslash P)$. Note now that A_P is a local ring with maximal ideal P. Since B is Noetherian there is a correspondence between primes in $\mathrm{Ass}_B(E)$ and $\mathrm{Ass}_{B_P}(E_P)$. So we may assume A is a local ring with maximal ideal P and replace B with B_P and E with E_P. Since $P \in \mathrm{Ass}_A(E)$, we know $P = \mathrm{ann}(x)$, $x \in E$. Now consider Bx. This is a Noetherian B-module since B is Noetherian. Thus $\mathrm{Ass}_B(Bx)$ is nonempty so take $Q \in \mathrm{Ass}_B(Bx)$. Then consider $f^{-1}(Q) = \{a \in A \mid f(a)y = 0, y = bx, b \in B\}$ where $Q = \mathrm{ann}(y)$. Then $f^{-1}(Q) = \{a \in A \mid f(a)bx = 0\} \supseteq \mathrm{ann}(x) = P$. But P is maximal, so we must have $f^{-1}(Q) = P$ and $P \in f^*(\mathrm{Ass}_B(E))$. □

Problem 4. *Let X be a topological space with $X = \bigcup_{i=1}^r X_i$, X_i closed subsets of X. Show that* $\dim(X) = \max(\dim(X_1), \dots, \dim(X_r))$*. Deduce that* $\dim(A/(I_1 \cap \cdots \cap I_r)) = \max(\dim(A/I_1), \dots, \dim(A/I_r))$*.*

Proof. We have $\dim(X_i) + \operatorname{codim}(X_i) \le \dim(X)$, so we must have $\dim(X) \ge \max(\dim(X_1), \dots \dim(X_r))$. Conversely, suppose $\dim(X) = n$ and take F_n to be the largest closed irreducible set in the longest chain. Then note that F_n must be contained in some X_i, since if it's contained in X_i and X_j, then $X_i \cap F_n$ and $X_j \cap F_n$ are two proper closed sets which union to F_n, a contradiction. Thus any chain of closed irreducible sets in X must be contained in some X_i, so $\dim(X) \le \max(\dim(X_1), \dots, \dim(X_r))$.

Note that $\operatorname{Spec}(A/I_i) = V(I_i)$ so

$$\operatorname{Spec}(A/I_1 \cap \cdots \cap I_r) = V(I_1 \cap \cdots \cap I_r) = \bigcup_{i=1}^r V(I_i) = \bigcup_{i=1}^r \operatorname{Spec}(A/I_i).$$

Since $\dim(A/I_1 \cap \cdots \cap I_r) = \dim(\operatorname{Spec}(A/I_1 \cap \cdots \cap I_r))$, we can apply the above result where $X = \operatorname{Spec}(A/I_1 \cap \cdots \cap I_r)$ and $X_i = \operatorname{Spec}(A/I_i)$. □

Problem 5. *(a) Let A be a commutative ring and I, J, ideals of A such that $\sqrt{I} = \sqrt{J}$ show that* $\operatorname{ht} I = \operatorname{ht} J$ *and* $\dim(A/I) = \dim(A/J)$*.*
(b) Deduce from (a) that $\dim(A/(I_1 \cdots I_r)) = \dim(A/I_1 \cap \cdots \cap I_r)$*, where I_i, $1 \le i \le r$ are ideals of A.*

Proof. (a) Note that $V(I) = V(\sqrt{I}) = V(\sqrt{J}) = V(J)$. Since $\operatorname{ht} I = \inf\{\operatorname{ht} P \mid P \in V(I)\}$, we know this is the same as $\operatorname{ht} J = \inf\{\operatorname{ht} P \mid P \in V(J)\}$. Further, $\dim(A/I) = \dim(V(I)) = \dim(V(J)) = \dim(A/J)$.

(b) We simply note that $\sqrt{I_1 \cap \cdots \cap I_r} = \sqrt{I_1 \cdots I_r}$ since if a prime ideal contains $I_1 \cdots I_r$ then P contains $I_1 \cap \cdots \cap I_r$. To see this, take a product $a_1 \cdots a_r \in I_1 \cdots I_r$ and note that if P contains this element, it must contain at least one of the a_i, so it must contain the intersection of all the I_i. □

Problem 6. *Let A be a commutative ring and I an ideal of A. Let $S \subseteq A$ a multiplicative set such that $I \cap S = \emptyset$. Show that*
i) $\dim(S^{-1}A) \le \dim(A)$*.*
ii) $\operatorname{ht} I \le \operatorname{ht} S^{-1}I$*.*
iii) $\operatorname{ht} P = \operatorname{ht} S^{-1}P$ *if* $P \in \operatorname{Spec}(A)$ *and* $P \cap S = \emptyset$*.*
Give an example of A, S and I with $I \cap S = \emptyset$ and $\operatorname{ht} I < \operatorname{ht} S^{-1}I$*.*

Proof. i) Note that $\operatorname{Spec}(S^{-1}A) = \{S^{-1}P \mid P \in \operatorname{Spec}(A), P \cap S = \emptyset\}$. Thus we have a map $\operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$ which takes $S^{-1}P$ to P. This map is a homeomorphism onto its image so $\operatorname{Spec}(S^{-1}A)$ is homeomorphic to a subspace of $\operatorname{Spec}(A)$. Thus $\dim(S{-}1A) \le \dim(A)$.

ii) We have $\operatorname{ht} I = \inf\{\operatorname{ht} P \mid P \in \operatorname{Spec} A, P \supseteq I\}$ and $\operatorname{ht} S^{-1}I = \inf\{\operatorname{ht} S^{-1}P \mid S^{-1}P \in \operatorname{Spec}(S^{-1}A), P \supseteq S^{-1}I\}$. Note that $\{S^{-1} \in \operatorname{Spec}(S^{-1}A) \mid S^{-1}P \supseteq S^{-1}I\} = \{P \in \operatorname{Spec}(A) \mid P \cap S = \emptyset, P \supseteq I\} \subseteq \{P \in \operatorname{Spec}(A) \mid P \supseteq I\}$, so the first infimum is less than the second. To make this claim, we need to know that $S^{-1}P \supseteq S^{-1}I$, then $P \supseteq I$. So take $a \in I$ so that $a/1 \in S^{-1}P$. Then $a/1 = b/s$ for $b \in P$ and $s \notin P$. So there exists $t \notin P$ such that $sta = tb$. Since $tb \in P$, we know either $st \in P$ or $a \in P$, but $st \notin P$ so we have $a \in P$ and $P \supseteq I$.

iii) From part ii) we have $\operatorname{ht} P \le \operatorname{ht} S^{-1}P$. To show the reverse, we simply note that any chain of prime ideals $S^{-1}P_0 \subseteq S^{-1}P_1 \subseteq \cdots \subseteq S^{-1}P$ corresponds to a chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P$ in $\operatorname{Spec}(A)$. Thus the longest such chain in $\operatorname{Spec}(S^{-1}A)$ is shorter than the longest chain in $\operatorname{Spec}(A)$.

Consider the ring $\mathbb{C}[x, y, z]$ and the ideal $I = (x)(y, z)$. Note that $\operatorname{ht} I = 1$ since $(x) \supseteq I$ and $(x) \supseteq 0$ is a chain of primes in $\operatorname{Spec}(\mathbb{C}[x, y, z])$. Now localize at $S = \mathbb{C}[x, y, z] \backslash (x - 1, y, z)$. Then we have a chain of primes $(y, z) \supseteq (z) \supseteq 0$ in $\mathbb{C}[x, y, z]$. But these ideals all avoid S so $S^{-1}(y, z) \supseteq S^{-1}(z) \supseteq S^{-1}0$ is a chain of length 2 in $S^{-1}\mathbb{C}[x, y, z]$. □

Problem 7. *A commutative ring A is called a* semi-local ring *if* $\operatorname{Max}(A)$ *is a finite set.*
(a) Let A be a semi-local ring and E an A-module such that E_M is Noetherian (respectively Artinian) for all $M \in \operatorname{Max}(A)$*. Show that E is a Noetherian A-module (Artinian A-module).*
(b) Let A be a commutative ring such that for all $a \in A$, $a \ne 0$, the ring A/Aa is a Noetherian A-module.

Show that A is a Noetherian ring.
(c) Let A be a commutative ring such that

i) *A_M is a Noetherian ring for all $M \in \mathrm{Max}(A)$,*

ii) *$a \in A$, $a \neq 0$, $a \notin A^*$ implies a is contained in only finitely many maximal ideals.*

Show that A is a Noetherian ring.

Proof. (a) Take an infinite ascending sequence of submodules $E_1 \subseteq E_2 \supseteq \cdots \subseteq E$. We can localize this sequence at each $M \in \mathrm{Max}(A)$ to get a sequence $(E_1)_M \subseteq (E_2)_M \subseteq \cdots \subseteq E_M$. Each E_M is Noetherian, so we can find some n_i such that $(E_{n_i})_{M_i} = (E_{n_i+1})_{M_i}$ for all $M_i \in \mathrm{Max}(A)$. Let $N = \max\{n_i \mid M_i \in \mathrm{Max}(A)\}$. Then we have $(E_n/E_{n+1})_M = (E_n)_M/(E_{n+1})_M = (E_n)_M/(E_n)_M = 0$ for all $M \in \mathrm{Max}(A)$ and all $n \geq N$. Thus $E_n = E_{n+1}$ for all $n \geq N$ and E is Noetherian. The proof for the Artinian case follows by considering a descending chain in place of an ascending chain.

(b) Take an ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots A$. Pick $a \in I_1$ with $a \neq 0$ (if $I_1 = 0$, throw it out and pick $a \in I_2$). Then we can quotient the chain by Aa to get $I_1/Aa \subseteq I_2/Aa \subseteq I_3/Aa \subseteq \cdots A/Aa$. This ring is Noetherian by assumption, so we have $I_n/Aa = I_{n+1}/Aa$ for some n. But there is a one to one correspondence of ideals in A/Aa and A, so each of these ideals corresponds to I_n and I_{n+1} in the preimage. Thus $I_n = I_{n+1}$ and A is Noetherian.

(c) If $a \in A^*$ then $Aa = A$ so $A/Aa = 0$, which is Noetherian. Now suppose $a \neq 0$ and $a \notin A^*$. Then A/Aa is a semi-local ring since the maximal ideals are just those maximal ideals of A which contain a. Note that A_M is Noetherian for each of these maximal ideals, by assumption. Then $A_M/(Aa)_M = (A/Aa)_M$ is Noetherian for each of these finitely many maximal ideals. By part (a) we know A/Aa is a Noetherian A/Aa-module. Since Aa is finitely generated, we know A/Aa is a Noetherian A-module since any submodule can be generated by the generators in A/Aa plus a. Then by part (b) we know A is a Noetherian ring. □

Problem 8. *Let K be a field and*

$$B = K[x_{11}, x_{21}, x_{22}, \ldots, x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots]$$

be a polynomial ring in an infinite number of variables indexed as above. Let $P_n = \sum_{i=1}^n Bx_{ni}$, $n = 1, 2, \ldots$, $P_n \in \mathrm{Spec}(B)$. Let $B_n = K[x_{11}, x_{21}, x_{22}, \ldots, x_{n1}, \ldots, x_{nn}]$. Thus $B_n \subsetneq B_{n+1}$ and $B = \bigcup_{n=1}^\infty B_n$. Note that $P_n \cap B_r = 0$ if $n > r$.
(a) Let $I \subseteq B$ be an ideal such that $I \subseteq \bigcup_{n=1}^\infty P_n$. Show that there exists an $i \geq 1$, such that $I \subseteq P_i$.
(b) Let $S = B \backslash \bigcup_{i=1}^\infty P_i$. Then S is a multiplicative set. Let $A = S^{-1}B$.

i) *Show that $\mathrm{Max}(A) = \{S^{-1}P_i \mid i = 1, 2, \ldots\}$.*

ii) *Show that $(S^{-1}B)_{S^{-1}P_i} = B_{P_i}$ is a Noetherian ring.*

iii) *Show that $\mathrm{ht}\, S^{-1}P_i = \dim(AP_i) = i$.*

iv) *Show that A is a Noetherian ring of infinite dimension.*

Proof. (a) Pick an $r > 0$ with $B_r \cap I \neq 0$ and take $a \in B_r \cap I$. Then $a \in P_{n_1} \cup \cdots \cup P_{n_k}$ for some $k > 0$. Further, we know that $P_n \cap B_r = 0$ for all $n > r$, so we have $a \notin P_n$ for $n > r$. Now take $b \in I$ and suppose $b \in P_{m_1}$ for some $m > r$. Then we consider $a - b$. Since $a \notin P_{m_1}$, but $b \in P_{m_1}$, we must have $a - b \notin P_{m_1}$. Then consider P_{m_2} containing b. By the same logic, we have $a - b - b \notin P_{m_1}$ and $a - b - b \notin P_{m_2}$. We can continue in this way for each P_{m_k} which contains b. So some chain $a - b - \cdots - b$ must be in a prime which contains a. But then $b \in P_{n_i}$ for some $1 \leq i \leq k$. Thus $b \in \bigcup_{i=1}^r P_i$ and $I \subseteq \bigcup_{i=1}^r P_i$. Now just apply the prime avoidance lemma to see that $I \subseteq P_i$ for some $1 \leq i \leq r$.

(b) i) Let $S^{-1}I$ be a proper ideal in A. Then note that $I \subseteq \bigcup_{n=1}^\infty P_n$ in B (since $S^{-1}I$ contains nothing in S). By part (a) we know $I \subseteq P_n$ for some n. But then $S^{-1}I \subseteq S^{-1}P_n$. Therefore any proper ideal in A is contained some $S^{-1}P_i$ which means $\mathrm{Max}(A) \subseteq \{S^{-1}P \mid i = 1, 2, \ldots\}$ since all the maximal ideals are proper and thus equal to some $S^{-1}P_i$. Now take $S^{-1}P_i$ and suppose we have $S^{-1}I \supseteq S^{-1}P_i$ for some $I \in \mathrm{Spec}(B)$.

This gives that $I \supseteq P_i$, but by the above, if I is proper, then $I \subseteq P_j$ for some j. Since P_i and P_j are disjoint if $i \neq j$, we must have $P_i = I$. Thus $S^{-1}P_i = S^{-1}I$ and $S^{-1}P_i \in \text{Max}(A)$.

ii) Let $L_i = K[x_{11}, \ldots, x_{i-1,1}, \ldots, x_{i-1,i-1}, x_{i+1,1}, \ldots, x_{i+1,i+1}, \ldots]$. Then $L_i[x_{i1}, \ldots, x_{ii}]$ is a Noetherian ring. But B_{P_i} is a localization of $L_i[x_{i1}, \ldots, x_{ii}]$ since we've inverted all the x_{ij}, $1 \leq j \leq i$. Since a localization of a Noetherian ring is Noetherian, we have B_{P_i} is Notherian.

iii) We can make a chain in $S^{-1}P_i$ as $(0) \subseteq (x_{i1}/1) \subseteq \cdots \subseteq (x_{ii}/1)$. So ht $S^{-1}P \geq i$. By Krull's generalized PID theorem, we know ht $S^{-1}P \leq i$ since $S^{-1}P$ has i generators. It follows that $\dim(B_{P_i}) = i$ as well.

iv) By part i) we know $S^{-1}P_i$ is a maximal ideal for $i > 0$. By part ii) we know $B_{S^{-1}P_i}$ is a Noetherian ring. Then using the previous problem, we know A is Noetherian ring. But its dimension must be infinite since we have an infinite number of primes, each with a height larger than the previous, so the supremum is not finite. □

Problem 9. *Let A be a ring and $a \in A$, a not in any minimal prime ideal of A. Show that* $\dim(A/Aa) \leq \dim(A) - 1$.

Proof. Since a is not in any minimal prime ideal, we know ht $Aa \geq 1$ since Aa is contained in some prime ideal P and this contains some minimal prime ideal Q. Note that the codimension of Aa is $\dim(V(Aa)) = \dim(A/Aa)$. But we know that $\dim(Aa) + \text{codim}(Aa) \leq \dim(A)$. Making the substitutions above and subtracting gives the inequality. □