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Homework 6

Exercise 3 *Show that Theorem 1 does not hold for the intersection of an infinite number of open sets.*

Proof. We see that for all $a \in C$ we have $\{a\} = C \backslash (C \backslash a)$ is closed since $\{a\}$ is a finite set and so $C \backslash a$ must be open. Now consider a point $p \in C$ and consider the intersection

$$\bigcap_{a \in C, a \neq p} C \backslash a = \{p\}.$$

Since $C \backslash a$ is infinite, this is an intersection of an infinite number of open sets. But their intersection is $\{p\}$ which is closed. □

Exercise 4 *Show that Theorem 2 does not hold for the union of an infinite number of closed sets.*

Proof. Similarly, we take a point $p \in C$ and then consider all the sets containing a single point other than p. Then we have

$$\bigcup_{a \in C, a \neq p} \{a\} = C \backslash p.$$

Since $\{a\}$ is finite, it is closed for all $a \in C$ and since $C \backslash p$ is open and so we have a union of an infinite number of closed sets equaling an open set. □

Corollary 9 *For all $a < b$ both a and b are limit points of the region $(a; b)$.*

Proof. Suppose that there exist $a < b$ such that a is not a limit point of $(a; b)$. Then there exists a region $R = (p; q)$ such that R contains a but contains no points in $(a; b)$. But then $p < a < q$ and we see that $q < b$, otherwise $p < a < b < q$ and so $(a; b) \subseteq R$. Since $a < q$, we see there exists a $c \in C$ such that $a < c < q$. Thus $p < c < q$ and so $c \in R$, but also $a < c < b$ and so $c \in (a; b)$. This is a contradiction.

Similarly, if b is not a limit point of $(a; b)$ then there exists a region $R = (p, q)$ which contains b, but no points in $(a; b)$. But then $p < b < q$ and we see that $a < p$ otherwise $p < a < b < q$ and so $(a; b) \subseteq R$. So we see there exists a $c \in C$ such that $p < c < b$. Thus, $p < c < q$ and so $c \in R$, but also $a < c < b$ and so $c \in (a; b)$. This is a contradiction. □

Corollary 10 *Every point of a region is a limit point of that region.*

Proof. Let A be a region and let $p \in A$. Then we see that for all regions R such that $p \in R$, we have $R \cap A = (a; b) \neq \emptyset$. We know that $p \in (a; b)$ and so there exists a $c \in (a; b)$ such that $a < c < p$. But then for all regions R we have $R \cap (A \backslash p) \neq \emptyset$ and so p is a limit point of A. □

Corollary 11 *Every nonempty region contains infinitely many points*

Proof. Suppose to the contrary that a nonempty region contains a finite number of points. Then it has no limit points. But by Corollary 10 we know that every point is a limit point and so this is a contradiction. □

Corollary 12 *Every point in C is a limit point of C*

Proof. Let $p \in C$. Then we see that every region R which contains p contains infinitely many points and so for all regions R which contain p, we have $R \cap (C \backslash p) \neq \emptyset$. □