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Homework 5

Problem 1. *Find the residue of the following function at* 0*:* $e^z/\sin z$*.*

Proof. Since $(\sin z)' = \cos z \neq 0$ at 0, we know $\text{Res}(e^z/\sin z; 0) = e^0\text{Res}(1/\sin z; 0) = 1/\cos(0) = 1$. □

Problem 2. *(a) Find the integral*

$$\int_C \frac{1}{z^2 - 3z + 5}dz,$$

where C *is a rectangle oriented clockwise, as shown in the figure.*
(b) Find the integral $\int_C 1/(z^2 + z + 1)dz$ *over the same* C*.*
(c) Find the integral $\int_C 1/(z^2 - z + 1)dz$ *over this same* C*.*

Proof. (a) Factoring we find that $z_1 = 3/2 - i/2\sqrt{11}$ and $z_2 = 3/2 + i/2\sqrt{11}$ are the the two zeros of the polynomial. Thus, these are the places where the integrand has a simple pole. Only z_2 is in the interior of C. Then we have $\text{Res}(1/(z^2 - 3z + 5); z_2) = 1/(z_2 - z_1) = -i/\sqrt{11}$. Since C is oriented clockwise we have

$$\int_C \frac{1}{(z^2 - 3z = 5)}dz = -2\pi i\left(\frac{-i}{\sqrt{11}}\right) = \frac{-2\pi}{\sqrt{11}}.$$

(b) We find that the solutions to $z^2 + z + 1$ are $\pm(-1)^{2/3}$, both of which have negative real parts. Thus $1/(z^2 + z + 1)$ is holomorphic on the interior of C and thus $\int_C 1/(z^2 + z + 1)dz = 0$.

(c) Factoring we see that the solutions to $z^2 - z + 1$ are $z_1 = 1/2 - i/2\sqrt{3}$ and $z_2 = 1/2 + i/2\sqrt{3}$. Since only z_2 is in the interior of C we have

$$\int_C \frac{1}{(z^2 - z + 1)}dz = -2\pi i\text{Res}\left(\frac{1}{z^2 - z + 1}; z_2\right) = -2\pi i\left(\frac{1}{z_2 - z_1}\right) = \frac{-2\pi}{\sqrt{3}}.$$

□

Problem 3. *Find the integrals, where* C *is the circle of radius* 8 *centered at the origin.*
(a) $\int_C \frac{1}{\sin z}dz$*.*
(c) $\int_C \frac{1+z}{1-e^z}dz$*.*

Proof. (a) We showed in Problem 1 that $\text{Res}(1/\sin z; 0) = 1$. Then $\int_C 1/\sin z dz = 2\pi i$.

(c) Since $1+z$ is holomorphic, and $(1-e^z)' = -e^z \neq 0$, we have $\text{Res}((1+z)/(1-e^z); 0) = (1+0)\text{Res}(1/(1-e^z); 0) = -e^0 = -1$, $\text{Res}((1+z)/(1-e^z); 2\pi i) = (1+2\pi i)(-1)$ and $\text{Res}((1+z)/(1-e^z); -2\pi i) = (1-2\pi i)(-1)$. Therefore $\int_C (1 + z)(1 - e^z)dz = 2\pi i(-3) = -6\pi i$. □

Problem 4. *Let* a *be real* > 1*. Prove that the equation* $ze^{a-z} = 1$ *has a single solution with* $|z| \leq 1$*, which is real and positive.*

Proof. Let $f(z) = ze^{a-z}$ and $g(z) = f(z) - 1$. Then $|f(z) - g(z)| = 1$ and if $|z| = 1$ with $z = x + iy$, we have $|f(z)| = e^{a-x}$. Since $a > 1$ we see that $|f(z)| > 1$ for $|z| = 1$ which means $|f(z) - g(z)| < |f(z)|$ on unit circle. Now apply Rouché's theorem to see that $ze^{a-x} = 0$ and $ze^{a-x} = 1$ have the same number of solutions on the unit disk. The second equation only has one solution ($z = 0$), and so we see that $ze^{a-z} = 1$ has a single solution with $|z| \leq 1$. Since $0 = f(0) < f(1) > 1$ we know by the intermediate value theorem that this solution must be real and positive. □

Problem 5. *Let f, h be analytic on the closed disc of radius R, and assume that $f(z) \neq 0$ for z on the circle of radius R. Prove that there exists $\epsilon > 0$ such that $f(z)$ and $f(z) + \epsilon h(z)$ have the same number of zeroes inside the circle of radius R. Loosely speaking, we may say that f and a small perturbation of f have the same number of zeros inside the circle.*

Proof. For $\epsilon > 0$ and let $g_\epsilon(z) = f(z) - \epsilon h(z)$. Then $|g_\epsilon(z) - f(z)| \leq \epsilon |h(z)|$. Let C be the boundary of the disk of radius R. We know f is continuous and nonzero on C, so f is bounded away from 0 on C. That is, there exists $\delta > 0$ such that $|f(z)| > \delta$ on C. But also, h is continuous on C and so there exists $\epsilon > 0$ such that $\epsilon|h(z)| < \delta$ for $z \in C$. But now

$$|g_\epsilon(z) - f(z)| < \epsilon|h(z)| < \delta < |f(z)|$$

on C. But then $f(z)$ and $g_\epsilon = f(z) - \epsilon h(z)$ have the same number of zeros on $D_R(0)$ and its easy to see that this also applies to $f(z) + \epsilon h(z)$. □

Problem 6. *Let $P_n(z) = \sum_{k=0}^{n} z^k/k!$. Given R, prove that P_n has no zeros in the disc of radius R for all n sufficiently large.*

Proof. Let $f(z) = e^z$. Note that $f(z)$ has no zeros in $\mathbb{C}$ so $|f(z)| > \delta_R > 0$ for some δ_R and all $z \in \overline{D}_R(0)$. But since P_n converges to f uniformly on $\overline{D}_R(0)$. Thus for sufficiently large n, $|f(z) - P_n(z)| \leq \delta_R < |f(z)|$. Since f has no zeros on $D_R(0)$, we see that P_n doesn't either. □

Problem 7. *(a) $\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx = 2\pi/3$.*
(b) Show that for a positive integer $n \geq 2$,

$$\int_0^\infty \frac{1}{1+x^n} dx = \frac{\pi/n}{\sin \pi/n}.$$

Proof. (a) We know the integrand is bounded by $B/|x|^6$ for some constant B. We thus calculate the residues in the upper half plane of $1/(1+x^6)$. This function has poles at the 6th roots of unity, and $e^{i\pi/6}$, $e^{i\pi/2}$ and $e^{5i\pi/6}$ are the three in the upper half plane. Since these are all simple poles, we can take the derivative of $1+x^6$ to find the residues. In particular

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx = 2\pi i \left(\frac{1}{6(e^{i\pi/6})^5} + \frac{1}{6(e^{i\pi/3})^5} + \frac{1}{6(e^{5i\pi/6})^5} \right) = \frac{\pi i}{3} \left(-\frac{\sqrt{3}}{2} - \frac{i}{2} - i - \frac{i}{2} + \frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3}.$$

(b) Let γ be the segment from 0 to R, η be the arc from R to $Re^{2\pi i/n}$ and γ' be the segment from $Re^{2\pi i/n}$ to 0. The integral over η tends to 0 as R becomes large since it's bounded by $||f||_\eta$ and the length of η (which goes to 0 as R gets large since $n \geq 2$). The only pole of $1/(1+z^n)$ in the interior of $\gamma + \eta + \gamma'$ is $e^{i\pi/n}$ which is a simple pole. Taking the derivative of $1 + x^n$ and putting this value in we see that $\mathrm{Res}(1/(1+x^n); e^{i\pi/n}) = (1/n)e^{-(n-1)i\pi/n} = -1/ne^{i\pi/n}$.

On the other hand, we can parameterize γ' by $te^{2\pi i/n}$ where $0 \leq t \leq R$ and then we find

$$\int_{\gamma'} = \frac{1}{1+x^n} dx = -e^{2\pi i/n} \int_\gamma \frac{1}{1+x^n} dx.$$

Now use the residue formula and the fact that $\gamma + \eta + \gamma'$ is a closed path so that

$$(1 - e^{2\pi i/n}) \int_0^\infty \frac{1}{1+x^n} dx = 2\pi i \left(\frac{-1}{n} e^{\pi i/n} \right).$$

After dividing we have

$$\frac{e^{i\pi/n} - e^{-i\pi/n}}{2i} \int_0^\infty \frac{1}{1+x^n} dx = \frac{\pi}{n}.$$

Use the fact that $\sin z = (e^{iz} - e^{-iz})/(2i)$ to conclude the result. □

Problem 8. *(a)* $\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \pi e^{-a}$ *if* $a > 0$.
(b) For any real number $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \pi e^{-a}/a.$$

Proof. (a) Note that the integrand is bounded by $K/|x|^2$ for some constant K. We now must find the residues for poles in the upper half plane. But this function only has poles at i and $-i$, and the pole at $-i$ has residue $e^{iai}(1/(2i))$. Thus the integral is $2\pi i e^{-a}/(2i) = \pi e^{-a}$.

(b) Let $x = ay$. Then using part (a) we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{1}{a} \int_{-\infty}^{\infty} \frac{\cos(ay)}{y^2 + 1} dy = \frac{1}{a} \mathrm{Re} \left(\int_{-\infty}^{\infty} \frac{e^{iay}}{y^2 + 1} dy \right) = \frac{\pi e^{-a}}{a}.$$

□

Problem 9. *(a)* $\int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx = \pi^3/8$.
(b) $\int_0^{\infty} \frac{\log x}{(x^2+1)^2} dx = -\pi/4$.

Proof. (a) Let γ be the segment from δ to R, γ' be the segment from $-R$ to $-\delta$, $S(\delta)$ be the half circle in the upper half plane centered at 0 with radius δ and let $S(R)$ be defined similarly. Then let $C = \gamma + S(R) + \gamma' + S(\delta)$. Note that the integrand only has a pole at i in C and this pole is simple. We find the residue to be $(\log(i))^2(1/2i) = -\pi^2/8i$. Then we have

$$\int_C \frac{(\log x)^2}{1 + x^2} dx = \frac{-\pi^3}{4}.$$

Now we can break up C into it's component paths so that

$$\int_{\delta}^{\infty} \frac{(\log x)^2}{1 + x^2} dx + \int_{-\delta}^{-\infty} \frac{(\log x)^2}{1 + x^2} dx = \int_{S(\delta)} \frac{(\log x)^2}{1 + x^2} dx - \int_{S(R)} \frac{(\log x)^2}{1 + x^2} dx + \frac{\pi^3}{4}.$$

If we take the limit as R goes to infinity and as δ goes to 0, we get twice the desired integral, so dividing by 2 gives us $\pi^3/8$.

(b) We use the same method as in part (a). The only pole inside C is at i. Splitting the integral into it's component paths, we see that the integral evaluates to $-\pi/4$. □

Problem 10. *Let* $U \subseteq C$ *be open and connected and let* $f_n \in H(U)$ *converge to* f *uniformly on compact subsets of* U. *If none of* f_n *has a root in* U, *prove that either* f *has no root in* U *or* $f \equiv 0$ *on* U.

Proof. Suppose that f is not constantly 0 on U. Then f is bounded away from 0 by some δ on any compact subset $K \subseteq U$. Since f_n converges uniformly to f on K, there exists N such that $|f_N(z) - f(z)| < \delta$ for all $z \in K$. But then $|f_N(z) - f(z)| < |f(z)|$ for all $z \in K$ and so $f(z)$ has no roots in K since f_N doesn't either. Since we can take compact subsets to be arbitrarily large in U, this must be true for all points in U. □