Kris Harper

MATH 25900

Quiz 3

Consider $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Referring to #17 and #18 in D&F page 582, compute each of the following and explain your answers briefly:

Problem 1. *(a)* $\mathrm{N}_{K/\mathbb{Q}}(\sqrt{2})$.
(b) $\mathrm{N}_{K/\mathbb{Q}}(\sqrt{6})$.
(c) $\mathrm{N}_{K/\mathbb{Q}}(\sqrt{2} + \sqrt{3})$.
(d) $\mathrm{N}_{K/\mathbb{Q}}(2)$.

Proof. (a) The subgroup we're interested in is simply all of $\mathrm{Gal}(K/\mathbb{Q}) = \{1, \sigma, \tau, \sigma\tau\}$ where $\sigma : \sqrt{2} \mapsto -\sqrt{2}$ and $\tau : \sqrt{3} \mapsto -\sqrt{3}$. Then

$$\mathrm{N}_{K/\mathbb{Q}}(\sqrt{2}) = \sqrt{2}\sigma(\sqrt{2})\tau(\sqrt{2})\sigma(\tau(\sqrt{2})) = (\sqrt{2})(-\sqrt{2})(\sqrt{2})(-\sqrt{2}) = 4.$$

(b) Note $\sqrt{6} = \sqrt{2}\sqrt{3}$ so

$$\mathrm{N}_{K/\mathbb{Q}}(\sqrt{6}) = \sqrt{6}\sigma(\sqrt{6})\tau(\sqrt{6})\sigma(\tau(\sqrt{6})) = (\sqrt{6})(-\sqrt{2}\sqrt{3})(-\sqrt{3}\sqrt{2})(-\sqrt{2})(-\sqrt{3}) = 36.$$

(c) We have

$$\mathrm{N}_{K/\mathbb{Q}}(\sqrt{2} + \sqrt{3}) = (\sqrt{2} + \sqrt{3})(\sigma(\sqrt{2} + \sqrt{3}))(\tau(\sqrt{2} + \sqrt{3}))(\sigma(\tau(\sqrt{2} + \sqrt{3})))$$
$$= (\sqrt{2} + \sqrt{3})(-\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})(-\sqrt{2} - \sqrt{3}) = 1.$$

(d) Note $2 \in \mathbb{Q}$ so it's fixed by all the elements of $\mathrm{Gal}(K/\mathbb{Q})$. Thus $\mathrm{N}_{K/\mathbb{Q}}(2) = 16$. $\square$

Problem 2. *(a)* $\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{2})$.
(b) $\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{6})$.
(c) $\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{2} + \sqrt{3})$.
(d) $\mathrm{Tr}_{K/\mathbb{Q}}(2)$.

Proof. (a) We have

$$\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{2}) = \sqrt{2} + \sigma(\sqrt{2}) + \tau(\sqrt{2}) + \sigma(\tau(\sqrt{2})) = \sqrt{2} - \sqrt{2} + \sqrt{2} - \sqrt{2} = 0.$$

(b) Note $\sqrt{6} = \sqrt{2}\sqrt{3}$ so

$$\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{6}) = \sqrt{6} + \sigma(\sqrt{6}) + \tau(\sqrt{6}) + \sigma(\tau(\sqrt{6})) = \sqrt{6} - \sqrt{2}\sqrt{3} - \sqrt{3}\sqrt{2} + \sqrt{2}\sqrt{3} = 0.$$

(c) We have

$$\mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{2} + \sqrt{3}) = (\sqrt{2} + \sqrt{3}) + \sigma(\sqrt{2} + \sqrt{3}) + \tau(\sqrt{2} + \sqrt{3}) + \sigma(\tau(\sqrt{2} + \sqrt{3}))$$
$$= (\sqrt{2} + \sqrt{3}) + (-\sqrt{2} + \sqrt{3}) + (\sqrt{2} - \sqrt{3}) + (-\sqrt{2} - \sqrt{3}) = 0.$$

(d) Note $2 \in \mathbb{Q}$ so it's fixed by all the elements of $\mathrm{Gal}(K/\mathbb{Q})$. Thus $\mathrm{Tr}_{K/\mathbb{Q}}(2) = 8$. $\square$