Sheet 13: Sequences

Definition 1 (Sequence) *A sequence of real numbers is a function from* $\mathbb{N}$ *to* $\mathbb{R}$.

Definition 2 (Limit) *We say that a sequence* (a_n) *converges to* $a \in \mathbb{R}$ *or*

$$\lim_{n\to\infty} a_n = a$$

if for every region R *containing* a*, there are only finitely many* $n \in \mathbb{N}$ *with* $a_n \notin R$*. We call* a *the limit of* (a_n).

Lemma 3 *For a sequence* (a_n) *we have* $\lim_{n\to\infty} a_n = a$ *if and only if for all* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for all* $n > N$ *we have* $|a_n - a| < \varepsilon$.

Proof. Let (a_n) be a sequence and suppose that $\lim_{n\to\infty} a_n = a$. Then for every region R containing a there exist finitely many $n \in \mathbb{N}$ with $a_n \notin R$. Let $\varepsilon > 0$ and consider the region $(a - \varepsilon; a + \varepsilon)$. We know there are finitely many $n \in \mathbb{N}$ such that $a_n \notin (a - \varepsilon; a + \varepsilon)$. Since there are finitely many of these elements we know there exists a greatest $N \in \mathbb{N}$ such that $a_N \notin (a - \varepsilon; a + \varepsilon)$. Thus, for all $n \in \mathbb{N}$ such that $n > N$ we have $a_n \in (a - \varepsilon; a + \varepsilon)$ and so $|a_n - a| < \varepsilon$.

Conversely, suppose that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n - a| < \varepsilon$. Let R be a region and let $a \in R$. Let $R = (a - p; a + q)$. Let $\varepsilon = \min(p, q)$ so that there exists some $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n \in (a - \varepsilon; a + \varepsilon)$. But then there exists only finitely many $n \in \mathbb{N}$ such that $a_n \notin (a - \varepsilon; a + \varepsilon)$ and therefore finitely many $a_n \notin R$. $\square$

Exercise 4 *Are the following sequences convergent? If yes, what do they converge to?*
1) $a_n = c$ *for* $c \in \mathbb{R}$*;*
2) $a_n = (-1)^n$*;*
3) $a_n = 1/n$*;*
4) $a_n = (-1)^n/n$.

1) Convergent.

Proof. For all $n \in \mathbb{N}$ we have $a_n = c$ and so every region containing c will include every element of (a_n). Thus, for every region R such that $c \in R$ we have a finite number of $n \in \mathbb{N}$ such that $a_n \notin R$. $\square$

2) Divergent.

Proof. For all $a \in \mathbb{R}$ there exists a region R with $a \in R$ such that $-1 \notin R$ or $1 \notin R$. Consider the case where $-1 \notin R$. Then for all $n \in \mathbb{N}$ such that n is odd we have $a_n \notin R$. But there are an infinite number of odd naturals. A similar case holds for $1 \notin R$ and even naturals. $\square$

3) Convergent.

Proof. We have for all $n \in \mathbb{N}$, $a_n \in (0; 1]$. Let $\varepsilon > 0$. In the case where $\varepsilon > 1$ then for all $n \in \mathbb{N}$ we have $|a_n| < \varepsilon$. If $\varepsilon = 1$ then for all $n \in \mathbb{N}$ with $n > 1$ we have $|a_n| < \varepsilon$. In the case where $\varepsilon \leq 1$ we have $1/\varepsilon \geq 1$ and by the Archimedean Property and the Well Ordering Principle there exists a least $k \in \mathbb{N}$ such that $k > 1/\varepsilon > k - 1$. Then $1/k < \varepsilon < 1/(k-1)$ and so for all $n \in \mathbb{N}$ with $n > k - 1$ we have $|a_n| < \varepsilon$. Thus, $\lim_{n\to\infty} a_n = 0$. $\square$

4) Convergent.

Proof. We have for all $n \in \mathbb{N}$, $a_n \in [-1; 1]$. Thus for all $n \in \mathbb{N}$, $|a_n| \in (0; 1]$. From here we use a similar proof to 3) since we need to show that there exists some $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n| < \varepsilon$. This is exactly what we did in 3). Thus, $\lim_{n\to\infty} a_n = 0$. $\square$

Theorem 5 *The following hold. 1) If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} a_n = a'$ *then* $a = a'$*;*
2) If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$ *then* $\lim_{n\to\infty}(a_n + b_n) = a + b$*;*
3) If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$ *then* $\lim_{n\to\infty}(a_n b_n) = ab$*;*
4) If $\lim_{n\to\infty} a_n = a$ *and* $c \in \mathbb{R}$ *then* $\lim_{n\to\infty}(ca_n) = ca$*;*
5) If $\lim_{n\to\infty} a_n = a \neq 0$ *and* $a_n \neq 0$ *for all* n *then* $\lim_{n\to\infty}(1/a_n) = 1/a$*;*
6) If $a_n \leq b_n$ *for all* n *and both* (a_n) *and* (b_n) *are convergent then* $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Proof. 1) Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = a'$ and suppose $a \neq a'$. Without loss of generality let $a < a'$. Consider $0 < (a' - a)/2$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n > N_1$ we have $|a - a_n| < (a' - a)/2$ and for all $n > N_2$ we have $|a' - a_n| < (a' - a)/2$. Let $N = \max N_1, N_2$ so that for all $n > N$ we have $a_n \in (a - (a' - a)/2; a + (a' - a)/2)$ and $a_n \in (a' - (a' - a)/2; a + (a' - a)/2)$. But these regions are disjoint so this is a contradiction and $a = a'$. ☐

Proof. 2) Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ and consider $\varepsilon/2 > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n > N_1$ we have $|a - a_n| < \varepsilon/2$ and for all $n > N_2$ we have $|b - b_n| < \varepsilon/2$. Let $N = \max(N_1, N_2)$ so that for all $n > N$ we have $|a - a_n| < \varepsilon/2$ and $|b - b_n| < \varepsilon/2$. But by Lemma 11.8 this means we have $|(a + b) - (a_n + b_n)| < \varepsilon$ for all $n > N$. This implies that $\lim_{n\to\infty}(a_n + b_n) = a + b$. ☐

Proof. 3) Let (a_n) converge to a and (b_n) converge to b. Let $\varepsilon > 0$ and consider $\min\left(1, \frac{\varepsilon}{2(|b|+1)}\right) > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that for all $n > N_1$ we have $|a - a_n| < \min\left(1, \frac{\varepsilon}{2(|b|+1)}\right)$. Also, there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$ we have $|b - b_n| < \frac{\varepsilon}{2(|a|+1)}$. Let $N = \max(N_1, N_2)$ so that for all $n > N$ we have $|a - a_n| < \min\left(1, \frac{\varepsilon}{2(|b|+1)}\right)$ and $|b - b_n| < \frac{\varepsilon}{2(|a|+1)}$. But then we know that for all $n > N$ we have $|ab - a_n b_n| < \varepsilon$. Thus $\lim_{n\to\infty}(a_n b_n) = ab$. ☐

Proof. 4) Let (a_n) converge to a. From Exercise 4 we know that $\lim_{n\to\infty} c = c$ and so from 3) we have $\lim_{n\to\infty}(ca_n) = ca$. ☐

Proof. 5) Let (a) converge to $a \neq 0$ such that $a_n \neq 0$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and consider $\min\left(\frac{|a|}{2}, \frac{\varepsilon|a|^2}{2}\right) > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|a - a_n| < \min\left(\frac{|a|}{2}, \frac{\varepsilon|a|^2}{2}\right)$. But then we have $\left|\frac{1}{a} - \frac{1}{a_n}\right| < \varepsilon$ for all $n > N$. Thus $\lim_{n\to\infty}(1/a_n) = 1/a$. ☐

Proof. Let (a) converge to a and (b_n) converge to b such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Suppose to the contrary that $a > b$. Let $\varepsilon = (a - b)/2 > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n > N_1$ we have $a_n \in (a - \varepsilon; a + \varepsilon)$ and for all $n > N_2$ we have $b_n \in (b - \varepsilon; b + \varepsilon)$. Let $N = \max(N_1, N_2)$ so that for all $n > N$ we have $a_n \in (a - (a - b)/2; a + (a - b)/2) = ((a + b)/2; (3a - b)/2)$ and $b_n \in (b - (a - b)/2; b + (a - b)/2) = ((3b - a)/2; (a + b)/2)$. But then $b_n < (a + b)/2 < a_n$ for all n which is a contradiction therefore $a \leq b$. ☐

Theorem 6 *Let* $A \subseteq \mathbb{R}$ *be a subset. Then* $a \in \overline{A}$ *if and only if there exists a sequence* $a_n \in A$ *that converges to* a.

Proof. Let $a \in \overline{A}$. Then we have $a \in A$ or a is a limit point of A. If $a \in A$ then we let $a_n = a$. This converges to a using a similar proof to 1) of Exercise 4. If a is a limit point of A and R is a region containing a then from Theorem 3 we have $R \cap A$ is infinite. Define (a_n) as follows. Choose $a_1 < a$ from $R \cap A$. Now let $a_2 \in (a - \frac{a-a_1}{2}; a + \frac{a-a_1}{2})$ such that $a_1 < a_2 < a$. Continue in this way so that $a_n \in (a - \frac{a-a_{n-1}}{2}; a + \frac{a-a_{n-1}}{2})$ and $a_{n-1} < a_n < a$. Now consider some region $(p; q)$ such that $a \in (p; q)$. In the case where $p < a_1$ there are no elements of (a_n) outside of $(p; q)$. If $a_1 < p$ then take the smallest $k \in \mathbb{N}$ such that $p < a_k$. Then $a_{k-1} \leq p$. Since there are a finite number of naturals less than k and every other natural maps to something between a_{k-1} and a, there are a finite number of $n \in \mathbb{N}$ such that $a_n \notin (p; q)$. We see that in all cases $\lim_{n\to\infty} a_n = a$.

Conversely suppose there exists a sequence $a_n \in A$ that converges to a. If $a = a_k$ for some $k \in \mathbb{N}$ then we have $a \in A$ and we're done. If $a \neq a_k$ for $k \in \mathbb{N}$ then for a region R with $a \in R$ there exists a finite number of $n \in \mathbb{N}$ such that $a_n \notin R$. But then there are an infinite number of $n \in \mathbb{N}$ with $a_n \in R$ and since a is not equal to any of these a_n we have a is a limit point of A which means $a \in \overline{A}$. □

Theorem 7 *Let f be a real function. Then f is continuous at a if and only if for all sequences (a_n) with $\lim_{a\to\infty} a_n = a$ we have $\lim_{n\to\infty} f(a_n) = f(a)$.*

Proof. Let f be continuous at a and consider some sequence (a_n) which converges to a. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $a_n \in \mathbb{R}$ when $|a - a_n| < \delta$ we have $|f(a) - f(a_n)| < \varepsilon$. But also for $\delta > 0$ there exists some $N \in \mathbb{N}$ such that for all $n > N$ we have $|a - a_n| < \delta$. But then for $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n > N$ we have $|f(a) - f(a_n)| < \varepsilon$.

To show the converse, we use the contrapositive. Assume that f is not continuous at a. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists some $x \in \mathbb{R}$ so that when $|a - x| < \delta$ we have $|f(a) - f(x)| \geq \varepsilon$. For this ε there exists some $a_1 \in (a - 1; a + 1)$ such that $|f(a) - f(a_1)| \geq \varepsilon > 0$. Then let (a_n) be a sequence such that $a_n \in (a - 1/n; a + 1/n)$ such that $|f(a) - f(a_n)| \geq \varepsilon$. We know that a_i exists for all $i \in \mathbb{N}$ because for each $\delta > 0$ there always exists an $x \in (a - \delta; a + \delta)$ such that $|f(a) - f(x)| \geq \varepsilon$. Let $(p; q)$ be a region with $a \in (p; q)$. If $p \leq a - 1$ and $q \geq a + 1$ then for all n we have $a_n \in (p; q)$ and so there are finitely many n with $a_n \notin (p; q)$. Consider the case where $p \in (a - 1; a)$. Using the Archimedean Property and the Well Ordering Principle there exists a least k such that $a - 1/k \leq p < a$. Then there are finitely many $n \leq k$ such that $a_n \leq p$. We can make a similar argument about q so that there are finitely many n with $a_n \notin (p; q)$. Then (a_n) converges to a but this means there exists a sequence which converges to a, but $\lim_{n\to\infty} f(a_n) \neq f(a)$. □

Definition 8 *Let (a_n) be a sequence. A point $a \in \mathbb{R}$ is called an accumulation point of (a_n) if for every region R containing a there are infinitely many n with $a_n \in R$.*

Lemma 9 *For a sequence (a_n), the set A of all its accumulation points is a closed set.*

Proof. Let (a_n) be a sequence and let A be the set of its accumulation points. Let $x \in \mathbb{R}\backslash A$. Then x is not an accumulation point of (a_n) and so there exists some region R such that there are finitely many $n \in \mathbb{N}$ with $a_n \in R$. Note that none of the points in R are accumulation points because there are only finitely many $a_n \in R$. But this means that $R \subseteq \mathbb{R}\backslash A$ and since such a region exists for all $x \in \mathbb{R}\backslash A$ we know that this set is open. But then A is closed. □

Theorem 10 *Let (a_n) be a sequence which converges to a. Then a is the only accumulation point of (a_n).*

Proof. Let (a_n) be a sequence such that $\lim_{n\to\infty} a_n = a$ and suppose that (a_n) has an accumulation point a' such that $a' \neq a$. Let R and R' be disjoint regions containing a and a' respectively. Then there are finitely many $n \in \mathbb{N}$ with $a_n \notin R$ but also there are infinitely many $n \in \mathbb{N}$ with $a_n \in R'$. Since R and R' are disjoint this is a contradiction. □

Definition 11 (Subsequence) *Let (a_n) be a sequence. A subsequence of (a_n) is a sequence $(b_k = a_{n_k})$ (meaning that $b_1 = a_{n_1}$, $b_2 = a_{n_2}$, $b_3 = a_{n_3}$ and so on), where $n_1 < n_2 < n_3 < \ldots$.*

Lemma 12 *If (a_n) converges to a, then so do all of it's subsequences.*

Proof. Let (a_n) be a sequence which converges to a and let $(b_k = a_{n_k})$ be a subsequence. Every element of $(b_k = a_{n_k})$ is an element of (a_n) and for every region R with $a \in R$ there are finitely many $n \in \mathbb{N}$ such that $a_n \notin R$. But then For every region R containing a, there must be finitely many $k \in \mathbb{N}$ such that $b_k \notin R$. Thus $(b_k = a_{n_k})$ converges to a. □

Lemma 13 *Let (a_n) be a sequence. Then a is an accumulation point of (a_n) if and only if there is a subsequence $(b_k = a_{n_k})$ which converges to a.*

Proof. Let (a_n) be a sequence which has a subsequence $(b_k = a_{n_k})$ which converges to a. Then for all regions R with $a \in R$ there are finitely many $k \in \mathbb{N}$ with $b_k \notin R$. Then there are infinitely many k with $b_k \in R$. But for all $k \in \mathbb{N}$ we have $b_k = a_{n_k}$ which means there are infinitely many $n \in \mathbb{N}$ with $a_n \in R$. Thus, a is an accumulation point of (a_n).

Conversely, let a be an accumulation point of (a_n). Create a subsequence $(b_k = a_{n_k})$ where $b_k = a_{n_k}$ and $a_{n_k} \in (a - 1/k; a + 1/k)$. We know that b_k will exist because for each $k \in \mathbb{N}$ there are infinitely many n such that $a_n \in (a - 1/k; a + 1/k)$ because a is an accumulation point. Let $(p; q)$ be a region containing a. Then if $p \leq a - 1$ and $q \geq a + 1$ then we have $a_n \in (p; q)$ for all n and so there are a finite number of n such that $n \notin (p; q)$. In the case where $a - 1 < p < a$, using the Archimedean Property and the Well Ordering Principle we know there exists a least $k \in \mathbb{N}$ such that $a - 1/k \leq p < a$. But then there are a finite number of $n \leq k$ such that $a_n \leq p$. Using a similar argument for $a < q < a + 1$ we have a finite number of n such that $a_n \notin (p; q)$. Then (b_k) converges to a. $\square$

Definition 14 (Bounded Sequence) *A sequence (a_n) is bounded above if there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. It is bounded below if there exists $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \in \mathbb{N}$. We have (a_n) is bounded if it is bounded above and bounded below.*

Lemma 15 *Every convergent sequence is bounded.*

Proof. Let (a_n) be a sequence which converges to a. Let $(p; q)$ be a region with $a \in (p; q)$. In the case that for all $n \in \mathbb{N}$, $p < a_n$ or $a_n < q$ we have p or q are upper or lower bounds for (a_n). Consider the case where there exists $n \in \mathbb{N}$ such that $a_n \leq p$. We have (a_n) converges to a so there are finitely many n with $a_n \notin (p; q)$. Thus, there exists $k \in \mathbb{N}$ such that $a_k \leq a_n$ for all $n \in \mathbb{N}$. Then this a_k is a lower bound for (a_n). A similar proof holds to find and upper bound for (a_n) if there exists $n \in \mathbb{N}$ with $a_n \geq q$. $\square$

Theorem 16 (Bolzano-Weierstrass for Sequences) *Any bounded sequence has a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence. Then there exists $l, u \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have $l \leq a_n \leq u$. Now suppose that (a_n) has no accumulation points. Then for all points $a \in \mathbb{R}$ there exists a region R_a such that there are finitely many $n \in \mathbb{N}$ with $a_n \in R_a$. Let $\mathcal{A} = \{R_a \mid a \in [l; u]\}$. Then $\mathcal{A}$ is an open cover for $[l; u]$ and $[l; u]$ is compact so let $\mathcal{B}$ be a finite subcover for $\mathcal{A}$. Then $\mathcal{B}$ covers $[l; u]$ with a finite number of regions R which each have a finite number of $n \in \mathbb{N}$ with $a_n \in R$. But (a_n) is bounded between l and u so there are an infinite number of $n \in \mathbb{N}$ with $a_n \in [l; u]$. This is a contradiction and so (a_n) must have some accumulation point a. Then by Lemma 13 there must exist a convergent subsequence of (a_n) which converges to a. $\square$

Corollary 17 *Let (a_n) be a bounded sequence. Then (a_n) is convergent if and only if it has only one accumulation point.*

Proof. If (a_n) is convergent at a then by Lemma 10 a is the only accumulation point of (a_n). Suppose now that (a_n) has only one accumulation point a. Note that (a_n) so there exist $l, u \in \mathbb{R}$ such that $a_n \in [l; u]$ for all $n \in \mathbb{N}$. Take an arbitrary region $(p; q) \subseteq [l; u]$ such that $a \in (p; q)$. Consider $[l; u] \backslash (p; q) = [l; p] \cup [q; u] = S$. Every element of S is not an accumulation point of (a_n). Thus for all $x \in S$ there exists some region R_x such that there are finitely many $n \in \mathbb{N}$ with $a_n \in R_x$. Let $\mathcal{A} = \{R_x \mid x \in S\}$ be an open cover for S. We have S is closed and bounded and so there exists a finite subcover $\mathcal{B}$ for $\mathcal{A}$. Then $\mathcal{B}$ covers S with a finite number of regions, R, each of which have a finite number of n with $a_n \in R$. Thus, there are finitely many $n \in \mathbb{N}$ with $a_n \notin (p; q)$. In the case where $[l; u] \subseteq (p; q)$ then we have every element of (a_n) is in $(p; q)$ are so there are finitely many n with $a_n \notin (p; q)$. In all cases we see that (a_n) must converge to a. $\square$

Theorem 18 (Increasing Bounded Sequences are Convergent) *Let (a_n) be a bounded above sequence, such that $a_n \leq a_{n+1}$ for all n. Then (a_n) converges and*

$$\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{N}\}.$$

Proof. Let $s = \sup\{a_n \mid n \in \mathbb{N}\}$. Consider some region $(p;q)$ with $s \in (p;q)$. In the case where $p < a_n$ for all $n \in \mathbb{N}$, we have a finite number of n with $a_n \notin (p;q)$. Suppose that $a_n \leq q$ for all n. Then there exists $c \in (q;s)$ such that $c > a_n$ for all n. But this is a contradiction because $c < s$ and c is an upper bound for (a_n). Thus there exists $i \in \mathbb{N}$ such that $a_i \leq p < a_{i+1}$. So now we have $q < a_n$ for all $n > i$ and since there are a finite number of naturals less than $i + 1$, there are a finite number of n with $a_n \notin (p;q)$. But this is true for every region R with $s \in R$. Thus $\lim_{n\to\infty} a_n = s$. $\square$

Theorem 19 *Every sequence has an increasing or decreasing subsequence.*

Proof. Let (a_n) be a sequence. Define n to be a peak point if for all $m > n$ we have $a_m < a_n$. Suppose there are infinitely many peak points for (a_n) and let n_1 be the least peak point. We can do this because peak points are natural numbers. Define the next largest peak point to be n_2 and so on. Note that $a_{n_i} > a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Thus, we have created a decreasing subsequence $(b_k = a_{n_k})$.

If there are no peak points then for all $n \in \mathbb{N}$, there exists $m > n$ such that $a_n \leq a_m$. Then we can make an increasing subsequence by letting $b_1 = a_1$. Then there exists $m_2 > 1$ such that $a_1 \leq a_{m_2}$. Let $b_2 = a_{m_2}$. Now there exists $m_3 > m_2$ such that $a_{m_2} \leq a_{m_3}$. Let $b_3 = a_{m_3}$. Thus $(b_k = a_{m_k})$.

Now suppose that there are finitely many peak points for (a_n) and that there exists at least one peak point. Let $n \in \mathbb{N}$ be the largest peak point for (a_n). Then for all $m > n$ we have $a_m < a_n$, but also m is not a peak point and so there exists $m' > m$ with $a_m \leq a_{m'}$. Then create an increasing sequence as before by choosing an arbitrary $m_1 > n$ and letting $b_1 = a_{m_1}$. Then there exists $m_2 > m_1$ such that $a_{m_1} \leq a_{m_2}$. Thus $(b_k = a_{m_k})$. $\square$