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MATH 26200
February 18, 2010

Homework 6

Problem 1. *Let X and Y be metric spaces with metrics d_X and d_Y respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X,*

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called an isometric imbedding *of X in Y.*

Proof. We need to show that f is a homeomorphism onto its image in Y. Note that f is injective since $x \neq y$ in X means $d_X(x, y) \neq 0$ and so $d_Y(f(x), f(y)) \neq 0$ which implies $f(x) \neq f(y)$. Also f is clearly surjective onto its image so f is a bijection onto it's image. Let $B = B_{d_Y}(y, \varepsilon)$ be a ε-ball around $y \in Y$. Then consider an element $x \in f^{-1}(B)$ and note that $f(x) \in B$ so $d_Y(y, f(x)) < \varepsilon$ so $d_X(f^{-1}(y), x) < \varepsilon$. Thus, the ball $B_{d_X}(f^{-1}(y), \varepsilon) \subseteq f^{-1}(B)$ and $f^{-1}(B)$ is open. The proof that f is an open mapping follows similarly. □

Problem 2. *Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)*

Proof. Let $x \in \mathbb{R}$ and consider the set of open sets $A_x = \{[x - (1/n), x + (1/n)) \mid n \in \mathbb{N}\}$. Now consider any neighborhood of x. This will necessarily contain some basis element of the form $[x - \varepsilon, x + \varepsilon)$. Picking $1/n < \varepsilon$ shows that this contains an element of A_x and so A_x serves as a countable basis at the point x. Since this is true for each point of $\mathbb{R}$ we see that $\mathbb{R}_\ell$ satisfies the first countability axiom.

Now let $x \times y \in I_0^2$. Consider the set of basis elements $\{(x \times y - (1/n), x \times y + (1/n)) \mid n \in \mathbb{N}\}$. Now any open set containing $x \times y$ will necessarily contain some interval of the form $(x \times y - \varepsilon, x \times y + \varepsilon)$ and choosing $1/n < \varepsilon$ gives the desired result. Thus I_0^2 is first countable. □

Problem 3. *Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to to f, then $(f_n(x_n))$ converges to $f(x)$.*

Proof. Let $\varepsilon > 0$. If (f_n) converges uniformly to f then there exists some N_1 such that for all $n > N_1$ and for each $x \in X$ we have $d(f_n(x), f(x)) < \varepsilon/2$. In particular for each $n > N_1$ we have $d(f_n(x_n), f(x_n)) < \varepsilon/2$. Note also that f is continuous since each f_n is continuous so there exists some N_2 such that for all $n > N_2$ we have $d(f(x_n), f(x)) < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Now using the triangle inequality we have $d(f_n(x_n), f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n > N$. Thus $(f_n(x_n))$ converges to $f(x)$. □

Problem 4. *Check the details of Example 3.*

Proof. There are only 6 nontrivial subsets to check, so we'll just do them individually. The preimages of the sets $\{a\}$ and $\{b\}$ are the open rays $(0, \infty)$ and $(-\infty, 0)$ so these sets must be open in A. The preimage of the set $\{c\}$ is the one point set $\{0\}$ so this set is not open in A. The preimage of the set $\{a, b\}$ is the set $\mathbb{R}\backslash\{0\}$ which is open so this set is open in A. The preimages of $\{a, c\}$ and $\{b, c\}$ are closed rays in $\mathbb{R}$ so these sets are not open. Clearly A and the empty set are both open in A. Thus, the open sets are $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$ and A. □

Problem 5. *(a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.*
(b) If $A \subseteq X$, a retraction *of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.*

Proof. (a) Note that f is a right inverse for p so p must be surjective. Let U be an open set in Y. Then since p is continuous, $p^{-1}(U)$ is open in X. Conversely, let V be a subset of Y such that $p^{-1}(V)$ is open in X. Since f is continuous, $f^{-1}(p^{-1}(V)) = (p \circ f)^{-1}(V)$ is open. But we know that $p \circ f$ is the identity map on Y so V must be open in Y. Therefore p is a quotient map.

(b) This is just a special case of part (a) where $p = r$ and $f : A \to X$ is the identity map. Then f is continuous and $r \circ f$ is the identity on A. Therefore r must be a quotient map. $\square$

Problem 6. *(a) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:*

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \textit{if} \quad x_0 + y_0^2 = x_1 + y_1^2.$$

Let X^* be the corresponding quotient space. It is homeomorphic to a familiar space; what is it?
(b) Repeat (a) for the equivalence relation

$$x_0 \times y_0 \sim x_1 \times y_1 \quad \textit{if} \quad x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

Proof. (a) Note that the equivalence classes are sets of the form $\{(x, y) \mid x + y^2 = a, a \in \mathbb{R}\}$. That is, they are concentric parabolas parallel to the x-axis. Let $f : X^* \to \mathbb{R}$ be defined by taking the value of x when the parabola intersects the x-axis. Note that f is injective since two different equivalence classes will have two different intersection points. Also f is surjective since for a point $a \in \mathbb{R}$ the equivalence class $x + y^2 = a$ is mapped to a.

Consider an open interval $(a, b) \subseteq \mathbb{R}$. Then consider the preimage of $f^{-1}((a, b))$ under the induced quotient map from X to X^*. This is the union of all the those parabolas $x + y^2 = c$ with $c \in (a, b)$. Note that this set is open in X, so $f^{-1}((a, b))$ is open in X^*. Thus f is continuous. Now consider an open set U in X^* and pick $f(a) \in U$. The preimage of this set under the quotient map is a union of parabolas which is open in X. In particular, if we consider the parabola which intersects the x-axis at a, then there's some $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon)$ is contained in this union. Then this interval must be in the image $f(U)$ so $f(U)$ is open and f is an open map. Therefore f is a homeomorphism. Thus X^* is homeomorphic to $\mathbb{R}$ in the usual topology.

(b) Now the equivalence classes are concentric circles. We can map these classes to the nonnegative reals in the subspace topology by mapping the radius of a circle to a number in $\mathbb{R}^+ \cup \{0\}$. A similar proof to the one in part (a) shows that f is a homeomorphism. Thus X^* is homeomorphic to the nonnegative reals in the subspace topology. $\square$

Problem 7. *Recall that $\mathbb{R}_K$ denote the real line in the K-topology. (See §13.) Let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point; let $p : \mathbb{R}_K \to Y$ be the quotient map.*
(a) Show that Y satisfies the T_1 axiom, but is not Hausdorff.
(b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map.

Proof. (a) We can view Y as a collection of equivalence classes where each point of $\mathbb{R}\backslash K$ is in its own equivalence class and K is its own equivalence class which we'll denote by the point y. An open set in Y is then a collection of these equivalence classes whose union is open in $\mathbb{R}_K$. Let $x \in Y$ such that $x \neq y$. Then $p^{-1}(Y\backslash\{x\})$ is a union of equivalence classes which contains every point in $\mathbb{R}$ other than x. This set is open in $\mathbb{R}_K$, so $Y\backslash\{x\}$ is open in Y and $\{x\}$ is closed. Now consider the set $Y\backslash\{y\}$. The union of these equivalence classes is $\mathbb{R}\backslash K = (-\infty, 0) \cup ((-1, 2)\backslash K) \cup (1, \infty)$ which is open in $\mathbb{R}_K$. Therefore $Y\backslash\{y\}$ is open in Y and $\{y\}$ is closed. Thus Y is T_1.

Now suppose U is an open set of Y which contains y. Then U is a union of equivalence classes whose union contains K and is open in $\mathbb{R}_K$. Note also that if V is an open set in Y which contains the equivalence class containing $\{0\}$, then $p^{-1}(V)$ must contain an interval of the form $(0 - \varepsilon, 0 + \varepsilon)$ (possibly without K). Then choose $1/n < \varepsilon$. Note that $p^{-1}(U)$ must contain some interval around $1/n$ since this point is in K and K is contained in $p^{-1}(U)$. Thus, $p^{-1}(U)$ and $p^{-1}(V)$ necessarily intersect at some point in $\mathbb{R}\backslash K$ less than $1/n$. The equivalence class for this point is thus in both U and V an so it's impossible to separate U and V with open sets. Therefore Y is not Hausdorff.

(b) Since Y is not Hausdorff, we know the diagonal Δ is not closed in $Y \times Y$. Consider $(p \times p)^{-1}(\Delta)$ in $\mathbb{R}_K \times \mathbb{R}_K$. This is the union of all elements of the form $x \times x$ where x is an equivalence class in Y. But this is just the diagonal of $\mathbb{R}_K \times \mathbb{R}_K$. Since $\mathbb{R}_K$ is Hausdorff, this set is closed. So $p \times p$ takes a set which is not closed to a closed set so it can't be a quotient map. $\square$