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MATH 26300
May 25, 2010

Homework 7

Problem 1. *Given a map $f : S^{2n} \to S^{2n}$, show that there is a point $x \in S^{2n}$ with wither $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.*

Proof. Suppose $\varphi : S^{2n} \to S^{2n}$ with $\varphi(x) \neq x$ and $\varphi(x) \neq -x$ for all $x \in S^{2n}$. Since $\varphi(x) \neq -x$ for all $x \in S^{2n}$ we know $(1-t)\varphi(x)+tx \neq 0$ for $t \in [0,1]$ so we get a homotopy $H(t,x) = ((1-t)\varphi(x)+tx)/|(1-t)\varphi(x)+tx|$ from φ to the identity map. Thus $\deg(\varphi) = 1$. But since φ has no fixed points we already know $\deg(\varphi) = -1^{2n+1} = -1$. This is a contradiction, so no such map can exist.

Suppose now $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ is any map. Compose f with the quotient map $g : S^{2n} \to \mathbb{R}P^{2n}$ to get $fg : S^{2n} \to \mathbb{R}P^{2n}$. Since S^{2n} is a covering space with trivial fundamental group for $\mathbb{R}P^{2n}$ we see that fg lifts to some map $h : S^{2n} \to S^{2n}$. This means that $gh = fg$ and since $h : S^{2n} \to S^{2n}$ we know there is some $x \in S^{2n}$ such that $h(x) = \pm x$. Then $f(g(x)) = g(h(x)) = g(\pm x) = g(x)$ since g identifies antipodal points. Thus f has a fixed point $g(x)$.

Let $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear transformation defined as $T(x_1, \ldots, x_{2n}) = (-x_{2n}, x_1, x_2, \ldots, x_{2n-1})$. Note that $-T^{2n}$ is the identity transformation so $x^{2n}+1$ divides the characteristic polynomial for T and since this has degree $2n$ it must be the characteristic polynomial. But this polynomial has no real roots and thus no real eigenvalues or eigenvectors. Since T has no eigenvectors we have $T(x) \neq x$ and $T(x) \neq -x$ for all $x \in S^{2n-1}$ where $S^{2n-1} \subseteq \mathbb{R}^{2n}$. Furthermore, since $T(-x) = -T(x)$ we see that T gives a map $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ which has no fixed points. $\square$

Problem 2. *A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.*

Proof. Let $z_1, \ldots z_r$ be the distinct roots of f with multiplicities $m_1, \ldots m_r$. There are disjoint neighborhoods of $U_1, \ldots, U_r$ of each z_i in S^2 such that $f(U_i) \subseteq V_i$ where V_i is a neighborhood of $0 \in S^2$. We then have the induced map on homology $\hat{f}_* : H_2(U_i, U_i \backslash \{z_i\}) \to H_2(V_i, V_i \backslash \{0\})$. Both these groups are $\mathbb{Z}$ so $\hat{f}_*$ is multiplication by some integer d, which by construction we know to be the local degree of $\hat{f}$ at z_i. Note that $\hat{f}$ restricted to a local neighborhood of z_i is an m_i-to-1 map onto V_i. But then a generator of $H_2(U_i, U_i \backslash \{z_i\})$ is mapped to m_i times a generator of $H_2(V_i, V_i \backslash \{0\})$. Thus the local degree of $\hat{f}$ is m_i. Now we know $\deg \hat{f} = \sum_i \deg \hat{f}|z_i = \sum_i m_i = \deg f$. $\square$

Problem 3. *Compute the homology groups of the following 2-complexes:*
(a) The quotient of S^2 obtained by identifying north and south poles to a point.
(b) $S^1 \times (S^1 \vee S^1)$.
(c) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.
(d) The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Proof. (a) This structure can be constructed using one 0-cell, one 1-cell and one 2-cell so we have the chain complex

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0.$$

Attach the 1-cell to the 0-cell and then attach half of the two cell boundary to the 1-cell, then attach the other half backwards so that our attaching map is of the form aa^{-1}. Since the attaching map for the 2-cell is trivial, we know $d_2 = 0$. Since there's only 1 0-cell, we also know $d_1 = 0$. Thus the homology groups are the same as the chain complex groups. Namely, $H_n \approx \mathbb{Z}$ for $n = 0$, $n = 1$ and $n = 2$ and $H_n = 0$ for $n > 2$.

(b) We can draw this space using the following diagram. There is one 0-cell v, three 1-cells a, b and c and two 2-cells U and L. This gives the chain complex

$$\mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0.$$

First we identify the three line segments labeled c in the diagram which forms the 1-skeleton for $I \times (S^1 \vee S^1)$.

Then we identify the sides labeled a and b so that we get $S^1 \times (S^1 \vee S^1)$. We then see the 2-cell U is attached via the identification $aca^{-1}c^{-1}$ and the 2-cell L is attached via the identification $bcb^{-1}c^{-1}$. Since there's only 1 0-cell we must have $d_1 = 0$. Also d_2 is 0 because each a_i, b_i or c_i appears with its inverse in $aca^{-1}c^{-1}$ and $bcb^{-1}c^{-1}$. Thus the homology groups are the same as the chain complex groups. Namely, $H_2 \approx \mathbb{Z}^2$, $H_1 \approx \mathbb{Z}^3$, $H_0 \approx \mathbb{Z}$ and $H_n = 0$ for $n > 2$.

(c) This space can be constructed using one 0-cell, one 1-cell, a, and one 2-cell, f, so we get the following chain complex

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0.$$

We know d_1 is 0 because there's only one 1-cell. Furthermore, f is attached to a 3-fold and since the orientation is preserved each time we know the generator of this attaching map is a^3. Therefore d_2 takes a generator of $\mathbb{Z}$ to 3 times that generator so $\deg(d_2) = 3$. Thus $H_1 \approx \ker d_1/\operatorname{im} d_2 \approx \mathbb{Z}/3\mathbb{Z}$. Also $H_2 \approx \ker d_2/\operatorname{im} d_1 \approx 0$ and $H_0 \approx \ker d_0/\operatorname{im} d_1 \approx \mathbb{Z}$. Clearly $H_n = 0$ for $n > 2$.

(d) The space can be described using the following diagram. There is one 0-cell, two 1-cells and one 2-cell. This gives the chain complex

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0.$$

Note that $d_1 = 0$ as before since there's only one 0-cell. The 2-cell is attached via the product $a^m b^n (-a)^m (-b)^n$. But the abelianization of this is clearly 0 which implies $d_2 = 0$ as well. This means our chain complex forms the actual homotopy groups. In particular $H_n = H_n(S^1 \times S^1)$. □

Problem 4. *Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.*

Proof. Note that $(S^1 \times S^1, S^1 \vee S^1)$ is a good pair and we know all the homology groups in question, so we have the long exact sequence

$$0 = H_2(S^1 \vee S^1) \to H_2(S^1 \times S^1) \approx \mathbb{Z} \to H_2(S^1 \times S^1, S^1 \vee S^1) \approx H_2(S^1 \times S^1/S^1 \vee S^1) \approx H_2(S^2) \approx \mathbb{Z}$$

Since the first term is 0, we see that the map $H_2(S^1 \times S^1) \to H_2(S^2)$ is injective and since it's into the same space, it must be an isomorphism. Thus the quotient map cannot be nullhomotopic.

Let $f : S^2 \to S^1 \times S^1$ and let $p : \mathbb{R}^2 \to S^1 \times S^1$ be the universal cover of $S^1 \times S^1$. Since S^2 has trivial fundamental group, we can use the lifting criterion to get a lift $h : S^2 \to \mathbb{R}^2$ such that $gh = f$. But then since $\mathbb{R}^2$ is contractable, we know h is nullhomotopic. It follows that f must be nullhomotopic as well. $\square$

Problem 5. *A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an* even map*. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show there exist even maps of any given even degree.*

Proof. Let f be even. Since $f(x) = f(-x)$ we can view f as a function $\mathbb{R}P^n \to S^n$ where x and $-x$ are identified. So f factors as $S^n \xrightarrow{g} \mathbb{R}P^n \xrightarrow{h} S^n$ where g is the quotient map. Then $\deg(f) = \deg(g)\deg(h)$ and we've already seen that $\deg(g) = 1 + (-1)^n$. Thus $\deg(f)$ is even and if n is even then $\deg(g) = 0$ so $\deg(f) = 0$ as well. When n is odd $\deg(g) = 2$. Since there are maps $h : \mathbb{R}P^n \to S^n$ of any degree we see that $\deg(f) = \deg(g)\deg(h) = 2\deg(h)$ can be any even number. $\square$

Problem 6. *Show the isomorphism between cellular and singular homology is natural in the following sense: A map $f : X \to Y$ that is cellular — satisfying $f(X^n) \subseteq Y$ for all n — induces a chain map f_* between the cellular chain complexes of X and Y, and the map $f_* : H_n^{CW}(X) \to H_n^{CW}(Y)$ induced by this chain map corresponds to $f_* : H_n(X) \to H_n(Y)$ under the isomorphism $H_n^{CW} \approx H_n$.*

Proof. Since f is a cellular map we know that for each n, the restriction of f to the n-skeleton of X gives a map of pairs $(X^n, X^{n-1}) \to (Y^n, Y^{n-1})$ which gives a map on relative homology $f_* : H_n(X^n, X^{n-1}) \to H_n(Y^n, Y^{n-1})$. These are precisely the cellular chain groups so f induces a chain map between the cellular chain complexes for X and Y. Note here that f_* commutes with the boundary maps d_n because f_* commutes with the relative homology boundary maps ∂_n and j_n.

Given this, we know f_* induces a map $f'_* : H_n^{CW}(X) \to H_n^{CW}(Y)$ and we already have a map $f_* : H_n(X) \to H_n(Y)$. Let $\gamma : H_n(X) \to H_n^{CW}(X)$ and $\delta : H_n(Y) \to H_n^{CW}(Y)$ be the isomorphisms between the singular and cellular homology groups. We're reduced to showing that the following diagram commutes

$$\begin{tikzcd} H_n(X) \arrow[r, "f_*"] \arrow[d, "\gamma"] & H_n(Y) \arrow[d, "\delta"] \\ H_n^{CW}(X) \arrow[r, "f'_*"] & H_n^{CW}(Y). \end{tikzcd}$$

Note that we already have the following commutative diagram from from the proof that γ is an isomorphism.

$$\begin{tikzcd}
 & & H_n(X) & \\
 & H_n(X^n) \arrow[ur, "\alpha"] \arrow[dr, "j_n"] & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] \arrow[d, "f_*"] & & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[d, "f_*"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "f_*"] \\
H_{n+1}(Y^{n+1}, Y^n) \arrow[dr, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(Y^n, Y^{n-1}) \arrow[r, "d_n"] & H_{n-1}(Y^{n-1}, Y^{n-2}) \\
 & H_n(Y^n) \arrow[ur, "j_n"] \arrow[dr, "\beta"] & & \\
 & & H_n(Y) &
\end{tikzcd}$$

If we look at the two three-term diagonal sequences we get the following diagram

$$\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_*"] & H_n(X^n) \arrow[r, "\alpha"] & H_n(X) \arrow[d, dashed] \\
H_{n+1}(Y^{n+1}, Y^n) \arrow[r, "\partial_{n+1}"] & H_n(Y^n) \arrow[r, "\beta"] & H_n(Y).
\end{tikzcd}$$

But note that the horizontal sequences are the long exact sequences of the pairs (X^{n+1}, X^n) and (Y^{n+1}, Y^n), so by the naturality of the long exact sequence, the dotted arrow must be f_* and the first diagram actually commutes. $\square$