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Homework 4

Problem 1. *Let A be a wff which does not contain $\neg$. Show that the length of A is odd. Show that no proper initial segment of A is a wff.*

Proof. We induct on the complexity of A. For the base case, let A be a single sentence symbol. Then (A) has length 3. Furthermore none of (, $(A$ or the empty string are wffs. Now suppose that $A = ((B) \wedge (C))$ for some wffs B and C which satisfy the stated properties. Then B and C both have odd length, and there are 7 more elements of $\mathcal{L}$ added to create A. Therefore A has odd length. Furthermore, since no proper initial segment of B is a wff, we know that no initial segment of A which doesn't include all of B will not be a wff. This follows because adding parentheses to the beginning of any proper initial segment of B will not make it a wff. The same is true for the initial segments $((B$, $((B)$, $((B)\wedge$ and proper initial segments which contain initial segments of C using a similar argument. Also $((B) \wedge (C)$ is not a wff since the first parenthesis is never closed, and the empty string is not a wff. Therefore no proper initial string of A is a wff. Since A doesn't contain $\neg$, we have shown by induction that the statement is true for all wffs which don't contain $\neg$. □

Problem 2. *Let T, Γ be sets of wffs. Suppose $T \vdash A$ for all $A \in \Gamma$.*
(a) If $T \cup \Gamma \vdash B$, then $T \vdash B$.
(b) If T is consistent then Γ is consistent. In particular the set of all wffs which can be deduced from T is consistent.

Proof. (a) Let $C_1, C_2, \ldots, C_n$ be a deduction of B from $T \cup \Gamma$. For each $C_i \in \Gamma$, replace C_i with the deduction $C_{i_1}, C_{i_2}, \ldots, C_{i_{m_i}}$ of C_i from T. Then $C_{1_1}, \ldots, C_{1_{m_1}}, \ldots, C_{n_1}, \ldots, C_{n_{m_n}}$ is a deduction of B from T so $T \vdash B$.

(b) Suppose T is consistent. Then there exists M, a model for T. Let C_{i_j} be an element of the deduction of C_i as in part (a). Then C_{i_j} is either in T, in which case $M \models C_{i_j}$, a tautology, so that once again $M \models C_{i_j}$ or the result of modus ponens from two earlier elements in the deduction. In the last case, $M \models C_{i_j}$ since $\to$ can be written using $\neg$ and $\wedge$. Since Γ can be written entirely as deductions of elements from T, we see that $M \models \Gamma$ as well. □

Problem 3. *Let T, Σ be sets of wffs and let A, B be wffs. Prove or refute the following statements:*
(a) If $T, \Sigma \models A$ then either $T \models A$ or $\Sigma \models A$.
(b) If $T \models A \vee B$ then either $T \models A$ or $T \models B$.
Do either of the answers change if we assume T is maximal consistent?

Proof. (a) Let $T = S_1$, $\Sigma = S_2$ and $A = S_1 \wedge S_2$. Then $T, \Sigma \models A$, but T does not model A and Σ does not model A.

(b) Suppose $T \models A \vee B$ and let M be a model of T. Then $M \models A \vee B$ and thus $M \models \neg(\neg A \wedge \neg B)$. But then M does not model $\neg A \wedge \neg B$, which means M does not model $\neg A$ or M does not model $\neg B$. Therefore $M \models A$ or $M \models B$.

If T is maximal consistent then if $T, \Sigma \models A$ then either $T \models A$ or $\Sigma \models A$. This follows from the fact that either $T \cup \Sigma$ is not maximally consistent, or $T \cup \Sigma = T$. The answer to part (b) is the same. □

Problem 4. *Let IP_x be the statement that:*
Let $P(x)$ be some property and suppose that $k \in \mathbb{N}$ is fixed. If
(a) $P(k)$ holds, and
(b) For all $n \geq k$, if $P(n)$ holds then $P(n+1)$ holds
then $P(n)$ holds for all natural numbers $n \geq k$.
Prove that, for fixed k, our first induction principle implies IP_k. What is IP_0?

Proof. Let $Q_k(x)$ be the statement such that $Q_k(x-k)$ holds whenever $P(x)$ holds. Then $Q_k(0)$ is true if $P(k)$ is true. If $Q_k(n)$ is true for $n \geq 0$, then $P(k+n)$ is true. Thus $P(k+n+1)$ is true implies that $Q_k(n+1)$ is true. Therefore $Q_k(x)$ holds for all $x \in \mathbb{N}$ and therefore $P(x+k)$ holds for $x+k \geq k$. Thus IP_k is implied by induction. IP_0 is the first induction principle. □

Problem 5. *Suppose $\mathcal{L}$ contains two ternary relations, one binary relation, and two constants and consider the model $\langle \mathbb{N}, +, \times, <, 0, 1 \rangle$ (with the usual meanings). Give a formula which defines:*
(a) $\{0\}$.
(b) $\{m \mid m$ *is divisible by* $3\}$.
(c) $\{(m,n) \mid m, n$ *have no common divisors besides* $1\}$.
Give an example of a set which is not definable (you do not need to justify your answer).

Proof. (a) $\exists x \forall y((x < y) \wedge (\neg(x = y)))$.
(b) $\exists m \exists n(m = ((1+1+1) \times n))$.
(c) $\exists m \exists n(\neg(\exists d \exists p \exists q((\neg(d = 1)) \wedge (m = dp) \vee (n = dq))))$
.
The set $\{2\}$ is not definable. □