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Homework 1

Problem 1. *Recall the three point geometry we mentioned in class, with three distinct points P, Q and R and three distinct lines $P+Q$, $P+R$ and $Q+R$. This satisfies the first and third axioms but not the second.*
(a) Place the points in $\mathbb{R}^2$ as $P = (0,0)$, $Q = (1,0)$ and $R = (0,1)$. Show that any subgeometry of $\mathbb{R}^2$ containing these three points contains all of $\mathbb{Q}^2$. You should work out exactly what "subgeometry" should mean.
(b) Now consider the above consideration abstractly, not contained in $\mathbb{R}^2$. How few points and lines must we add to get a geometry satisfying all three axioms?

Proof. (a) By axiom 2 we must have lines through P, Q and R which are parallel to $P+Q$, $P+R$ and $Q+R$ respectively. Call the line which is parallel to $P+Q$ through R, ℓ. We will inductively show that there must be points $(n,0)$ and $(0,n)$ for $n \in \mathbb{Z}$ in our set of points. Note that the base case $n = 0$ is done for us. Suppose we have points $(n,0)$, $(-n,0)$, $(0,n)$ and $(0,-n)$ for some integer n. Note that the line $(n,0) + (0,-n)$ intersects ℓ in some point. Now draw the line parallel to $Q+R$ which goes through this point. This necessarily intersects the line $P+Q$ in the point $(n+1,0)$. The cases for $(0,n+1)$, $(-(n+1),0)$ and $(0,-(n+1))$ are similar. Note now that the lines through each of these points which are parallel to $P+R$ and $P+Q$ form a lattice on the plane and their intersections give us the set of points $\mathbb{Z}^2$.

Now we can construct any point of the form $(a/b,0)$ where $a,b \in \mathbb{Z}$ and $a/b < 1$. Simply find the point $(a, b-a)$ (which we've just shown must exist) and draw the line through P and $(a,b-a)$. Note that the equation of this line is $y = ((b-a)/a)x$ and it's easy to see that its intersection with the line $Q+R$ occurs at the point $(a/b,(b-a)/b)$. Drawing the line through this point parallel to $P+R$ we find a point at $(a/b,0)$ as desired. An argument to construct the points $(0,a/b)$, $(-a/b,0)$ and $(0,-a/b)$ is similar. Furthermore, we could have used the lines $(n+1,0)+(0,n+1)$ and $(n,0)+(b+n,a)$ to construct $(n+a/b,0)$ for some integer

n. In this way we see that we've constructed all the points $(a/b, 0)$ and $(0, a/b)$ for $a/b \in \mathbb{Q}$. Taking the lines through these points parallel to $P+R$ and $P+Q$ and all their intersections we get all points $(a/b, c/d)$, that is, all points in $\mathbb{Q}^2$.

(b) From the second axiom we need to add at least three lines. Namely, we need a line containing P parallel to $Q+R$, a line containing Q parallel to $P+R$ and a line containing R parallel to $P+Q$. Any two of these lines intersect in some point because otherwise they would be parallel which would imply two of $P+Q$, $Q+R$ and $P+R$ are parallel since parallelism is transitive. Suppose each of our new lines intersect in some point S. Axiom 1 is satisfied for S by the lines $P+S$, $Q+S$ and $R+S$ which are the lines we just added. Axiom 2 is satisfied for any line not containing S and the point S by the lines we added. For a line containing S, say $P+S$ and a point not on S, say Q, the line $Q+R$ contains Q and is parallel to S by our own constructions. The other possibilities following similarly. Finally, axiom 3 was already satisfied before we added any new lines or points. Thus we must add at least three lines and one point to satisfy all three axioms. $\square$

Problem 2. *Show that the group T of translations is normal in the group D of dilatations. If $\tau \neq 1$ is a translation, show that $\sigma\tau\sigma^{-1}$ is a translation with the same direction as τ.*

Proof. Let $\tau \in T$ and $\sigma \in D$. Suppose $\sigma\tau\sigma^{-1}$ has a fixed point P. Then $\tau\sigma^{-1}P = \sigma^{-1}P$ so $\sigma^{-1}P$ is a fixed point of τ. Thus $\tau = 1$ and $\sigma\tau\sigma^{-1} = 1$ as well. Therefore $\sigma\tau\sigma^{-1}$ is necessarily in T so that $\sigma T\sigma^{-1} = T$ and $T \trianglelefteq D$.

Now suppose $\tau \neq 1$ and let π be the direction of τ. Note that $\sigma^{-1}P + \tau\sigma^{-1}P$ is a trace of τ and therefore in π. Since σ is a dilatation $\sigma^{-1}P + \tau\sigma^{-1}P \parallel \sigma\sigma^{-1}P + \sigma\tau\sigma^{-1}P = P + \sigma\tau\sigma^{-1}P$. Therefore $P + \sigma\tau\sigma^{-1}P$ is in π as well and so τ and $\sigma\tau\sigma^{-1}$ have the same direction. $\square$

Problem 3. *Let π be a pencil of parallel lines. Show that the set T_π of translations with direction π or equal to 1 is a subgroup of T.*

Proof. Note that our set is nonempty as identity is assumed to be included. Let $\tau, \tau' \in T_\pi$. If $\tau \neq 1$ then $P + \tau P = \tau^{-1}\tau P + \tau^{-1}P$ is both a τ trace and a τ^{-1} trace so $\tau^{-1} \in T_\pi$ as well.

If τ' or τ is the identity then we're done since $\tau\tau' = \tau$ or $\tau\tau' = \tau'$. Suppose $\tau \neq 1$ and $\tau' \neq 1$. Then $P+\tau P$ is a trace of τ. Note that $P+\tau P$ also contains $\tau'\tau P$. If $\tau'\tau P = P$ then $\tau'\tau = 1$ since T is a group and nontrivial translations have no fixed points. Otherwise $P + \tau P = P + \tau'\tau P$ so $\tau'\tau \in T_\pi$ as well. Therefore T_π is closed under products and inverses and is nonempty so it must be a subgroup of T. $\square$

Problem 4. *Consider the Moulton Plane introduced in class: the points are the points of $\mathbb{R}^2$, and the lines are the subsets defined by the following equations:*

- $x = c$ *(vertical lines);*
- $y = mx + b$ *for* $m \geq 0$ *(lines with nonnegative slope); and*
- $y = \begin{cases} mx + b & \text{if } x \leq 0 \\ 2mx + b & \text{if } x \geq 0 \end{cases}$ *for* $m \leq 0$ *(broken lines).*

Show that this satisfies axioms 1, 2 and 3, but not 4a.

Proof. Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two distinct points. If it happens that $p_1 = q_1$ or $p_2 = q_2$ then the lines $x = p_1$ or $y = p_2$ will respectively contain P and Q. If $p_1 > q_1$ and $p_2 > q_2$ or $p_1 < q_1$ and $p_2 < q_2$ then the line

$$y = \frac{p_2 - q_2}{p_1 - q_1}x + \frac{p_1q_2 - p_2q_1}{p_1 - q_1}$$

contains P and Q. It's easy to see each of these lines is the unique such line since the slope is uniquely determined by P and Q. Now suppose $p_1 < q_1$ and $p_2 > q_2$ or $p_1 > q_1$ and $p_2 < q_2$. If $p_1 < 0$ and $q_1 < 0$ then the same line as above (but broken this time) will contain P and Q. Likewise if $p_1 > 0$ and $q_1 > 0$ then the line above (but broken) will contain P and Q. Now suppose, without loss of generality, that $p_1 \leq 0 < q_1$ and $p_2 > q_2$. Then find the point on the y-axis such that the line drawn from that point to P has half the slope as the line drawn from that point to Q. The intermediate value theorem ensures that such a point will exist and will be unique. This line contains P and Q and is the unique line which does so. Thus the Moulton plane satisfies axiom 1.

Let ℓ be a line and let $P = (p_1, p_2)$ be a point not on ℓ. Note that if ℓ is vertical, then ℓ necessarily intersects every non-vertical line. Likewise, if ℓ has nonnegative slope then ℓ intersects every vertical line as well as every broken line. Finally, if ℓ is not vertical or with nonnegative slope then it necessarily intersects every vertical line and every line with nonnegative slope. This shows that we only need to consider lines of a particular category when finding a unique line parallel to ℓ.

Now if ℓ is vertical then the line $x = p_1$ parallel to ℓ and contains P. Clearly no other vertical line will contain P and by the above argument all other lines intersect ℓ so this line is unique. Suppose ℓ has positive slope m. Then the line $y = mx + (p_2 - mp_1)$ contains P (as is easily verified) and is parallel to ℓ (since it has the same slope). All other lines with the same slope will not contain P because they have different constant terms. Using the above argument again we see that this line is then unique. Finally, suppose ℓ is of the third category with slopes m and $2m$. If $p_1 \leq 0$ then the line with slopes m and $2m$ and constant term $p_2 - mp_1$ contains P and is parallel to ℓ. If $p_1 \geq 0$ then the line with slopes m and $2m$ with constant term $p_2 - 2mp_1$ is parallel to ℓ and contains P. No other line of this category which is parallel to ℓ can contain P since the corresponding line pieces will not contain P. Also, by the above argument this line intersects all lines of the other two categories so this is indeed unique. Therefore the Moulton plane satisfies axiom 2.

Finally note that axiom 3 is nearly trivially satisfied by (for example) the points $(0, 0)$, $(1, 0)$ and $(1, 1)$. Then $(1, 1)$ is clearly not on the line $(0, 0) + (1, 0)$, i.e. $y = 0$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a (nontrivial) dilatation of the Moulton plane and write $f(x, y) = (f_1(x, y), f_2(x, y))$. Note that this implies that lines with nonnegative slope are taken to parallel lines with nonnegative slope, i.e., lines with the same slope. Let $P = (p_1, p_2)$ be a point on such a line with slope m and equation $y = mx + b_1$. Suppose $f(P)$ lies on the line $y = mx + b_2$. Consider the function $f_r : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f_r(x, y) = (rx, ry)$. Choose $r = b_2/b_1$. It follows that $rp_2 = mrp_1 + b_2$ so $f_r(P)$ lies on the same line as $f(P)$. Now let $a = f_1(p_1, p_2) - rp_1$ and $b = f_2(p_1, p_2) - rp_2$. Then $f(p_1, p_2) = (rp_1 + a, rp_2 + b)$. Furthermore, it's not hard to see that this is actually independent of P so we arrive at the result $f(x, y) = (rx + a, ry + b)$ for some $r, a, b \in \mathbb{R}$.

Note that if $r = 1$ and $b/a = m$ then f is the identity, so assume $r \neq 1$. Then note that the point $(a/(1 - r), b/(1 - r))$ is defined and is a fixed point for f since

$$\frac{ra}{1 - r} + a = \frac{ra + a(1 - r)}{1 - r} = \frac{a}{1 - r} = x$$

and similarly for y. Therefore, for f to be a translation we need $r = 1$ so that $f(x, y) = (x + a, y + b)$. Note that this is the unique translation taking a point $P = (p_1, p_2)$ to the point $(p_1 + a, p_2 + a)$. Suppose $b = 0$ and $a > 0$ so that f simply shifts points to the right. Suppose we have a line with slopes m and $2m$. Note that points on the this line with negative x-coordinate greater than $-a$ will be sent to a line which has slope m for points with nonnegative x-coordinate. But this image line is clearly not parallel to our original line since it has the wrong slope for positive x-coordinates. Thus there is no translation of the Moulton plane which takes, for example, the point $(-1, 0)$ to the point $(2, 0)$ since the translation $f(x, y) = (x + 3, y)$ is not a dilatation. Therefore the Moulton plane doesn't satisfy axiom 4a. $\square$