Kris Harper
MATH 20900
May 25, 2009

Homework 8

**** Problem 1.** *If $g \in N$, show that*

$$\int_{\mathbb{R}^n} f(g \cdot x)dx = \int_{\mathbb{R}^n} f(x)dx.$$

Proof. Let

$$\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & & & \vdots \\ 0 & a_{n2} & \dots & 1 \end{pmatrix}.$$

Then if $x = (x_1, x_2, \dots, x_n)^T$ we have

$$g \cdot x = \begin{pmatrix} x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2 + a_{23}x_3 + \dots + a_{2n}x_n \\ \dots \\ x_n \end{pmatrix}.$$

Thus, x_i will always appear without a multiple in the ith row of $g \cdot x$. Recall dx_i is translation invariant. Since we integrate component-wise, and all other terms in this row are constants, we have the desired result. □

**** Problem 2.** *What is the Haar measure on $\mathbb{C}^\times = \mathbb{T} \times \mathbb{R}_+^\times$?*

Proof. We know that the Haar measure on $\mathbb{R}_+^\times$ is dx/x. Since we integrate component wise, we only need to find the Haar measure on $\mathbb{T}$ and then the product of the measures will suffice. Let f be a function on $\mathbb{T}$. Then for a constant $a \in \mathbb{T}$ we have

$$\int_{\mathbb{T}} f(ax)e^{i\theta}dx = \frac{1}{a}\int_{\mathbb{T}} f(ax)ae^{i\theta}dx = \int_{\mathbb{T}} f(x)e^{i\theta}dx.$$

The desired measure is then $e^{i\theta}dxdy/x$. □

**** Problem 3.** *For $\mathbb{C}$ and $\alpha \in \mathbb{C}$ what is $d(\alpha z)$?*

Proof. Treat $\mathbb{C}$ as $\mathbb{R}^2$. Then $dz = dxdy$. Moreover, since we're only concerned with scaling areas in $\mathbb{R}^2$, we can consider only $|\alpha|$. Then we have

$$d(\alpha z) = d(|\alpha|x)d(|\alpha|y) = |\alpha|^2 dxdy = |\alpha|^2 dz.$$

□

**** Problem 4.** *Show that every character is unitary over the group $(\mathbb{Q}_p, +)$.*

Proof. Let $\chi : \mathbb{Q}_p \to \mathbb{C}^\times$ be an additive character and let $a \in \mathbb{Q}_p$. Note that by p-adic expansion, $a = \sum_{k=\gamma}^{\infty} a_k p^k$ where $|a|_p = p^{-\gamma}$ and $0 \le a_k \le p-1$. Then we have

$$\chi(a) = \chi\left(\sum_{k=\gamma}^{\infty} a_k p^k\right) = \prod_{k=\gamma}^{\infty} \chi(a_k p^k) = \prod_{k=\gamma}^{-1} e^{2\pi i a_k p^k}.$$

The last result uses the tail of a, $\sum_{k=\gamma}^{-1} a_k p^k$. Note that $|\chi(a)| = 1$ by Euler's identity. □

** **Problem 5.** *Show $\hat{G}$ is a group.*

Proof. Let $\chi_1, \chi_2, \chi_3 \in \hat{G}$. Then, since associativity holds in $\mathbb{C}$, we have

$$((\chi_1\chi_2)\chi_3)(x) = (\chi_1\chi_2)(x)\chi_3(x) = (\chi_1(x)\chi_2(x))\chi_3(x) = \chi_1(x)(\chi_2(x)\chi_3(x)) = \chi_1(x)(\chi_2\chi_3(x)) = (\chi_1(\chi_2\chi_3))(x).$$

Now consider the 1 function which sends all values to 1. Then for $\chi \in \hat{G}$, $(1 \cdot \chi)(x) = 1\chi(x) = \chi(x)$. Finally, for $\chi \in \hat{G}$, let $\chi^{-1} : G\mathbb{C}^{\times}$ such that $\chi^{-1}(x) = (\chi(x))^{-1}$. The fact that this map is continuous follows from the fact that $\chi(x) \neq 0$. To show χ^{-1} is a homomorphism, let $x, y \in G$. Then

$$\chi^{-1}(xy) = (\chi(xy))^{-1} = (\chi(x)\chi(y))^{-1} = (\chi(x))^{-1}(\chi(y))^{-1} = \chi^{-1}(x)\chi^{-1}(y).$$

Since associativity, identity and inverses hold, $\hat{G}$ is a group. □

** **Problem 6.** *Read about the compact-open topology.*

Proof. For the case of $\hat{G}$, let K be a compact subset of G and U be an open subset of $\mathbb{C}$. If $V(K, U) \subseteq \hat{G}$ is the set of all elements of $\hat{G}$ such that $\chi(K) \subseteq U$, then $V(K, U)$ is a subbase for the compact-open topology. □

** **Problem 7.** *Suppose $G = \mathbb{T}$ and $f \in L^1(G)$ such that $f(e^{i\theta}) = 1$. Find $\hat{f}$.*

Proof. By definition we know

$$\hat{f}(\chi) = \int_{\mathbb{T}} f(x)\overline{\chi(x)}dx$$

and since $f(e^{i\theta}) = 1$, this simplifies to

$$\hat{f}(\chi) = \int_{\mathbb{T}} \overline{\chi(x)}dx.$$

Since $|\chi(x)| = 1$ we can write $\chi(x) = e^{i(x+a)}$ for some constant $a \in \mathbb{R}$. Then we have

$$\hat{f} = \int_{\mathbb{T}} e^{-i(x+a)}dx = \int_0^{2\pi} e^{-i(x+a)}dx = 0.$$

□