Kris Harper
STAT 24400
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Homework 8

Problem 1. *Which of the following hypotheses are simple, and which are composite?*
(a) X follows a uniform distribution on $[0, 1]$.
(b) A die is unbiased.
(c) X follows a normal distribution with mean 0 and variance $\sigma^2 > 10$.
(d) X follows a normal distribution with mean $\mu = 0$.

(a) Simple because the distribution is completely determined.

(b) Simple, assuming the die has six sides. Otherwise composite because we can't determine the distribution completely from the hypothesis.

(c) Simple because the distribution is given by the hypothesis.

(d) Composite because the distribution isn't determined by the hypothesis. The variance is unknown.

Problem 2. *Suppose that $X \sim \text{bin}(100, p)$. Consider the test that rejects $H_0 : p = .5$ in favor of $H_A : p \neq .5$ for $|X - 50| > 10$. Use the normal approximation to the binomial distribution to answer the following:*
(a) What is α?
(b) Graph the power as a function of p.

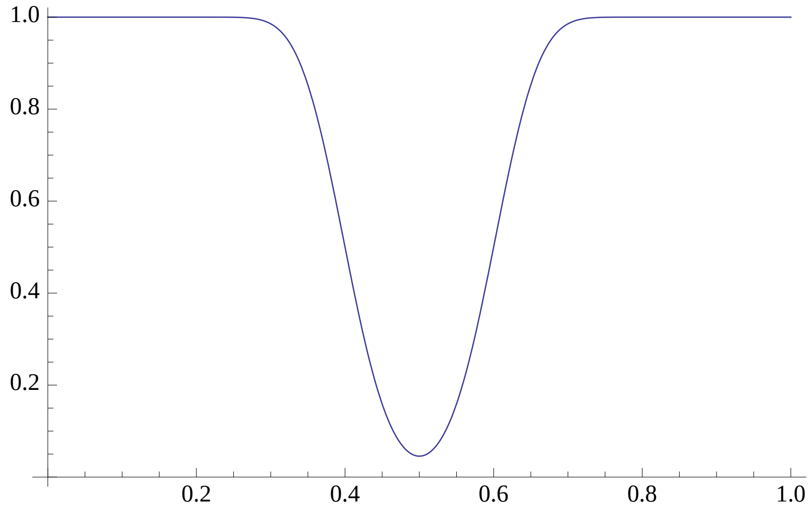
(a) We know α is $P(\text{reject } H_0 \mid H_0)$ which is $P(|X - 50| > 10 \mid p = .5)$. The normal approximation to this distribution is $N(np, np(1-p)) = N(50, 25)$ in this case. Then

$$\begin{aligned}
\alpha &= P(|X - 50| > 10 \mid p = .5) \\
&= P(X < 40) + P(X > 60) \\
&= P\left(\frac{X - 50}{5} > \frac{40 - 50}{5}\right) + P\left(\frac{X - 50}{5} > \frac{60 - 50}{5}\right) \\
&= \Phi(-2) + (1 - \Phi(2)) \\
&\approx .0456.
\end{aligned}$$

(b) We use the same normal approximation $N(np, np(1-p)) = N(100p, 100p(1-p))$ to get

$$\begin{aligned}
1 - \beta &= P(X < 40) + P(X > 60) \\
&= P\left(\frac{X - 100p}{\sqrt{100p(1-p)}} < \frac{40 - 100p}{\sqrt{100p(1-p)}}\right) + P\left(\frac{X - 100p}{\sqrt{100p(1-p)}} < \frac{60 - 100p}{\sqrt{100p(1-p)}}\right) \\
&= \Phi\left(\frac{40 - 100p}{\sqrt{100p(1-p)}}\right) + 1 - \Phi\left(\frac{60 - 100p}{\sqrt{100p(1-p)}}\right).
\end{aligned}$$

The graph of this for $0 \leq p \leq 1$ is



Problem 3. *Consider the coin tossing example of Section 9.1. Suppose that instead of tossing the coin 10 times, the coin was tossed until a head came up and the total number of tosses, X, was recorded.*
(a) If the prior probabilities are equal, which outcomes favor H_0 and which favor H_1?
(b) Suppose $P(H_0)/P(H_1) = 10$. What outcomes favor H_0?
(c) What is the significance level of a test that rejects H_0 if $X \geq 8$?
(d) What is the power of this test?

(a) The posterior probabilities are

$$P(H_0 \mid X) = \frac{(.5)P(X \mid H_0)}{(.5)P(X \mid H_0) + (.5)P(X \mid H_1)} = \frac{(1-.5)^{x-1}(.5)}{(1-.5)^{x-1}(.5) + (1-.7)^{x-1}(.7)}$$

$$P(H_1 \mid X) = \frac{(.5)P(X \mid H_1)}{(.5)P(X \mid H_0) + (.5)P(X \mid H_1)} = \frac{(1-.7)^{x-1}(.7)}{(1-.5)^{x-1}(.5) + (1-.7)^{x-1}(.7)}.$$

Then

$$\frac{P(H_0 \mid X)}{P(H_1 \mid X)} = \frac{(.5)^x}{(.3)^{x-1}(.7)} > 1$$

when

$$x > \frac{\log(7) - \log(3)}{\log(5) - \log(3)} \approx 1.66.$$

(b) Now we have

$$\frac{P(H_0 \mid X)}{P(H_1 \mid X)} = 10\frac{(.5)^x}{(.3)^{x-1}(.7)} > 1$$

when

$$x > \frac{\log(30) - \log(7)}{\log(5) - \log(3)} \approx -2.85$$

so all realistic outcomes.

(c) This is

$$\alpha = P(\text{reject } H_0 \mid H_0) = P(X \geq 8 \mid H_0) = 1 - P(X < 7 \mid H_0) = 1 - \sum_{i=1}^{6} (.5)^i = .015625$$

(d) The power is

$$1 - \beta = 1 - P(X < 7 \mid H_1) = 1 - \sum_{i=1}^{6} (.3)^{i-1}(.7) = .000729.$$

Problem 4. *Show that the test of Problem 7 is uniformly most powerful for testing $H_0 : \lambda = \lambda_0$ versus $H_A : \lambda > \lambda_0$.*

Proof. In Exercise 7 the likelihood ratio will be of the form

$$\frac{\lambda_0^{\sum_{i=1}^n X_i} e^{-n\lambda_0}}{\lambda_1^{\sum_{i=1}^n X_i} e^{-n\lambda_1}} = \lambda_0^{\sum_{i=1}^n X_i} \lambda_1^{-\sum_{i=1}^n X_i} e^{n(\lambda_1 - \lambda_0)} = \exp\left(n(\log(\lambda_0) - \log(\lambda_1))\overline{X} + n(\lambda_1 - \lambda_0)\right).$$

The likelihood ratio is then small if $\overline{X}$ is large since $\lambda_1 > \lambda_0$. So the most powerful test rejects for $\overline{X} > x_0$ for some x_0. The null distribution given H_0 in this case is Poisson with parameter $(1/n)n\lambda_0 = \lambda_0$. Thus we can choose x_0, specified for a significance α, using this distribution. But note that this is independent of λ_1 so it must hold true for all $\lambda_1 > \lambda_0$. Therefore this test is uniformly most powerful. □

Problem 5. *Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with the density function $f(x \mid \theta) = \theta \exp[-\theta x]$. Derive a likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$, and show that the rejection region is of the form $\{\overline{X} \exp[-\theta_0 \overline{X}] \leq c\}$.*

Proof. First we find the mle for this distribution. The log likelihood is

$$l(\theta) = n\log(\theta) - \theta \sum_{i=1}^n X_i$$

so

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n X_i$$

and

$$\hat{\theta} = 1/\overline{X}.$$

Now we form the likelihood ratio as

$$\frac{\theta_0^n \exp\left(-\theta_0 \sum_{i=1}^n X_i\right)}{\hat{\theta}^n \exp\left(-\hat{\theta} \sum_{i=1}^n X_i\right)} = \left(\frac{\theta_0}{\hat{\theta}}\right)^n \exp\left(-n(\theta_0 - \hat{\theta})\overline{X}\right) = (\theta_0 \overline{X} \exp(-\theta_0 \overline{X} + 1))^n.$$

This gives a rejection region of the form

$$\begin{aligned}(\theta_0 \overline{X} \exp(-\theta_0 \overline{X} + 1))^n &\leq c \\ \theta_0 e^{-1} \overline{X} \exp(-\theta_0 \overline{X}) &\leq c^{1/n} \\ \overline{X} \exp(-\theta_0 \overline{X}) &\leq \theta_0^{-1} e c^{1/n}.\end{aligned}$$

Now note that θ_0 is simply a constant and n is fixed, so we can redefine c to be the constant on the right and the desired result follows. □

Problem 6. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a double exponential distribution with density $f(x) = \frac{1}{2}\lambda \exp(-\lambda |x|)$. Derive a likelihood ratio test of the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda = \lambda_1$, where λ_0 and $\lambda_1 > \lambda_0$ are specified numbers. Is the test uniformly most powerful against the alternative $H_1 : \lambda > \lambda_0$?*

The likelihood ratio will be of the form

$$\begin{aligned}\frac{\lambda_0^n \exp\left(-\lambda_0 \sum_{i=1}^n |X_i|\right)}{\lambda_1^n \exp\left(-\lambda_1 \sum_{i=1}^n |X_i|\right)} &= \left(\frac{\lambda_0}{\lambda_1}\right)^n \exp\left(-(\lambda_0 - \lambda_1) \sum_{i=1}^n |X_i|\right) \\ &= \exp\left(n(\log(\lambda_0) - \log(\lambda_1)) - (\lambda_0 - \lambda_1) \sum_{i=1}^n |X_i|\right).\end{aligned}$$

Since $\lambda_1 > \lambda_0$, this ratio is small when $\sum_{i=1}^n |X_i|$ is small. Note that $n|\overline{X}| = |\sum_{i=1}^n X_i| \leq \sum_{i=1}^n |X_i|$ so when the right hand side is small, so is the left. Now note that the null hypothesis has a double exponential distribution with parameter λ_0. Thus $n|\overline{X}|$ is a known distribution with a parameter depending only on λ_0. It's independence of λ_1 shows that it's uniformly most powerful.

Problem 7. *Consider two probability density functions on* $[0,1]$*:* $f_0(x) = 1$*, and* $f_1(x) = 2x$*. Among all tests of the null hypothesis* $H_0 : X \sim f_0(x)$ *versus the alternative* $X \sim f_1(x)$*, with significance level* $\alpha = 0.10$*, how large can the power possibly be?*

By the Lemma, we know the likelihood ratio test will give the maximum power for a given significance level α. The likelihood ratio is of the form

$$\frac{f_0(x)}{f_1(x)} = \frac{1}{2x} > c$$

where $c \in (0,1)$. We are interested in

$$\alpha = .1 = P(\text{reject } H_0 \mid H_0) = P\left(\frac{1}{2x} \leq c \mid H_0\right) = P\left(\frac{1}{2c} \leq X \mid H_0\right) = 1 - P\left(X \leq \frac{1}{2c} \mid X_0\right).$$

So $1 - 1/2c = 1/10$ and $c = 5/9$. Then

$$1 - \beta = 1 - P(\text{accept } H_0 \mid H_1) = 1 - P\left(\frac{1}{2x} > c \mid H_1\right) = 1 - P\left(\frac{1}{2c} > X \mid H_1\right) = P\left(X \leq \frac{1}{2c} \mid H_1\right).$$

The cdf for $f_1(x)$ is $F_X(x) = x^2$ so

$$1 - \beta = \left(\frac{1}{2c}\right)^2 = \frac{1}{4c^2} = .81.$$