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Homework 3

Problem 1. *A set S is called star-shaped if there exists a point z_0 in S such that the line segment between z_0 and any point z in S is contained in S. Prove that a star-shaped set is simply connected, that is, every closed path is homotopic to a point.*

Proof. Let γ be a closed path in S. Consider the function $\psi(t,s) = sz_0 + (1-s)\gamma(t)$. It's easy to see that ψ_s is a closed curve for each s and that ψ is continuous. Also $\psi_0(t) = \gamma(t)$ and $\psi_1(t) = z_0$. Therefore each closed curve in S is homotopic to a point. □

Problem 2. *Show that the set $\mathbb{C}\backslash\{z \mid \mathrm{Re}(z) \leq 0$ and $|\mathrm{Im}(z)| \leq 1\}$ is simply connected (provide an explicit homotopy between any closed curve and a point).*

Proof. Let $S = \mathbb{C}\backslash\{z \mid \mathrm{Re}(z) \leq 0 \text{ and } |\mathrm{Im}(z)| \leq 1\}$ and let γ be a closed curve in S. Note that we may take γ to be continuous by reparametrization. Thus γ is the continuous image of a compact set is thus compact. Now let $a = \inf\{\mathrm{Re}(z) \mid z \in \gamma\}$ and $b = \inf\{|\mathrm{Im}(z)| - 1 \mid z \in \gamma\}$. Since γ is compact, these sets are bounded and nonempty and so a and b exist. Note that a is the "most negative" real part of γ and b is the closest γ gets to $\{z \mid |\mathrm{Im}(z)| \leq 1\}$. Furthermore, since γ is compact, it has two points a' and b' such that $\mathrm{Re}(a') = a$ and $\mathrm{Im}(b') = b$. That is, it realizes these values.

Now consider the real-valued function $f(x) = (-a/b)x + c$. Let $c = \mathrm{Im}(a') + (a/b)\mathrm{Re}(a')$. Then f is a line in one variable. Furthermore, for $x \leq 0$, we see that $f(x) > 1$. This follows from how a and b are defined. Now let z be the point such that $f(z) = 0$ and let $z_0 > z$. Define $\psi(s,t) = sz_0 + (1-s)\gamma(t)$ as in Problem 1. It follows that ψ is continuous and that $\psi_0(t) = \gamma(t)$ and $\psi_1(t) = z_0$. Additionally, for each $t \in [a,b]$, the line between $\gamma(t)$ and z_0 does not contain points in $\{z \mid \mathrm{Re}(z) \leq 0 \text{ and } |\mathrm{Im}(z)| \leq 1\}$. This follows because of how $f(x)$ is defined, and consequently how z_0 is defined. Therefore $\psi_s(t) \in S$ for all s and t. Since we can find a z_0 for each closed curve, we see that each one is homotopic to a point and therefore S is simply connected. □

Problem 3. *Let U be a simply connected open set and let f be a holomorphic function on U. Is $f(U)$ simply connected?*

Proof. Consider the set $H = \{z \mid \mathrm{Im}(z) > 0\}$ and let $f(z) = e^{2\pi i z}$. If $z = x + iy$ then we have $f(z) = e^{-2\pi y}e^{2\pi i x}$. If $y > 0$ then $0 < e^{-2\pi y} < 1$ and so $f(H) = D_1(0)\backslash\{0\}$ which is not simply connected. Any circle containing the origin is not homotopic to a point. Since H is simply connected (it is an open convex set), we see that $f(U)$ is not always simply connected for a holomorphic function f and a simply connected set U. □

Problem 4. *Prove: If $f \in C(\mathbb{C}$ and $f(z) \to 0$ as $|z| \to \infty$, then f is bounded.*

Proof. Let $\varepsilon > 0$. From the statement of the result, we know there exists $m > 0$ such that $|f(x)| < \varepsilon$ whenever $|z| > m$. Thus, f is bounded on the set $\{z \mid |z| > m\}$. But the set $\{z \mid |z| \leq m\}$ is a compact set, and since f is continuous, $f(\{z \mid |z| \leq m\})$ is compact, and thus bounded. Therefore f is bounded on all of $\mathbb{C}$. □

Problem 5. *Find the integrals over the unit circle γ:*
(a) $\int_\gamma \frac{\cos z}{z} dz$.
(b) $\int_\gamma \frac{\sin z}{z} dz$.
(c) $\int_\gamma \frac{\cos(z^2)}{z} dz$.

Proof. (a) Use the Local Cauchy Theorem letting $f(z) = \cos z$ and $z_0 = 0$. Then

$$1 = \cos(0) = f(z_0) = \frac{1}{2\pi i} = \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_\gamma \frac{\cos z}{z} dz.$$

Therefore $\int_\gamma \frac{\cos z}{z} dz = 2\pi i$.

(b) Use the method of part (a) letting $f(z) = \sin z$ and $z_0 = 0$. Since $f(z_0) = 0$, we know $\int_\gamma \frac{\sin z}{z} dz = 0$.

(c) Use the method of part (a) letting $f(z) = \cos(z^2)$ and $z_0 = 0$. Since $f(z_0) = 1$ we know $\int_\gamma \frac{\cos(z)^2}{z} dz = 2\pi i$. □

Problem 6. *Let $f \in H(U)$ and $g \in H(f(U))$ be such that f' has no zero in the open set U while g has a zero of order k at $w_0 = f(z_0)$ for some $z_0 \in U$. Show that $h = g \circ f$ has a zero of order k at z_0.*

Proof. Note that $h(z_0) = g(f(z_0)) = g(w_0) = 0$. Furthermore, note that each term of $h^{(n)}(z_0)$ for $1 \leq n < k$ has at least one power of $g^{(m)}(f(z_0)) = 0$ where $1 \leq m < n$. That is, every term is 0. This can be verified by using the chain rule and product rule repeatedly and noting that each term must contain $g^{(m)}$ for some $1 \leq m < n$. But now note that $h^{(k)}(z_0)$ will contain the term $g^{(k)}(f(z_0))f'(z_0)^k$. Again, this term can be found by differentiating $g(f(z_0))$ k times using the product and chain rules and always taking the first term of the result. But since $g(f(z_0))$ is a zero of order k and $f'(z_0) \neq 0$, we see that $h^{(k)}(z_0) \neq 0$ and so h has a zero of order k at z_0. □

Problem 7. *Let $\mathbb{D} = D_1(0)$ and $f \in H(\mathbb{D})$ be such that $|f(z)| < 1$ for all $z \in \mathbb{D}$. Show that $|f'(0)| \leq 1$ (notice that f need not be defined on $\partial\mathbb{D}$). How about if "$|f(z)| < 1$" is replaced by "$|f(z) - 10i| < 1$"?*

Proof. Let $R < 1$. Then $f \in H(\overline{D}_R(0))$ and thus f is analytic on $\overline{D}_R(0)$. Now let $0 < R_1 < R$. Note that $||f||_R < 1$ by hypothesis. Now recall that for each $c \in \mathbb{C}$ we have

$$|f'(0)| \leq \frac{R}{(R - R_1)^2} ||f - c||_R.$$

This must be true for all $0 < R_1 < R < 1$ and for $c = 0$ as R_1 approaches 0 and R approaches 1, the term on the right approaches 1. Therefore $|f'(0)| \leq 1$. Letting $c = -10i$ handles the second case in the same manner. □

Problem 8. *Let $f \in H(\mathbb{D})$ be such that $\text{Re} f(z) > 0$ for all $z \in \mathbb{D}$ and $f(0) = 1$. Show that $|f'(0)| \leq 2$.*

Proof. Let $R = \{z \mid \text{Re} z > 0\}$. Let $g : R \to \mathbb{D}$ be a function such that $g(z) = \frac{1-z}{1+z}$. Then note that $|g(z) = \frac{|z-1|}{|z+1|} < 1$ for $z \in R$. This map is clearly injective, and is also surjective since $g^{-1}(z) = \frac{z+1}{1-z}$ as can easily be seen. Thus g is a bijection from R into $\mathbb{D}$. Let $h = g \circ f$. From Problem 7 we know $1 \geq |h'(0)| = |g'(f(0))f'(0)|$. We know $f(0) = 1$ and $g'(z) = \frac{2}{(z+1)^2}$ so $g'(f(0)) = \frac{1}{2}$. Therefore $|f'(0)| \leq 2$. □

Problem 9. *Find U open and $f \in H(U)$ such that f is 2-to-1 on U (i.e., for all $w \in f(U)$ we have $|\{z \in U \mid f(z) = w\}| = 2$).*

Proof. Let $U = \mathbb{C} \backslash 0$ and let $f = z^2$. We've shown that z^n is an n-to-1 function and this is the case $n = 2$. Note that 0 is not included in the set since $0^2 = 0$. Then for $w \neq 0$ with $w = r^{i\theta}$ we have $w_1 = |w|e^{i\theta/2}$ and $w_2 = |w|e^{i\theta/2}e^{2\pi i\theta/2}$. □

Problem 10. *Show that if f is as in Problem 9, then f' has no zeros in U.*

Proof. If $f(z) = z^2$ then $f'(z) = 2z$. But then $f'(z) = 0$ only if $z = 0$ and $0 \notin U$. □