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Homework 6

Problem 1. *Let f be analytic on the unit disk D, and assume that $|f(z)| < 1$ on the disk. Prove that if there exist two distinct points a, b in the disc which are fixed points, that is, $f(a) = a$ and $f(b) = b$, then $f(z) = z$.*

Proof. Note that $|f(z)| < 1$ implies $f(D) \subseteq D$. Let $h = g_a \circ f \circ g_a$. Note that $h(0) = g_a(f(g_a(0))) = g_a(f(a)) = g_a(a) = 0$. There exists some $z_0 \in D$ such that $g(z_0) = b$ (as g is an automorphism of the unit disk). Note that $h(z_0) = g_a(f(g_a(z_0))) = g_a(f(b)) = g_a(b) = g_a^{-1}(b) = z_0$. Furthermore, $z_0 \neq 0$ since $a \neq b$. So now using the Schwartz Lemma, we know that there exists α such that $h(z) = \alpha z$ and we must have $\alpha = 1$ since $h(z_0) = z_0$. Therefore $h(z) = z$ for all $z \in D$ which means $g_a \circ f(z) = g_a(z)$ and $f(z) = z$. $\square$

Problem 2. *Let α be real, $0 \leq \alpha < 1$. Let U_α be the open set obtained from the unit disk by deleting the segment $[\alpha, 1]$, as shown on the figure.*
(a) Find an isomoprphism of U_α with the unit disk from which the segment $[0, 1]$ has been deleted.
(b) Find an isomorphism of U_0 with the upper half of the disk. Also find an isomorphism of U_α with this upper half disk.

Proof. (a) Choose the automorphism $e^{i\pi}\frac{\alpha - z}{1-\overline{\alpha}z} = \frac{z-\alpha}{1-\alpha z}$ where $\overline{\alpha} = \alpha$ since α is real. It's clear that $f(\alpha) = 0$ and $f(1) = 1$. Furthermore, for $y \in (\alpha, 1)$, $f(y)$ is certainly a real number and $0 < f(y) < 1$ so $[\alpha, 1]$ gets mapped to $[0, 1]$ under f. But since f is an automorphism of the unit disk, we must have f maps U_α onto U_0 isomorphically.

(b) Use $g(z) = \sqrt{z}$. If $z = re^{i\theta}$ then $g(z) = e^{(1/2)(\log r + i\theta)} = \sqrt{r}e^{i\theta/2}$. Since $\sqrt{r} > 0$ and $0 < \theta/2 < \pi$ (remember $[0, 1]$ is not in our set so θ can't be 0 or 2π) we must have $g(z)$ is in the upper half disk. The function $g \circ f$ will take U_α to the upper half disk. $\square$

Problem 3. *Show that f_M gives a map of H into H.*

Proof. If $z = x + iy$ is in H then $y > 0$ and it suffices to show that $\mathrm{Im} f_M(z) = \mathrm{Im}(az + b)/(cz + d) > 0$. We have

$$\begin{aligned}
f_M(z) &= f_M(x+iy) \\
&= \frac{ax+iay+b}{cx+icy+d} \\
&= (ax+b+iay)\frac{cx+d-icy}{(cx+d)^2+c^2y^2} \\
&= ((ax+b)+iay)\left(\frac{cx+d}{(cx+d)^2+c^2y^2} - i\frac{cy}{(cx+d)^2+c^2y^2}\right).
\end{aligned}$$

Multiplying and taking the imaginary part we see

$$\mathrm{Im} f_M(x+iy) = \frac{acxy - ady}{(cx+d)^2+c^2y^2} - \frac{acxy+bcy}{(cx+d)^2+c^2y^2} = \frac{ady-bcy}{(cx+d)^2+c^2y^2}.$$

The denominator is clearly positive, and by assumption $ad - bc > 0$. Since $x + iy \in H$, $y > 0$ as well and so $ady - bcy = y(ad - bc) > 0$ and we're done. $\square$

Problem 4. *(a) Given an element $z = x + iy \in H$, show that there exists an element $M \in SL_2(\mathbb{R})$ such that $f_M(i) = z$.*
(b) Given $z_1, z_2 \in H$, show that there exists $M \in SL_2(\mathbb{R})$ such that $F_M(z_1) = z_2$. In light of (b), one then says that $SL_2(\mathbb{R})$ acts transitively *on H.*

Proof. (a) With $y > 0$, the matrix

$$M_1 = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

takes i to iy. Now the matrix

$$M_2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

takes iy to $x + iy$. Thus $f_{M_2} \circ f_{M_1}(i) = z$ and $M_1, M_2 \in SL_2(\mathbb{R})$.

(b) Use modified matrices in (a) to first take z_1 to i. That is, apply M_2 with a $-x$, then M_1 with the elements inverted. Now use the same matrices in (a) to take i to z_2. $\square$

Problem 5. *Let K denote the subset of elements $M \in SL_2(\mathbb{R})$ such that $f_M(i) = i$. Show that if $M \in K$, then there exists a real θ such that*

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Proof. Let $M \in SL_2(\mathbb{R})$ such that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Suppose we have $(ai + b)/(ci + d) = i$. Then $ai + b = i(ci + d)$ (c and d can't both be 0 because $ad - bc = 1$). So $b + ia = -c + id$ and $b = -c$, $a = d$. Putting these equalities into the determinant equation we have $a^2 + c^2 = 1$ so that the point (a, c) is on the unit circle. Thus there exists some θ such that $a = \cos\theta$ and $c = \sin\theta$. Given that $a = d$ and $-b = c$ immediately gives the result. $\square$

Problem 6. *Let $f : H \to D$ be the isomorphism of the text, that is*

$$f(z) = \frac{z - i}{z + i}.$$

Note that f is represented as a fractional linear map, $f = F_M$ where M is the matrix

$$M = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

Of course, this matrix does not have determinant 1.
Let K be the set of Exercises 5. Let $\mathrm{Rot}(D)$ denote the set of rotations of the unit disk, i.e. $\mathrm{Rot}(D)$ consists of all automorphisms

$$R_\theta : w \mapsto e^{i\theta} w \text{ for } w \in D.$$

Show that $fKf^{-1} = \mathrm{Rot}(D)$, meaning that $\mathrm{Rot}(D)$ consists of all elements $f \circ f_M \circ f^{-1}$ with $M \in K$.

Proof. Let M be the matrix in the problem statement, and let M' be the matrix from Problem 5. Note that

$$M^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{pmatrix}$$

and $f_M^{-1} = f_{M^{-1}}$. Furthermore, note that $f_M \circ f_{M'} \circ f_{M^{-1}} = f_{MM'M^{-1}}$. We have

$$\begin{aligned} MM'M^{-1} &= \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta - i\sin\theta & -i\cos\theta - \sin\theta \\ \cos\theta + i\sin\theta & i\cos\theta - \sin\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{pmatrix}. \end{aligned}$$

Thus $f_M \circ f_{M'} \circ f_M^{-1} = e^{-i\theta} z / e^{i\theta z} = e^{-2i\theta} z$. Thus $fKf^{-1} \subseteq \mathrm{Rot}(D)$. But if $R_\theta \in \mathrm{Rot}(D)$, then $R_\theta = R_{-\varphi/2}$ for some appropriate φ, and the argument above works backwards. Thus both inclusions hold and we're done. $\square$

Problem 7. *Every automorphism of H is of the form f_M for some $M \in SL_2(\mathbb{R})$.*

Proof. Let $g \in \text{Aut}(H)$. From Problem 4 we know that there exists $M \in SL_2(\mathbb{R})$ such that $f_M(g(i)) = i$. By Problem 5 we know that $f_M \circ g \in K$. Let $f_M \circ g = f_{M'}$ for $M' \in K$. Then we have $g = f_{M^{-1}} \circ f_{M'} = f_{M^{-1}M'}$. Since $SL_2(\mathbb{R})$ is a group under matrix multiplication, $M^{-1}M' \in SL_2(\mathbb{R})$ and we're done. □

Problem 8. *Let a be a real number. Let U be the open set obtained from the complex plane by deleting the infinite segment $[a, \infty)$. Find explicitly an analytic isomorphism of U with the unit disk. Give this isomorphism as a composition of simpler ones.*

Proof. Translate the plane by $-a$ to get the plane with $[0, \infty)$ removed. Now take $\sqrt{z}$ to get the upper half plane. Use $(z-i)/(z+i)$ to get the unit disk. So the isomorphism will be $(\sqrt{z-a}-i)/(\sqrt{z-a}+i)$. □

Problem 9. *Let $w = u + iv = f(z) = z + \log z$ for z in the upper half plane H. Prove that f gives an isomorphism of H with the open set U obtained from the upper half plane by deleting the infinite half line of numbers*

$$u + i\pi \text{ with } u \leq -1.$$

Proof. Let γ be the path defined by the line segment from R to ε, then the semicircle from ε to $-\varepsilon$, then the line segment from $-\varepsilon$ to $-R$. Then the semicircle from $-R$ to R. We must consider $f \circ \gamma$. For the first piece, $\theta = 0$ and so $f(z) = r + \log r$. This is a strictly increasing function which maps the real line to itself. Thus $f([\varepsilon, R]) = [\varepsilon + \log \varepsilon, R + \log R]$. For a point z on the semicircle of radius ε, we have $f(z) = \varepsilon \cos\theta + \log\varepsilon + i(\theta + \varepsilon \sin\theta)$. Taking the limit as ε approaches 0 gives this image to be a line segment from $\varepsilon + \log\varepsilon$ to $-\varepsilon + \log\varepsilon + i\pi$. For z on the segment from $-\varepsilon$ to $-R$, $f(z) = -r + \log r + i\pi$. Thus, the image of this segment under f is the line segment from $-\varepsilon + \log\varepsilon + i\pi$ to $-1 + i\pi$ and then the segment from $-1 + i\pi$ to $-R + \log R + i\pi$. The final semicircle completes the path $f \circ \gamma$ which shows that $f \circ \gamma$ has an interior. Moreover, this interior (as R goes to infinity and ε goes to 0) is simply U. Since γ also has an interior and both of these interiors are connected, we have an isomorphism between them. Now let ε tend to 0 and R tend to infinity to conclude the proof. □