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MATH 16300
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Homework 5

Exercise 1 *Show that for all $\alpha \in K$ and $v \in V$ we have*

$$\alpha \cdot_V 0 = 0 \cdot_V v = 0$$

and if $\alpha \cdot_V v = 0$ then $\alpha = 0$ or $v = 0$.

Proof. Let $\alpha \in K$ and $v \in V$. Then we have $0 +_V 0 = 0$ using V2 so $\alpha \cdot_V (0 +_V 0) = \alpha \cdot_V 0$. Then using V5 we have $(\alpha \cdot_V 0) +_V (\alpha \cdot_V 0) = \alpha \cdot_V 0$ and so

$$\alpha \cdot_V 0 = \alpha \cdot_V 0 +_V 0 = \alpha \cdot_V 0 +_V (-\alpha \cdot_V 0) = 0.$$

Similarly, $0 +_K 0 = 0$ using $V1$ and so $(0 +_K 0) \cdot_V v = 0 \cdot_V v$. Then using $V4$ we have $(0 \cdot_V v) +_V (0 \cdot_V v) = 0 \cdot_V v$ and so

$$0 \cdot_V v = 0 \cdot_V +_V 0 = 0 \cdot_V v +_V (-0 \cdot_V v) = 0.$$

□

Exercise 5 *X is linearly dependent if and only if there exists $x \in X$ that depends on $X \backslash x$.*

Proof. Suppose that X is linearly dependent. Then there exist $a_1, \ldots, a_n$ and $v_1, \ldots, v_n$ with some $a_j \neq 0$ and $v_i \neq v_j$ such that

$$\sum_{i=1}^{n} a_i v_i = 0.$$

Then

$$\sum_{i=1}^{j-1} a_i v_i + \sum_{i=j+1}^{n} a_i v_i = -a_j v_j$$

and since $a_j \neq 0$ we have

$$v_j = \sum_{i=1}^{j-1} \frac{a_i}{a_j} v_i + \sum_{i=j+1}^{n} \frac{a_i}{a_j} v_i.$$

Since $v_j \in X$ and the right hand side of this is a linear combination of distinct elements of X, we have v_j depends on $X \backslash x$.

Conversely suppose that there exists $x \in X$ that depends on $X \backslash x$. Then

$$x = \sum_{i=1}^{n} a_i v_i$$

for $v_i \in X \backslash x$. Note that if $v_i = v_j$ for some i, j then we can write $a_i v_i + a_j v_j = (a_i + a_j) v_i$ so that each term in $\sum_{i=1}^{n} a_i v_i$ is distinct. Then we have

$$\sum_{i=1}^{n} a_i v_i - x = 0$$

and since $v_i \neq x$ for all i, each term is distinct and the coefficient for x is not 0 so we have a non-trivial linear combination of distinct elements of X which equals 0 so X is linear dependent. □

Exercise 6 *Let X be a linearly independent subset. Then $v \in V$ is depends on X if and only if $X \cup \{v\}$ is linearly dependent.*

Proof. Let v depend on X. Then

$$v = \sum_{i=1}^{n} a_i v_i$$

for $v_i \in X$. Note that since X is linearly independent every non-trivial linear combination of elements of X is nonzero. Thus it's not the case that $v = v_i$ for some i. If $v = a_j v_j$ for some j and $a_j \neq 1$ then we can write $v - a_j v_j = (1 - a_j)v = \sum_{i=1}^{j-1} a_i v_i + \sum_{i=j+1}^{n} a_i v_i$. Thus we can assume that $v \neq a_j v_j$. Then

$$\sum_{i=1}^{n} a_i v_i - v = 0$$

and since $a_i \neq a_j$ for all i, j and $v \neq v_i$ for all i we have a linear combination of distinct elements of $X \cup \{v\}$ which equals 0. Thus $X \cup \{v\}$ is linearly dependent.

Conversely assume that $X \cup \{v\}$ is linearly dependent. Then there exists distinct elements of $X \cup \{v\}$, $v_1, \ldots, v_n$, and $a_1, \ldots, a_n$ with some $a_j \neq 0$ such that

$$\sum_{i=1}^{n} a_i v_i = 0.$$

Since X is linearly independent, every non-trivial linear combination of elements of X is nonzero. Thus we must have $v = v_j$ for some j and $a_j \neq 0$. Then we have

$$-\left(\sum_{i=1}^{j-1} \frac{a_i}{a_j} v_i + \sum_{i=j+1}^{n} \frac{a_i}{a_j} v_i\right) = v$$

and so v depends on X since $v_i \neq v_j$ for all i, j. $\square$

Exercise 8 *The intersection of an arbitrary number of subspaces is a subspace.*

Proof. Let $S = \{U_1, U_2, \ldots\}$ be a set of subspaces and consider

$$T = \bigcap_{U_i \in S} U_i.$$

Let $u, v \in T$ and let $a \in K$. Then we have $u, v \in U_i$ for all i and so $u + v \in U_i$ for all i which means $u + v \in T$. Also $au \in U_i$ for all i and so $au \in T$. Thus T is a subspace. $\square$

Exercise 10 *For an subset $X \subseteq V$ we have*

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\}.$$

Proof. Let

$$x \in \left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\}.$$

Then x is a linear combination of elements of X. But then for all subspaces U such that $X \subseteq U$ we have $x \in U$ since U is closed under addition and scalar multiplication. Thus

$$\left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\} \subseteq \bigcap_{\substack{U \text{ is a subspace in } V \\ X \subseteq U}} U = \langle X \rangle.$$

Now let

$$x \in \bigcap_{\substack{U \text{ is a subspace in } V \\ X \subseteq U}} U.$$

Then since all subspaces U are closed under addition and scalar multiplication we know x is some linear combination of elements which are in U for all $X \subseteq U$. But since this is true for all U such that $X \subseteq U$, we must have

$$x \in \left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\}$$

and so

$$\left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\} = \langle X \rangle.$$

□

Exercise 13 *A subset $X \subseteq V$ is a basis if and only if every element of V can be obtained as a unique linear combination of elements of X.*

Proof. Suppose $X \subseteq V$ is a basis. Then

$$V = \langle X \rangle = \left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\}$$

which means that each element, v, of V is some linear combination of elements of X. But then $X \cup \{v\}$ is linearly dependent and so the linear combination must be unique. Conversely suppose that for all $v \in V$ we have v is a unique linear combination of elements in X. Then

$$v \in \left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\} = \langle X \rangle.$$

Since this is true for all $v \in V$ we have $V \subseteq \langle X \rangle$. But $X \subseteq V$ and so we must have $V = \langle X \rangle$ so X is a basis for V. □

Exercise 14 *The following are equivalent for a subset $X \subseteq V$:*
1) X is a basis;
2) X is a maximal independent subset;
3) X is a minimal spanning subset.
Exercise 15 *Let $X \subseteq V$ be a finite linearly independent subset and let $Y \subseteq V$ be a finite spanning subset. Then $|X| \leq |Y|$.*
Exercise 16 *If V is finitely generated then it has a basis and every basis has the same number of elements.*

Proof. Suppose that V is finitely generated by X. Then we have

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} a_i v_i \mid n \in \mathbb{N},\, a_i \in K,\, v_i \in X \right\} = V$$

and so every element in V is linear combination of elements of X which means that a subset of X serves as a basis for V. Also, if two basis, X and Y, have different numbers of elements then we can use X to write one element of Y as a linear combination of the other elements of Y which means Y is not linearly independent. □

Exercise 18 *Assume that V is finitely generated. Then any linearly independent subset of V can be extended to be a basis and any spanning subset of V contains a basis.*

Proof. Let V be finitely generated by X. Exercise 16 shows that a spanning subset of V contains a basis since a spanning subset of V will be linear combinations of elements of X. A linearly independent subset of V, Y can be made a basis by adding the vectors of $X \backslash Y$ to Y. □