Sheet 3: Attack of the Continuum

Definition 1 (Disjoint) *Two sets A and B are disjoint if $A \cap B = \emptyset$. A set of sets S is pairwise disjoint if for all sets $A, B \in S$ we have $A = B$ or $A \cap B = \emptyset$.*

Theorem 2 *If $p, q \in C$ and $p < q$, then there exist disjoint regions containing p and q.*

Proof. Let $a, c, p, q \in C$ such that $a < p$, $p < q$ and $q < c$ (A2.1, A2.2, A2.3). Then there are two possibilities. There may be another point $b \in C$ which is between p and q. We see that this implies $p < b$ and $b < q$ and so the region $(a; b)$ contains p and the region $(b; c)$ contains q but $(a; b) \cap (b; c) = \emptyset$. There is also the possibility that there are no points between p and q. Then the region $(a; q)$ contains p but not q and the region $(p; c)$ contains q but not p and $(a; q) \cap (p; c) = (p; q) = \emptyset$. □

Corollary 3 *A set consisting of one point has no limit points.*

Proof. Let $A \subseteq C$ be a set with one point x. If $p \in C$ is to be a limit point of A, then every region which contains p must also contain a point in A which is not p. So we see $p \neq x$. But then $p < x$ or $p > x$. In either case, Theorem 2 shows that there are disjoint regions containing p and x which means there exist regions containing p, but no points in A so p cannot be a limit point of A (3.2). □

Theorem 4 *A nonempty finite set of points has no limit points.*

Proof. By Corollary 3 we see that a set with one point has no limit points (3.3). Use induction on n and assume that a subset of C with $n \in \mathbb{N}$ points has no limit points. Consider the set S where $|S| = n + 1$ and let $a \in S$. We know that $|S \backslash a| = n$ and so $S \backslash a$ has no limit points. But $S = (S \backslash a) \cup \{a\}$ and so we know that a limit point of S is a limit point of $S \backslash a$ or a limit point of $\{a\}$ (2.17). By the inductive hypothesis and Corollary 3 we know that both $S \backslash a$ and $\{a\}$ have no limit points (3.3). Therefore S has no limit points. Thus, by induction, all nonempty finite sets have no limit points. □

Corollary 5 *If $A \subseteq C$ is a finite set and $x \in A$, then there exists a region R such that $A \cap R = \{x\}$.*

Proof. Let $A \subseteq C$ be a finite set of n elements such that $x \in A$. We know that x cannot be a limit point of A by Theorem 4 and so there exists a region R such that $x \in R$ and $R \cap (A \backslash x) = \emptyset$ (3.4). But then we have $R \cap A = \{x\}$. □

Theorem 6 *If p is a limit point of a set A and R is a region containing p, then the set $R \cap A$ is infinite.*

Proof. Assume that p is a limit point of a set $A \subseteq C$ and R is a region containing p. Assume to the contrary that $R \cap A$ is finite. Then p is not a limit point of $R \cap A$ by Theorem 4 (3.4). But since $(A \backslash (R \cap A)) \cup (R \cap A) = A$, and p is a limit point of the union, we see that p must be a limit point of $A \backslash (R \cap A)$ (2.17). We also have $R \cap (A \backslash (R \cap A)) = \emptyset$ and $p \in R$ so p is not a limit point of $A \backslash (R \cap A)$. This is a contradiction and so $R \cap A$ must be infinite. □

Definition 7 (Closed Set) *A set $A \subseteq C$ is closed if it contains all its limit points.*

Corollary 8 *Finite sets are closed.*

Proof. Finite sets have no limit points and so they vacuously contain all of their limit points (3.4). □

Definition 9 (Closure) *Let $M \subseteq C$ be a set. Let $\overline{M}$, the closure of M, be the set consisting of M and all of its limit points:*

$$\overline{M} = M \cup \{x \in C \mid x \text{ is a limit point of } M\}.$$

Theorem 10 *A set is $M \subseteq C$ is closed if and only if $M = \overline{M}$.*

Proof. We see that if $M \subseteq C$ is closed, then it contains all its limit points. That is $\{x \in C \mid x \text{ is a limit point of } M\} \subseteq M$. So we have $M = M \cup \{x \in C \mid x \text{ is a limit point of } M\} = \overline{M}$. Conversely, if $M = \overline{M}$ then $M = M \cup \{x \in C \mid x \text{ is a limit point of } M\}$. Therefore $\{x \in C \mid x \text{ is a limit point of } M\} \subseteq M$, and so M contains all its limit points. Thus M is closed. $\square$

Theorem 11 *For all $M \subseteq C$ we have $\overline{M} = \overline{\overline{M}}$*

Proof. We wish to show that the set of limit points of $\overline{M}$ is a subset of $\overline{M}$ for $M \subseteq C$. Consider a limit point $p \in C$ of $\overline{M}$. Since $\overline{M} = M \cup \{x \mid x \text{ is a limit point of } M\}$ we see that p is a limit point of M or p is a limit point of the set of limit points of M because p is a limit point of their union (2.17). If p is a limit point of M, then $p \in \overline{M}$. If p is a limit point of the set of limit points of M, then every region containing p contains a limit point of M. But every region containing a limit point of M contains a point in M and so for all regions $R \subseteq C$ such that $p \in R$ we have $R \cap M \neq \emptyset$. But then either p is in M or p is a limit point of M and so $p \in \overline{M}$. Thus we see $\{x \mid x \text{ is a limit point of } \overline{M}\} \subseteq \overline{M}$ and so

$$\overline{M} = M \cup \{x \mid x \text{ is a limit point of } M\} \cup \{x \mid x \text{ is a limit point of } \overline{M}\} = \overline{\overline{M}}.$$

$\square$

Corollary 12 *If M is a set of points, then $\overline{M}$ is closed.*

Proof. Let $M \subseteq C$ be a set of points. By Theorem 11 we know $\overline{M} = \overline{\overline{M}}$ and so by Theorem 10, $\overline{M}$ is closed (3.10, 3.11). $\square$

Theorem 13 *The sets C and $\emptyset$ are closed.*

Proof. All limit points are elements of C and so C must contain all its limit points and is closed. The empty set can have no limit points since there are no regions which contain a point in $\emptyset$. Therefore it vacuously contains all its limit points and is closed. $\square$

Definition 14 (Open Set) *A set of points M is open if the complement $C \backslash M$ is closed.*

Theorem 15 *The sets C and $\emptyset$ are open.*

Proof. We see that the complement $C \backslash C = \emptyset$ and $\emptyset$ is closed so C is open (3.13). Likewise $C \backslash \emptyset = C$ and C is closed so $\emptyset$ is open (3.13). $\square$

Theorem 16 *Every region is open and its complement is closed.*

Proof. We wish to show that for all regions R, the complement $C \backslash R$ is closed. So we assume that for some region R there exists a limit point p of $C \backslash R$ such that $p \notin C \backslash R$. Thus, $p \in R$. But then, since $R \cap (C \backslash R) = \emptyset$, p is not a limit point of $C \backslash R$ and this is a contradiction. Thus, $C \backslash R$ contains all its limit points and so it is closed which means R is open for all $R \subseteq C$. $\square$

Theorem 17 (Open Condition) *A set $A \subseteq C$ is open if and only if for all $x \in A$, there exists a region $R \subseteq A$ such that $x \in R$.*

Proof. Let $A \subseteq C$ be open. Then $C \backslash A$ is closed. Assume there exists $x \in A$ such that for all regions R containing x, R is not a subset of A. Then for all regions R containing x, R contains at least one point in $C \backslash A$ and so x is a limit point of $C \backslash A$. But x is in A and $C \backslash A$ is closed and so we have a contradiction. Thus for all $x \in A$, there exists a region $R \subseteq A$ such that $x \in R$.

Conversely, let $A \subseteq C$ be a subset such that for all $x \in A$ there exists a region $R \subseteq A$ such that $x \in R$. Assume A is not open. Then $C \backslash A$ is not closed and so it doesn't include all its limit points. But then there exists a limit point p of $C \backslash A$ such that $p \in A$. And so there exists a region $R \subseteq A$ which contains p and so p is not a limit point of $C \backslash A$. This is a contradiction and so A must be open. $\square$

Corollary 18 *Every nonempty open set is the union of regions.*

Proof. Let $R_a \subseteq A$ denote a region containing an element $a \in A$ for some open set A. Then we see that $\bigcup_{a \in A} R_a$ is a union of subsets of A which contains every element of A so it must be equal to A. $\square$