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MATH 16100
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Homework 2

1. *Show that for all $n, k \in \mathbb{N}$ we have*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

First we prove a lemma showing that for two sets A and B, if $A \cap B = \emptyset$ then $|A| + |B| = |A \cup B|$.

Proof. We use contradiction. Suppose, to the contrary, that if A and B are sets and $A \cap B = \emptyset$ then $|A| + |B| \neq |A \cup B|$. Then there are two cases.

Case 1: $|A| + |B| > |A \cup B|$. Then there exists an element which is in A and is in B since all elements in A or in B are in $A \cup B$. But this is a contradiction since $A \cap B = \emptyset$.

Case 2: $|A| + |B| < |A \cup B|$. Then there exists an element in $A \cup B$ which is not in A or in B. But this goes against the definition for $A \cup B$ and is a contradiction.

In both cases we have contradictions thus if $A \cap B = \emptyset$ then $|A| + |B| = |A \cup B|$. $\square$

Now we prove the original result.

Proof. Let S be a set with n elements and let $A \subseteq S$ such that A has k elements. Then for a given element $a \in S$, we see that either $a \in A$ or $a \notin A$ for all $A \subseteq S$. Now let $X = \{A \subseteq S \mid |A| = k,\ a \in A\}$ and let $Y = \{A \subseteq S \mid |A| = k,\ a \notin A\}$. Because it is never the case that for some $a \in S$, $a \in A$ and $a \notin A$ for any $A \subseteq S$, X and Y have no common elements and so $X \cap Y = \emptyset$. Additionally, every subset of S with k elements is either in X or in Y and X and Y by definition only contain subsets of S with k elements. We see that $X \cup Y$ contains all the subsets of S of size k and since $|S| = n$, we have $|X \cup Y| = \binom{n}{k}$.

Now consider the set X. For every element $A \in X$, $A \subseteq S$, $|A| = k$ and $a \in A$ for some $a \in S$. Then for every $A \in X$ there exists a set $B \subseteq S \backslash \{a\}$ such that $a \notin B$ and $|B| = k - 1$. Since X only contains subsets $A \subseteq S$ and $|S \backslash \{a\}| = n - 1$, we see that the number of elements of X is equal to the number of sets with $k - 1$ elements which are subsets of a set with $n - 1$ elements. Thus $|X| = \binom{n-1}{k-1}$.

Finally consider the set Y. For every element $A \in Y$, $A \subseteq S$, $|A| = k$ and $a \notin A$. But if for all $A \subseteq S$, $a \notin A$, then $A \subseteq S \backslash \{a\}$. This is true for all $A \in Y$ since by definition, $A \in Y$ if $a \notin A$ for some $a \in S$. Then, since $|S \backslash \{a\}| = n - 1$, Y contains all the sets with k elements which are subsets of set with $n - 1$ elements. Thus, $|Y| = \binom{n-1}{k}$. But since $X \cap Y = \emptyset$, $|X \cup Y| = |X| + |Y|$ and so $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. $\square$

2. *(Binomial Theorem) Show that for all a, b and $n \in \mathbb{N}$ we have*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. We use induction on n. We see that the theorem holds for $n = 1$ since

$$\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = \binom{1}{0} a^0 b^1 + \binom{1}{1} a^1 b^0 = a + b = (a+b)^1.$$

Now we assume that $(a+b)^j = \sum_{k=0}^{j} \binom{j}{k} a^k b^{j-k}$ for some $j \in \mathbb{N}$ and show that it holds for $j+1$. We see that

$$\begin{aligned}
(a+b)^{j+1} &= (a+b)^j(a+b) \\
&= \left(\sum_{k=0}^{j} \binom{j}{k} a^k b^{j-k} \right)(a+b) \\
&= \sum_{k=0}^{j} \binom{j}{k} a^{k+1} b^{j-k} + \sum_{k=0}^{j} \binom{j}{k} a^k b^{j+1-k} \\
&= \sum_{k=0}^{j-1} \binom{j}{k} a^{k+1} b^{j-k} + \sum_{k=1}^{j} \binom{j}{k} a^k b^{j+1-k} + \binom{j}{0} a^0 b^{j+1} + \binom{j}{j} a^{j+1} b^0 \\
&= \sum_{k=1}^{j} \binom{j}{k-1} a^k b^{j+1-k} + \sum_{k=1}^{j} \binom{j}{k} a^k b^{j+1-k} + \binom{j}{0} a^0 b^{j+1} + \binom{j}{j} a^{j+1} b^0 \\
&= \sum_{k=1}^{j} \left(\binom{j}{k-1} + \binom{j}{k} \right) a^k b^{j+1-k} + \binom{j}{0} a^0 b^{j+1} + \binom{j}{j} a^{j+1} b^0 \\
&= \sum_{k=1}^{j} \binom{j+1}{k} a^k b^{j+1-k} + \binom{j+1}{0} a^0 b^{j+1} + \binom{j+1}{j+1} a^{j+1} b^0 \\
&= \sum_{k=0}^{j} \binom{j+1}{k} a^k b^{j+1-k}.
\end{aligned}$$

Since the theorem is true for $n = 1$ and it's true for $j+1$ when it is true for j for all $j \in \mathbb{N}$ then we can conclude it is true for all $n \in \mathbb{N}$. ☐

3. *Prove that for all $n, k \in \mathbb{N}$ with $0 \le k \le n$ we have*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. We use induction on n. We see that when $n = 1$, k can either equal 0 or 1. When $k = 0$ we have $\binom{1}{0} = 1 = \frac{1!}{(0!)(1-0)!}$ and when $k = 1$ we have $\binom{1}{1} = 1 = \frac{1!}{(1!)(1-1)!}$. We now assume that $\binom{j}{k} = \frac{j!}{k!(j-k)!}$ for some $j \in \mathbb{N}$ and $0 \leq k \leq j$ and show that this implies the statement is true for $j+1$. Note that

$$\begin{aligned}
\binom{j+1}{k} &= \binom{j}{k} + \binom{j}{k-1} && \text{For } k \neq 0 \text{ and } k \neq j+1 \\
&= \frac{j!}{k!(j-k)!} + \frac{j!}{(k-1)!(j-(k-1))!} \\
&= \frac{j!(j+1-k) + j!(k)}{k!(j+1-k)!} \\
&= \frac{j!(j+1-k+k)}{k!(j+1-k)!} \\
&= \frac{j!(j+1)}{k!(j+1-k)!} \\
&= \frac{(j+1)!}{k!((j+1)-k)!}.
\end{aligned}$$

We must now show that if $k=0$ or $k=j+1$ the equality still holds. We see that $\binom{j+1}{0} = 1$ since there is only one way to choose the empty set from a set with $j+1$ elements. But also $\frac{(j+1)!}{0!(j+1-0)!} = 1$. So the equality holds. Additionally, $\binom{j+1}{j+1} = 1$ since there is only one subset with $j+1$ elements in a set with $j+1$ elements and $\frac{(j+1)!}{(j+1)!(j+1-(j+1))!} = 1$ and so the equality holds as well. Thus, the statement is true for all $0 \leq k \leq j+1$. Since we have shown the base case for $n=1$ and shown that the statement holds for $j+1$ when j is true for all $j \in \mathbb{N}$, we can conclude that it's true for all $n \in \mathbb{N}$. $\square$

4. *Prove that for all* $n \in \mathbb{N}$ *we have*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. This is a special case of the Binomial Theorem. Let $a = b = 1$. Then we have

$$\begin{aligned}
2^n &= (1+1)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} (1)^k (1)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k}
\end{aligned}$$

since $1^k = 1$ for all $k \in \mathbb{N}$. $\square$

5. *Is it true that for all* $n \in \mathbb{N}$ *we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0?$$

Proof. This is another special case of the Binomial Theorem. Let $a = -1$ and $b = 1$. Then

$$\begin{aligned} 0 &= (-1+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \end{aligned}$$

since $1^{n-k} = 1$ and $0^n = 0$ for all $k, n \in \mathbb{N}$. $\square$