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Homework 5

** **Problem 1.** *For all maps $f : X \to Y$, for any collection of subsets of Y, $\mathcal{E} \subseteq \mathcal{P}(Y)$, we have $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$.*

Proof. Note that $f^{-1}(\sigma(\mathcal{E}))$ is a σ-algebra of subsets of X including $f^{-1}(\mathcal{E})$, so $\sigma(f^{-1}(\mathcal{E})) \subseteq f^{-1}(\sigma(\mathcal{E}))$. Now let

$$\mathcal{F} = \{A \in \sigma(\mathcal{E}) \mid f^{-1}(A) \in \sigma(f^{-1}(\mathcal{E}))\}$$

Then $\mathcal{F}$ is a σ-algebra of subsets of X which includes $\mathcal{E}$ and so $\sigma(\mathcal{E}) = \mathcal{F}$. Thus $f^{-1}(\sigma(\mathcal{E})) = f^{-1}(\mathcal{F}) \subseteq \sigma(f^{-1}(\mathcal{E}))$. Since both inclusions hold, we have equality of the sets. □

** **Problem 2.** *If f and g are measurable functions on $(X, \mathcal{F})$, then fg is measurable.*

Proof. If either $f = 0$ or $g = 0$ then we have $fg = 0$, which is clearly measurable. Suppose that neither f nor g is the 0 function. Then we have

$$\{x \in X \mid fg(x) < a, a \in \mathbb{R}\} = \{x \in X \mid f(x) < \frac{a}{g(x)}, a \in \mathbb{R}\}.$$

Since $h(x) = a/g(x)$ is a measurable function as $g \neq 0$, we have then reduced the problem to

$$\{x \in X \mid f(x) < h(x), a \in \mathbb{R}\}$$

which we know is a measurable set. Thus, fg is measurable. □

** **Problem 3.** *Let (f_n) be a sequence of pointwise convergent measurable functions which converge to f almost everywhere. Show that $f = \lim_{n\to\infty} f_n$ is measurable.*

Proof. Suppose each function $f_n : X \to Y$ such that d is a metric on Y. Let A be a nonempty closed set in Y. It suffices to show that $f^{-1}(A)$ is measurable in X. Define $G_n = \{y \in Y \mid d(y, A) < 1/n\}$ for every n. Note that each G_n is open and $\bigcap_n G_n = A$. We want to show

$$f^{-1}(A) = \bigcap_k \bigcup_m \bigcap_{n \geq m} f_n^{-1}(G_k).$$

First suppose that $x \in f^{-1}(A)$. Then $f(x) \in A$. Since $f_n(x)$ converges to $f(x)$ and for every k, G_k is a neighborhood of x, there exists an m such that for all $n \geq m$ we have $f_n(x) \in G_k$. Therefore $x \in \bigcap_k \bigcup_m \bigcap_{n\geq m} f_n^{-1}(G_k)$. Now suppose that $x \in \bigcap_k \bigcup_m \bigcap_{n\geq m} f_n^{-1}(G_k)$. Then for each k, the point $f_n(x)$ eventually lies in G_k. Therefore $f(x) = \lim_{n\to\infty} f_n(x)$ and $f(x) \in \overline{G_k}$. But since $\overline{G_{k+1}} \subseteq G_k$ we can write $\bigcap_k \overline{G_k} = \bigcap_k G_k$ and so $f(x) \in \bigcap_k G_k = F$. Thus $x \in f^{-1}(F)$ and so both inclusions have been shown. □