Kris Harper
MATH 25700
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Problem 1. *Let* $H \cong Z_8$, $K \cong Z_2$.
(a) Find $\mathrm{Aut}(H)$.
(b) Describe the four different groups of order 16*, given by* $H \rtimes K$.

Proof. (a) $\mathrm{Aut}(H) \cong (\mathbb{Z}/8\mathbb{Z})^\times$. Since 2, 4 and 6 are not relatively prime to 8, $\varphi(8) = 4$ and $|\mathrm{Aut}(H)| = 4$. The only choices are then $\mathrm{Aut}(H) = Z_4$ or $\mathrm{Aut}(H) = Z_2 \times Z_2$. But note that $\overline{3}^2 = \overline{5}^2 = \overline{7}^2 = \overline{1}$. Thus $\mathrm{Aut}(H)$ has three elements of order 2 and we have $\mathrm{Aut}(H) \cong Z_2 \times Z_2$.

(b) Let $H = \langle x \rangle$ and $K = \langle y \rangle$. Define the three nonidentity elements of $\mathrm{Aut}(H)$ as follows. Let a be the automorphism which takes x to x^3, b the automorphism which takes x to x^5 and ab be the automorphism which takes x to x^7. Note that these must be automorphisms because we know $\mathrm{Aut}(H) \cong (\mathbb{Z}/8\mathbb{Z})^\times$, where the isomorphism between them takes $a \in (\mathbb{Z}/8\mathbb{Z})^\times$ to ψ_a, an automorphism taking x to x^a.

First note that if $\varphi : K \to \mathrm{Aut}(H)$ is the trivial homomorphism, then we simply have $H \rtimes K \cong H \times K \cong Z_8 \times Z_2$. Now suppose that $\varphi(y) = a$. Then we have $y \cdot x = yxy^{-1} = x^3$. Note that $y = y^{-1}$ and so we have $xy = yx^3$. Thus in this case

$$H \rtimes K \cong \langle x, y \mid x^8 = y^2 = 1, xy = yx^3 \rangle.$$

This is QD_{16}, the quasidihedral group of order 16. Now suppose that $\varphi(y) = b$. In this case $y \cdot x = yxy^{-1} = x^5$. We see that now

$$H \rtimes K \cong \langle x, y \mid x^8 = y^2 = 1, xy = yx^5 \rangle.$$

This is the modular group of order 16. Finally, suppose that $\varphi(y) = ab$. Then we have $y \cdot x = yxy^{-1} = x^7$. But note that $x^7 = x^{-1}$ which gives $xy = yx^{-1}$. The presentation is now

$$H \rtimes K \cong \langle x, y \mid x^8 = y^2 = 1, xy = yx^{-1} \rangle$$

which is precisely the presentation of D_{16}.

In general, if $H \cong Z_n = \langle x \rangle$, $K \cong Z_2 = \langle y \rangle$, $\psi_a \in \mathrm{Aut}(H)$ takes x to x^a for $a \in (\mathbb{Z}/n\mathbb{Z})^\times$, and $\varphi_a : K \to \mathrm{Aut}(H)$ takes y to ψ_a, then

$$H \rtimes_{\varphi_a} K \cong \langle x, y \mid x^n = y^2 = 1, xy = yx^a \rangle.$$

$\square$