Sheet 5: A New Continuum

Theorem 1 (Intersections) *The intersection of any set of closed sets is closed and the intersection of a finite number of open sets is open.*

Proof. Consider the set S of closed sets $A \subseteq C$. Then let p be a limit point of $\bigcap_{A \in S} A$. Then since $\bigcap_{A \in S} \subseteq A$ for all $A \in S$ we see that p is a limit point of A for all $A \in S$ (2.10). But all $A \in S$ are closed so $p \in A$ for all $A \in S$. And so $p \in \bigcap_{A \in S} A$ and we have $\bigcap_{A \in S} A$ is closed.

To show that an intersection of finitely many open sets is open, use induction on the number of sets, n. For the base case we have a single open set. Assume that the intersection of any n open sets is open. Then consider the set of $n+1$ open sets $S = \{A_1, A_2, \ldots, A_{n+1}\}$. We see that the intersection $\bigcap_{A_i \in S \setminus A_{n+1}} A_i$ is open and A_{n+1} is open. Then for all $x \in \bigcap_{A_i \in S} A_i$, we have $x \in \bigcap_{A_i \in S \setminus A_{n+1}}$ and $x \in A_{n+1}$. By the open condition, for all $x \in \bigcap_{A_i \in S} A_i$ there exist regions $R_1 \subseteq \bigcap_{A_i \in S \setminus A_{n+1}} A_i$ and $R_2 \subseteq A_{n+1}$ such that $x \in R_1$ and $x \in R_2$ (3.17). But then x is in the region $R_3 = R_1 \cap R_2$ and $R_3 \subseteq \bigcap_{A_i \in S} A_i$ (2.15). So for all $x \in \bigcap_{A_i \in S} A_i$ there exists a region $R \subseteq \bigcap_{A_i \in S} A_i$ such that $x \in R$. Thus the intersection is open by the open condition (3.17). By mathematical induction, this must be true for all $n \in \mathbb{N}$. $\square$

Theorem 2 (Unions) *The union of any set of open sets is open, and the union of a finite set of closed sets is closed.*

Proof. Consider the set S of open sets $A \subseteq S$. By the open condition, for every $x \in A$ for some $A \in S$, there exists a region $R \subseteq A$ such that $x \in R$ (3.17). But if $x \in A$, then $x \in \bigcup_{A \in S} A$ and so there exists a region $R \subseteq A \subseteq \bigcup_{A \in S} A$ and $x \in R$ so the union must be open (3.17).

Now we use induction on a finite number of closed sets n. For the base case we have one closed set. Assume that the union of any n closed sets is closed. Consider the set of $n+1$ closed sets $S = \{A_1, A_2, \ldots, A_{n+1}\}$. We see $\bigcup_{A_i \in S \setminus A_{n+1}} A_i$ is closed and A_{n+1} is closed. Then if p is a limit point of $\bigcup_{A_i \in S} A_i$ then it is a limit point of $\bigcup_{A_i \in S \setminus A_{n+1}} A_i$ or it is a limit point of A_{n+1} (2.17). And since $\bigcup_{A_i \in S \setminus A_{n+1}} A_i$ and A_{n+1} are closed, then we have $p \in \bigcup_{A_i \in S \setminus A_{n+1}} A_i$ or $p \in A_{n+1}$. Thus $p \in \bigcup_{A_i \in S} A_i$ and so it is closed. So by mathematical induction we see that this is true for any $n \in \mathbb{N}$. $\square$

Axiom 1 (Connectedness) *The only point sets which are both closed and open are C and $\emptyset$.*

Exercise 3 *Show that Theorem 1 does not hold for the intersection of an infinite number of open sets.*

Proof. We see that for all $a \in C$ we have $\{a\} = C \setminus (C \setminus a)$ is closed since $\{a\}$ is a finite set and so $C \setminus a$ must be open (2.8). Now consider a point $p \in C$ and consider the intersection

$$\bigcap_{a \in C, a \neq p} C \setminus a = \{p\}.$$

Since $C \setminus p$ is infinite, this is an intersection of an infinite number of open sets. But their intersection is $\{p\}$ which is not open (2.8, A5.1). $\square$

Exercise 4 *Show that Theorem 2 does not hold for the union of an infinite number of closed sets.*

Proof. Similarly, we take a point $p \in C$ and then consider all the sets containing a single point other than p. Then we have

$$\bigcup_{a \in C, a \neq p} \{a\} = C \setminus p.$$

Since $\{a\}$ is finite, it is closed for all $a \in C$ (2.8). From Exercise 3 and Axiom 1 we know $C \setminus p$ is not closed (A5.1, 5.3). So we have a union of an infinite number of closed sets which equals a set that is not closed. $\square$

Let O be an open subset of C. Let us define the relation $\sim$ on O as follows: $a \sim b$ if there exists a region $R \subseteq O$ containing both a and b.

Theorem 5 *$\sim$ is an equivalence relation.*

First we prove a lemma showing that if two regions contain a common element x, then their union is also a region containing all points in either region.

Proof. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be regions such that $x \in A$ and $x \in B$. Then we see that $x \in A \cup B$. Without loss of generality, let $a_1 \leq b_1$. Then we see that $a_2 > b_1$, otherwise A and B would not both contain x. Thus there are two cases.

Case 1: Let $a_1 \leq b_1$ and $a_2 < b_2$ Then we have $a_1 \leq b_1 < a_2 < b_2$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $a_1 < x < a_2$. But $a_2 < b_2$ so $a_1 < x < b_2$ and $x \in (a_1, b_2)$. Likewise, if $x \in B$ then $b_1 < x < b_2$. But $a_1 \leq b_2$ so $a_1 < x < b_2$ and $x \in (a_1, b_2)$. Therefore $A \cup B \subseteq (a_1, b_2)$. Additionally, if $x \in (a_1, b_2)$ then $x < a_2$ or $x \geq a_2$. If $x < a_2$ then $a_1 < x < a_2$ and $x \in A$. If $x \geq a_2$ then $b_1 < x < b_2$ and $x \in B$. Therefore $x \in A$ or $x \in B$ and $x \in A \cup B$. Thus $(a_1, b_2) \subseteq A \cup B$ and so $A \cup B = (a_1, b_2)$.

Case 2: Let $a_1 \leq b_1$ and $a_2 \geq b_2$. Then we have $a_1 \leq b_1 < b_2 \leq a_2$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \in (a_1, a_2)$. Likewise, if $x \in B$ then $b_1 < x < b_2$. But $a_1 \leq b_2$ and $b_2 \leq a_2$ so $a_1 < x < a_2$ and $x \in (a_1, a_2)$. Therefore $A \cup B \subseteq (a_1, a_2)$. Additionally, if $x \in (a_1, a_2)$ then either $x > b_1$ and $x < b_2$ and so $x \in (b_1, b_2)$ or $x \leq b_1$ or $x \geq b_2$. If $x \in (b_1, b_2)$ then $x \in B$. If $x \leq b_1$ or $x \geq b_2$ then $a_1 < x < a_2$ and $x \in A$. Therefore $x \in A$ or $x \in B$ and $x \in A \cup B$. Thus $(a_1, a_2) \subseteq A \cup B$ and so $A \cup B = (a_1, a_2)$.

We see that in either case, $A \cup B$ is a region which contains every point in either A or B. $\square$

We now prove the original result.

Proof. Let O be an open subset of C. We see that if a $a \in O$, then by the open condition there exists a region $R \subseteq O$ such that $a \in R$ and so $a \sim a$ so we have reflexivity (3.17). Also if $a \sim b$ then $a, b \in R$ for a region $R \subseteq O$ and so $b, a \in R$ and $b \sim a$. So we have symmetry. Finally, if $a \sim b$ and $b \sim c$, then we have $a, b \in R_1$ and $b, c \in R_2$ where $R_1, R_2 \subseteq O$ are regions. But by the previous lemma $R_3 = R_1 \cup R_2 \subseteq O$ is a region and since $a, b, c \in R_3$ we have $a \sim c$ so we have transitivity. $\square$

Theorem 6 *For all $a \in C$ the sets $\{x \mid x < a\}$ and $\{x \mid a < x\}$ are open.*

Proof. Let $a, p \in C$ such that $p \in \{x \mid x < a\}$. Then there exists some point $q \in C$ such that $q < p$ since C has no first point and so $p \in (q; a)$ (A2.3). Since $(q; a) \subseteq \{x \mid x < a\}$ we see that there exists a region containing p which is a subset of $\{x \mid x < a\}$. So $\{x \mid x < a\}$ must be open by the open condition (3.17). A similar proof holds for $\{x \mid a < x\}$ because C has no last point (A2.3). $\square$

Corollary 7 *If $A, B \subseteq C$ are open subsets, $A \cap B = \emptyset$ and $A \cup B = C$, then $A = \emptyset$ or $B = \emptyset$.*

Proof. We have $A \cap B = \emptyset$ and so $B \subseteq C \backslash A$. But additionally we have $A \cup B = C$ and so $C \backslash A \subseteq B$. Then $B = C \backslash A$ and since A is open, $C \backslash A$ is closed and so B is both open and closed. But then either $B = C$ or $B = \emptyset$ by Axiom 1 (A5.1). If $B = \emptyset$ then we're done and if $B = C$ then $A = \emptyset$ because $A \cap B = \emptyset$. So either A or B is empty. $\square$

Theorem 8 (Regions are Nonempty) *For all $a < b$ there exists c such that $a < c < b$.*

Proof. Consider $a, b \in C$ such that $a < b$. Then the sets $\{x \mid x < b\}$ and $\{x \mid a < x\}$ are both open by Theorem 6 (5.6). For every $p \in C$ we have $p < a$, $p = a$ or $p > a$ and so $\{x \mid x < b\} \cup \{x \mid a < x\} = C$. But $\{x \mid x < b\} \cap \{x \mid a < x\} = (a; b)$ and using Corollary 7 and the fact that C has no first or last point we see that this intersection cannot be empty since $\{x \mid x < b\} \neq \emptyset$ and $\{x \mid x > a\} \neq \emptyset$ (A2.3, 5.7). $\square$

Corollary 9 *For all $a < b$ both a and b are limit points of the region $(a; b)$.*

Proof. Let $(p; q)$ be a region such that $a \in (p; q)$. Then $q \geq b$ or $q < b$. If $q \geq b$ then $(a; b) \subseteq (p; q)$ and because regions are nonempty there exists $c \in (a; b)$ such that $c \in (p; q)$ (5.8). If $q < b$ then there exists a point $c \in C$ such that $a < c < q$ and so $c \in (a; b)$ and $c \in (p; q)$ (5.8). We see that all regions containing a also contain a point in $(a; b)$ so a must be a limit point of $(a; b)$. A similar proof holds for b. □

Corollary 10 *Every point of a region is a limit point of that region.*

Proof. Let A be a region and let $p \in A$. Then for all regions R such that $p \in R$, we have $R \cap A = (a; b) \neq \emptyset$ where $(a; b)$ is a region containing p (2.15). Thus there exists a $c \in (a; b)$ such that $a < c < p$ (5.8). But then for all regions R containing p we have $R \cap (A \backslash p) \neq \emptyset$ and so p is a limit point of A. □

Corollary 11 *Every nonempty region contains infinitely many points*

Proof. Suppose to the contrary that a nonempty region contains a finite number of points. Then it has no limit points (3.4). But by Corollary 10 we know that every point is a limit point and so this is a contradiction (5.10). □

Corollary 12 *Every point in C is a limit point of C*

Proof. Let $p \in C$. Then we see that every region R which contains p contains infinitely many points and so for all regions R which contain p, we have $R \cap (C \backslash p) \neq \emptyset$ (5.11). □