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Homework 5

Problem 1. *Let G be a finite group with $|G| > 4$ and let N and N' be simple subgroups, both of index 2 in G (so in particular, they are normal in G). Show that $N = N$. This is the last step in our proof that $PSL(2,7) \cong PSL(3,2)$.*

Proof. Consider $N \cap N'$. Since N and N' are both normal, any conjugate of $N \cap N'$ is contained in both N and N', so $N \cap N'$ is normal in G. It is thus also normal in N and N'. Since N and N' are both simple, we see that $N \cap N'$ is either trivial or equal to N and N'. Suppose that $N \cap N'$ is trivial. Then since N is normal we know NN' is a subgroup of G and $|NN'| = |N||N'|/|N \cap N'| = |N||N'| = (|G|/2)(|G|/2) = |G|(|G|/4)$. But $|G| > 4$ so $|NN'| > |G|$, a contradiction. Thus $N = N \cap N' = N'$. □

Problem 2. *Let β be a bilinear form on a finite dimensional vector space V. Write $B_{\mathcal{E}}$ for its matrix with respect to a basis $\mathcal{E}$. Show that the following are equivalent:*

- $\det B_{\mathcal{E}} \neq 0$ *for some basis $\mathcal{E}$.*
- $\det B_{\mathcal{E}} \neq 0$ *for every basis $\mathcal{E}$.*
- *For every nonzero vector $v \in V$, there is some $v' \in V$ such that $\beta(v, v') \neq 0$*
- *The maps $v \mapsto \beta(v, \cdot)$ and $v \mapsto \beta(\cdot, v)$ are isomorphisms $V \to V^*$.*

If these equivalent conditions hold, we say β is nondegenerate.

Proof. Suppose $\det B_{\mathcal{E}} \neq 0$ for some basis $\mathcal{E}$ and let $\mathcal{F}$ be another basis. Then if A is the change of basis matrix from $\mathcal{E}$ to $\mathcal{F}$ we know $B_{\mathcal{F}} = A^T B_{\mathcal{E}} A$. Taking determinants we see $\det B_{\mathcal{F}} = \det(A^T)\det(B_{\mathcal{E}})\det(A) = \det B_{\mathcal{E}} \neq 0$.

Now assume $\det B_{\mathcal{E}} \neq 0$ for every basis $\mathcal{E}$. Let $v \in V$ be nonzero and let $\mathcal{E}$ be a basis containing v. Then some column of $B_{\mathcal{E}}$ contains $\beta(v, w)$ for each $w \in \mathcal{E}$. Since $\det B_{\mathcal{E}} \neq 0$ we see that this column cannot be 0 so there must be some $w \in \mathcal{E}$ with $\beta(v, w) \neq 0$.

Now assume for each nonzero $v \in V$ there exists $v' \in V$ such that $\beta(v, v') \neq 0$. Let $\varphi : V \to V^*$ be a map given by $\varphi : v \mapsto \beta(v, \cdot)$. Let $v \in \ker\varphi$ so that $\varphi(v)$ is the linear functional taking every element of V to 0. But by assumption, if $v \neq 0$ then there exists $v' \in V$ such that $\beta(v, v') \neq 0$. Therefore $v = 0$ and $\ker\varphi = 0$. Thus φ is injective. Now let $\gamma \in V^*$. Let $\beta(v, w) = \gamma(w)$ for each $w \in V$. Then β is clearly a linear functional and $\varphi(v) = \beta(v, \cdot)$ so φ is surjective and thus an isomorphism. The proof for $\beta(\cdot, v)$ is nearly identical.

Finally, suppose that the maps $v \mapsto \beta(v, \cdot)$ and $v \mapsto \beta(\cdot, v)$ are isomorphisms from V to V^*. Then the kernel of these maps are 0 so for $v \neq 0$ there must be some vector w such that $\beta(v, w) \neq 0$. Let $\mathcal{E}$ be a basis for V and note that for each basis vector v_i it's not possible that $\beta(v_i, v_j) = 0$ for all j because then $\beta(v_i, w) = 0$ for any vector w. Thus $\beta(v_i, v_j) \neq 0$ for at least value of j for each i which ensures that $\det B_{\mathcal{E}} \neq 0$. □