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Homework 6

**** Problem 1.** *Let V be a normed linear space over $\mathbb{R}$ and let W be a subspace of V. Let $f \in W^*$ and let $v_0 \in V \backslash W$ such that $W' = W + \{\lambda v_0 \mid \lambda \in \mathbb{R}\}$. We define $F : W' \to V$ such that $F(w + \lambda v_0) = f(w) + \lambda c$. The constant c is chosen as follows. Suppose that $||f|| = 1$. Then*

$$\sup_{w_1 \in W} -f(w_1) - ||w_1 - v_0|| \leq c \leq \inf_{w_2 \in W} ||w_2 - v_0|| - f(w_2).$$

Now we must show that $||F|| = 1$.

Proof. For $\lambda \neq 0$ we have

$$|F(w + \lambda v_0)| = |\lambda||F\left(\frac{1}{\lambda}w + v_0\right)| = |\lambda||f\left(\frac{1}{\lambda}w\right) + c|.$$

Thus

$$|\lambda||f\left(\frac{1}{\lambda}w\right) - c| = |F\left(\frac{1}{\lambda}w - v_0\right)| \leq ||F|||\frac{1}{\lambda}w - v_0|$$

and thus based on our choice of c, this forces $||F|| = 1$. ☐

**** Problem 2.** *Suppose V is a Banach space over $\mathbb{R}$ and that p is a subadditive functional on V. Take $v \neq 0$ in V and let $W = \{\alpha v \mid \alpha \in \mathbb{R}\}$. Define a function $f : W \to \mathbb{R}$ by $f(\alpha v) = \alpha p(v)$ for all $\alpha \in \mathbb{R}$. Show that, $f(w) \leq p(w)$ for all $w \in W$.*

Proof. Note that $0 = 0p(v) = p(0 \cdot v) = p(0)$. For $\alpha \geq 0$ we have $f(\alpha v) = \alpha p(v) = p(\alpha v)$. For $\alpha < 0$ we have $p(\alpha v - \alpha v) \leq p(\alpha v) + p(-\alpha v)$. Thus $0 \leq p(\alpha v) - \alpha p(v)$ and so $f(\alpha v) = \alpha p(v) \leq p(\alpha v)$. ☐

**** Problem 3.** *Let V be a normed linear space. Show that the Hahn-Banach Theorem implies a linear functional can be extended when the subadditive functional on V is the norm.*

Proof. We must show that the norm is subadditive for $v, w \in V$. But these are simply properties of the norm function. That is, for all $\alpha \geq 0$ we have $||\alpha v|| = \alpha||v||$. Additionally we have $||v + w|| \leq ||v|| + ||w||$. Since these properties are true for all $v, w \in V$, and because of the way the norm of $f \in V^*$ is defined, a linear functional on a subspace of V can be extended to a functional with the same norm. ☐

**** Problem 4.** *Let p be a subadditive functional on $\ell^\infty(\mathbb{R})$ such that for $c = (c_n) \in \ell^\infty(\mathbb{R})$*

$$p(c) = \inf\left\{\limsup_{n\to\infty} \frac{1}{k}\sum_{j=1}^{k} c_{n+i_j} \mid i_1, i_2, \ldots i_k \text{ is a finite sequence in } \mathbb{N}\right\}.$$

*Let $f \in (\ell^\infty)^*$ be the extended linear functional defined in ** Problem 2. Show that $f((c_{n+1})) = f((c_n))$. Show that if $c_n = 1$ for all n then $f((c_n)) = 1$.*

Proof. Let $c = (c_n)$ and $c' = (c_{n+1})$. Then we have

$$
\begin{aligned}
p(c) &= \inf\left\{ \limsup_{n\to\infty} \frac{1}{k}\sum_{j=1}^{k} c_{n+i_j+1} \mid i_1, i_2, \ldots i_k \text{ is a finite sequence in } \mathbb{N} \right\} \\
&= \inf\left\{ \limsup_{n\to\infty} \frac{1}{k}\sum_{j=1}^{k} c_{n+i_j} \mid i_1, i_2, \ldots i_k \text{ is a finite sequence in } \mathbb{N} \right\} \\
&= p(c').
\end{aligned}
$$

Note that $f(c) - f(c') = f(c - c') \leq p(c - c') \leq p(c) + p(-c') = 0$. Likewise $f(c') - f(c) = 0$. Therefore $f(c) = f(c')$.
Suppose that $c_n = 1$ for all n. Then the quantity

$$
\frac{1}{k}\sum_{j=1}^{\infty} c_{n+i_j} = 1
$$

for all n and all finite sequences of natural numbers. Thus $f(c) \leq p(c) = 1$. Moreover, $p(-c) = -1$ for the same reasons and $f(-c) \leq p(-c)$. Then $f(c) = -f(-c) \geq -p(-c) = 1$. Thus $f(c) \leq 1 \leq f(c)$ and $f(c) = 1$. □

**** Problem 5.** *Let $f : \ell^\infty \to \mathbb{R}$ be defined as in ** Problem 4 and let $c = (c_n) \in \ell^\infty(\mathbb{R})$. Show that*

$$
\liminf_{n\to\infty} c_n \leq f(c) \leq \limsup_{n\to\infty} c_n.
$$

Proof. Note that for arbitrary finite sequences of natural numbers $i_1, i_2, \ldots, i_j$ we have

$$
\limsup_{n\to\infty} \frac{1}{k}\sum_{j=1}^{k} c_{n+i_j} \leq \limsup_{n\to\infty} c_n
$$

because the terms on the left are averages of groups of terms on the right. Then it must be the case that

$$
f(c) \leq p(c) \leq \limsup_{n\to\infty} c_n.
$$

We know that $-\liminf_{n\to\infty} c_n = \limsup_{n\to\infty} -c_n$. Since (c_n) is an arbitrary element of $\ell^\infty(\mathbb{R})$ we have

$$
-f(c) = f(-c) \leq \limsup_{n\to\infty} -c_n = -\liminf_{n\to\infty}
$$

and thus $\liminf_{n\to\infty} \leq f(c)$. Therefore

$$
\liminf_{n\to\infty} c_n \leq f(c) \leq \limsup_{n\to\infty} c_n.
$$

□

**** Problem 6.** *1) Show that $p((c_n)) = \limsup_{n\to\infty} c_n$ defines a subadditive functional on $\ell^\infty(\mathbb{R})$.*
2) Use this subadditive functional, p, to construct a different functional, f, on $\ell^\infty(\mathbb{R})$ and show that $f((c_n)) \geq 0$ if $c_n \geq 0$ for all n and $f((c_n)) = 1$ if $c_n = 1$ for all n.
3) Show that f may be constructed in such a way that $f((c_{n+1})) \neq f((c_n))$.

Proof. 1) For $\alpha \geq 0$ in $\mathbb{R}$ we have

$$\begin{aligned} p(\alpha(c_n)) &= \limsup_{n\to\infty} \alpha c_n \\ &= \inf\{\sup\{\alpha c_m \mid m \geq n\} \mid n \geq 1\} \\ &= \inf\{\alpha \sup\{c_m \mid m \geq n\} \mid n \geq 1\} \\ &= \alpha \inf\{\sup\{c_m \mid m \geq n\} \mid n \geq 1\} \\ &= \alpha \limsup_{n\to\infty} c_n \\ &= \alpha p((c_n)). \end{aligned}$$

Let $(d_n) \in \ell^\infty(\mathbb{R})$. Then we have

$$\begin{aligned} p((c_n) + (d_n)) &= p((c_n + d_n)) \\ &= \limsup_{n\to\infty} c_n + d_n \\ &= \inf\{\sup\{c_m + d_m \mid m \geq n\} \mid n \geq 1\} \\ &\leq \inf\{\sup\{c_m \mid m \geq n\} + \sup\{d_m \mid m \geq n\} \mid n \geq 1\} \\ &= \inf\{\sup\{c_m \mid m \geq n\} \mid n \geq 1\} + \inf\{\sup\{d_m \mid m \geq n\} \mid n \geq 1\} \\ &= \limsup_{n\to\infty} c_n + \limsup_{n\to\infty} d_n \\ &= p((c_n)) + p((d_n)). \end{aligned}$$

2) Suppose that $c_n \geq 0$ for all n. Then we must have $p(c) \limsup_{n\to\infty} c_n \geq 0$. Likewise $p(-c) \leq 0$. Then $f(-c) \leq p(-c) \leq 0$ and so $f(c) = -f(-c) \geq -p(-c) \geq 0$. Now suppose that $c_n = 1$ for all n. We have $\limsup_{n\to\infty} c_n = 1$ and $-\limsup_{n\to\infty} c_n = -1$. Then $f(c) \leq p(c) = 1$. Additionally we have $f(-c) \leq p(-c) = -1$ and so $f(c) = -f(-c) \geq -p(-c) = 1$. Thus $1 \leq f(c) \leq 1$.

3) Because p no longer takes the average over terms in a sequence, it is possible to create a functional on $\ell^\infty(\mathbb{R})$ which maps to a different number if the sequence is shifted. A sequence such as $c_n = (-1)^{n+1}$ will either map to 1 or -1 depending on whether the sequence starts on $n = 1$ or $n = 2$. Thus $f(c_n) \neq f(c_{n+1})$. □