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Homework 1

Problem 1. *Show that a contraction mapping is continuous.*

Proof. Let X be a metric space and suppose that $f : X \to X$ is a contraction mapping such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$. Then let $\varepsilon > 0$ and choose $\delta = \varepsilon/k$. Then if $d(x, y) < \delta$ we have $d(f(x), f(y)) < kd(x, y) < k\delta < \varepsilon$. Therefore f is continuous. $\square$

Problem 2. *Let f be a polynomial function from $\mathbb{R}$ to $\mathbb{R}$. Give conditions on f such that f is a contraction mapping.*

Proof. The function f must have degree at most 1. To see this, suppose that f is a contraction mapping with constant k and that $\deg(f(x)) = n$ such that $n \geq 2$. But since $\deg(f(x)) \geq 2$, for large enough x, $|f(x) - f(x + 1)|$ will get arbitrarily large. Simply choose x large enough such that $1/k|f(x) - f(x + 1)| > 1$. This is a contradiction. Conversely, if $\deg(f(x)) < 1$ then f is constant and so $d(f(x), f(y)) = 0$ for all $x, y \in \mathbb{R}$. Similarly if $\deg(f(x)) = 1$, then $f(x) = kx + b$ for constants $k, b \in \mathbb{R}$. Then $|f(x) - f(y)| = k|x - y|$. In both cases, f is a contraction mapping. $\square$

Problem 3. *1) Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that f has a fixed point.*
2) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ that does not have a fixed point.

Proof. 1) Consider such a function f. Note that on $\mathbb{R}^2$, the function f cannot lie entirely below the line $g(x) = x$. If this were the case, then not all of the values in $[0, 1]$ would be taken on by f. Similarly, f must take on values above the line $g(x) = x$. Using a rotation of the plane, we can apply the Intermediate Value Theorem so that there must exist a point, x_0, on the line $g(x) = x$ so that $f(x_0) = x_0$.

2) The function $f(x) = x + 1$ has this property. For all $x \in \mathbb{R}$, $x + 1 \neq x$. Furthermore, f is clearly continuous. $\square$

Problem 4. *Define f and x_0 as in the proof for the Contraction Mapping Theorem. Show that x_0 is the unique fixed point of f.*

Proof. Consider some fixed point, $y \in X$ such that $f(y) = y$. Then note that

$$\begin{aligned} d(x_n, y) &= d(f^n(x_1), f(y)) \\ &\leq \alpha d(f^{n-1}(x_1), y) \\ &\leq \alpha d(f^{n-1}(x_1), f(y)) \\ &\leq \alpha^2 d(f^{n-2}(x_1), y). \end{aligned}$$

Continuing inductively, we see that $d(x_n, y) \leq \alpha^n(x_1, y)$. But then $\lim_{n\to\infty} x_n = y = x_0$. $\square$

Problem 5. *1) Let $B = B_1(0)$ be the unit ball in the usual metric on $\mathbb{R}^n$, and let f be a map from B to B. Suppose there exists a constant C such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in B$. Show that if $0 < C < 1$, then f is a contraction mapping. Show that, if $C \geq 1$, then f need not be a contraction mapping.*
2) Let $T : \ell_n^p(\mathbb{R}) \to \ell_n^q(\mathbb{R})$, with $1 \leq p, q \leq \infty$, be a linear transformation. When is T a contraction mapping?

Proof. 1) We have $d(f(x), f(y)) < |f(x) - f(y)| < C|x - y| < Cd(x, y)$. If $0 < C < 1$ then this is clearly a contraction mapping. If $C \geq 1$ then f is not a contraction mapping.

2) A linear transformation $T : \ell_n^p(\mathbb{R}) \to \ell_n^q(\mathbb{R})$ is continuous if and only if there exists $c \in \mathbb{R}$ such that $||T(x)||_q \leq c||x||_p$ for all $x \in \ell_n^p(\mathbb{R})$. Then a continuous linear transformation T is a contraction mapping whenever $0 < c < 1$. □

Problem 6. *Consider $\mathcal{C}([0,1], \mathbb{R})$ with the* sup *metric and let $k(x, y) : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function satisfying*

$$\sup_{0 \leq x \leq 1} \int_0^1 |k(x, y)| dy < 1.$$

Given a function $g(x) \in \mathcal{C}([0, 1], \mathbb{R})$ show that there is a unique solution $f(x) \in \mathcal{C}([0, 1], \mathbb{R})$ to the equation

$$f(x) - \int_0^1 k(x, y) f(y) dy = g(x).$$

Proof. Using the Contraction Mapping Theorem we know that if there is a function $g(x) \in \mathcal{C}([0, 1], \mathbb{R})$, then there exists a unique fixed point which can be used to create a unique function satisfying the equality. □

Problem 7. *1) Show that polynomial functions in $\mathcal{C}([0, 1], \mathbb{R})$ separates points.*
2) Does the class of functions $\{\sin(2\pi nx) \mid n \in \mathbb{N}\}$ in $\mathcal{C}([0, 1], \mathbb{R})$ separate points?

Proof. 1) Consider two points $x_1, x_2 \in [0, 1]$. Then there exists x_3 such that $x_1 < x_3 < x_2$. Then since all linear functions are polynomials, the function $f(x) = x - x_3$ is a polynomial in $\mathcal{C}([0, 1], \mathbb{R})$. But then $f(x_1) < 0 < f(x_2)$.

2) No. Consider the points $0, 1/2 \in [0, 1]$. Then

$$\sin(2\pi n \cdot 0) = \sin(0) = 0 = \sin(n\pi) = \sin(2n\pi \cdot 1/2).$$

□

Problem 8. *1) Show that $R[x_1, x_2, \ldots, x_n]$ is a commutative ring with 1 for $R = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Find the units in each of these rings.*
2) Find the possible images of a polynomial in $\mathbb{R}[x_1, x_2, \ldots, x_n]$.

Proof. 1) Let R be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Let

$$p(x) = \sum_{i=1}^{k} \prod_{j=1}^{n} x_j^{i_j}$$

and

$$q(x) = \sum_{i=1}^{l} \prod_{j=1}^{n} x_j^{i_j}$$

be two real polynomial functions. Then note that

$$p(x) + q(x) = \sum_{i=1}^{k} \prod_{j=1}^{n} x_j^{i_j} + \sum_{i=1}^{l} \prod_{j=1}^{n} x_j^{i_j}$$

which still has the form of a finite linear combination of expressions of the form $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$. Thus, $R[x_1, x_2, \ldots, x_n]$ is closed under addition. A similar proof holds to show it's closed under multiplication. Since the coefficients for each linear combination are in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, it follows that associativity and commutativity of addition and multiplication as well as distributivity follow as they do in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. The 0 polynomial, where all coefficients are 0, serves as the additive identity. Then taking the additive inverses of each coefficient serves as an additive inverse for a given real polynomial function. The 1 polynomial serves as the multiplicative identity. Since there are no zero-divisors in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, there are no invertible elements in $R = \mathbb{Z}$ and only constants, that is, elements of R, are invertible if $R = \mathbb{Q}$ or $R = \mathbb{R}$.

2) The image of a polynomial in $\mathbb{R}[x_1, x_2, \ldots, x_n]$ will either be $\mathbb{R}$, if n is odd, or all real numbers greater than or equal to or less than or equal to some constant, if n is even. $\square$

Problem 9. *Let V be a lattice on a metric space X. If f, g are in V, set $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$. Show that $f \wedge g, f \vee g \in V$.*

Proof. Certainly, $f \wedge g$ is a real valued function on X. Let $\varepsilon > 0$ and let $x \in X$. Then there exists δ such that for all $y \in X$, if $d(x, y) < \delta$ we have $|f(x) - f(y)| < \varepsilon/2$ and $|g(x) - g(y)| < \varepsilon/2$. But then $|f \wedge g(x) - f \wedge g(y)| < |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon$. Note that this requires using $|f|$ is continuous as well. A similar proof holds for $f \vee g$. $\square$

Problem 10. *Let X, Y be compact metric spaces. Show that the set $A = \{(x, y) \to f(x)g(y) \mid f \in \mathcal{C}(X, \mathbb{R})$ and $g \in \mathcal{C}(Y, \mathbb{R})\}$ is uniformly dense in $\mathcal{C}(X \times Y, \mathbb{R})$.*

Proof. Note that the product space $X \times Y$ is compact since both X and Y are compact. Since f and g are both continuous, an element of A is continuous. We need only to show that A separates points. Consider two points $(x_1, y_1), (x_2, y_2) \in X \times Y$. Suppose, for the moment, that $x_1 y_1 \neq x_2 y_2$. Then take the functions $f(x) = x$ and $g(y) = y$ where $f \in \mathcal{C}(X, \mathbb{R})$ and $g \in \mathcal{C}(Y, \mathbb{R})$. Then consider the function $h(x, y) = f(x)g(y)$ so that $h(x_1, y_1) = x_1 y_1 \neq x_2 y_2 = h(x_2, y_2)$. In the case that $x_1 y_1 = x_2 y_2$, taking the square of f or g in place of either function will suffice. By the Stone-Weierstrass Theorem A is uniformly dense in $\mathcal{C}(X \times Y, \mathbb{R})$. $\square$

Problem 11. *1) Let X be a compact metric space. Let A be an algebra of continuous complex valued functions on X with the property that, if $f \in A$ then $\overline{f} \in A$. Assume A separates points and there is no point $x \in X$ such that $f(x) = 0$ for all $f \in A$. Show that the uniform closure of A is $\mathcal{C}(X, \mathbb{C})$.*
2) Show that the set of trigonometric polynomials is uniformly dense in $\mathcal{C}(\mathbb{T}, \mathbb{C})$.

Proof. 1) Suppose for any $x, y \in X$ with $x \neq y$ and $a, b \in \mathbb{C}$, there is a function $f_{xy} \in A$ such that $f_{xy}(x) = a$ and $f_{xy}(y) = b$. Then, for every $f \in \mathcal{C}(X, \mathbb{R})$, there is a sequence $(f_n) \in A$ such that (f_n) converges uniformly to f. This fact, combined with the fact that $\overline{f} \in A$ whenever f is, proves the theorem.

2) Note that trigonometric polynomials are all continuous. Furthermore, if $f(e^{i\theta}) \in \mathbb{T}$ then clearly $\overline{f}(e^{i\theta}) \in \mathbb{T}$ because

$$f(e^{i\theta}) = \sum_{j=-n}^{n} a_n e^{ij\theta} = \sum_{j=-n}^{n} a_n e^{-ij\theta} = f(e^{i\theta}).$$

Additionally, given $x_1, x_2 \in \mathbb{T}$ we can find a polynomial which maps them to different points, simply by choosing adequate coefficients. Since all points in $\mathbb{T}$ are nonzero, there is no point in $\mathbb{T}$ for which every polynomial maps to 0. Then by the Stone-Weierstrass Theorem, the set of trigonometric polynomials is uniformly dense in $\mathbb{C}(\mathbb{T}, \mathbb{C})$. $\square$