Sheet 30: Uniform Limits

Definition 1 *Let (f_n) be a sequence of functions defined on A and let f be defined on A. Then f is the uniform limit of (f_n) (or $\lim_{n\to\infty} f_n = f$) if for all $\varepsilon > 0$ there exists N such that for all $n > N$ and for all $x \in A$ we have $|f(x) - f_n(x)| < \varepsilon$.*

Theorem 2 *Let (f_n) be a sequence of continuous functions on $[a;b]$ that uniformly converges to f on $[a;b]$. Then f is continuous on $[a;b]$.*

Proof. Let $\varepsilon > 0$ and consider $\varepsilon/3$. We know (f_n) uniformly converges to f so there exists N such that for all $n > N$ and for all $x, y \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon/3$ and $|f(y) - f_n(y)| < \varepsilon/3$. Also f_n is continuous for all n so for all $n > N$ and for all $x \in [a;b]$ there exists $\delta_n > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta_n$ we have $|f_n(x) - f_n(y)| < \varepsilon/3$. Consider δ_{N+1}. Then for all $x \in [a;b]$ there exists $\delta_{N+1} > 0$, which may depend on x, such that for all $y \in [a;b]$ with $|x - y| < \delta_{N+1}$ we have $|f_{N+1}(x) + f_{N+1}(y)| < \varepsilon/3$. By the triangle inequality we have $|f(x) - f_{N+1}(y)| \leq |f_{N+1}(x) - f_{N+1}(y)| + |f(x) - f_{N+1}(x)| < 2\varepsilon/3$ and then $|f(x) - f(y)| < |f(x) - f_{N+1}(y)| + |f(y) - f_{N+1}(y)| < \varepsilon$. Thus for all $x \in [a;b]$ there exists some $\delta > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Therefore f is continuous on $[a;b]$. $\square$

Theorem 3 *Let (f_n) be a sequence of functions which are integrable on $[a;b]$ and that (f_n) uniformly converges to f on $[a;b]$, which is integrable on $[a;b]$. Then*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Let $\varepsilon > 0$. Since (f_n) uniformly converges to f on $[a;b]$, then there exists N such that for all $n > N$ and all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon/(b-a)$. Note that

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \left| \int_a^b f_n - f \right| < \int_a^b \frac{\varepsilon}{(b-a)} = \varepsilon$$

for all $n > N$ (22.14). Thus we have

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

$\square$

Exercise 4 *Let (f_n) be a sequence of functions which are integrable on $[a;b]$ and that (f_n) uniformly converges to f on $[a;b]$. Is f integrable on $[a;b]$?*

Yes.

Proof. Let $\varepsilon > 0$. Since f_n is integrable on $[a;b]$ for all n we know there exists some partition $P = \{t_0, \ldots, t_n\}$ such that

$$U(f_n, P) - L(f_n, P) < \varepsilon.$$

Since (f_n) uniformly converges on $[a;b]$ there exists N such that for all $n > N$ and all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$. Let

$$m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$
$$m_{i_n} = \inf\{f_n(x) \mid t_{i-1} \leq x \leq t_i\}$$
$$M_i = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}.$$

and

$$M_{i_n} = \sup\{f_n(x) \mid t_{i-1} \leq x \leq t_i\}.$$

Then since $|f(x) - f_n(x)| < \varepsilon$ for all $n > N$ and all $x \in [a;b]$ then we have $|m_i - m_{i_n}| < \varepsilon/(3(b-a))$ for all $i \leq i \leq n$. Thus

$$|L(f,P) - L(f_n,P)| = \left|\sum_{i=1}^{n} m_i(t_i - t_{i-1}) - \sum_{i=1}^{n} m_{i_n}(t_i - t_{i-1})\right| = \left|\sum_{i=1}^{n}(m_i - m_{i_n})(t_i - t_{i_n})\right| < \varepsilon/3.$$

And a similar statement can be made to show $|U(f,P) - U(f_n,P)| < \varepsilon/3$. Also since

$$0 < U(f_n,P) - L(f_n,P) < \frac{\varepsilon}{3} < \varepsilon$$

we have

$$|U(f_n,P) - L(f_n,P)| < \varepsilon 3.$$

Combining the second of these inequalities with the last we have

$$|U(f,P) - L(f_n,P)| \leq |U(f,P) - U(f_n,P)| + |U(f_n,P) - L(f_n,P)| < \frac{2\varepsilon}{3}$$

and then

$$|U(f,P) - L(f,P)| \leq |U(f,P) - L(f_n,P)| + |L(f,P) - L(f_n,P)| < \varepsilon$$

and since $0 < U(f,P) - L(f,P)$ we have

$$U(f,P) - L(f,P) < \varepsilon$$

which means f is integrable on $[a;b]$. $\square$

Exercise 5 *Find a sequence of differentiable functions that uniformly converge to $f(x) = |x|$ on $[-1;1]$.*

Let

$$f(x) = \begin{cases} (-x)^{\frac{1+n}{n}} & \text{if } x < 0 \\ x^{\frac{1+n}{n}} & \text{if } x \geq 0. \end{cases}$$

Exercise 6 *Let*

$$f_n = \frac{1}{n}\sin(n^2 x).$$

Then f_n uniformly converges to $f = 0$ but $\lim_{n\to\infty} f_n'$ does not exist.

Proof. Let $\varepsilon > 0$. Note that $-1 \leq \sin(n^2 x) \leq 1$ for all n and all x. Then note that there exists some N such that $1/N < \varepsilon$. Thus, for all $n > N$ we have $|1/n| < \varepsilon$ and since $|\sin(n^2 x)| < 1$, for all $n > N$ we have

$$\left|\frac{1}{n}\sin(n^2 x)\right| < \varepsilon.$$

Thus we have

$$\lim_{n\to\infty} \frac{1}{n}\sin(n^2 x) = 0.$$

Now note that f_n' were to converge uniformly to some function f, then f is also the pointwise limit of (f_n') (19.7). We have $f_n' = 2\cos(n^2 x)$. Thus for $x = \pi/2$ we have $2\cos(n^2 x) = 0$ for even n and $2\cos(n^2) = 1$ for odd n. Then there are infinitely many n with $f_n'(\pi/2) = 0$ and likewise for 1 which means 0 and 1 are accumulations points for $(f_n'(\pi/2))$. Thus $\lim_{n\to\infty} f_n'(\pi/2)$ does not exist (13.10). $\square$

Theorem 7 *Let (f_n) be a sequence of functions which are differentiable on $[a;b]$, with integrable derivatives f_n' and that (f_n) pointwise converges to f on $[a;b]$. Suppose that f_n' uniformly converges on $[a;b]$ to some continuous function g. Then f is differentiable on $[a;b]$ and for all $x \in [a;b]$ we have*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof. Since g is continuous we know it's integrable on $[a;b]$ (22.9). Also because (f_n) pointwise converges to f on $[a;b]$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a;b]$. Thus we have

$$\int_a^x g = \lim_{n\to\infty} \int_a^x f_n' = \lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a)$$

for all $x \in [a;b]$ by the Second Fundamental Theorem of Calculus and Theorem 3 (22.18, 30.3). If we let

$$G(x) = \int_a^x g$$

then $G'(x) = g(x)$ and so we have $G'(x) = (f(x) - f(a))' = f'(x)$ for all $x \in [a;b]$. Then it must be the case that $g = f'$ and so we have

$$f'(x) = g(x) = \lim_{n\to\infty} f_n'(x).$$

□

Definition 8 *The series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on A if the sequence of partial sums $s_n = \sum_{i=1}^{n} f_n$ converges to f uniformly.*

Theorem 9 *Let $\sum_{n=1}^{\infty} f_n$ converge uniformly to f on $[a;b]$. If f_n is continuous on $[a;b]$ for all n, then f is continuous on $[a;b]$. If f_n is integrable on $[a;b]$ for all n and f is integrable on $[a;b]$ then*

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n.$$

If f_n has an integrable derivative for all n and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a;b]$ to some continuous function then for all $x \in [a;b]$ we have

$$f'(x) = \sum n = 1^{\infty} f_n'(x).$$

Proof. Let f_n be continuous on $[a;b]$ for all n. Then since the sum of two continuous functions is still continuous, we have the partial sums of $\sum_{n=1}^{\infty} f_n$ are continuous. Thus (s_n) is a sequence of continuous functions on $[a;b]$ which uniformly converges to f on $[a;b]$. Thus f is continuous on $[a;b]$ (30.2).

Let f_n be integrable on $[a;b]$ for all n and f be integrable on $[a;b]$. Then since the sum of two integrable functions is still integrable, we have the partial sums, s_n are a sequence of integrable functions on $[a;b]$ (22.11). Thus we have

$$\sum_{n=1}^{\infty} \int_a^b f_n = \lim_{n\to\infty} \int_a^b s_n = \int_a^b f$$

from Theorem 3 (30.3).

Let f_n have an integrable derivative for all n and $\sum_{n=1}^{\infty} f_n'$ converge uniformly on $[a;b]$ to some continuous function then for all $x \in [a;b]$. By the same argument as before, since the sum of integrable functions is still integrable we have the partial sums of $\sum_{n=1}^{\infty} f_n'$ are integrable (22.11). Thus we have

$$f'(x) = \sum n = 1^{\infty} f_n'(x).$$

from Theorem 7 (30.7). □

Theorem 10 (Weierstrass M-Test) *Let (f_n) be a sequence of functions defined on A and suppose $|f_n|$ is bounded by M_n on A. Suppose that $\sum_{n=1}^{\infty} M_n$ converges. Then for all $x \in A$ the series $\sum_{n=1}^{\infty} f_n(x)$ absolutely converges and $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to the function*

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Proof. Let

$$M = \sum_{n=1}^{\infty} M_n.$$

Since for all n we have $|f_n| \leq M_n$, we have

$$\sum_{i=1}^{n} |f_n| \leq \sum_{i=1}^{n} M_n \leq M$$

for all n. But since $0 \leq |f_n|$, the series of partial sums of $\sum_{n=1}^{\infty} |f_n|$ is a bounded increasing sequence so it must converge. Thus $\sum_{n=1}^{\infty} f_n$ is absolutely convergent on A. Note that since an absolutely convergent series implies a convergent series we have

$$\sum_{i=1}^{n} f_n$$

is convergent. Then we can write

$$\left| \sum_{n=1}^{\infty} f_n - \sum_{n=1}^{k} f_n \right| = \left| \sum_{n=k+1}^{\infty} f_n \right| \leq \sum_{n=k+1}^{\infty} |f_n| \leq \sum_{n=k+1}^{\infty} M_n$$

and taking the limit as k goes to ∞ we see that

$$\lim_{k \to \infty} \left| \sum_{n=1}^{\infty} f_n - \sum_{n=1}^{k} f_n \right| = 0$$

so

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

□