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Homework 5

Exercise 1 *Write up the matrices for the following:*
1) Rotation by degree a;
2) Reflection about the line spanned by (a, b);
3) Reflection about 0.

Proof. 1) Let

$$A = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}.$$

This uses the fact that on the plane, the cos of an angle corresponds to the distance on the horizontal axis from the origin and the sin of an angle corresponds to the distance on the vertical axis from the origin.
2) Make the angle measured from the horizontal axis to the line spanned by (a, b) by

$$\arctan\left(\frac{b}{a}\right) = \alpha.$$

Then $f(b_2) = \cos 2\alpha b_2 + \sin 2\alpha b_1$ and $f(b_1) = \cos(2(\pi/2 - \alpha))b_2 - \sin(2(\pi/2 - \alpha))b_1$.
3) This is just a rotation by an angle π so we have

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$\square$

Exercise 2 *Find a 2 by 2 matrix A such that f_A is bijective, but is not a linear transformation.*

Proof. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then we have $b_1 = (0, 1)$ and $b_2 = (1, 0)$ so $f(b_1) = 2b_1 + 0b_2 = (0, 2)$ and $f(b_2) = 0b_1 + 2b_2 = (2, 0)$. Then if $v = a_1b_1 + a_2b_2$ then $f_A(v) = 2a_1b_1 + 2a_2b_2 = 2(a_1b_1 + a_2b_2) = 2v$. Let $v_1, v_2 \in V$ such that $f_A(v_1) = f_A(v_2)$. Then $2v_1 = f_A(v_1) = f_A(v_2) = 2v_2$ and multiplying by 2^{-1} we have $v_1 = v_2$. Thus f_A is injective. Now let $v \in V$ and consider $1/2v$. Then $f_A(1/2v) = 2/2v = v$. Thus f_A is surjective and therefore bijective. But f_A is not Euclidean because it doesn't preserve lengths. $\square$

Exercise 3 *Let W be a 1 dimensional vector space over K. Let $f : W \to W$ be a linear transformation. Then there exists $a \in K$ such that $f(v) = av$ for all $v \in W$.*

Proof. Let $v \in W$ such that $v \neq 0$. Then since v is linearly independent we know it is a basis for W. Since f is a linear map from W to W we have $f(v) = w$ for $w \in W$ and since v is a basis for W we know $w = av$ for some $a \in K$. Thus $f(v) = av$. Note that since f is a linear transformation, $\ker f = \{0\}$ and since $v \neq 0$ we have $a \neq 0$. Now consider some other element of W, $v' \neq 0$. We know $f(v') = a'v$ for some $a' \in K$ because v is a basis. Then $av = a'v$ so $v(a - a') = 0$ and since $v \neq 0$ we must have $a = a'$. $\square$

Exercise 6 *Find a real matrix that has no eigenvalues.*

Proof. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that this matrix changes the direction of every vector put into it by a 90° rotation. Thus for all vectors v we have $\langle v \rangle$ is not an invariant subspace under f_A. $\square$

Exercise 7 *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an arbitrary complex matrix.
1) When is 0 *an eigenvalue of* A*? 2) Find the eigenvalues of* A*.*

Proof. 1) If $a = b = c = d = 0$ then $f(v) = 0v$ for all v and so 0 is an eigenvalue of A. $\square$

Exercise 9 *We have*

$$M_{g \circ f} = M_f M_g.$$

Exercise 10 *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and assume that 0 *is not an eigenvalue of* A*. Find the inverse of* A*.*

Proof. We want

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

such that

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Which means we're left with the equations

$$aw + by = 1$$

$$ax + bz = 0$$

$$cw + dy = 0$$

$$cx + dz = 1.$$

Solving these we have

$$w = \frac{d}{ad - bc} \; x = \frac{b}{bc - ad} \; y = \frac{c}{bc - ad} \; z = \frac{a}{ad - bc}.$$

and so

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{b}{bc-ad} \\ \frac{c}{bc-ad} & \frac{a}{ad-bc} \end{bmatrix}.$$

$\square$

Exercise 11 *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and let $v = (a, b)$*. What is* vA*? What is* $(1,1)A^n$*?*

Proof. We have

$$vA = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \end{bmatrix}.$$

Since this is true for $v = (1,1)$, for $(1,1)A^n$ we obtain the Fibonacci sequence in vectors. That is, $(1,1)A^n = (f_{n+1}, f_{n+2})$ where f_n is the nth number in the Fibonacci sequence. $\square$