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Homework 7

Problem 1. *Let T be a countable theory in a countable language $\mathcal{L}$ and let $M \models T$, $A \subseteq |M|$. Say that $p(x_1, \ldots, x_n)$ is an* n-type of T over A *if it is a consistent in the language $\mathcal{L}'$ in which we add constants for the elements of A and it is realized in some model of T.*
(a) Let $M = \langle \mathbb{Q}, < \rangle$. Show that there are continuum many types of $\mathrm{Th}(\mathbb{Q})$ over $\mathbb{Q}$.
(b) Give an example of a model in a countable language in which there are continuum many distinct 1-types of $\mathrm{Th}(M)$ over $\emptyset$. Can they all be principal?
(c) Give an example of a model in which there are only countably many 1-types of $\mathrm{Th}(M)$ over $\emptyset$.

Proof. (a) Let $r \in \mathbb{R}$ and consider the 1-type $p_r(q) = \{x > q \mid q < r\} \cup \{x < q \mid q > r\}$. Every finite subset of $p(q)$ is realized in M. Additionally, since there exists a p_r for each real number r, there are continuum many of them. To see that they're all distinct, consider two p_r and p_s. Without loss of generality, let $r < s$ and let $q \in (r, s)$. Then q realizes a formula in p_r, but not in p_s.

(b) Let $M = \langle \mathbb{N}, <, \cdot, +, 0, 1 \rangle$. Let P be the set of primes and let $X \subseteq P$. Define $p_X(y) = \{p \mid p \mid y, p \in X\}$. We can define the prime divisors of y and so this is finitely satisfiable. But since P is countable, there are continuum many types p_X. They can't all be principal, since a countable language implies countably many formulas and thus each formula can only label countably many things.

(c) Let $M = \langle \mathbb{N}, S, 0 \rangle$. Let $p_n = \{x \neq S^n(0)\}$. Since $\mathbb{N}$ is countable, there are countably many p_n. $\square$

Problem 2. *Let S_0 be the following topological space: the points T are the maximal consistent sets of $\mathcal{L}$-sentences and for each $\mathcal{L}$-sentence φ, $O_\varphi = \{T \mid \varphi \in T\}$ is a basic open set.*
(a) Show that the complement of each basic open set is open. Show that for any two points T_1, T_2, there are φ, $\psi = \neg\varphi$ such that $T_1 \in O_\varphi$, $T_2 \in O_\psi$. (So S_0 is Hausdorff.)
(b) Show that the compactness theorem is equivalent to the statement that S_0 is a compact space.

Proof. (a) Let O_φ be an open set and note that $S_0 \backslash O_\varphi = \{T \mid \varphi \notin T\}$. But since the points T are maximally consistent, this is the same as $\{T \mid \neg\varphi \in T\}$. Thus $S_0 \backslash O_\varphi = O_{\neg\varphi}$. Let T_1 and T_2 be two distinct points in S_0. Then there exists $\varphi \in T_1$ such that $\varphi \notin T_2$. But since T_2 is maximally consistent, $\neg\varphi \in T_2$ and thus $T_1 \in O_\varphi$ and $T_2 \in O_{\neg\varphi}$.

(b) Assume that a theory T is maximally consistent if and only if it is finitely satisfiable. Let $F = \{f_i \mid i \in I\}$ be a family of closed sets with the finite intersection property. From part (a) we can write each f_i as the intersection of open sets

$$f_i = S_0 \backslash \bigcup_j O_{\varphi_j} = \bigcap_j S \backslash O_{\varphi_j} = \bigcap_j O_{\varphi'_j}.$$

We know that for each $k < \omega$, we have

$$\bigcap_{i \leq k} f_i = \bigcap_{i \leq k} \bigcap_j O_{\varphi'_j} \neq \emptyset.$$

Thus there exists a theory T_k in this intersection which is maximally consistent. But then since T_k is finitely satisfiable for each k, we can extend the intersection to all i so that

$$\bigcap_i f_i = \bigcap_i \bigcap_j O_{\varphi_j} \neq \emptyset$$

and S_0 is a compact space. Now suppose the converse, that S_0 is compact. Then every collection of closed sets F with the finite intersection property has nonempty intersection. Let T be a theory which is finitely satisfiable. Let T_i be a subset of T and extend T_i to a maximal consistent theory T'_i. Each T'_i is a point in S_0, so each one corresponds to a closed set in S_0. This family of closed sets has the finite intersection property, and thus the entire family has nontrivial intersection. But this means precisely that $T \in S_0$ and is thus maximally consistent. $\square$

Problem 3. *Explain how to modify the proof of the Omitting Types Theorem to omit* two *nonprincipal types simultaneously.*

Proof. The statement of the theorem will now be "Suppose $\mathcal{L}$ is a countable language and T is a set of $\mathcal{L}$-structures. If Σ_1 and Σ_2 are nonprincipal types then there exists a model $M \models T$ which omits Σ_1 and Σ_2. Steps 1, 2 and 3 of the proof remain the same. In step 4, we need to write, "There are $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ such that $\neg\sigma_1(c_i) \in T_{i+1}$ and $\neg\sigma_2(c_i) \in T_{i+1}$. □

Problem 4. *Let T be a complete countable theory and let p_i ($i \in \mathbb{N}$) be a countable set of 1-types of T over $\emptyset$. Show that there exists a countable model of T in which each p_i is realized (i.e., for each p_i there exists $a_i \in |M|$ such that $M \models \varphi(a_i)$ for each $\varphi \in p_i$).*

Proof. Let T be over a language $\mathcal{L}$ and let $\mathcal{L}' = \mathcal{L} \cup \{c_i \mid i \in \mathbb{N}\}$. Let $T' = T \cup \{\varphi(c_i) \mid \varphi \in p_i, i \in \mathbb{N}\}$. Since T is countable and complete, we know there exists a countable model $M \models T$. Furthermore for each $i \in \mathbb{N}$ and each $k < \omega$, we know $T \models \exists x \bigwedge_{j=1}^{k} \varphi_j(x)$ where $\varphi_j \in p_i$. But now just let c_i be interpreted as elements of M which satisfy φ_k for each p_i. Then each p_i is realized in M and M is still countable. □

Problem 5. *A model M is said to be* countably saturated *if for all finite $A \subseteq |M|$ and all 1-types p of* $\mathrm{Th}(M)$ *over A, p is realized in M. Suppose M, N are countable and countably saturated. Write $(M, a_1, \ldots, a_k) \equiv (N, b_1, \ldots, b_k)$ to indicate $M \equiv N$ in the language where we add new constant symbols $c_1, \ldots, c_k$ and c_i is interpreted as a_i in M and as b_i in N.*
(a) Suppose $(M, a_1, \ldots, a_n) \equiv (N, b_1, \ldots, b_n)$. For each $a_{n+1} \in |M|$, there exists $b_{n+1} \in |N|$ such that $(M, a_1, \ldots, a_{n+1}) \equiv (N, b_1, \ldots, b_{n+1})$.
(b) Restate (a) in terms of a condition about realizing types over finite sets.
(c) Show that any two countable countably saturated models which are elementary equivalent are isomorphic.
(d) If a countable countably saturated model of T exists, there cannot be more than countably many 1-types of T over $\emptyset$.

Proof. (a) Let $a_{n+1} \in |M|$ where a_{n+1} is the interpretation of c_{n+1} in M. Since N is countably saturated, let $A = \{b_1, \ldots, b_n\}$ so that every 1-type over A is realized in N. In particular, $p(x) = \{x \neq b_1, x \neq b_2, \ldots, x \neq b_n\}$ is realized by some b_{n+1}. Then we must have b_{n+1} is the interpretation of c_{n+1} in N and $(M, a_1, \ldots, a_{n+1}) \equiv (N, b_1, \ldots, b_{n+1})$.

(b) Let $M \equiv N$. For each finite subset $A \subseteq |M|$ and let $x \notin A$ be the realization of a 1-type over A. Then if $B \subseteq |N|$ with every element in A corresponding to an element of B, there exists y, the realization of a 1-type over B.

(c) Let M and N be two elementary equivalent countably saturated countable models for $\mathcal{L}$. Add countably many constants c_i to $\mathcal{L}$. Let a_1 be the interpretation of c_1 in M. From part (a) we know that there exists $b_1 \in |N|$ which is the interpretation of c_1 in N. Choosing $b_2 \in |N|$ as the interpretation of c_2, we again know there exists $a_2 \in |M|$ which is the interpretation of c_2 in M. Since M and N are countable, we can enumerate every element as an interpretation of some constant c_i, and so they must be isomorphic.

(d) Let $M \models T$ be a countably saturated countable model of T. Then there are only countably many formulas. Thus each 1-type can only be countable and there can only be countably many of them. □