Kris Harper
MATH 20800
February 16, 2009

Homework 6

**** Problem 1.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and let L be the derivative of f at $x_0 \in \mathbb{R}^n$. Show that L is unique.*

Proof. Let L and M be the derivatives of f at x_0. Then we have

$$\begin{aligned}\lim_{h\to 0}\frac{|Lh - Mh|}{|h|} &= \lim_{h\to 0}\frac{|(Lh - f(x_0+h) + f(x_0)) + (f(x_0+h) - f(x_0) - Mh)|}{|h|}\\ &\le \lim_{h\to 0}\frac{|f(x_0+h) - f(x_0) - Lh|}{|h|} + \lim_{h\to 0}\frac{|f(x_0+h) - f(x_0) - Mh|}{|h|}.\end{aligned}$$

If $x \in \mathbb{R}^n$, then $tx \to 0$ as $t \to 0$. Thus for $x \neq 0$ we have

$$0 = \lim_{t\to 0}\frac{|M(tx) - L(tx)|}{|tx|} = \frac{M(x) - L(x)|}{|x|}$$

and $M = L$. □

**** Problem 2.** *Suppose $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at $x \in \mathbb{R}^n$. Show that $f + g$ and αf are differentiable at x for $\alpha \in \mathbb{R}$ and*

$$D(f+g)(x) = Df(x) + Dg(x)$$

and

$$D(\alpha f)(x) = \alpha Df(x).$$

Proof. We have

$$\begin{aligned}Df(x) + Dg(x) &= \lim_{h\to 0}\frac{|f(x+h) - f(x)|}{|h|} + \lim_{h\to 0}\frac{|g(x+h) - g(x)|}{|h|}\\ &= \lim_{h\to 0}\frac{|f(x+h) - f(x)| + |g(x+h) - g(x)|}{|h|}\\ &= \lim_{h\to 0}\frac{|f(x+h) + g(x+h) - f(x) - g(x)|}{|h|}\\ &= D(f+g)(x)\end{aligned}$$

for small enough values of h. Also

$$\begin{aligned}\alpha Df(x) &= \lim_{h\to\infty}\frac{\alpha|f(x+h) - f(x)|}{|h|}\\ &= \lim_{h\to\infty}\frac{|\alpha(f(x+h) - f(x))|}{|h|}\\ &= \lim_{h\to\infty}\frac{|(\alpha f)(x+h) - (\alpha f)(x)|}{|h|}\\ &= D(\alpha f)(x)\end{aligned}$$

and this limit exists because scalar multiples apply to limits. □

**** Problem 3.** *Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at x then $D_v f(x)$ exists for all $v \in \mathbb{R}^n$.*

Proof. Let $v \in \mathbb{R}^n$. Since f is differentiable at x we have for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{|f(x+h) - f(x)|}{|h|} < \varepsilon$$

whenever $|h| < \delta$. So choose $|h| < \delta$ and choose $t \in \mathbb{R}$ such that $0 < |h| < |t| < \delta$. Then we have

$$\frac{f(x+tv) - f(x)}{t} < \frac{|f(x+h) - f(x)|}{|h|} < \varepsilon.$$

Thus $D_v f(x)$ exists. $\square$

**** Problem 4.** *Find $f : \mathbb{R}^n \to \mathbb{R}$ defined on an open set $U \subseteq \mathbb{R}^n$ and $x \in U$ such that $D_v f(x)$ exists for all $v \in \mathbb{R}^n$ but f is not differentiable at x.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the norm on ℓ_n^2 denoted by $|\cdot|$. Let U be the open unit ball in $\mathbb{R}^n$ and let $x = 0$. Then we have

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - Df(h)|}{|h|} = \lim_{h \to 0} \frac{||h| - Df(h)|}{|h|}.$$

Thus, Df must be a linear transformation which is always positive, if $|h| - Df(h) = 0$. But a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ must take on values less than 0. Thus $Df(0)$ doesn't exist. Now consider

$$D_v f(0) = \lim_{t \to 0} \frac{|tv|}{t} = \lim_{t \to 0} \frac{t|v|}{t} = |v|.$$

Thus $D_v f(0)$ exists for all $v \in \mathbb{R}^n$. $\square$

**** Problem 5.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ then*

$$Df(a) = (D_1 f(a), D_2 f(a), \ldots, D_n f(x)).$$

Proof. Define $g : \mathbb{R} \to \mathbb{R}^n$ by $g(x) = (a_1, a_2, \ldots, x, \ldots, a_n)$ where x is in the jth place. Then $D_j f(a) = D(f \circ g)(a_j)$ and $D(f \circ g)(a_j) = Df(a) \cdot Dh(a_j)$. But this last term is a vector with only one nonzero entry. Thus $D_j f(a)$ is the jth entry in the matrix $(D_1 f(a), D_2 f(a), \ldots, D_n f(x))$. $\square$