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Homework 10

**** Problem 1.** *For $z \in \mathbb{C}$ we have*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. This series converges for all z by the ratio test. We know that $f'(z)$ can be found by differentiating the series term by term. Then

$$f'(z) = 0 + \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Thus $f'(z) = f(z)$. Moreover, $f(0) = 1$. This means that f must be the unique function e^z. $\square$

**** Problem 2.** *For $x \in \mathbb{R}$ we have*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Proof. This follows from ** Problem 1 since $\mathbb{R} \subseteq \mathbb{C}$. $\square$

**** Problem 3.** *Let $U \subseteq \mathbb{R}^{n+p}$ be open and let $F : U \to \mathbb{R}^p$ be C^1. Suppose there exists $(x_0, y_0) \in U$ such that $F(x_0, y_0) = 0$ and $\det D_y F(x_0, y_0) \neq 0$. Also suppose there exists an open neighborhood of (x_0, y_0), $D \times E$ and a function $f : D \to E$ such that $F(x, f(x)) = 0$ for all $x \in D$. Then if $F \in C^r$ then $f \in C^r$ for all $r \geq 1$.*

Proof. Use induction on r. We already have the base case for $r = 1$. Suppose now that $F \in C^r$ for $r \in \mathbb{N}$. Consider the difference

$$\begin{aligned}
|f^{(r)}(x+h) - f^{(r)}(x)| &= |\phi^{(r)}(x+h, f^{(r)}(x+h)) - \phi^{(r)}(x, f^{(r)}(x))| \\
&\leq |\phi^{(r)}(x+h, f^{(r)}(x+h)) - \phi^{(r)}(x+h, f^{(r)}(x))| + |\phi^{(r)}(x+h, f^{(r)}(x+h)) - \phi^{(r)}(x, f^{(r)}(x))| \\
&\leq \frac{1}{2}|f^{(r)}(x+h) - f(x)| + \beta_r |h|
\end{aligned}$$

where

$$\beta_r = \sum_{j=1}^{p} \sum_{i=1}^{n} \sup_{D \times E} \left| \frac{\partial \phi_i^{(r)}}{\partial x_j}(x, y) \right|.$$

Thus $|f^{(r)}(x+h) - f^{(r)}(x)| \leq 2\beta|h|$ and so $f^{(r)}$ is continuous. Now we consider

$$|F^{(r)}(x+h, y+k) - F^{(r)}(x, y) - D_x F^{(r)}(x, y)h - D_y F^{(r)}(x, y)k| < \varepsilon |(h, k)|$$

for small $|(h, k)|$. Letting $k = f^{(r)}(x+h) - f^{(r)}(x)$ and $y = f^{(r)}(x)$ we have the given result. $\square$

Problem 1. *Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} e^{-x^{-2}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is a C^∞ function and that $f^{(i)}(0) = 0$ for all i.

Proof. Note that

$$\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$$

and so this function is continuous and thus differentiable at $x = 0$. For $x \neq 0$, using the chain rule we have

$$Df(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^3}$$

where a_1 is some integer constant. Now suppose that the kth derivative for $x \neq 0$ is

$$D^k f(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^{(k+2)}} + \frac{a_2 e^{-\frac{1}{x^2}}}{x^{(k+4)}} + \cdots + \frac{a_k e^{-\frac{1}{x^2}}}{x^{(k+2k)}}$$

where $a_1, \ldots, a_k$ are integer constants. Using the chain rule and the product rule, we can differentiate again to obtain

$$D^{k+1} f(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^{(k+3)}} + \frac{a_2 e^{-\frac{1}{x^2}}}{x^{(k+5)}} + \cdots + \frac{a_k e^{-\frac{1}{x^2}}}{x^{(k+1+2(k))}} + \frac{a_{k+1} e^{-\frac{1}{x^2}}}{x^{(k+1+2(k+1))}}$$

for all $x \neq 0$ where $a_1, \ldots, a_k$ are different integer constants. Thus, by induction, the is the kth nonzero derivative. To show that each derivative is continuous at 0, note that the first derivative for $x \neq 0$ is

$$Df(x) = \frac{a_1 e^{-\frac{1}{x^2}}}{x^3}.$$

Taking $\lim_{x\to 0} Df(x)$ we see that l'Hopital's Rule applies, and we end up with $\lim_{x\to 0} Df(x) = 0$. We can assume inductively that the kth derivative is continuous at 0, and then use that fact and l'Hopital's Rule to show the $k+1$st derivative is continuous at 0. Thus $D^k f(0)$ exists for all k and $D^k f(0) = 0$ for all k. $\square$

Problem 2. *Let*

$$f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1) \\ 0 & x \notin (-1,1). \end{cases}$$

1) Show that $f : \mathbb{R} \to \mathbb{R}$ is C^∞ function which is positive on $(-1,1)$ and 0 elsewhere.
2) Show that there exists a C^∞ function $g : \mathbb{R} \to [0,1]$ such that $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \varepsilon$.
3) If $a \in \mathbb{R}^n$ define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) = f(\frac{(x_1 - a_1)}{\varepsilon}) \ldots f(\frac{(x_n - a_n)}{\varepsilon}).$$

Show that g is a C^∞ function which is positive on

$$(a_1 - \varepsilon, a_1 + \varepsilon) \times \cdots \times (a_n - \varepsilon, a_n + \varepsilon)$$

and 0 elsewhere.
4) If $A \subseteq \mathbb{R}^n$ is open and $C \subseteq A$ is compact, show that there is a nonnegative C^∞ function $f : A \to \mathbb{R}$ such that $f(x) > 0$ for $x \in C$ and $f = 0$ outside of some closed set contained in A.
5) Show that we can choose an f so that $f : A \to [0,1]$ and $f(x) = 1$ for $x \in C$.

Proof. 1) For all points other than 1 and -1 the result is clear. At $x = 1$ and $x = -1$ we can take the left and right hand derivatives, and use Problem 1. This shows that the derivative exists there.

2) For $0 < \varepsilon < 1$ let

$$g(x) = \begin{cases} 0 & x < 0 \\ \frac{\int_0^x f}{\int_0^\varepsilon} & 0 \leq x \leq \varepsilon \\ 1 & x > \varepsilon. \end{cases}$$

Clearly $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \varepsilon$.

3) This follows almost immediately from Part 1). As in part one, all points easily satisfy the statement except for $(a_1 \pm \varepsilon, a_2 \pm \varepsilon, \ldots, a_n \pm \varepsilon)$. At these points the left or right hand derivative and Problem 1 give the desired result.

4) Let d be the distance between C and ${}^cA$. Let $\varepsilon = d/(2\sqrt{n})$. For all $x \in C$ let R_x be the open rectangle around x with side length 2ε. Now let f_x be the function defined in Part 3). These rectangles form an open cover, so since C is compact a finite number of them, say $R_{x_1}, \ldots, R_{x_k}$ cover C. Let $f = \sum_{i=1}^{k} f_{x_i}$. Since these rectangles cover C, by Part 3) we know that f is positive on C. By the way we chose ε we know that the closure of all the rectangles is contained in C, and f is defined to be 0 outside of this union.

5) Since C is compact we know that $f(C)$ attains a minimum value, $\varepsilon > 0$. Thus $f(x) \geq \varepsilon$ for $x \in C$. Now consider $g \circ f$ where g is the function defined in Part 2). Then $g \circ f(x) = 1$ for all $x \in C$. $\square$

Problem 3. *Define $g, h : \{x \in \mathbb{R}^2 \mid |x| \leq 1\} \to \mathbb{R}^3$ by*

$$g(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), h(x, y) = (x, y, -\sqrt{1 - x^2 - y^2}).$$

Show that the maximum of f on $\{x \in \mathbb{R}^3 \mid |x| = 1\}$ is either the maximum of $f \circ g$ or the maximum of $f \circ h$ on $\{x \in \mathbb{R}^2 \mid |x| \leq 1\}$.

Proof. Let $A = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ and $B = \{x \in \mathbb{R}^3 \mid |x| = 1\}$. Consider $P = (x, y, z) \in B$ and note that $x^2 + y^2 + z^2 = 1$. Then $z^2 = 1 - x^2 - y^2$. This shows that $B = g(A) \cup h(A)$. Thus, the maximum of f on B is either the maximum of f on $g(A)$ or the maximum of f on $h(A)$. Therefore the maximum of f on B is the maximum of $f \circ g$ on A or $f \circ h$ on A. $\square$

Problem 4. *Find the partial derivatives for the following:*
1) $F(x, y) = f(g(x)k(y), g(x) + h(y))$
2) $F(x, y, z) = f(g(x + y), h(y + z))$
3) $F(x, y, z) = f(x^y, y^z, z^x)$
4) $F(x, y) = f(x, g(x), h(x, y))$.

Proof. 1) We have

$$D_1F(x, y) = (D_1f(g(x)k(y), g(x) + h(y)))(k(y)g'(x)) + (D_2f(g(x)k(y), g(x) + h(y)))g'(x)$$

$$D_2F(x, y) = (D_1f(g(x)k(y), g(x) + h(y)))(g(x)k'(y)) + (D_2f(g(x)k(y), g(x) + h(y)))h'(y).$$

2) We have

$$D_1F(x, y, z) = (D_1f(g(x + y), h(y + z)))g'(x + y)$$
$$D_2F(x, y, z) = (D_1f(g(x + y), h(y + z)))g'(x + y) + (D_2f(g(x_y), h(y + z)))h'(y + z)$$
$$D_3F(x, y, z) = (D_2f(g(x + y), h(y + z)))h'(y + z).$$

3) We have

$$D_1F(x, y, z) = (D_1f(x^y, y^z, z^x))(yx^{y-1}) + (D_3f(x^y, y^z, z^x))(\ln z z^x)$$
$$D_2F(x, y, z) = (D_1f(x^y, y^z, z^x))(\ln x x^y) + (D_2f(x^y, y^z, z^x))(zy^{z-1})$$
$$D_3F(x, y, z) = (D_2f(x^y, y^z, z^x))(\ln y y^z) + (D_3f(x^y, y^z, z^x))(xz^{x-1}).$$

4) We have

$$D_1F(x,y) = (D_1f(x,g(x),h(x,y))) + (D_2f(x,g(x),h(x,y)))g'(x) + (D_3f(x,g(x),h(x,y)))(D_1h(x,y))$$

$$D_2F(x,y) = (D_3f(x,g(x),h(x,y)))(D_2h(x,y)).$$

□

Problem 5. *1) Show that $D_{e_i}f(a) = D_if(a)$.*
2) Show that $D_{tx}f(a) = tD_xf(a)$.
3) If f is differentiable at a then show that $D_xf(a) = Df(a)(x)$ and therefore $D_{x+y}f(a) = D_xf(a) + D_yf(a)$.

Proof. 1) We have

$$\begin{aligned} D_if(a) &= \lim_{h\to 0} \frac{f(a_1,\dots,a_i+h,\dots,a_n) - f(a_1,\dots,a_n)}{h} \\ &= \lim_{h\to 0} \frac{f((a_1,\dots,a_i,\dots,a_n) + (0,\dots,h_j,\dots,0)) - f(a_1,\dots,a_n)}{h} \\ &= \lim_{h\to 0} \frac{f(a+he_i) - f(a)}{h} \\ &= D_{e_i}f(a). \end{aligned}$$

2) We have

$$D_{tx}f(a) = \lim_{s\to 0} \frac{f(a+stx) - f(a)}{s} = \lim_{st\to 0} t\frac{f(a+stx) - f(a)}{st} = tD_xf(a).$$

3) We have

$$0 = \lim_{tx\to 0} \frac{|f(a+tx) - f(a) - Df(a)(tx)|}{|tx|} = \lim_{tx\to 0} \left| \frac{f(a+tx) - f(a)}{t} - Df(a)(x) \right| /|x|$$

which gives the desired result for $x \neq 0$. The case when $x = 0$ is trivial. The fact that $D_{x+y}f(a) = D_xf(a) + D_yf(a)$ follows from the additivity of $Df(a)$. □

Problem 6. *Let g be a continuous real-valued function on the unit circle $\{x \in \mathbb{R}^2 \mid |x| = 1\}$ such that $g(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x) = \begin{cases} |x|g\left(\frac{x}{|x|}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that $D_xf(0,0)$ exists for all x, but if $g \neq 0$, then $D_{x+y}(0,0) = D_x(0,0) + D_y(0,0)$ is not true for all x and y.

Proof. Define $h(t) = f(tx)$. Then either $h(t) = t|x|g(x/|x|)$ or $h(t) = 0$. In either case, h is linear and thus differentiable. Thus $D_xf(0,0)$ exists for all x. Suppose that $g(a,b) \neq 0$. Then we have $D_{a+b}f(0,0) = g(a,b) \neq 0$. But $D_af(0,0) + D_bf(0,0) = 0 + 0 = 0$. □

Problem 7. *Let $A = \{(x, y) \in \mathbb{R}^2 \mid x > 0$ and $0 < y < x^2\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A. \end{cases}$$

Show that $D_x f(0,0)$ exists for all x although f is not continuous at $(0,0)$.

Proof. We have the result every straight line $y = ax$ through $(0,0)$ contains an interval around $(0,0)$ which is in $\mathbb{R}^2 \backslash A$. To see this, note that if $a \leq 0$, the line is disjoint from A. If $a > 0$ the line intersects the graph at (a, a^2) and $(0,0)$. Letting $g(x) = ax - x^2$ we see that $y = ax$ cannot intersect A anywhere left of $x = a$. Now let $g_h(t) = f(th)$ for all $h \in \mathbb{R}^2$. Each of these is identically 0 in some neighborhood of the origin, which shows it's continuous there. Therefore $D_x f(0,0)$ exists for all x. Moreover, we have that f is not continuous at $(0,0)$ since any rectangle around the origin will contain a point $x \in A$ such that $|f(x) - f((0,0))| = 1$. □

Problem 8. *1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable at 0 but f' is not continuous at 0.
2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$ but $D_i f$ is not continuous at $(0,0)$.

Proof. 1) It's clear that f is differentiable at $x \neq 0$. At $x = 0$ we have

$$Df(0) = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

since $|h \sin 1/h| \leq |h|$. For $x \neq 0$ we have $f'(x) = 2x \sin(1/x) - \sin(1/x)$. The first term goes to 0 as x goes to 0. The second term takes on every value between -1 and 1 in each neighborhood of 0. Thus $\lim_{x \to 0} f'(0)$ doesn't exist.

2) The fact that f is differentiable at 0 follows exactly as in Part 1). Taking $f(x,0) = f(0,y) = x^2 \sin(1/|x|)$ we see that this is $g(|x|)$ where g is the function in Part 1). It's clear then that $D_1 f(x,0)$ and $D_2 f(0,y)$ are defined, as they are in Part 1). Moreover, the partial derivatives are equivalent within a sign of g' and so are not continuous at 0 as in Part 1). □

Problem 9. *If $f : \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f_i(x)$ exist in an open set containing a and if each function $D_j f_i$ is continuous at a except for $D_1 f_i$.*

Proof. The proof follows exactly as in the proof of Theorem 2-9, for all $i > 1$. In the case for $i = 1$, we already know $Df(a)$ exists so we have

$$\lim_{h \to 0} \frac{|f(a_1 + h_1, a_2, \ldots, a_n) - f(a_1, a_2, \ldots, a_n) - D_i f(a) h_1|}{|h|} = 0.$$

This completes the proof. □

Problem 10. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree m if $f(tx) = t^m f(x)$ for all x. If f is also differentiable show that*

$$\sum_{i=1}^{n} x_i D_i f(x) = m f(x).$$

Proof. Let $g(t) = f(tx)$. We know then that

$$g'(t) = \sum_{i=1}^{n} x_i D_i f(tx).$$

More over, $g(t) = f(tx) = t^m f(x)$. Thus $g'(t) = m t^{m-1} f(x)$. Letting $t = 1$ gives the result. □

Problem 11. *If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $f(0) = 0$, prove there exist $g_i : \mathbb{R}^n \to \mathbb{R}$ such that*

$$f(x) = \sum_{i=1}^{n} x^i g_i(x).$$

Proof. Let $h_x(t) = f(tx)$. Then

$$\int_0^1 h_x'(t)dt = h_x(1) - h_x(0) = f(x) - f(0) = f(x).$$

Similarly, using the method in Problem 10 we have

$$\int_0^1 h'(t)dt = \int_0^1 \left(\sum_{i=1}^{n} x_i D_i f(tx) \right) dt = \sum_{i=1}^{n} x_i \int_0^1 D_i f(tx)dt.$$

Letting $g_i = \int_0^1 D_i f(tx)dt$ gives the result. □