Kris Harper
MATH 27300
February 25, 2011

Homework 5

**Problem 1.** *Consider the linear initial-value problem $w' = a(z)w + b(z)$, $w(0) = 0$. Suppose that the functions $a$ and $b$ are analytic in a disk centered at the origin and of radius $R$. Compare the estimate of equation (4.32) with that which know from the linear theory.*

*Proof.* Note that in equation (4.29), $\rho$ and $\sigma$ are picked to be within the radius of convergence for $f(z, w)$. In our case this corresponds to picking two points in $(0, R)$. Then since $\rho < R$ and $\rho \exp(-\sigma/2M\rho) > 0$, we know $r < R$ in equation (4.32). So in this case we're guaranteed a radius of convergence at least as large as some $r < R$. On the other hand, in the linear case we're guaranteed a radius of convergence at least as large as $R$, so the later is a stronger statement. □

**Problem 2.** *Show that, for any complex constant $\mu$, the functions $z^\mu$ and $z^\mu \ln z$ are linear independent over the complex numbers.*

*Proof.* Suppose we have $c_1 z^\mu + c_2 z^\mu \ln z = 0$. Differentiate to get

$$0 = c_1\mu z^{\mu-1} + c_2\mu z^{\mu-1} \ln z + c_2 z^{\mu-1} = z^{\mu-1}(c_1\mu + c_2(\mu \ln z + 1)).$$

Assuming $z \neq 0$, we can divide by $z^\mu$ in the first equation and $z^{\mu-1}$ in the second to obtain $0 = c_1 + c_2 \ln z$ and $0 = c_1\mu + c_2(\mu \ln z + 1)$. If $\mu = 0$ then the functions 1 and $\ln z$ are clearly linearly independent since $\ln z$ is nonconstant. Thus we may assume otherwise and solve for $c_1$ as $c_1 = -c_2 \ln z + c_2/\mu$. Putting this into the first equation gives $c_2/\mu = 0$ and $c_2 = 0$. Using the first equation again we have $c_1 = 0$ as well so the two functions are linearly independent. □

**Problem 3.** *Find the influence function for the differential equation*

$$u'' + 2x^{-1}u' - 2x^{-2}u = r(x)$$

*on the interval* $[1, \infty]$ *of the real axis.*

The indicial equation is given by $\mu^2 + (2 - 1)\mu - 2 = 0$, which has roots $\mu = -2$ and $\mu = 1$. Then the influence function is defined as

$$\begin{aligned} G(z, s) &= \begin{cases} \frac{u_1(s)u_2(z)-u_2(s)u_1(z)}{W(u_1,u_2;s)} & s < z \\ 0 & s \geq z \end{cases} \\ &= \begin{cases} \frac{zs^{-2}-sz^{-2}}{W(u_1,u_2;s)} & s < z \\ 0 & s \geq z \end{cases} \end{aligned}$$

where $W(u_1, u_2; s) = \exp\left(-2\int_{z_0}^{z} \zeta^{-1} d\zeta\right) = Kz^2$.

**Problem 4.** *In equation (5.1) put $p(z) = 1/z^2$ and verify that the solution $w(z)$ cannot have the form given in equation (5.6).*

*Proof.* Making the substitution in equation (5.1) gives $w(z) + w'(z)/z^2 = 0$. If we write $w(z) = z^c \sum_{k=0}^{\infty} a_k z^k$ then

$$z^{-2}w'(z) = z^{-2}\sum_{k=0}^{\infty}(k + c)a_k z^{k+c-1} = \sum_{k=0}^{\infty}(k + c)a_k z^{k+c-3}.$$

Since $w(z) + w'(z)/z^2 = 0$, we must have $a_k + (k + c)a_{k+3} = 0$. Then $a_k/a_{k+3} = -(k + c)$. But this means $w(z)$ is only convergent at $z = 0$. □

**Problem 5.** *Find the indicial equation for Bessel's equation (5.10) and find the indices. For the case when $n$ is a non-negative integer, obtain the series solution for the index with maximum real part. Do the indices differ by an integer?*

In this case $P_0 = 1$ and $Q_0 = -n^2$ so the indicial equation is $I(t) = t(t-1)+t-n^2 = t^2-n^2 = (t+n)(t-n)$. The solutions are thus $\pm n$ which clearly differ by an integer.

The larger root of the indicial equation is $\mu = n$. Then $I(\mu+m) = m^2 + 2mn$. Now we can use the recursion relation

$$a_m = \frac{-1}{I(\mu+m)} \sum_{k=0}^{m-1} ((\mu+k)P_{m-k} + Q_{m-k})a_k$$

to find $a_m$. Note that we can arbitrarily pick $a_0 \neq 0$. Further, $P_0 = 1$ and $P_k = 0$ for all $k \neq 0$ and $Q_2 = 1$ and $Q_k = 0$ for all $k \neq 0$, $k \neq 2$. Using this we can deduce that

$$a_m = \frac{-a_{m-2}}{m^2 + 2mn}$$

**Problem 6.** *Suppose the coefficients of equation (5.7) are analytic and single-valued in a punctured neighborhood of the origin. Suppose it is known that the function $w(z) = f(z)\ln z$ is a solution, where $f$ is analytic and single-valued in the punctured neighborhood. Deduce that $f$ is also a solution.*

*Proof.* Note that $w'(z) = f'(z)\ln z + f(z)/z$ and $w''(z) = f''(z)\ln z + f'(z)/z + f'(z)/z - f(z)/z^2$. Then

$$\begin{aligned}
0 &= w'' + p(z)w' + q(z)w \\
&= f''(z)\ln z + 2f'(z)/z - f(z)/z^2 + p(z)f'(z)\ln z + p(z)f(z)/z + q(z)f(z)\ln z \\
&= \ln z(f''(z) + p(z)f'(z) + q(z)f(z)) + 2f'(z)/z - f(z)/z^2 + p(z)f(z)/z.
\end{aligned}$$

But now note that we must have the term inside the parentheses equal to 0 since the terms outside of it must evaluate to 0. This means precisely that $f$ satisfies the equation. □

**Problem 7.** *Find the recursion relation for the coefficients $\{b_k\}$ in the power series expansion of the function $f_2$ appearing in equation (5.18). What determines the constant $C$?*

We have $w_2 = z^{\mu_2}f_2(z) + Cw_1(z)\ln(z)$. Then $w_2'(z) = \mu_2 z^{\mu_2-1}f_2(z) + z^{\mu}f_2'(z) + Cw_1'(z)\ln(z) + Cw_1(z)/z$ and

$$w_2''(z) = \mu_2(\mu_2-1)z^{\mu_2-2}f_2(z) + 2\mu_2 z^{\mu_2-1}f_2'(z) + z^{\mu_2}f_2''(z) + Cw_1''(z)\ln(z) + 2Cw_1'(z)/z - Cw_1(z)/z^2.$$

Since $z^2w_2'' + zP(z)w_2' + Q(z)w_2 = 0$, we can now plug the above values in to get a recursion relation. Note that $f_2(z) = \sum_{k=0}^{\infty} b_k z^k$. Then we have

$$\begin{aligned}
0 &= z^2w_2''(z) + zP(z)w_2'(z) + Q(z)w_2(z) \\
&= \mu_2(\mu_2-1)z^{\mu_2-2}\sum_{k=0}^{\infty} b_k z^{k+2} + 2\mu_2 z^{\mu_2-1}\sum_{k=0}^{\infty} b_{k+1}(k+1)z^{k+2} + z^{\mu_2}\sum_{k=0}^{\infty} b_{k+2}(k+2)(k+1)z^{k+2} \\
&\quad + z^2Cw_1''(z)\ln(z) + 2zCw_1'(z) - Cw_1(z) + \mu_2 z^{\mu_2-1}\left(\sum_{k=0}^{\infty} b_{k+1}z^{k+2}\right)\left(\sum_{k=-2}^{\infty} P_{k+2}z^{k+2}\right) \\
&\quad + z^{\mu_2}\left(\sum_{k=0}^{\infty} b_k k z^k\right)\left(\sum_{k=0}^{\infty} P_k z^k\right) + Czw_1'(z)\ln(z)P(z) + Cw_1(z)P(z) + z^{\mu_2}\left(\sum_{k=0}^{\infty} b_k z^k\right)\left(\sum_{k=0}^{\infty} Q_k z^k\right) \\
&\quad + Cw_1(z)\ln(z)Q(z).
\end{aligned}$$

Now we can use the expansion of $w_1(z) = \sum_{k=0}^{\infty} c_k z^k$ and find the coefficients of the expanded and combined power series. Since each term must be zero, we can solve for the one $b_{k+2}$ term to get a recursion relation. The constant $C$ is $c_n$ in the power series expansion for $\psi/f_1^2$, where $\psi(z) = z^{P_0}W(z)$. So $C$ depends on the Wronskian, $P_0$ and $f_1 = z^{-\mu_1}w_1(z)$.

**Problem 8.** *Consider the equation*

$$(1-z)z^2w'' + (z-4)zw' + 6w = 0.$$

*(a) Verify that this equation has a regular singular point at the origin.*
*(b) Find the indicial equation and the indices relative to this point.*
*(c) For the index with the greater real part, find the recursion relation for the coefficients in the series solution.*
*(d) Determine whether the second solution is given purely by a series solution (as in equation (5.12)) or involves in addition a logarithmic term (as in equation (5.18)).*

(a) Divide by $(1-z)$ to get $z^2w'' + ((z-4)/(1-z))zw' + 6/(1-z) = 0$. Since $(z-4)/(1-z)$ and $6/(1-z)$ are both analytic, this equation is of the form $z^2w'' + zP(z)w' + Q(z)w = 0$, with $P$ and $Q$ analytic on a punctured disk, so the equation must have a regular singular point at the origin.

(b) The indicial equation is $I(\mu) = \mu(\mu-1) + P_0\mu + Q_0 = \mu(\mu-1) - 4\mu + 6 = 0$ This factors as $(\mu-2)(\mu-3)$.

(c) The index with greater real part is $\mu = 3$. Then $I(\mu+n) = n^2+n$. Further note that $Q(z) = \sum_{k=0}^{\infty} 6z^k$ and $P(z) = -4 + \sum_{k=1}^{\infty} -3z^k$. Then we can use the following formula for the recursion relation. We have

$$a_n = \frac{-1}{I(\mu+n)}\sum_{k=0}^{n-1}((\mu+k)P_{n-k} + Q_{n-k})a_k = \frac{-1}{n^2+n}\sum_{k=0}^{n-1}((3+k)(-3)+6)a_k = \frac{3}{n^2+n}\sum_{k=0}^{n-1}(1-k)a_k.$$

(d) Since the zeros of the indicial equation differ by an integer, the second solution has the structure of equation (5.18) and has a logarithmic term in it.

The following four problems relate to singular points at infinity. These are investigated by making the transformation $t = 1/z$ and investigating the singular points at $t = 0$. In each case determine whether the point in question is a point of analyticity, a regular singular point or an irregular singular point. In the case of a regular singular point, find the indices.

**Problem 9.** *Bessel's equation* $z^2w'' + zw' + (n^2 - z^2)w = 0$ *where* $n$ *is a constant.*

Making the substitution $t = 1/z$ we now have $w''/t^2 + w'/t + (n^2 - 1/t^2)w = 0$. Multiply through by $t^4$ to obtain $t^2w'' + t^3w' + (n^2t^4 - t^2)w = 0$. We now see that 0 is a regular singular point, since we can write this as $t^2w'' + tP(t)w' + Q(t)w = 0$ where $P(t) = t^2$ and $Q(t) = n^2t^4 - t^2$ are both analytic. The indices will be given by solving the equation $I(t) = t(t-1) + P_0t + Q_0 = t(t-1) + 0 + 0 = t^2 - t$. This has solutions $t = 0$ and $t = 1$.

**Problem 10.** *Legendre's equation* $(1-z^2)w'' - 2zw' + \lambda w = 0$ *where* $\lambda$ *is a constant.*

Making the substitution $t = 1/z$ we now have $(1 - 1/t^2)w'' - 2w'/t + \lambda w = 0$. If we divide by $(1-1/t^2)$ then we have $w'' - 2w'/(t(1-1/t^2)) + \lambda w/(1-1/t^2) = 0$. At this point we note that $-2/(t^2(1-1/t^2)) = -2/(t^2-1) = 2/(1-t^2)$ and $\lambda/(t^2(1-1/t^2)) = \lambda/(t^2-1)$ are both analytic functions. Letting $P(t) = 2/(1-t^2)$ and $Q(t) = \lambda/(t^2-1)$ we can write the equation as $t^2w'' + tP(t)w' + Q(t)w = 0$. Thus 0 is a regular singular point. To find the indices we solve the equation $I(t) = t(t-1) + P_0t + Q_0 = t^2 + t - \lambda$, where we've used the power series expansions for $P$ and $Q$ to fill in $P_0$ and $Q_0$. This has solutions $t = (-1 \pm \sqrt{1+4\lambda})/2$.