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Homework 4

**** Problem 1.** *Let V and W be normed linear spaces. On $V \times W$ define $||(v,w)|| = (||v||^p + ||w||^p)^{1/p}$ for $1 \le p < \infty$. Show that this is a norm.*

Proof. Since $||v|| \ge 0$ and $||w|| \ge 0$, it's clear that $||(v,w)|| \ge 0$. Suppose that $||(v,w)|| = 0$. Then $||v||^p + ||w||^p = 0$ and $||v|| + ||w|| = 0$. But since each term on the left is greater or equal to 0, we must have $||v|| = ||w|| = 0$ which implies $v = w = 0$ and $(v,w) = (0,0)$. Supposing that $(v,w) = (0,0)$ we clearly have $||(v,w)|| = (||0||^p + ||0||^p)^{1/p} = 0$.

For some $\alpha \in F$ we have

$$||\alpha(v,w)|| = ||(\alpha v, \alpha w)|| = (||\alpha v||^p + ||\alpha w||^p)^{1/p} = (|\alpha|^p(||v||^p + ||w||^p))^{1/p} = |\alpha|(||v||^p + ||w||^p)^{1/p} = |\alpha| \cdot ||(v,w)||.$$

Finally, for $(v_1, w_1), (v_2, w_2) \in V \times W$ note that

$$\begin{aligned} ||(v_1,w_1) + (v_2,w_2)|| &= ||(v_1 + v_2, w_1 + w_2)|| \\ &= (||v_1 + v_2||^p + ||w_1 + w_2||^p)^{1/p} \\ &\le ((||v_1|| + ||v_2||)^p + (||w_1|| + ||w_2||)^p)^{1/p} \\ &\le (||v_1||^p + ||w_1||^p)^{1/p} + (||v_2||^p + ||w_2||^p)^{1/p} \\ &= ||(v_1,w_1)|| + ||(v_2,w_2)|| \end{aligned}$$

which follows from the same use of Hölder's inequality as is used in the $\ell_n^p(F)$ metric. ☐

**** Problem 2.** *Let V and W be normed linear spaces. On $V \times W$ define $||(v,w)|| = \max(||v||, ||w||)$. Show that this is a norm.*

Proof. Since $||v|| \ge 0$ and $||w|| \ge 0$ it's clear that $||(v,w)|| \ge 0$. Suppose that $||(v,w)|| = 0$. Then without loss of generality suppose that $||v|| \ge ||w||$ so that $||(v,w)|| = 0 = ||v||$. Then we have $0 \le ||w|| \le ||v|| = 0$ which shows that $||v|| = ||w|| = 0$ and so $v = w = 0$. If $(v,w) = (0,0) = 0$ then it's clear that $||v|| = ||w|| = 0$ and so $||(v,w)|| = 0$.

For some $\alpha \in F$ we have

$$||\alpha(v,w)|| = ||(\alpha v, \alpha w)|| = \max(||\alpha v||, ||\alpha w||) = \max(|\alpha|||v||, |\alpha|||w||) = |\alpha| \max(||v||, ||w||) = |\alpha| \cdot ||(v,w)||.$$

Finally, for $(v_1, w_1), (v_2, w_2) \in V \times W$ note that

$$\begin{aligned} ||(v_1,w_1) + (v_2,w_2)|| &= ||(v_1 + v_2, w_1 + w_2)|| \\ &= \max(||v_1 + v_2||, ||w_1 + w_2||) \\ &\le \max(||v_1|| + ||v_2||, ||w_1|| + ||w_2||) \\ &< \max(||v_1||, ||w_1||) + \max(||v_2||, ||w_2||) \\ &= ||(v_1,w_1)|| + ||(v_2,w_2)||. \end{aligned}$$

Since all three properties are met, $|| \cdot ||$ is a norm. ☐

**** Problem 3.** *Let $\pi : V \to V/V_0$ such that $\pi(v) = v + V_0$. Show the following:*
1) $\pi(v_1 + v_2) = \pi(v_1) + \pi(v_2)$.
2) $\pi(\alpha v) = \alpha\pi(v)$.
3) π is an open map.
4) π is continuous.

Proof. 1) We have

$$\pi(v_1 + v_2) = (v_1 + v_2) + V_0 = (v_1 + V_0) + (v_2 + V_0) = \pi(v_1) + \pi(v_2).$$

2) We have

$$\pi(\alpha v) = (\alpha v) + V_0 = \alpha(v + V_0) = \alpha\pi(v).$$

3) Let $A \subseteq V$ be an open set. Clearly $\pi(\emptyset) = \emptyset$, so let $v \in A$. Since A is open, there exists some $r \in \mathbb{R}$ such that $B_r(v) \subseteq A$. Then for all $w \in B_r(v)$ we have $||v - w|| < r$. Note that

$$||\pi(v) - \pi(w)|| = ||\pi(v - w)|| = \inf\{||v|| \mid v \in \pi(v - w)\} < r$$

since $v - w \in \pi(v - w)$. Thus for all $v' \in \pi(A)$ there exists a ball around v' completely contained in $\pi(A)$ so $\pi(A)$ is open.

4) To show that π is continuous, consider some open set, $U \subseteq V/V_0$ and suppose that $\pi^{-1}(U)$ is not open. Then there exists $v \in \pi^{-1}(U)$ such that for all $r > 0$ there exists $u \in B_r(v)$ such that $u \notin \pi^{-1}(U)$. Note that from the definition of the norm on V/V_0, we have $||\pi(v) - \pi(u)|| \leq ||u - v|| < r$. But since U is open and $v \in U$, there exists some $r' > 0$ such that $B_{r'}(\pi(u)) \subseteq U$, and since r can be arbitrarily small, choose $r < r'$. Then $||\pi(v) - \pi(u)|| \leq ||u - v|| < r < r'$, which shows that $\pi(u) \in U$ and $u \in \pi^{-1}(U)$. This is a contradiction, and so π^{-1} maps open sets to open sets. Thus π is continuous. $\square$

**** Problem 4.** *Let V and W be normed linear spaces over F. Prove $\mathcal{BL}(V, W)$ is a vector space over F.*

Proof. Let $T, U \in \mathcal{BC}(V, W)$ and $\alpha, \beta \in F$. Define $(T + U)v = Tv + Uv$ and $(\alpha T)v = T\alpha v$. It's clear that commutativity and associativity of addition hold, since they do in V and W. The zero operator, which maps all vectors to 0, serves as the additive identity since $(T + 0)v = Tv + 0 = Tv$. The additive inverse of T maps a vector v to $-Tv$. Then $(T + (-T))v = Tv + (-Tv) = 0$. Thus $\mathcal{BC}(V, W)$ is an abelian group under addition. Additionally we have

$$(\alpha(T + U))v = (T + U)\alpha v = T\alpha v + U\alpha v = (\alpha T)v + (\alpha U)v,$$
$$((\alpha + \beta)T)v = T(\alpha + \beta)v = T(\alpha v + \beta v) = T(\alpha v) + T(\beta v) = (\alpha T)v + (\beta T)v,$$
$$(\alpha(\beta T))v = (\beta T)\alpha v = T\beta\alpha v = T\alpha\beta v = (\alpha\beta T)v$$

and $(1 \cdot T)v = T \cdot 1 \cdot v = Tv$ so that the remaining axioms of a vector space are all met. $\square$

**** Problem 5.** *Define $||T|| = \inf\{M \mid ||Tv|| \leq M||v||\}$. Show $||T_1 + T_2|| \leq ||T_1|| + ||T_2||$.*

Proof. We have

$$\begin{aligned}
||T_1 + T_2|| &= \inf\{M \mid ||(T_1 + T_2)v|| \leq M||v||\} \\
&= \inf\{M \mid ||T_1v + T_2v|| \leq M||v||\} \\
&\leq \inf\{M \mid ||T_1v|| + ||T_2v|| \leq M||v||\} \\
&\leq \{M \mid ||T_1v|| \leq M||v||\} + \inf\{M \mid ||T_2v|| \leq M||v||\} \\
&= ||T_1|| + ||T_2||
\end{aligned}$$

where the first inequality arises from the triangle inequality in a vector space and the second is a property of greatest lower bounds. □

**** Problem 6.** *Consider $\{e_j\}$, a linearly independent set of $\ell^\infty(\mathbb{R})$. Let B be the space of all finite linear combinations of $\{e_j\}$ over $\mathbb{R}$. Show B is dense in $\ell^p(\mathbb{R})$ for $1 \le p < \infty$. Show that this is not dense in $\ell^\infty(\mathbb{R})$.*

Proof. Let $1 \le p < \infty$ and note that for $(x_n) \in \ell^p(\mathbb{R})$, we must have $\lim_{n\to\infty} x_n = 0$. This is a consequence of the sequence satisfying the p-norm on the space. Then consider a ball of radius r around (x_n). We know that there exists N such that for all $n > N$ we have $|x_n| < \varepsilon$ for any $\varepsilon > 0$. Consider the sequence $(y_n) = (x_1, x_2, \ldots, x_{N-1}, x_N, 0, 0, \ldots)$ which has the first N terms of (x_n) and then terminates in 0s. Note that $(y_n) \in B$. Since we can choose ε to be arbitrarily small, it follows that $||(x_n) - (y_n)|| < r$. Thus any open ball in $\ell^p(\mathbb{R})$ must contain some element of B which shows that B is dense in $\ell^p(\mathbb{R})$.

Now let $p = \infty$. Consider the sequence (x_n) where $x_n = (-1)^{n+1}$ and the ball $B_{1/2}((x_n))$. Then since every sequence $(y_n) \in B$ must terminate in 0s, we must have $||(x_n) - (y_n)|| \ge 1$. But then there exists an open set in $\ell^\infty(\mathbb{R})$ with empty intersection with B. Thus B is not dense in $\ell^\infty(\mathbb{R})$. □

**** Problem 7.** *Show $\mathcal{BL}(V, W)$ is complete if W is complete.*

Proof. Let W be a complete normed linear space. Let (T_n) be a cauchy sequence in $\mathcal{BL}(V, W)$. Note that $||T_n - T_m|| = \inf\{M \mid ||(T_n - T_m)v|| \le M||v||, v \in V\} = \sup\{||(T_n - T_m)v|| \mid ||v|| = 1\}$. For each $v \in V$ with $||v|| = 1$, we have a sequence in W where $w_n = T_n v$. Then

$$||w_n - w_m|| = ||T_n v - T_m v|| = ||(T_n - T_m)v|| \le ||T_n - T_m||$$

which shows that (w_n) is Cauchy in W. Since W is complete, we have $\lim_{n\to\infty} w_n = w$ for some $w \in W$. Because every nonzero vector in V can be rescaled to have a norm of 1, such a sequence and limit can be created for all $v \in V$. We can then define a bounded linear operator T such that $Tv = w$ where $v \in V$ and w is the limit of the Cauchy sequence in W generated by V. Then note that

$$||T_n - T|| = \sup\{||(T_n - T)v|| \mid ||v|| = 1\} = \sup\{||T_n v - Tv|| \mid ||v|| = 1\} = \sup\{||w_n - w||\}$$

and since (w_n) converges to w, we must have T_n converges to T. Thus $\mathcal{BL}(V, W)$ is complete. □