Sheet 16: Metric Spaces

Definition 1 *Let X be a set. A topology on X is a set $\mathcal{A}$ of subsets of X, that we call open sets, satisfying the following:*
1) $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$;
2) if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$;
3) if $\mathcal{B} \subset \mathcal{A}$ then

$$\bigcup_{B \in \mathcal{B}} B \in \mathcal{A}.$$

Definition 2 *A topological space is a pair $(X, \mathcal{A})$ such that $\mathcal{A}$ is a topology on X.*

Definition 3 *Let X be a set and let $d : X \times X \to \mathbb{R}$ be a function. We say that (X, d) is a metric space if the following hold:*
1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$;
3) $d(x, y) + d(y, z) \geq d(x, z)$.

Definition 4 *For $c \in X$ and $r \in \mathbb{R}$ with $r > 0$ let*

$$B(c, r) = \{x \in X \mid d(c, x) < r\}$$

be the ball of radius r centered at c.

Definition 5 *A subset $A \subseteq X$ is open if for every $a \in A$ there exists $r > 0$ such that $B(a, r) \subseteq A$. This topology is the topology generated by d.*

Theorem 6 *For all $c \in X$ and $r > 0$ the ball $B(c, r)$ is open.*

Proof. Let $a \in B(c, r)$. Then $d(c, a) < r$. Consider the ball $B(a, r - d(c, a))$. For $x \in B(a, r - d(c, a))$ we have $d(a, x) < r - d(c, a)$ so $d(c, a) + d(a, x) < r$. By the triangle inequality we have $d(c, x) < r$ so $x \in B(c, r)$. Thus, $B(a, r - d(c, a)) \subseteq B(c, r)$ and $B(c, r)$ is open. $\square$

Proposition 7 *There is a topology on $\{0, 1\}$ that cannot be generated by any metric on $\{0, 1\}$.*

Proof. Consider the topology $\mathcal{A} = \{\emptyset, \{0, 1\}\}$ and consider some arbitrary metric on $\{0, 1\}$, $d(0, 1) = a$ for $a \in \mathbb{R}$. Then the ball $B(0, a)$ will be in the topology generated by this metric, but $B(0, a) = \{0\}$ which is not in $\mathcal{A}$. $\square$

Theorem 8 (Metric Spaces are Hausdorff) *Let (X, d) be a metric space and let $a, b \in X$ with $a \neq b$. Then there exist $A, B \subseteq X$ open such that $a \in A$, $b \in B$ and $A \cap B = \emptyset$.*

Proof. Consider the two balls $B(a, d(a, b)/2)$ and $B(b, d(a, b)/2)$. Suppose there exists $x \in X$ such that $x \in B(a, d(a, b)/2)$ and $x \in B(b, d(a, b)/2)$. Then $d(a, x) < d(a, b)/2$ and $d(b, x) < d(a, b)/2$ so $d(a, x) + d(x, b) < d(a, b)$ which contradicts the triangle inequality. Thus $B(a, d(a, b)/2) \cap B(b, d(a, b)/2) = \emptyset$. We also have $B(a, d(a, b)/2)$ and $B(a, d(a, b)/2)$ are open (16.6). $\square$

Definition 9 *Let $A \subseteq X$ be a subset. We say that $x \in X$ is a limit point of A if for all open sets $B \subseteq X$ with $x \in B$ the intersection $A \cap B$ is infinite.*

Lemma 10 *Let $A \subseteq X$ be a subset. Then $x \in X$ is a limit point of A if for all $r > 0$ the intersection $A \cap B(x, r)$ is infinite.*

Proof. Suppose that for $x \in X$ and all $r > 0$ we have $A \cap B(x, r)$ is infinite. Consider some open set $B \subseteq X$ with $x \in B$. Then there exists $B(x, r) \subseteq B$ because B is open. But then $B \cap A$ is infinite since $B(x, r) \cap A$ is infinite. $\square$

Theorem 11 *A subset of X is closed if and only if it contains all its limit points.*

Proof. Let $A \subseteq X$ be closed and consider some point $p \in X \backslash A$. Since $X \backslash A$ is open, there exists some ball $B(p, r) \subseteq X \backslash A$. But since this ball is open and disjoint from X we have p is not a limit point of A (16.6). Thus there are no limit points of A in $X \backslash A$ so A must contain all its limit points. Conversely let $A \subseteq X$ be a subset which contains all its limit points and let $p \in X \backslash A$. Since p is not a limit point of A, there exists some ball $B(p, r)$ such that $B(p, r) \cap A$ is finite. Then consider the point $x \in B(p, r) \cap A$ such that $d(p, x) = \min\{d(p, y) \mid y \in B(p, r) \cap A\}$. The ball $B(p, x)$ will then contain no points of A which means $B(p, x) \subseteq X \backslash A$ and thus $X \backslash A$ is open. Then A is closed. $\square$

Theorem 12 (Metric Spaces are T3) *Let $C \subseteq X$ be closed and let $b \in X$ such that $b \notin C$. Then there exist $A, B \subseteq X$ open such that $C \subseteq A$, $b \in B$ and $A \cap B = \emptyset$.*

Proof. Since C is closed, $X \backslash C$ is open and so there exists a ball $B = B(b, r) \subseteq X \backslash C$. Consider the set $S = \{B(a, (d(a, b) - r)/2) \mid a \in C\}$. Then let

$$A = \bigcup_{B \in S} B$$

so that $C \subseteq A$. Now let $x \in A$. Then there exists some ball $B(a, (d(a, b) - r)/2) \subseteq A$ such that $a \in C$ and $x \in B(a, (d(a, b) - r)/2)$. Then $d(x, a) < (d(a, b) - r)/2$ so $r < d(a, b) - d(a, x) \leq d(x, b)$. Thus $x \notin B(b, r)$ and so $A \cap B = \emptyset$. $\square$

Definition 13 *A subset $C \subseteq X$ is compact if every open cover of C has a finite subcover.*

Definition 14 *A sequence on X is a function from $\mathbb{N}$ to X. The sequence (a_n) converges to a (or $\lim_{n\to\infty} a_n = a$) if for every open set $A \subseteq X$ with $a \in A$ there are only finitely many n with $a_n \notin A$.*

Proposition 15 *There is a topological space on every set where every sequence converges to every element.*

Proof. Consider the trivial topology, $\{\emptyset, X\}$. Consider some sequence $(a_n) \in X$ and let $a \in X$. The only open set which contains a is X, but there are no terms of (a_n) not in X so we have for all open sets A with $a \in A$, there are finitely many terms of (a_n) not in A. Thus (a_n) converges to a. This is true of all sequences and points in X. $\square$

Proposition 16 *There is a topological space on every set where the only convergent sequences are the ones that are constant up to finitely many elements.*

Proof. Consider the full topology where every subset is open. Then for all $x \in X$, the set $\{x\}$ is open. Thus for a sequence (a_n), there are finitely many n such that $a_n \notin \{x\}$ which means there are finitely many n such that $a_n \neq x$. $\square$

Definition 17 *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces. A function $f : X \to Y$ is continuous if for all $B \in \mathcal{B}$ the preimage $f^{-1}(B) \in \mathcal{A}$*

Theorem 18 *Let $(X, \mathcal{A})$ be a Hausdorff topological space and let (a_n) be a sequence on X. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$ then $a = b$.*

Proof. Suppose that $a \neq b$. Then there exist two open sets A and B such that $a \in A$ and $b \in B$ and $A \cap B = \emptyset$ by the Hausdorff property. There are finitely many n with $a_n \notin A$ so there are infinitely many n with $a_n \in A$. But then there are finitely many n with $a_n \notin B$ which is a contradiction because $\lim_{n\to\infty} a_n = b$. Thus $a = b$. $\square$

Theorem 19 *Let (X,d) and (X',d') be metric spaces and let $f : X \to X'$ be a function. Then the following are equivalent:*
1) f is continuous;
2) for all $x \in X$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in X$ with $d(x,y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$;
3) for all convergent sequences $a_n \in X$ we have

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

Proof. Let f be continuous and let $x \in X$ and consider the ball $B(f(x), \varepsilon)$ for $\varepsilon > 0$. Then since f is continuous, $f^{-1}(B(f(x), \varepsilon))$ is open. And since $x \in f^{-1}(B(f(x), \varepsilon))$ there exists some ball $B(x, \delta) \subseteq B(f(x), \varepsilon)$. But then for all $y \in B(x, \delta)$, $f(y) \in B(f(x), \varepsilon)$. Thus for all $y \in X$ such that $d(x,y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$.

Now suppose that for all $x \in X$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in X$ with $d(x,y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$. Let $a_n \in X$ be a sequence which converges to a and let $\varepsilon > 0$. Consider $B(a, \delta)$. Since $\lim_{n\to\infty} a_n = a$, there are finitely many n with $a_n \notin B(a, \delta)$. But then there are finitely many n such that $d(a, a_n) \geq \delta$ which means there are finitely many n with $d'(f(a), f(a_n)) \geq \varepsilon$. Therefore there are finitely many n with $f(a_n) \notin B(f(a), \varepsilon)$ and since this is true for all $\varepsilon > 0$, we have $\lim_{n\to\infty} f(a_n) = f(a)$.

Finally use the contrapositive and assume that f is not continuous. Then there exists some set $A \subseteq X'$ such that $f^{-1}(A)$ is not open. Then there exists $a \in f^{-1}(A)$ such that for all $r > 0$ there exists $x \in B(a,r)$ such that $x \notin A$. Create a sequence $a_n \in X$ where $a_n \in B(a, 1/n)$, but $a_n \notin A$. We know that a_n exists for all n because $f^{-1}(A)$ is not open. Note that for the ball $B(a,r)$ with $r > 1$ there are no terms of (a_n) not in $B(a,r)$ and for $r \leq 1$ we can use the Archimedean Property to show that there are finitely many terms not in $B(a,r)$. Thus (a_n) converges to a. Note that for all n, $a_n \notin f^{-1}(A)$ and thus $f(a_n) \notin A$, while $a \in f^{-1}(A)$ and so $f(a) \in A$. But A is open so there exists some ball $B(a,r) \subseteq A$ for which $a_n \notin B(a,r)$ for all n. But then $\lim_{n\to\infty} f(a_n) \neq f(a)$. $\square$

Theorem 20 *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and let $f : X \to Y$ be continuous. Then for every compact subset $C \subseteq X$ the image $f(C)$ is also compact.*

Proof. Let $\mathcal{E} \subseteq \mathcal{B}$ be an open cover of $f(C)$. For all $x \in C$ we have $f(x) \in f(C)$ and so for all $x \in C$ there exist an open set $B \in \mathcal{E}$ such that $f(x) \in B$. But then for all $x \in C$, $x \in f^{-1}(B)$ for some $B \in \mathcal{E}$. So we have $C \subseteq \bigcup_{B\in\mathcal{E}} f^{-1}(B)$ and since f is continuous $\{f^{-1}(B) \mid B \in \mathcal{E}\} \subseteq \mathcal{A}$ is an open cover for C. But C is compact so there exists a finite subcover, $\{f^{-1}(B_1), f^{-1}(B_2), \ldots, f^{-1}(B_n)\}$ which covers C. So for all $x \in C$ there exists some $B_i \in \mathcal{E}$ such that $x \in f^{-1}(B_i)$. But then $f(x) \in B_i$ and since $f(C) = \{y \in Y \mid x \in C, y = f(x)\}$, we have for all $y \in f(C)$, $y \in B_i$ for some i. Since every $B_i \in \mathcal{E}$ we have found a finite subcover of $\mathcal{E}$ which covers $f(C)$. Thus $f(C)$ is compact. $\square$

Theorem 21 *Let (X,d) be a metric space. Then every compact subset of X is bounded and closed.*

Proof. Let C be a compact subset of X and suppose that C is not bounded. Let $\mathcal{A}$ be the set of all balls with centers at elements of C. Then $\mathcal{A}$ covers C and since C is compact there exists a finite subcover $\mathcal{B} \subseteq \mathcal{A}$ which covers C. Then $\mathcal{B} = \{B(c_1, r_1), B(c_2, r_2), \ldots, B(c_n, r_n)\}$. Take the largest r_i such that $B(c, r_i) \in \mathcal{B}$. But we have C is not bounded so there exists $x \in C$ such that $d(x,c) > r_i$. Thus $C \nsubseteq \bigcup_{B\in\mathcal{B}} B$ and so $\mathcal{B}$ doesn't cover C. This is a contradiction and so compact sets are bounded.

Now suppose that $C \subseteq X$ is compact and C is not closed. Let $p \notin C$ be a limit point of C. Let $\mathcal{A} = \{X \backslash \overline{B(p,r)} \mid r \in \mathbb{R}\}$. Since $p \notin C$ we see that $\mathcal{A}$ covers C. Since C is compact, let $\mathcal{B}$ be a finite subset of $\mathcal{A}$ which covers C. We have X is open and $X \backslash \emptyset$ is closed so $X \neq \emptyset$. Thus if $\mathcal{B} = \emptyset$, $\mathcal{B}$ does not cover X.

Then $\mathcal{B} = \{X \backslash \overline{B(p, r_1)}, X \backslash \overline{B(p, r_2)}, \ldots, X \backslash \overline{B(p, r_n)}\}$. Take the smallest r_i such that $X \backslash \overline{B(p, r_i)} \in \mathcal{B}$ and consider $B(p, r_i/2)$. This ball contains p, which is a limit point of C, and since balls are open, $B(p, r_i/2) \cap C \neq \emptyset$. But $B(p, r_i/2)$ is defined such that $B(p, r_i/2) \nsubseteq \bigcup_{B \in \mathcal{B}} B$ and so $C \nsubseteq \bigcup_{B \in \mathcal{B}} B$. But then $\mathcal{B}$ doesn't cover C which is a contradiction. Therefore compact sets are closed. $\square$

Proposition 22 *Let X be an infinite set. Then there is a metric on X such that there exists a bounded and closed set that is not compact.*

Proof. Consider the metric $d(x, y) = a$ for $x \neq y$ and for some $a \in \mathbb{R}$ with $a > 0$. Let $Y \subseteq X$ be a bounded closed infinite set and let $\mathcal{A} = \{B(y, a) \mid y \in Y\}$. This set covers Y, but each element contains only one element of Y so a finite subset of $\mathcal{A}$ will only contain finitely many elements of Y. $\square$

Definition 23 *Let (X, d) and (X', d') be metric spaces and let $f : X \to X'$ be a function. We say that f is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$.*

Theorem 24 *Let (X, d) and (X', d') be metric spaces and let $f : X \to X'$ be a continuous function. If X is compact then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and consider $\varepsilon/2 > 0$. We have f is continuous so for all $x \in X$ there exists $\delta(x) > 0$ such that for all $y \in X$ with $d(x, y) < \delta(x)$ we have $d'(f(x), f(y)) < \varepsilon/2$ (16.19). Consider the set of balls $\mathcal{A} = \{B(x, \delta(x)) \mid x \in X\}$ and let $\mathcal{A}' = \{B(x, \delta(x)/2) \mid B(x, \delta(x)) \in \mathcal{A}\}$. $\mathcal{A}'$ is an open cover for X and since X is compact there exists a finite subcover, $\mathcal{B} \subseteq \mathcal{A}'$. Let $\delta = \min\{\delta(x)/2 \mid B(x, \delta(x)/2) \in \mathcal{B}\}$. Then consider two points $x, y \in X$ such that $d(x, y) < \delta$. $\mathcal{B}$ is an open cover for X so there exists some ball $B(z, \delta(z)/2) \in \mathcal{B}$ such that $x \in B(z, \delta(z)/2)$. Then $d(x, z) < \delta(z)/2 < \delta(z)$ and $d(x, y) < \delta \leq \delta(z)/2$ so $d(z, y) \leq d(z, x) + d(x, y) < \delta(z)$. But then $d'(f(z), f(x)) < \varepsilon/2$ and $d'(f(z), f(y)) < \varepsilon/2$ so $d'(f(x), f(y)) \leq d'(f(x), f(z)) + d'(f(z), f(y)) < \varepsilon$. Therefore for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$. $\square$