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MATH 16300
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Sheet 21: Derivatives

Definition 1 *A function f is differentiable at a if*

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

exists.

Definition 2 *The function f', called the derivative of f, is defined as the function whose domain is all a such that f is differentiable at a and whose value at a is*

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

The function $f''=(f')'$ is the second derivative of f. Similarly $f'''=(f'')'$. We denote $f^{(n)}$ as the nth derivative of f for $n\geq 4$.

Theorem 3 *If f is differentiable at a, then f is continuous at a.*

Proof. We have

$$\lim_{h\to 0} f(a+h)-f(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}h=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\lim_{h\to 0}h=f'(a)\cdot 0=0.$$

Thus $\lim_{h\to 0} f(a+h)=f(a)$ which means that f is continuous at a. □

Exercise 4 *Give and prove an example of a function that is continuous but not differentiable.*

Proof. Let $f(x)=|x|$ and consider $x=0$. Let $\varepsilon>0$ and let $\delta=\varepsilon$. Then if we have $|x|<\delta=\varepsilon$ we have $|f(x)|=||x||=|x|<\varepsilon$. Thus f is continuous at $x=0$. Then consider

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0^+}\frac{|h|}{h}=1$$

and

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0^-}\frac{|h|}{h}=-1$$

because $|h|\geq 0$. Since the left and right hand limits are not the same the limit does not exist and f is not differentiable at 0. □

Exercise 5 *If $g(x)=f(x+c)$ then $g'(x)=f'(x+c)$. Also if $g(x)=f(cx)$ then $g'(x)=cf'(cx)$.*

Proof. Both of these can be proved with the Chain rule. Let $h(x)=x+c$. Then $f(h(x))'=f'(h(x))h'(x)=f'(x+c)$ (21.16). If $h(x)=cx$. Then $f(h(x))'=f'(h(x))h'(x)=cf'(cx)$ (21.16). □

Exercise 6 *Let f be a function such that $|f(x)|\leq x^2$ for all x. Show that f is differentiable at 0.*

Proof. Note that $f(0)=0$ because $0\leq|f(0)|\leq 0^2=0$. We have $|f(h)/h|\leq|h^2/h|\leq|h|$ which means that $\lim_{h\to 0} f(h)/h=0$. Thus $f'(0)=0$. □

Theorem 7 *If $f(x) = c$ then $f'(x) = 0$.*

Proof. We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} \lim_{h\to 0} \frac{0}{h} = 0.$$

□

Theorem 8 *If $f(x) = ax + b$ then $f'(x) = a$.*

Proof. We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(a(x+h)+b) - (ax+b)}{h} \lim_{h\to 0} \frac{ax + ah + b - ax - b}{h} = \lim_{h\to 0} \frac{ah}{h} = \lim_{h\to 0} a = a.$$

□

Theorem 9 *If f and g are differentiable at a then $f + g$ is also differentiable at a and*

$$(f+g)'(a) = f'(a) + g'(a).$$

Proof. Since f and g are both differentiable at a we know

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

and

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$$

both exist. Then

$$\begin{aligned} f'(a) + g'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a) + g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} & = (f+g)'(a). \end{aligned}$$

We know this limit exists because the sum of the limits of two functions is the limit of their sum. □

Theorem 10 *If f and g are differentiable at a then*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof. Since f and g are both differentiable at a we know

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

and

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$$

both exist. Then $f(a)$ and $g(a)$ are both constants so

$$\begin{aligned}
f'(a)g(a) + f(a)g'(a) &= \lim_{h\to 0} g(a) \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h\to 0} f(a+h) \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a)g(a)}{h} + \lim_{h\to 0} \frac{g(a+h)f(a+h) - g(a)f(a+h)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a+h)g(a) + g(a+h)f(a+h) - g(a)f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&= (fg)'(a).
\end{aligned}$$

□

Theorem 11 *If $g(x) = cf(x)$ and f is differentiable at a then g is differentiable at a and*

$$g'(a) = cf'(a).$$

Proof. We have f is differentiable at a so

$$\begin{aligned}
cf'(a) &= c\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{cf(a+h)-cf(a)}{h} \\
&= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\
&= g'(a).
\end{aligned}$$

We know this limit exists because the limit of the product of two functions is the product of their limits. □

Theorem 12 *If $f(x) = x^n$ for some $n \in \mathbb{N}$ then*

$$f'(a) = na^{n-1}.$$

Proof. Note that for $n = 1$ we have $f'(a) = 1 \cdot a^0 = 1$ by Theorem 8 (21.8). Use induction on n and suppose that if $f(x) = x^n$ for $n \in \mathbb{N}$ we have $f'(a) = na^{n-1}$. Consider a function $f(x) = x^{n+1} = x \cdot x^n$. Then from Theorem 10 we have

$$f'(a) = x^n + x \cdot (nx^{n-1}) = x^n + nx^n = (n+1)x^n$$

as desired. □

Theorem 13 *If f is differentiable at a and $f(a) \neq 0$ then $1/f$ is differentiable at a and*

$$\left(\frac{1}{f}\right)'(a) = \frac{-f'(a)}{(f(a))^2}.$$

Proof. We have

$$\begin{aligned}
\left(\frac{1}{f}\right)'(a) &= \lim_{h\to 0}\frac{\frac{1}{f(a+h)}-\frac{1}{f(a)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{f(a)-f(a+h)}{f(a+h)f(a)}}{h}\\
&= \lim_{h\to 0}\frac{1}{f(a+h)f(a)}\frac{f(a)-f(a+h)}{h}\\
&= \lim_{h\to 0}\frac{1}{f(a+h)f(a)}\lim_{h\to 0}\frac{f(a)-f(a+h)}{h}\\
&= \frac{1}{(f(a))^2}\left(-\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right)\\
&= \frac{-f'(a)}{(f(a))^2}.
\end{aligned}$$

Note that $1/f$ is differentiable at a because of the product rules for limits and $f'(a)$ exists. $\square$

Corollary 14 *If f and g are differentiable at a and $g(a)\neq 0$ then f/g is differentiable at a and*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2}.$$

Proof. We have

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)'(a)\\
&= \frac{f'(a)}{g(a)} + \frac{-g'(a)f(a)}{(g(a))^2}\\
&= \frac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2}.
\end{aligned}$$

using Theorems 10 and 13 (21.10, 21.13). $\square$

Lemma 15 *Let g be continuous at a and let f be differentiable at $g(a)$. Let*

$$\phi(h) = \begin{cases} \frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)} & \text{if } g(a+h)-g(a)\neq 0\\ f'(g(a)) & \text{if } g(a+h)-g(a)=0. \end{cases}$$

Then $\phi(x)$ is continuous at 0.

Proof. Since $f'(g(a))$ exists we have

$$\lim_{k\to 0}\frac{f(g(a)+k)-f(g(a))}{k} = f'(g(a))$$

which means that for all $\varepsilon>0$ there exists $\delta_1>0$ such that if $0<|m|<\delta_1$ we have

$$\left|\frac{f(g(a)+m)-f(g(a))}{k} - f'(g(a))\right| < \varepsilon.$$

Since $g'(a)$ exists then g is continuous at a (21.3). Thus for all $\delta_1 > 0$ there exists $\delta_2 > 0$ such that for all h if $|h| < \delta_2$ we have $|g(a+h) - g(a)| < \delta_1$. Now let$|h| < \delta_2$. If $k = g(a+h) - g(a) \neq 0$ then we have

$$\phi(h) = \frac{f(g(a+h) - f(g(a))}{g(a+h) - g(a)} = \frac{f(g(a)+k) - f(g(a))}{k}.$$

We know from our second continuity statement that $|k| < \delta_1$ and from our first continuity statement that $|\phi(h) - f'(g(a))| < \varepsilon$. If $g(a+h) - g(a) = 0$ then $\phi(h) = f'(g(a)$ and so we have $0 = |\phi(h) - f'(g(a)| < \varepsilon$. Thus

$$\lim_{h \to 0} \phi(h) = f'(g(a))$$

which means ϕ is continuous at 0. $\square$

Theorem 16 (Chain Rule) *If g is differentiable at a and f is differentiable at $g(a)$ then $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. Use the function from Lemma 15 and note that if $h \neq 0$ we have

$$\frac{f(g(a+h)) - f(g(a))}{h} = \phi(h)\frac{g(a+h) - g(a)}{h}.$$

Then

$$(f \circ g)'(a) = \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h} = \lim_{h \to 0} \phi(h) \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = f'(g(a))g'(a)$$

which exists because $g'(a)$ exists and because of the product rules for limits. $\square$

Exercise 17 *Differentiate*

$$f(x) = \sin\left(\frac{x^3}{\cos(x^3)}\right).$$

Proof. Using the chain rule we have

$$f'(x) = \cos\left((x^3)(\cos x^3)^{-1}\right)\left(3(x^5)(\cos x^3)^{-2}(\sin x^3) + 3(x^2)(\cos x^3)^{-1}\right).$$

$\square$

Exercise 18 *Let a be a double root of the polynomial function f if $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Show that a is a double root of f if and only if a is a root of both f and f'.*

Proof. Let a be a double root of f. Then $f(x) = (x-a)^2 g(x)$ for some polynomial function g. Then $f(a) = (a-a)^2 g(a) = (0)g(x) = 0$ so a is a root of f. Also using the product and chain rules we have $f'(x) = (x-a)^2 g'(x) + 2(x-a)g(x) = (x-a)((x-a)g'(x) + 2g(x))$. Then $f'(a) = (a-a)((a-a)g'(a) + 2g(a)) = 0$ so a is a root of f'. Conversely assume that a is a root of both f and f'. Then $f(a) = f'(a) = 0$. Thus $f(x) = (x-a)g(x)$ for some polynomial function $g(x)$ and $f'(x) = (x-a)g'(x) + g(x)$. But since $f'(a) = 0$ we have $g(a) = 0$. Thus $g(a) = (x-a)h(x)$ for some polynomial function h. But then $f(x) = (x-a)^2 h(x)$. Therefore a is a double root of f. $\square$

Definition 19 *Let f be a function and A a set of numbers contained in the domain of f. A point $x \in A$ is a maximum point for f on A if $f(x) \geq f(y)$ for all $y \in A$. The number $f(x)$ itself is called the maximum value of f on A and we say that f has its maximum value on A at x.*

Theorem 20 *Let f be a function defined on $(a;b)$. If x is a maximum or minimum point for f on $(a;b)$ and f is differentiable at x then $f'(x) = 0$.*

Proof. Consider h such that $x + h \in (a; b)$. Then $f(x + h) - f(x) \leq 0$. If $h > 0$ then we have

$$\frac{f(x+h) - f(x)}{h} \leq 0$$

which means

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \leq 0.$$

If $h < 0$ then we have

$$\frac{f(x+h) - f(x)}{h} \geq 0$$

which means

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \geq 0.$$

Since f is differentiable at x these two limits must be equal to $f'(x)$ which means $0 \leq f'(x) \leq 0$ and so $f'(x) = 0$. If x is a minimum point for f on $(a; b)$ then consider $-f$ and we end up with the equality $0 \leq -f'(x) \leq 0$ as well. ☐

Definition 21 *Let f be a function and A a set of numbers contained in the domain of f. A point x in A is a local maximum or minimum point for f on A if there is some $\delta > 0$ such that x is a maximum or minimum point for f on $A \cap (x - \delta; x + \delta)$.*

Theorem 22 *Let f be a function defined on $(a; b)$. If x is a local maximum or local minimum point for f on $(a; b)$ and f is differentiable at x then $f'(x) = 0$.*

Proof. Let x be a local maximum or minimum for f on $(a; b)$ then there exists $\delta > 0$ such that x is a maximum or minimum for f on $(a; b) \cap (x - \delta; x + \delta)$. But this set is a subset of the domain of f and so $f'(x) = 0$ (21.20). ☐

Definition 23 *A critical point of a function f is a number x such that $f'(x) = 0$. The number $f(x)$ itself is called a critical value of f.*

Exercise 24 *Prove that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has at most $n - 1$ critical points.*

Proof. Taking the derivative of f we have $f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1$. This is a polynomial of degree $n - 1$ and so it must have at most $n - 1$ roots which means that $f'(x) = 0$ at at most $n - 1$ points (19.9). Thus f has at most $n - 1$ critical points. ☐

Theorem 25 (Rolle's Theorem) *If f is continuous on $[a; b]$, differentiable on $(a; b)$ and $f(a) = f(b)$ then there is some $x \in (a; b)$ such that $f'(x) = 0$.*

Proof. Since f is continuous on $[a; b]$ there exists $x_1, x_2 \in [a; b]$ such that $f(x_1) \geq f(x)$ and $f(x_2) \leq f(x)$ for all $x \in [a; b]$ (10.9). If $x_1 \in (a; b)$ or $x_2 \in (a; b)$ then we have a maximum or minimum point for f on $(a; b)$ in $(a; b)$. Thus $f'(x_1) = 0$ or $f'(x_2) = 0$ and we're done. If $x_1, x_2 \notin (a; b)$ then x_1 and x_2 are the values a and b, not necessarily respectively. Then since $f(a) = f(b)$ the maximum and minimum values of f are the same so f must be constant on $[a; b]$. Then $f'(x) = 0$ for all $x \in [a; b]$. ☐

Corollary 26 (Mean Value Theorem) *If f is continuous on $[a; b]$ and differentiable on $(a; b)$ then there exists some $x \in (a; b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let
$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$
Then $g(x)$ is continuous on $[a; b]$ and differentiable on $(a; b)$ and we have $g(a) = f(a)$, $g(b) = f(a) = g(a)$. Then we know that there exists some $x \in (a; b)$ such that
$$0 = g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$
from Rolle's Theorem (21.25). Thus we have
$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$
□

Exercise 27 *If f is defined on an interval and $f'(x) = 0$ for all x in the interval then f is constant on the interval.*

Proof. Consider two points a and b in the interval with $a \neq b$. We know that there exists $x \in (a; b)$ such that
$$0 = f'(x) = \frac{f(b) - f(a)}{b - a}$$
which means that $f(a) = f(b)$ (21.26). So for any two points in the interval the value of f is the same which means f is constant on the interval. □

Exercise 28 *If f and g are defined on the same interval and $f'(x) = g'(x)$ for all x in the interval then there is some number c such that $f = g + c$.*

Proof. For all x in the interval we have $f'(x) - g'(x) = (f - g)'(x) = 0$. Then we must have $(f - g)(x) = c$ for some constant c (21.27). Thus $f = g + c$. □

Definition 29 *A function is increasing on an interval if $f(a) < f(b)$ for all a and b in the interval with $a < b$. The function f is decreasing on an interval if $f(a) > f(b)$ for all a and b in the interval with $a < b$.*

Exercise 30 *If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval. If $f'(x) < 0$ for all x in the interval then f is decreasing on the interval.*

Proof. Let $f'(x) > 0$ for all x in the interval and let a and b be two points in the interval with $a < b$. Then there exists $x \in (a; b)$ such that
$$0 < f'(x) = \frac{f(b) - f(a)}{b - a}$$
and so $f(b) - f(a) > 0$ (21.26). But then $f(b) > f(a)$ and so f is increasing on the interval. A similar proof holds for decreasing f. □

Theorem 31 *Suppose $f'(a) = 0$. If $f''(a) > 0$ then f has a local minimum at a. If $f''(a) < 0$ then f has a local maximum at a.*

Proof. Suppose that $f''(a) > 0$. Since $f'(a) = 0$ we have
$$f''(a) = \lim_{h \to 0} \frac{f'(a + h)}{h} > 0.$$
Then $f'(a + h)/h > 0$ for small enough values of h. Thus for small values of $h > 0$ we have $f'(a + h) > 0$ which means f is increasing on an interval to the right of a. Similarly f is decreasing on an interval to the left of a. Then f must have a minimum at a. A similar proof holds for $f''(a) < 0$. □

Exercise 32 *Let $a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} = 0$. Show that the polynomial $p(x) = a + bx + cx^2 + dx^3 + ex^4$ has at least one real zero.*

Proof. Let $P(x) = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4 + \frac{e}{5}x^5$ and note that $P'(x) = p(x)$. Also note that $P(0) = P(1) = 0$. Then we know there exists some $x \in (0; 1)$ such that

$$p(x) = P'(x) = \frac{P(1) - P(0)}{1 - 0} = 0$$

from the Mean Value Theorem (21.26). □

Theorem 33 *Suppose that f is continuous at a and that $f'(a)$ exists for all x in some interval containing a, except perhaps for $x = a$. Suppose, moreover, that $\lim_{x\to a} f'(x)$ exists. Then $f'(a)$ also exists and*

$$f'(a) = \lim_{x\to a} f'(x).$$

Proof. Note that if $h > 0$ is small enough then f is continuous on $[a; a + h]$ and differentiable on $(a; a + h)$. We know there exists some value y such that

$$f'(y) = \frac{f(a + h) - f(a)}{h}$$

by the Mean Value Theorem (21.26). Note that y goes to a as h goes to 0 because $y \in (a; a + h)$. Then

$$f'(a) = \lim_{h\to 0^+} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0^+} f'(y) = \lim_{x\to a^+} f'(x).$$

If $h < 0$ is small enough then f is continuous on $[a + h; a]$ and differentiable on $(a + h; a)$. We know there exists some value y such that

$$f'(y) = \frac{f(a) - f(a + h)}{-h} = \frac{f(a + h) - f(a)}{h}$$

by the Mean Value Theorem (21.26). Note that y goes to a as h goes to 0 because $y \in (a; a + h)$. Then

$$f'(a) = \lim_{h\to 0^-} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0^-} f'(y) = \lim_{x\to a^-} f'(x).$$

Since the left and right hand limits are the same we must have

$$f'(a) = \lim_{x\to a} f'(x).$$

□

Theorem 34 (Cauchy Mean Value Theorem) *If f and g are continuous on $[a; b]$ and differentiable on $(a; b)$ then there exists $x \in (a; b)$ such that*

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

If $g(b) \neq g(a)$ and $g'(x) \neq 0$ this equation can be written

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x}{g'(x)}.$$

Proof. Let
$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$
Then h is continuous on $[a; b]$, differentiable on $(a; b)$ and $h(a) = h(b)$. Then $h'(x) = 0$ for some $x \in (a; b)$ (21.25). Thus
$$0 = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a))$$
giving the desired equality. $\square$

Theorem 35 (L'Hôpital's Rule) *Suppose that*
$$\lim_{x\to a} f(x) = 0,$$
$$\lim_{x\to a} g(x) = 0$$
and $\lim_{x\to a} f'(x)/g'(x)$ *exists. Then* $\lim_{x\to a} f(x)/g(x)$ *exists and*
$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x).$$

Proof. Note that $f(a)$ and $g(a)$ need not necessarily defined so let $f(a) = g(a) = 0$. Then f and g are continuous on $[a; x]$ and differentiable on $(a; x)$. Then there exists some $y \in (a; x)$ such that
$$(f(x) - f(a))g'(y) = (g(x) - g(a))f'(y)$$
which means
$$\frac{f(x)}{g(x)} = \frac{f'(y)}{g'(y)}$$
after using the Cauchy Mean Value Theorem on f and g (21.34). But then y goes to a as x goes to a because $y \in (a; x)$. Then we have
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(y)}{g'(y)} = \lim_{z\to a} \frac{f'(z)}{g'(z)}.$$
$\square$

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Sheet 30: Uniform Limits

Definition 1 *Let (f_n) be a sequence of functions defined on A and let f be defined on A. Then f is the uniform limit of (f_n) (or $\lim_{n\to\infty} f_n = f$) if for all $\varepsilon > 0$ there exists N such that for all $n > N$ and for all $x \in A$ we have $|f(x) - f_n(x)| < \varepsilon$.*

Theorem 2 *Let (f_n) be a sequence of continuous functions on $[a;b]$ that uniformly converges to f on $[a;b]$. Then f is continuous on $[a;b]$.*

Proof. Let $\varepsilon > 0$ and consider $\varepsilon/3$. We know (f_n) uniformly converges to f so there exists N such that for all $n > N$ and for all $x, y \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon/3$ and $|f(y) - f_n(y)| < \varepsilon/3$. Also f_n is continuous for all n so for all $n > N$ and for all $x \in [a;b]$ there exists $\delta_n > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta_n$ we have $|f_n(x) - f_n(y)| < \varepsilon/3$. Consider δ_{N+1}. Then for all $x \in [a;b]$ there exists $\delta_{N+1} > 0$, which may depend on x, such that for all $y \in [a;b]$ with $|x - y| < \delta_{N+1}$ we have $|f_{N+1}(x) + f_{N+1}(y)| < \varepsilon/3$. By the triangle inequality we have $|f(x) - f_{N+1}(y)| \leq |f_{N+1}(x) - f_{N+1}(y)| + |f(x) - f_{N+1}(x)| < 2\varepsilon/3$ and then $|f(x) - f(y)| < |f(x) - f_{N+1}(y)| + |f(y) - f_{N+1}(y)| < \varepsilon$. Thus for all $x \in [a;b]$ there exists some $\delta > 0$ such that for all $y \in [a;b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. Therefore f is continuous on $[a;b]$. □

Theorem 3 *Let (f_n) be a sequence of functions which are integrable on $[a;b]$ and that (f_n) uniformly converges to f on $[a;b]$, which is integrable on $[a;b]$. Then*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Let $\varepsilon > 0$. Since (f_n) uniformly converges to f on $[a;b]$, then there exists N such that for all $n > N$ and all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon/(b - a)$. Note that

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \left| \int_a^b f_n - f \right| < \int_a^b \frac{\varepsilon}{(b-a)} = \varepsilon$$

for all $n > N$ (22.14). Thus we have

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

□

Exercise 4 *Let (f_n) be a sequence of functions which are integrable on $[a;b]$ and that (f_n) uniformly converges to f on $[a;b]$. Is f integrable on $[a;b]$?*

Yes.

Proof. Let $\varepsilon > 0$. Since f_n is integrable on $[a;b]$ for all n we know there exists some partition $P = \{t_0, \ldots, t_n\}$ such that

$$U(f_n, P) - L(f_n, P) < \varepsilon.$$

Since (f_n) uniformly converges on $[a;b]$ there exists N such that for all $n > N$ and all $x \in [a;b]$ we have $|f(x) - f_n(x)| < \varepsilon$. Let

$$m_i = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

$$m_{i_n} = \inf\{f_n(x) \mid t_{i-1} \le x \le t_i\}$$
$$M_i = \sup\{f(x) \mid t_{i-1} \le x \le t_i\}.$$

and

$$M_{i_n} = \sup\{f_n(x) \mid t_{i-1} \le x \le t_i\}.$$

Then since $|f(x) - f_n(x)| < \varepsilon$ for all $n > N$ and all $x \in [a; b]$ then we have $|m_i - m_{i_n}| < \varepsilon/(3(b-a))$ for all $i \le i \le n$. Thus

$$|L(f,P) - L(f_n,P)| = \left|\sum_{i=1}^{n} m_i(t_i - t_{i-1}) - \sum_{i=1}^{n} m_{i_n}(t_i - t_{i-1})\right| = \left|\sum_{i=1}^{n}(m_i - m_{i_n})(t_i - t_{i_n})\right| < \varepsilon/3.$$

And a similar statement can be made to show $|U(f,P) - U(f_n,P)| < \varepsilon/3$. Also since

$$0 < U(f_n,P) - L(f_n,P) < \frac{\varepsilon}{3} < \varepsilon$$

we have

$$|U(f_n,P) - L(f_n,P)| < \varepsilon 3.$$

Combining the second of these inequalities with the last we have

$$|U(f,P) - L(f_n,P)| \le |U(f,P) - U(f_n,P)| + |U(f_n,P) - L(f_n,P)| < \frac{2\varepsilon}{3}$$

and then

$$|U(f,P) - L(f,P)| \le |U(f,P) - L(f_n,P)| + |L(f,P) - L(f_n,P)| < \varepsilon$$

and since $0 < U(f,P) - L(f,P)$ we have

$$U(f,P) - L(f,P) < \varepsilon$$

which means f is integrable on $[a; b]$. □

Exercise 5 *Find a sequence of differentiable functions that uniformly converge to $f(x) = |x|$ on $[-1; 1]$.*

Let

$$f(x) = \begin{cases} (-x)^{\frac{1+n}{n}} & \text{if } x < 0 \\ x^{\frac{1+n}{n}} & \text{if } x \ge 0. \end{cases}$$

Exercise 6 *Let*

$$f_n = \frac{1}{n}\sin(n^2 x).$$

Then f_n uniformly converges to $f = 0$ but $\lim_{n\to\infty} f_n'$ does not exist.

Proof. Let $\varepsilon > 0$. Note that $-1 \le \sin(n^2 x) \le 1$ for all n and all x. Then note that there exists some N such that $1/N < \varepsilon$. Thus, for all $n > N$ we have $|1/n| < \varepsilon$ and since $|\sin(n^2 x)| < 1$, for all $n > N$ we have

$$\left|\frac{1}{n}\sin(n^2 x)\right| < \varepsilon.$$

Thus we have

$$\lim_{n\to\infty} \frac{1}{n}\sin(n^2 x) = 0.$$

Now note that f_n' were to converge uniformly to some function f, then f is also the pointwise limit of (f_n') (19.7). We have $f_n' = 2\cos(n^2 x)$. Thus for $x = \pi/2$ we have $2\cos(n^2 x) = 0$ for even n and $2\cos(n^2) = 1$ for odd n. Then there are infinitely many n with $f_n'(\pi/2) = 0$ and likewise for 1 which means 0 and 1 are accumulations points for $(f_n'(\pi/2))$. Thus $\lim_{n\to\infty} f_n'(\pi/2)$ does not exist (13.10). □

Theorem 7 *Let (f_n) be a sequence of functions which are differentiable on $[a;b]$, with integrable derivatives f_n' and that (f_n) pointwise converges to f on $[a;b]$. Suppose that f_n' uniformly converges on $[a;b]$ to some continuous function g. Then f is differentiable on $[a;b]$ and for all $x \in [a;b]$ we have*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof. Since g is continuous we know it's integrable on $[a;b]$ (22.9). Also because (f_n) pointwise converges to f on $[a;b]$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a;b]$. Thus we have

$$\int_a^x g = \lim_{n\to\infty} \int_a^x f_n' = \lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a)$$

for all $x \in [a;b]$ by the Second Fundamental Theorem of Calculus and Theorem 3 (22.18, 30.3). If we let

$$G(x) = \int_a^x g$$

then $G'(x) = g(x)$ and so we have $G'(x) = (f(x) - f(a))' = f'(x)$ for all $x \in [a;b]$. Then it must be the case that $g = f'$ and so we have

$$f'(x) = g(x) = \lim_{n\to\infty} f_n'(x).$$

□

Definition 8 *The series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on A if the sequence of partial sums $s_n = \sum_{i=1}^{n} f_n$ converges to f uniformly.*

Theorem 9 *Let $\sum_{n=1}^{\infty} f_n$ converge uniformly to f on $[a;b]$. If f_n is continuous on $[a;b]$ for all n, then f is continuous on $[a;b]$. If f_n is integrable on $[a;b]$ for all n and f is integrable on $[a;b]$ then*

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n.$$

If f_n has an integrable derivative for all n and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a;b]$ to some continuous function then for all $x \in [a;b]$ we have

$$f'(x) = \sum n = 1^{\infty} f_n'(x).$$

Proof. Let f_n be continuous on $[a;b]$ for all n. Then since the sum of two continuous functions is still continuous, we have the partial sums of $\sum_{n=1}^{\infty} f_n$ are continuous. Thus (s_n) is a sequence of continuous functions on $[a;b]$ which uniformly converges to f on $[a;b]$. Thus f is continuous on $[a;b]$ (30.2).

Let f_n be integrable on $[a;b]$ for all n and f be integrable on $[a;b]$. Then since the sum of two integrable functions is still integrable, we have the partial sums, s_n are a sequence of integrable functions on $[a;b]$ (22.11). Thus we have

$$\sum_{n=1}^{\infty} \int_a^b f_n = \lim_{n\to\infty} \int_a^b s_n = \int_a^b f$$

from Theorem 3 (30.3).

Let f_n have an integrable derivative for all n and $\sum_{n=1}^{\infty} f_n'$ converge uniformly on $[a;b]$ to some continuous function then for all $x \in [a;b]$. By the same argument as before, since the sum of integrable functions is still integrable we have the partial sums of $\sum_{n=1}^{\infty} f_n'$ are integrable (22.11). Thus we have

$$f'(x) = \sum n = 1^{\infty} f_n'(x).$$

from Theorem 7 (30.7). □

Theorem 10 (Weierstrass M-Test) *Let (f_n) be a sequence of functions defined on A and suppose $|f_n|$ is bounded by M_n on A. Suppose that $\sum_{n=1}^{\infty} M_n$ converges. Then for all $x \in A$ the series $\sum_{n=1}^{\infty} f_n(x)$ absolutely converges and $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to the function*

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Proof. Let

$$M = \sum_{n=1}^{\infty} M_n.$$

Since for all n we have $|f_n| \leq M_n$, we have

$$\sum_{i=1}^{n} |f_n| \leq \sum_{i=1}^{n} M_n \leq M$$

for all n. But since $0 \leq |f_n|$, the series of partial sums of $\sum_{n=1}^{\infty} |f_n|$ is a bounded increasing sequence so it must converge. Thus $\sum_{n=1}^{\infty} f_n$ is absolutely convergent on A. Note that since an absolutely convergent series implies a convergent series we have

$$\sum_{i=1}^{n} f_n$$

is convergent. Then we can write

$$\left| \sum_{n=1}^{\infty} f_n - \sum_{n=1}^{k} f_n \right| = \left| \sum_{n=k+1}^{\infty} f_n \right| \leq \sum_{n=k+1}^{\infty} |f_n| \leq \sum_{n=k+1}^{\infty} M_n$$

and taking the limit as k goes to ∞ we see that

$$\lim_{k \to \infty} \left| \sum_{n=1}^{\infty} f_n - \sum_{n=1}^{k} f_n \right| = 0$$

so

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

□

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Sheet 31: Taylor Series

Definition 1 *A function of the form*
$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$
is called a power series centered at a.

Theorem 2 *Suppose that the series*
$$\sum_{n=0}^{\infty} a_n x_0^n$$
converges and let $0 < a < |x_0|$. Then on $B(0,a)$ the series
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
and
$$g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$
uniformly and absolutely converge. Also f is differentiable and
$$f'(x) = g(x)$$
for all $x \in B(0,a)$.

Proof. Note that for $x \in B(0,a)$ we have $|x/x_0| < 1$ and so
$$\sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n$$
is convergent since it's a geometric series. Then by the Comparison Criterion we have
$$\sum_{n=0}^{\infty} |a_n| \left| \frac{x}{x_0} \right|^n = \sum_{n=0}^{\infty} \left| a_n \frac{x^n}{x_0^n} \right|$$
is convergent and so
$$\sum_{n=0}^{\infty} |a_n x^n|$$
is convergent. A similar proof holds to show that $g(x)$ is absolutely convergent using the fact that $1/n$ converges to 0. Also we have $a_n x^n$ is bounded by $|a_n a^n|$ on $B(0,a)$ and $n a_n x^{n-1}$ is bounded by $|n a_n a^{n-1}|$ on $B(0,a)$ and since the series absolutely converge, we can use the Weierstrass M-test to show that f and g are uniformly convergent (30.10). Finally since $n a_n x^{n-1}$ is integrable on $[a;b]$, $n a_n x^{n-1}$ uniformly converges and $n a_n x^{n-1}$ is continuous so g is continuous, we have $f'(x) = g(x)$ for all $x \in B(0,a)$ (30.9). $\square$

Theorem 3 *For a power series* $\sum_{n=0}^{\infty} a_n x^n$ *let*

$$A = \left\{ x \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}$$

be the set of converge for the power series. Then either A *is everything or there exists* a *such that*

$$B(0,a) \subseteq A \subseteq \overline{B(0,a)}.$$

This a *is called the radius of convergence of the power series.*

Proof. Suppose that A is not everything. Then there exists $b \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} a_n b^n$ diverges. Note then that for all $x \in \mathbb{R}$ with $x \geq b$ we have $\sum_{n=1}^{\infty} a_n x^n$ diverges. Note also that $\sum_{n=1}^{\infty} a_n (0)^n$ converges. Then note that b is an upper bound for A and A is nonempty so let $a = \sup A$. Then we have $B(0,a) \subseteq A$. If we have $c > a$ then $\sum_{n=1}^{\infty} a_n c^n$ diverges so it must also be the case that $A \subseteq \overline{B(0,a)}$. $\square$

Exercise 4 *Find real power series centered at* 0 *with sets of convergence* 0, $\mathbb{R}$, $(-1;1)$, $[-1;1)$ *and* $[-1;1]$.

0:

$$\sum_{n=0}^{\infty} n! x^n.$$

$\mathbb{R}$:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$(-1;1)$:

$$\sum_{n=0}^{\infty} -x^{2n}.$$

$[-1;1)$:

$$\sum_{n=0}^{\infty} x^n.$$

$[-1;1]$:

$$\sum_{n=1}^{\infty} (-1)^n x^{2n}.$$

Theorem 5 *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge absolutely and* (c_n) *is a sequence containing the products* $a_i b_j$ *for each pair* (i,j) *then*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

Proof. Note that

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, we can rearrange the terms and they will still converge to the same thing. Then the partial sums of $\sum_{n=0}^{\infty} b_n$ can be rearranged in the same way as c_n so that the partials sums of $\sum_{n=0}^{\infty} c_n$ are just the product of the partial sums of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Then since the product of limits is the limit of products we have the desired relation. $\square$

Theorem 6 (Cauchy Product) *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *be the power series with radius of convergence at least* a. *Let*

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

Then the power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

has radius of convergence of at least a *and for* $x \in B(0,a)$ *we have*

$$h(x) = f(x)g(x).$$

Proof. We know that $f(x)$ and $g(x)$ are absolutely convergent on $B(0,a)$ (31.2). Also we know that $h(x)$ is uniformly and absolutely convergent on $B(0,a)$ because $f(x)$ and $g(x)$ are (31.2, 31.5). Also using Theorem 5 we know that for $x \in B(0,a)$ we have $h(x) = f(x)g(x)$. $\square$

Definition 7 *Let* f *be a real function such that* $f^{(n)}(a)$ *exists for all* n. *Then the Taylor series of* f *at* a *is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem 8 *For all real* x *we have*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Proof. Consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^n x^{2n}}{(2n)!}$$

and note that

$$f'(x) = \sum_{n=0}^{\infty} -\frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{(-1)^n x^{2n}}{(2n)!}$$

and

$$f''(x) = \sum_{n=0}^{\infty} -\frac{(-1)^n x^{2n+1}}{(2n+1)!} - \frac{(-1)^n x^{2n}}{(2n)!}.$$

Then we can easily verify $f + f'' = 0$, $f(0) = 1$ and $f'(0) = 1$. Then we must have $f = \cos + \sin$ (27.14). Then since $\sin' = \cos$ it must be the case that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Also we have $(e^x)' = e^x$ and $e^0 = 1$ so the Taylor series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

But note then that for all n, the remainder terms in the Taylor polynomial will converge to zero because of the $n!$ factor. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

□

Theorem 9 *For $x \in (-1; 1)$ we have*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Proof. We have $1/(1-x)$ is a geometric series (15.6). Also, using the Taylor polynomial definition we have the Taylor series for log is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

Note that for $x < 1$ we know this series converges so the remainder terms must go to zero. Thus

$$\log x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

□

Theorem 10 *Let $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ be a convergent sequence in $B(a, r)$ for some $r > 0$. Then the Taylor series of $f(x)$ at a equals $\sum_{n=0}^{\infty} a_n (x-a)^n$.*

Proof. Note that since

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

we have

$$f'(x) = f(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1}$$

and in general

$$f^{(j)}(x) = \sum n = 0^{\infty} \frac{n!}{(n-j)!} a_n (x-a)^{n-j}$$

using Theorem 2 (31.2). But then each term in $f^{(j)}(a)$ is zero unless $n = j$ in which case we have

$$f^{(j)}(a) = \frac{j!}{(j-j)!} a_j (a-a)^{j-j} = j! a_j (0)^0 = j! a_j$$

Thus $f^{(n)}(a) = n! a_n$. Using this in the Taylor Series definition we have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{n! a_n}{n!}(x-a)^n = \sum_{n=0}^{\infty} a_n (x-a)^n = f(x).$$

$\square$