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MATH 25800
Quiz 5
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Problem 1. *Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and x, considered as a submodule of R. Show that $\{2, x\}$ is not a basis of M. Show that the rank of M is 1 but that M is not free of rank 1.*

Proof. Note that $x(2) + (-2)(x) = 0$ but $x \neq 0$ and $-2 \neq 0$ so $\{2, x\}$ cannot be a basis for M since this set is linearly dependent. This shows that the rank of M must be less than 2. Since M contains a nonzero element, it must have rank 1. But M is not free of rank 1 since we know the ideal $(2, x)$ is not principal. $\square$

Problem 2. *Let R be a P.I.D., let B be a torsion R-module and let p be a prime in R. Prove that if $pb = 0$ for some nonzero $b \in B$, then $\operatorname{Ann}(B) \subseteq (p)$.*

Proof. Since $b \neq 0$, we can form the nontrivial R-module Rb and note that $Rb \subseteq B$. This means that $\operatorname{Ann}(B) \subseteq \operatorname{Ann}(Rb)$. But note that $(p) \subseteq \operatorname{Ann}(Rb)$ since $pb = 0$ and R is commutative. Since p is prime and R is a P.I.D., we know (p) is maximal. Thus either $\operatorname{Ann}(Rb) = (p)$ or $\operatorname{Ann}(Rb) = R$. Suppose the latter. Then for each $r \in R$ we have $rm = 0$ for all $m \in Rb$. In particular, we have $rb = 0$ for all $r \in R$. Then $Rb = 0$ and we trivially have $\operatorname{Ann}(B) \subseteq (p) \subseteq \operatorname{Ann}(Rb) = 0$. On the other hand, if $\operatorname{Ann}(Rb) = (p)$ then we have $\operatorname{Ann}(B) \subseteq \operatorname{Ann}(Rb) = (p)$ as desired. $\square$