Sheet 25: Complex Numbers

Definition 1 *A complex number is a an ordered pair of real numbers. The set of complex numbers is denoted by* $\mathbb{C}$.

Definition 2 *For* $z_1 = (a_1, b_1) \in \mathbb{C}$ *and* $z_2 = (a_2, b_2) \in \mathbb{C}$ *let*

$$z_1 + z_2 = (a_1 + a_2, b_1 + b_2)$$

and let

$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1).$$

Theorem 3 $\mathbb{C}$ *endowed with* $+$ *and* $\cdot$ *is a commutative ring.*

Proof. Let $z_1 = (a_1, b_1) \in \mathbb{C}$, $z_2 = (a_2, b_2) \in \mathbb{C}$ and $z_3 = (a_3, b_3) \in \mathbb{C}$. Then note that

$$z_1 + z_2 = (a_1 + a_2, b_1 + b_2) = (a_2 + a_1, b_2 + b_1) = z_2 + z_1.$$

Also

$$\begin{aligned}
(z_1 + z_2) + z_3 &= (a_1 + a_2, b_1 + b_2) + (a_3, b_3) \\
&= (a_1 + a_2 + a_3, b_1 + b_2 + b_3) \\
&= (a_1 + (a_2 + a_3), b_1 + (b_2 + b_3)) \\
&= (a_1, b_1) + (a_2 + a_3, b_2 + b_3) \\
&= z_1 + (z_2 + z_3).
\end{aligned}$$

Furthermore let $0 = (0, 0)$ so we have

$$z_1 + 0 = (a_1, b_1) + (0, 0) = (a_1 + 0, b_1 + 0) = (a_1, b_1) = z_1$$

Supposing there are two distinct 0s we have $0 = 0 + 0' = 0' + 0 = 0'$ which shows that 0 is unique. Letting $-z_1 = (-a_1, -b_1)$ we have

$$z_1 + -z_1 = (a_1, b_1) + (-a_1, -b_1) = (a_1 + -a_1, b_1 + -b_1) = (0, 0) = 0.$$

Supposing there are two distinct values of $-z_1$ we have $z_1 + (-z_1) = 0$ and $z_1 + (-z_1') = 0$. Then

$$\begin{aligned}
-z_1 &= -z_1 + 0 \\
&= -z_1 + (0, 0) \\
&= -z_1 + (a_1 - a_1, b_1 - b_1) \\
&= -z_1 + (a_1, b_1) + (-a_1, -b_1) \\
&= -z_1 + z_1 + (-a_1, -b_1) \\
&= -z_1' + z_1 + (-a_1, -b_1) \\
&= -z_1' + (a_1, b_1) + (-a_1, -b_1) \\
&= -z_1' + (a_1 - a_1, b_1 - b_1) \\
&= -z_1' + (0, 0) \\
&= -z_1' + 0 \\
&= -z_1'
\end{aligned}$$

So we have shown additive commutativity, associativity, identity and inverse. Now consider

$$z_1 \cdot z_2 = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) = (a_2a_1 - b_2b_1, a_2b_1 + a_1b_2) = z_2 \cdot z_1.$$

Also

$$\begin{aligned}(z_1 \cdot z_2) \cdot z_3 &= (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) \cdot (a_3, b_3) \\ &= (a_3(a_1a_2 - b_1b_2) - b_3(a_1b_2 + a_2b_1), b_3(a_1a_2 - b_1b_2) + a_3(a_1b_2 + a_2b_1)) \\ &= (a_1a_2a_3 - a_3b_1b_2 - a_1b_2b_3 - a_2b_1b_3, a_1a_2b_3 - b_1b_2b_3 + a_1a_3b_2 + a_2a_3b_1) \\ &= (a_1(a_2a_3 - b_2b_3) - b_1(a_2b_1 + a_3b_2), b_1(a_2a_3 - b_2b_3) + a_1(a_2b_3 + a_3b_2)) \\ &= (a_1, b_1) \cdot (a_2a_3 - b_2b_3, a_2b_3 + a_3b_2) \\ &= z_1 \cdot (z_2 \cdot z_3)\end{aligned}$$

Let $1 = (1, 0)$ so we have

$$z_1 \cdot 1 = (a_1, b_1) \cdot (1, 0) = (a_1 \cdot 1 - b_1 \cdot 0, a_1 \cdot 0 + b_1 \cdot 1) = (a_1, b_1) = z_1.$$

Supposing there are two distinct 1s we have $1 = 1 \cdot 1' = 1' \cdot 1 = 1'$ which shows that 1 is unique. Finally note that

$$\begin{aligned}z_1 \cdot (z_2 + z_3) &= (a_1, b_1) \cdot ((a_2, b_2) + (a_3, b_3)) \\ &= (a_1, b_1) \cdot (a_2 + a_3, b_2 + b_3) \\ &= (a_1(a_2 + a_3) - b_1(b_2 + b_3), a_1(b_2 + b_3) + b_1(a_2 + a_3)) \\ &= (a_1a_2 + a_1a_3 - b_1b_2 - b_1b_3, a_1b_2 + a_1b_3 + b_1a_2 + b_1a_3) \\ &= (a_1a_2 - b_1b_2 + a_1a_3 - b_1b_3, a_1b_2 + a_2b_1 + a_1b_3 + a_3b_1) \\ &= (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) + (a_1a_3 - b_1b_3, a_1b_3 + a_3b_1) \\ &= (a_1, b_1) \cdot (a_2, b_2) + (a_1, b_1) \cdot (a_3, b_3) \\ &= z_1 \cdot z_2 + z_1 \cdot z_3.\end{aligned}$$

Thus we have shown multiplicative commutativity, associativity and identity as well as distributivity. Therefore $\mathbb{C}$ is a commutative ring. ☐

Definition 4 *The imaginary number*

$$i = (0, 1).$$

Theorem 5 *Define $\varphi : \mathbb{R} \to \mathbb{C}$ be defined by $\phi(x) = (x, 0)$. Then φ is injective and for all $x, y \in \mathbb{R}$ we have $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x) \cdot \varphi(y)$.*

Proof. Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$. Then $\varphi(x_1) = (x_1, 0)$ and $\varphi(x_2) = (x_2, 0)$. But since $x_1 \neq x_2$ we have $(x_1, 0) \neq (x_2, 0)$ and so φ is injective. Now let $x, y \in \mathbb{R}$. Then we have

$$\varphi(x + y) = (x + y, 0) = (x + y, 0 + 0) = (x, 0) + (y, 0) = \varphi(x) + \varphi(y).$$

Also

$$\varphi(xy) = (xy, 0) = (xy - 0 \cdot 0, x \cdot 0 + y \cdot 0) = (x, 0) \cdot (y, 0) = \varphi(x) \cdot \varphi(y).$$

☐

Lemma 6 *We have*

$$i \cdot i = -1$$

Proof. We have

$$i \cdot i = (0,1) \cdot (0,1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 0 \cdot 1) = (-1,0) = \varphi(-1) = 1.$$

$\square$

Definition 7 *Let $z = (a,b)$ be a complex number. Then the real part*

$$\mathit{Re}\, z = a$$

and the imaginary part

$$\mathit{Im}\, z = b.$$

Lemma 8 *Let z be a complex number. Then we have*

$$z = \mathit{Re}\, z + i \cdot \mathit{Im}\, z.$$

Proof. Let $z = (a,b)$. We have

$$\begin{aligned}
z &= (a,b) \\
&= (a+0, 0+b) \\
&= (a,0) + (0,b) \\
&= \varphi(a) + (0 \cdot b - 1 \cdot 0, 0 \cdot 0 + b \cdot 1) \\
&= a + (0,1) \cdot (b,0) \\
&= a + i \cdot \varphi(b) \\
&= a + i \cdot b \\
&= \mathrm{Re}\, z + i \cdot \mathrm{Im}\, z.
\end{aligned}$$

$\square$

Definition 9 *Let $z = a + bi$ be a complex number. Then the complex conjugate of z is*

$$\overline{z} = a - bi.$$

Lemma 10 *For $0 \neq z \in \mathbb{C}$ we have*

$$z \frac{\overline{z}}{z\overline{z}} = 1.$$

That is,

$$z^{-1} = \frac{\overline{z}}{z\overline{z}}.$$

Proof. Let $z = a + bi$. We have

$$\frac{z\overline{z}}{z\overline{z}} = \frac{(a+bi)(a-bi)}{(a+bi)(a-bi)} = \frac{a^2+b^2}{a^2+b^2} = 1.$$

$\square$

Exercise 11 *Show that* $(1+i)/(2+3i) = (5-i)/13$.

Proof. We have

$$\frac{1+i}{2+3i} = \frac{(1+i)(2-3i)}{(2+3i)(2-3i)} = \frac{2-i+3}{13} = \frac{5-i}{13}.$$

$\square$

Definition 12 *For* $z \in \mathbb{C}$ *let the absolute value of* z *be*

$$|z| = \sqrt{z\overline{z}}.$$

Theorem 13 *Let* z *and* w *be complex numbers. Then the following hold:*
1) $\overline{\overline{z}} = z$;
2) $\overline{z} = z$ *if and only if* z *is real;*
3) $\overline{z+w} = \overline{z} + \overline{w}$;
4) $-\overline{z} = \overline{-z}$;
5) $\overline{zw} = \overline{z} \cdot \overline{w}$;
6) $\overline{z^{-1}} = \overline{z}^{-1}$ *if* $z \neq 0$;
7) $|z| = 0$ *if and only if* $z = 0$;
8) $|z+w| \leq |z| + |w|$;
9) $|zw| = |z||w|$.

Proof. Let $z = a + bi$ and $w = c + di$.

1) $\overline{\overline{z}} = \overline{a - bi} = a - (-bi) = a + bi = z$.

2) Let $\overline{z} = z$. Then $a - bi = a + bi$ which means $-b = b$ so $b = 0$. Thus z has no imaginary part and is real. Now suppose z is real. Then $b = 0$ so we have $\overline{z} = a = z$.

3) $\overline{z+w} = \overline{a + c + (b+d)i} = a + c - (b+d)i = a - bi + c - di = \overline{z} + \overline{w}$.

4) $-\overline{z} = -(a - bi) = (-a + bi) = \overline{-z}$.

5) $\overline{zw} = \overline{ac - bd + (ad + bc)i} = ac - bd - adi - bci = (a - bi) \cdot (c - di) = \overline{z} \cdot \overline{w}$.

6) Let $z \neq 0$. Then

$$\overline{z^{-1}} = \overline{\frac{\overline{z}}{z\overline{z}}} = \frac{\overline{a-bi}}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{bi}{a^2+b^2} = \frac{a+bi}{a^2+b^2} = \frac{z}{z\overline{z}} = \overline{z}^{-1}.$$

7) Let $|z| = 0$. Then $0 = \sqrt{z\overline{z}} = \sqrt{a^2+b^2}$ so $a^2 + b^2 = 0$ and since a^2 and b^2 are both greater than or equal to 0, they must both be 0. Then $a = b = 0$ so $z = 0$. Now suppose $z = 0$. Then $|z| = \sqrt{z\overline{z}} = \sqrt{a^2+b^2} = \sqrt{0} = 0$.

8) We have

$$b^2c^2 + a^2d^2 - 2abcd = (ad - bc)^2 \geq 0$$

so

$$b^2c^2 + a^2d^2 \geq 2abcd$$

and

$$(a^2+b^2)(c^2+d^2) = a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \geq a^2c^2 + 2abcd + b^2d^2 = (ac+bd)^2.$$

Then we have

$$2\sqrt{(a^2+b^2)(c^2+d^2)} \geq 2(ac+bd)$$

so

$$\begin{aligned}(|z|+|w|)^2 &= (\sqrt{a^2+b^2}+\sqrt{c^2+d^2})^2 \\ &= a^2+b^2+2\sqrt{(a^2+b^2)(c^2+d^2)}+c^2+d^2 \\ &\geq a^2+b^2+2(ac+bd)+c^2+d^2 \\ &= (a+c)^2+(b+d)^2 \\ &= |z+w|^2.\end{aligned}$$

Thus $|z|+|w| \geq |z+w|$.

9) We have

$$\begin{aligned}|zw| &= |(ac-bd)+(ad+bc)i| \\ &= \sqrt{(ac-bd)^2+(ad+bc)^2} \\ &= \sqrt{a^2c^2-2abcd+b^2d^2+a^2d^2+2abcd+b^2c^2} \\ &= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{a^2+b^2}\sqrt{c^2+d^2} \\ &= |z||w|.\end{aligned}$$

□

Definition 14 *Let $z \in \mathbb{C}$. A real number α satisfying the equality $z = |z|(\cos\alpha + i\sin\alpha)$ is called an argument of z.*

Theorem 15 *If α and β are arguments of z then $\alpha - \beta = 2k\pi$ for some $k \in \mathbb{Z}$.*

Proof. Let α and β be arguments of z. Then $z = |z|(\cos\alpha + i\sin\alpha) = |z|(\cos\beta + i\sin\beta)$ so $\cos\alpha + i\sin\alpha = \cos\beta + i\sin\beta$. Multiplying both sides by $\cos\beta - i\sin\beta$ we have

$$\begin{aligned}\cos(\alpha-\beta)+i\sin(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta+i(\sin\alpha\cos\beta-\sin\beta\cos\alpha) \\ &= \cos\alpha\cos\beta+i\sin\alpha\cos\beta-i\sin\beta\cos\alpha+\sin\alpha\sin\beta \\ &= (\cos\alpha+i\sin\alpha)(\cos\beta-i\sin\beta) \\ &= (\cos\beta+i\sin\beta)(\cos\beta-i\sin\beta) \\ &= \cos^2\beta+\sin^2\beta \\ &= 1.\end{aligned}$$

Thus $(\cos(\alpha-\beta)+i\sin(\alpha-\beta)) = 1$ which only occurs if $\cos(\alpha-\beta) = 1$ and $i\sin(\alpha-\beta) = 0$. Thus $\alpha - \beta = 2k\pi$ for $k \in \mathbb{Z}$. □

Theorem 16 *Let $z = |z|(\cos\alpha + i\sin\alpha)$ and $w = |w|(\cos\beta + i\sin\beta)$. Then*

$$zw = |z||w|(\cos(\alpha+\beta)+i\sin(\alpha+\beta))$$

Proof. We have

$$\begin{aligned} zw &= (|z|(\cos\alpha + i\sin\alpha))(|w|(\cos\beta + i\sin\beta)) \\ &= |z||w|(\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)) \\ &= |z||w|(\cos(\alpha+\beta) + i\sin(\alpha+\beta)). \end{aligned}$$

□

Corollary 17 *Let $z = |z|(\cos\alpha + i\sin\alpha)$. Then*

$$z^n = |z|^n(\cos n\alpha + i\sin n\alpha).$$

Proof. Note that for $n = 1$ we have $z^1 = z = |z|(\cos\alpha + i\sin\alpha) = |z|^1(\cos(1\cdot\alpha) + i\sin(1\cdot\alpha))$. Induct on n and assume that for $n \in \mathbb{N}$ we have $z^n = |z|^n(\cos n\alpha + i\sin n\alpha)$. Then from Theorem 16 we have

$$\begin{aligned} z^{n+1} &= z \cdot z^n \\ &= (|z|(\cos\alpha + i\sin\alpha))(|z|^n(\cos n\alpha + i\sin n\alpha)) \\ &= |z|^{n+1}(\cos(n+1)\alpha + i\sin(n+1)\alpha) \end{aligned}$$

as desired (25.17). □

Definition 18 *A complex number z is an nth root of unity if it satisfies*

$$z^n = 1.$$

Theorem 19 *Let n be a natural number. Then there are exactly n nth roots of unity, namely*

$$\varepsilon_{n,k} = \cos\left(k\frac{2\pi}{n}\right) + i\sin\left(k\frac{2\pi}{n}\right) \text{ for } 0 \le k \le n-1.$$

Proof. Note that from Theorem 21 we know that there are at most n roots of the polynomial $x^n - 1$ which means there are at most n solutions to the equation $x^n = 1$. From Corollary 17 we know

$$\varepsilon_{n,k}^n = \cos(k2\pi) + i\sin(k2\pi) = 1$$

where $0 \le k \le n-1$ (25.17). Suppose that there are two values of k, $k_1 \neq k_2$, such that $\varepsilon_{n,k_1} = \varepsilon_{n,k_2}$. Then we have

$$\cos\left(k_1\frac{2\pi}{n}\right) + i\sin\left(k_1\frac{2\pi}{n}\right) = \cos\left(k_2\frac{2\pi}{n}\right) + i\sin\left(k_2\frac{2\pi}{n}\right).$$

But then $(2k_1\pi)/n - (2k_2\pi)/n = 2m\pi$ for some $m \in \mathbb{Z}$ (25.15). Thus $k_1 - k_2 = mn$ and $k_2 - k_1 = -mn$. Note that $m \neq 0$ because $k_1 \neq k_2$ and so the positive difference between k_1 and k_2 must be greater than or equal to n. But we have $0 \le k_1, k_2 \le n-1$, so $k_i - k_j < n$ for all values of k. Thus for distinct values of k we have distinct values of $\varepsilon_{n,k}$. Therefore there are least and at most n nth roots of unity which means there are n nth roots of unity. □

Theorem 20 *Let $0 \neq z \in \mathbb{C}$ and let $n \in \mathbb{N}$. Then there are exactly n complex numbers satisfying the equality*

$$w^n = z.$$

Proof. Let $z = |z|(\cos(\alpha) + i\sin(\alpha))$. Then from Theorem 21 we know that there are at most n values satisfying $w^n = z$. Consider

$$w = |z|^{1/n}\left(\cos\left(\frac{\alpha + 2k\pi}{n}\right) + i\sin\left(\frac{\alpha + 2k\pi}{n}\right)\right).$$

Then

$$\begin{aligned} w^n &= |z|(\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi)) \\ &= |z|(\cos(\alpha) + i\sin(\alpha)) \\ &= z. \end{aligned}$$

from Corollary 17 (25.17). We know that each of the values $0 \le k \le n-1$ is distinct using a similar argument as in Theorem 19. $\square$

Theorem 21 *Let $p(x) \in \mathbb{C}[x]$ be a complex polynomial of degree n. Then $p(x)$ has at most n roots.*

Proof. Suppose that $\deg(p) = n$ and p has m distinct roots with $m > n$. Let the m roots be $\alpha_1, \alpha_2, \ldots, \alpha_m$. In the case where $\alpha_i = \alpha_j$ for all $1 \le i, j < m$ we have $p(x) = (x - \alpha_1)^m$ which has degree higher than n. Thus we can assume that there exists two α_i and α_j such that $\alpha_i \neq \alpha_j$ and $i \neq j$. We know that $p = (x - \alpha_i)q_i$ for some $q \in \mathbb{C}[x]$ (19.8). We also know that since α_j is a root of p it is a root of $(x - \alpha_i)$ or q_i (19.7). Since $\alpha_j - \alpha_i \neq 0$, α_j is a root of q_i. Thus $q_i = (x - \alpha_j)q_j$ and $p = (x - \alpha_i)(x - \alpha_j)q_j$ (19.8). We can continue in this process m times until we have

$$p = \prod_{i=1}^{m}(x - \alpha_i)q_m.$$

But then $\deg(p) = m \neq n$ which is a contradiction. $\square$

Exercise 22 *Where is the mistake in the following?*

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = -1.$$

The square root function is only defined for non-negative real numbers. It makes no sense to say $\sqrt{-1 \cdot -1} = \sqrt{-1} \cdot \sqrt{-1}$ because $\sqrt{-1}$ is meaningless.

Exercise 23 *Let u, w be complex numbers. Find the complex numbers z such that u, w, z form a equilateral triangle. Express the centers of these triangles.*

Proof. Given the three points u, w, z, the centroid of the triangle formed by them should be

$$x = \frac{u + w + z}{3}.$$

Given this and the two points u and w we want the condition each of u w and z are a distance L from the center, x, and are separated by an angle of $2\pi/3$. Thus

$$u - x = L(\cos(\alpha) + i\sin(\alpha)),$$

$$z - x = L\left(\cos\left(\alpha - \frac{2\pi}{3}\right) + i\sin\left(\alpha - \frac{2\pi}{3}\right)\right)$$

and

$$w - x = L\left(\cos\left(\alpha + \frac{2\pi}{3}\right) + i\sin\left(\alpha + \frac{2\pi}{3}\right)\right)$$

for some angle α. This implies that $(u-x)(w-x) = (z-x)^2$ which after substituting for x and expanding gives us

$$u^2 + w^2 + z^2 = uw + uz + wz.$$

Using the quadratic formula to solve for z we end up with

$$z = \frac{u + w \pm i\sqrt{3}(u-w)}{2}.$$

The center of the triangle is then at

$$\frac{u+w}{2} \pm \frac{i\sqrt{3}(u-w)}{6}.$$

□

Exercise 24 *Take an arbitrary and draw an equilateral triangle on all sides looking outward. Prove that the centers of these triangles forms an equilateral triangle.*

Proof. Let a, b and c be vertices of an equilateral triangle and x, y and z be the centers of the outer equilateral triangles formed. Then

$$x = \frac{a+b}{2} \pm \frac{i\sqrt{3}(a-b)}{6},$$

$$y = \frac{b+c}{2} \pm \frac{i\sqrt{3}(b-c)}{6}$$

and

$$z = \frac{c+a}{2} \pm \frac{i\sqrt{3}(c-a)}{6}.$$

Then we can verify that

$$x^2 + y^2 + z^2 = xy + yz + xz$$

which is the condition we had earlier for an equilateral triangle. □

Exercise 25 *Compute* $(1+i)^{2006}$.

Let $z = 1+i$. Note that $|z| = \sqrt{z\bar{z}} = \sqrt{2}$. Then let $\alpha = \pi/4$ so that

$$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = |z|(\cos\alpha + i\sin\alpha).$$

Then

$$z^{2006} = \sqrt{2}^{2006}\left(\cos\left(\frac{1003\pi}{2}\right) + i\sin\left(\frac{1003\pi}{2}\right)\right) = -i2^{1003}$$

Exercise 26 *What is the sum of the nth roots of unity?*

Proof. Note that the kth root of unity is given by

$$\varepsilon_{n,k} = \cos\left(k\frac{2\pi}{n}\right) + i\sin\left(k\frac{2\pi}{n}\right).$$

Let $n > 1$ and let $k = 1$. Then

$$\varepsilon_{n,1} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) \neq 1$$

and the arguments of $\varepsilon_{n,k}$ are $(2\pi)/n$. But then

$$\begin{aligned}\varepsilon_{n,1}^{k} &= |\varepsilon_{n,1}|\left(\cos\left(k\frac{2\pi}{n}\right)+i\sin\left(k\frac{2\pi}{n}\right)\right)\\ &= \cos\left(k\frac{2\pi}{n}\right)+i\sin\left(k\frac{2\pi}{n}\right)\\ &= \varepsilon_{n,k}\end{aligned}$$

by Corollary 17 (25.17). Thus if we have one nontrivial root of unity we can find the rest by taking powers of the first for powers $0 \leq k \leq n-1$. But then

$$\sum_{k=0}^{n-1}\varepsilon_{n,k} = \sum_{k=0}^{n-1}\varepsilon_{n,1}^{k} = \frac{1-\varepsilon_{n,1}^{n}}{1-\varepsilon_{n,1}} = 0$$

because $\varepsilon_{n,1} = 1$. □

Exercise 27 *What is the product of the nth roots of unity?*

Proof. Similarly

$$\prod_{k=0}^{n-1}\varepsilon_{n,k} = \prod_{k=0}^{n-1}\varepsilon_{n,1}^{k} = \varepsilon_{n,1}^{\frac{n(n-1)}{2}}.$$

For n odd we can write this as

$$\left(\varepsilon_{n,1}^{n}\right)^{\frac{n-1}{2}} = 1.$$

For n even we can write

$$\left(\varepsilon_{n,1}^{\frac{n}{2}}\right)^{n-1}.$$

Note that

$$\varepsilon_{n,1}^{\frac{n}{2}} = |\varepsilon_{n,1}|^{\frac{n}{2}}\left(\cos\left(\frac{n}{2}\frac{2\pi}{n}\right)+i\sin\left(\frac{n}{2}\frac{2\pi}{n}\right)\right) = \cos\pi + i\sin\pi = -1.$$

Thus we have -1^{n-1} and since n is even this is -1. Therefore the product of the nth roots of unity is 1 for n odd and -1 for n even. □

Exercise 28 *What is the sum of the squares of the nth roots of unity?*

Proof. We have

$$\sum_{k=0}^{n-1}\varepsilon_{n,k}^{2} = \sum_{k=0}^{n-1}\varepsilon_{n,1}^{2k} = \varepsilon_{n,1}^{0}+\varepsilon_{n,1}^{2}+\cdots+\varepsilon_{n,1}^{2n-2}.$$

If we multiply both sides of this equation by $1-\varepsilon_{n,1}^{2}$ we have

$$\sum_{k=0}^{n-1}\varepsilon_{n,1}^{2k} = \frac{1-\varepsilon_{n,1}^{2n}}{1-\varepsilon_{n,1}^{2}} = \frac{1-\left(\varepsilon_{n,1}^{n}\right)^{2}}{1-\varepsilon_{n,1}} = 0.$$

□