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Homework 7

Problem 1. *Let f be a measurable function and $\int_X f d\mu = 0$. Then $f = 0$ almost everywhere.*

Proof. Suppose that $f \neq 0$ on a set A such that $\mu(A) \neq 0$. If f has a minimum on A, then take the characteristic function on A, χ_A. If f has no minimum on A, then we can take a subset $B \subseteq A$ which is closed and bounded such that f has a minimum on B. Then we can assume that f has a minimum on A. The function χ_A is a simple function, and given a scaling factor $\alpha \neq 0$, we have $\alpha\chi_A \leq f$ on A. But then

$$0 < \alpha\mu(A) \leq \int_A \alpha\chi_A d\mu \leq \int_A f d\mu \leq \int_X f d\mu.$$

This is a contradiction and so $f = 0$ almost everywhere. □

Problem 2. *Let f be measurable, $\mu(X) < \infty$ and f^q is integrable for $q > 0$. Show f^p is integrable if $0 \leq p \leq q$.*

Proof. Since f^q is integrable, we know that $|f|^q$ is integrable. It follows that since $|f|^p \leq |f|^q$ that $|f|^p$ is integrable, and thus f^p is integrable. □

Problem 3. *Let f be a non-decreasing function on $[0, 1]$. Show for all $t \in [0, 1]$ and for all $A \subseteq [0, 1]$ with $m(A) = t$,*

$$\int_{[0,t]} f dx \leq \int_A f dx.$$

Proof. We can take f to be positive by adding an appropriate constant. Note that since f is non-decreasing, if $A \backslash [0, t] \neq \emptyset$ then $\sup_{x \in [0,t]} f(x) \leq \sup_{x \in A} f(x)$. Then there exist simple functions s and s' such that $s \leq s'$ and $s \leq f$ on $[0, t]$ and $s' \leq f$ on A. Taking the supremum over these simple functions we have

$$\int_{[0,t]} f dx = \int_{[0,t]} \sup_{s \leq f} s dx \leq \int_A \sup_{s' \leq f} s' dx = \int_A f dx.$$

□

Problem 4. *Let f be integrable on X and $f > 0$ on X. Show*

$$\lim_{n \to \infty} \int_X f^{\frac{1}{n}} d\mu = \mu(X).$$

Proof. We know that f is integrable and that $|f^{1/n}| \leq f$ almost everywhere on X. Then by the dominated convergence theorem we have

$$\lim_{n \to \infty} \int_X f^{\frac{1}{n}} d\mu = \int_X \lim_{n \to \infty} f^{\frac{1}{n}} d\mu = \int_X d\mu = \mu(X).$$

□

Problem 5. *Let f be integrable on $\mathbb{R}$ and $p > 0$. Show*

$$\lim_{n \to \infty} n^{-p} f(nx) = 0$$

almost everywhere.

Proof. Note that $(n^{-p}f(nx))$ is a sequence of measurable functions. Moreover, since $n^{-p} < 1$ we have $|n^{-p}f(nx)| \leq |f(nx)| \leq Mf(x)$ for some large M. Then using the dominated convergence theorem we have

$$\int_X \lim_{n\to\infty} n^{-p}f(nx)d\mu = \lim_{n\to\infty} n^{-p}\int_X f(nx)d\mu = \lim_{n\to\infty} n^{-p-1}\int_X f(x)d\mu = 0.$$

Thus by Problem 1, we know that $\lim_{n\to\infty} n^{-p}f(nx) = 0$ almost everywhere. □

Problem 6. *Suppose (f_n) is a sequence of measurable functions and g is integrable. Suppose $f_n \geq g$ for all n almost everywhere. Then*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty}\int_X f_n d\mu.$$

Proof. Create a new sequence of functions $h_n = f_n - g$. Then (h_n) is a sequence of nonnegative measurable functions and so Fatou's Lemma holds. Then since g is independent of n in this sequence we have

$$\begin{aligned}
\int_X \liminf_{n\to\infty} f_n d\mu - \int_X g d\mu &= \int_X \liminf_{n\to\infty}(f_n - g)d\mu \\
&= \int_X \liminf_{n\to\infty} h_n d\mu \\
&\leq \liminf_{n\to\infty}\int_X h_n d\mu \\
&= \liminf_{n\to\infty}\int_X (f_n - g)d\mu \\
&= \liminf_{n\to\infty}\int_X f_n d\mu - \int_X g d\mu.
\end{aligned}$$

The result follows by adding $\int_X g d\mu$ to each side. □

Problem 7. *Suppose f_n converges to f uniformly, and f_n is integrable for all n.*
1) If $\mu(X) < \infty$, show f is integrable and $\int_X f_n d\mu$ converges to $\int_X f d\mu$.

Proof. Since $\mu(X) < \infty$ and since f_n is integrable, we know that f must be bounded because of uniform convergence. Then the bounded convergence theorem applies and so

$$\int_X f d\mu \int_X \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty}\int_X f_n d\mu.$$

□

2) If $\mu(X) = \infty$ show Part 1) is false.

Proof. Let $f_n = 1/n$. Then $\int_X f_n d\mu$ does not exist, as it's constantly infinite. But (f_n) converges to the zero function uniformly and $\int_X f d\mu = 0$ where $f = 0$. □

Problem 8. *Let $f \in L^p(X)$, then for all $\alpha > 0$, if $1 \leq p \leq \infty$ we have*

$$\mu(\{x \in X \mid |f(x)| \geq \alpha\}) \leq \left(\frac{||f||_p}{\alpha}\right)^p.$$

Proof. Define the set $A_\alpha = \{x \in X \mid f(x) \geq \alpha\}$. Then we have

$$0 \leq \alpha^p \chi_{A_\alpha} \leq f^p \chi_{A_\alpha} \leq f^p$$

and it follows that

$$\alpha^p \mu(A_\alpha) = \int_X \alpha^p \chi_{A_\alpha} d\mu \leq \int_{A_t} f^p d\mu \leq \int_X f^p d\mu = ||f||_p^p.$$

Dividing by α^p gives the result. □

Problem 9. *If $f \in L^1(X) \cap L^2(X)$ then*

$$\lim_{p \to 1^+} \int_X |f|^p d\mu = \int_X |f| d\mu.$$

Proof. Note that since $f \in L^2(X)$, $f \in L^q(X)$ for $1 \leq q \leq 2$ by Problem 2). Let $p = 1/n + 1$. Then as p approaches 1, n approaches infinity. Thus we have

$$\lim_{p \to 1^+} \int_X |f|^p d\mu = \lim_{n \to \infty} \int_X |f|^{1+\frac{1}{n}} d\mu$$

Since $|f|^{1/n}|f| \leq |f|^2$ for all n, we use the dominated convergence theorem and

$$\lim_{p \to 1^+} \int_X |f|^p d\mu = \int_X \lim_{n \to \infty} |f|^{1+\frac{1}{n}} d\mu = \int_X |f|.$$

$\square$

Problem 10. *If $\mu(X) < \infty$ and $0 \leq p_1 \leq p_2 \leq \infty$ then $L^{p_2}(X) \subseteq L^{p_1}(X)$.*

Proof. Let $f \in L^{p_2}(X)$. Then $\int_X |f|^{p_2} d\mu < \infty$. The result follows from Problem 2 and Hölder's Inequality. $\square$

Problem 11. *If $0 < r < p < s \leq \infty$ and $f \in L^r(X) \cap L^s(X)$ then $f \in L^p(X)$ and*

$$||f||_p \leq ||f||_r^{\lambda} ||f||_s^{1-\lambda}$$

where

$$\frac{1}{p} = \frac{\lambda}{r} + \frac{1-\lambda}{s}.$$

Proof. We use Hölder's inequality. We can choose $r' = p\lambda/r$ and $s' = p(1-\lambda)/s$ so that $||f||_1 \leq ||f||_{r'} ||f||_{s'}$. Then this inequality can be modified, by taking powers of λ so that we obtain $||f||_p \leq ||f||_r^{\lambda} ||f||_s^{1-\lambda} < \infty$. This shows that $f \in L^p$. $\square$