Assignment

EE23010: Probability and Random Processes
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Question: Let X be a random variable with probability density function

$$f(x;\lambda) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} & \text{if} x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\lambda > 0$ is an unknown parameter. Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a population having the same distribution as X^2.If

$$\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \tag{2}$$

then which of the following statements is true?

1) $\sqrt{\frac{\bar{Y}}{2}}$is a method of moments estimator ofλ
2) $\sqrt{\bar{Y}}$is a method of moments estimator ofλ
3) $\frac{1}{2}\sqrt{\bar{Y}}$is a method of moments estimator of λ
4) $2\sqrt{\bar{Y}}$is a method of moments estimator ofλ

Solution:
For random variables $Y_1, ..., Y_n$ chosen according to the probability distribution as X^2 for the parameter value λ and m a real valued function, then

$$\frac{1}{n} \sum_{i=1}^{n} m(Y_i) \to k(\theta), n \to \theta \tag{3}$$

The method of moments results from the choices $m(x) = x^m$,then

$$\mu_m = EY^m = k_m(\theta) \tag{4}$$

for m^{th} moment.
Now, PDF of X is :

$$f(x;\lambda) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} & \text{if} x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

For finding method of moment estimator we will find the PDF of X^2,denoted as $f_{X^2}(y)$.For $Y = X^2$, CDF of Y:

$$F_Y(y) = P(X^2 \leq y) \tag{6}$$
$$= P(0 \leq X \leq \sqrt{y}) \tag{7}$$

Now,

$$F_Y(y) = \int_0^{\sqrt{y}} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx \tag{8}$$
$$= 1 - e^{-\frac{\sqrt{y}}{\lambda}} \tag{9}$$

Differentiating to find PDF for Y:

$$f_{X^2}(y) = \frac{d(1 - e^{-\frac{\sqrt{y}}{\lambda}})}{dy} \tag{10}$$
$$= \frac{1}{2\lambda\sqrt{y}} e^{-\frac{\sqrt{y}}{\lambda}} \tag{11}$$

Now, we want to estimate the parameter λ, for which we can use method of moment estimator $(\hat{\lambda})$.
Defining Population moment

$$\mu_1(\lambda) = E(y) \tag{12}$$
$$= \int_0^{\infty} y f_{X^2}(y) dy \tag{13}$$
$$= \int_0^{\infty} \frac{1}{2\lambda} \sqrt{y} e^{-\frac{\sqrt{y}}{\lambda}} dy \tag{14}$$
$$= 2\lambda^2 \tag{15}$$

Defining sample moment as:

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} Y_i \tag{16}$$

Now, equating Population and sample moment:

$$M_1 = \mu_1(\lambda) \tag{17}$$

$$\bar{Y} = 2(\lambda^2) \tag{18}$$

$$\hat{\lambda} = \sqrt{\frac{\bar{Y}}{2}} \tag{19}$$

$\therefore$ Option (1) is correct.