Assignment

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I. Question:- GATE 2023.54

Suppose that X is a discrete random variable with the following probability mass

$$P(X = 0) = \frac{1}{2}\left(1 + e^{-1}\right) \tag{1}$$

$$P(X = k) = \frac{e^{-1}}{2k!} \; for \; k = 1, 2, 3, \ldots . \tag{2}$$

Which of the following is/are true?

1) $E(X) = 1$
2) $E(X) < 1$
3) $E(X|X > 0) < \frac{1}{2}$
4) $E(X|X > 0) > \frac{1}{2}$

Solution:

1) **Theory:**
   a) As we know,

$$E(X) = \sum k p_X(k) \tag{3}$$

Therefore,

$$E(X) = 0 \cdot \frac{1}{2}\left(1 + e^{-1}\right) + \sum_{k=1}^{\infty} \frac{k e^{-1}}{2k!} \tag{4}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-1}}{2(k-1)!} \tag{5}$$

$$= \frac{1}{2e} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \tag{6}$$

$$= \frac{1}{2e} \cdot e \qquad \text{(Using standard result of exponential series)}$$

$$= \frac{1}{2} \tag{7}$$

b) To find $E(X|X>0)$, first we need to find $\Pr(X|X>0)$ which can be given as:

$$\Pr(X|X>0) = \frac{\Pr(X=k)}{1-\Pr(X=0)} \tag{8}$$
$$= \frac{e^{-1}}{2k!}\cdot\frac{1}{(1-\frac{1}{2}(1+e^{-1}))} \tag{9}$$
$$= \frac{e^{-1}}{2k!}\cdot\frac{2}{(1-e^{-1})} \tag{10}$$
$$= \frac{e^{-1}}{k!(1-e^{-1})} \tag{11}$$
$$= \frac{1}{k!(e-1)} \tag{12}$$

Therefore,

$$E(X|X>0) = \sum_{k=1}^{\infty} k\frac{1}{k!(e-1)} \tag{13}$$
$$= \frac{1}{e-1}\sum_{k=1}^{\infty}\frac{1}{(k-1)!} \tag{14}$$
$$= \frac{1}{e-1}\cdot e \qquad \text{(Using standard result of exponential series)}$$
$$= 1.582 \tag{15}$$

Refering to equations (7) and (15), we get that option (2) and (4) are correct.

2) **Simulation:**

To make the simulation of the given question, generate a large set of random variables, say X, with the probability:

$$P(X=0) = \frac{1}{2}\left(1+e^{-1}\right) \tag{16}$$
$$P(X=k) = \frac{e^{-1}}{2k!}\ for\ k = 1,2,3,\ldots. \tag{17}$$

In the simulation process, we use the concept of Inverse Transform Sampling. This involves generating a uniform random variable U from the range $[0,1]$ and then inverting the generalized form of CDF, i.e. F_X, to obtain X. The inversion is done using the formula:

$$X = {F_X}^{-1}(U) \tag{18}$$

Since, this distribution is discrete, we use following method to find the discrete random variable:

$$\sum_{j=0}^{k-1} p_X(j) \le U < \sum_{j=0}^{k} p_X(j) \tag{19}$$

In simpler terms, you're using U to navigate through the distribution's cumulative probabilities to find the corresponding value of X .Using this method, random number is compared to cumulative probabilities (CDF) for various values of the random variable k until the CDF exceeds it. This process continues until a match is found and k is returned as the generated random variable.

a) To get $E(X)$, take the weighted sum of all possible values of X including 0, each multiplied by its respective probability. Expression for this can be given as:

$$E(X) = \sum k p_X(k) \tag{20}$$

b) As given in theory, to get $E(X|X > 0)$, take the weighted sum of all possible values of X excluding 0, each multiplied by its respective probability and dividing it by the probability of getting random variables greater than 0. Expression for this can be given as:

$$\Pr(X|X > 0) = \frac{\Pr(X = k)}{1 - \Pr(X = 0)} \tag{21}$$

$$\tag{22}$$

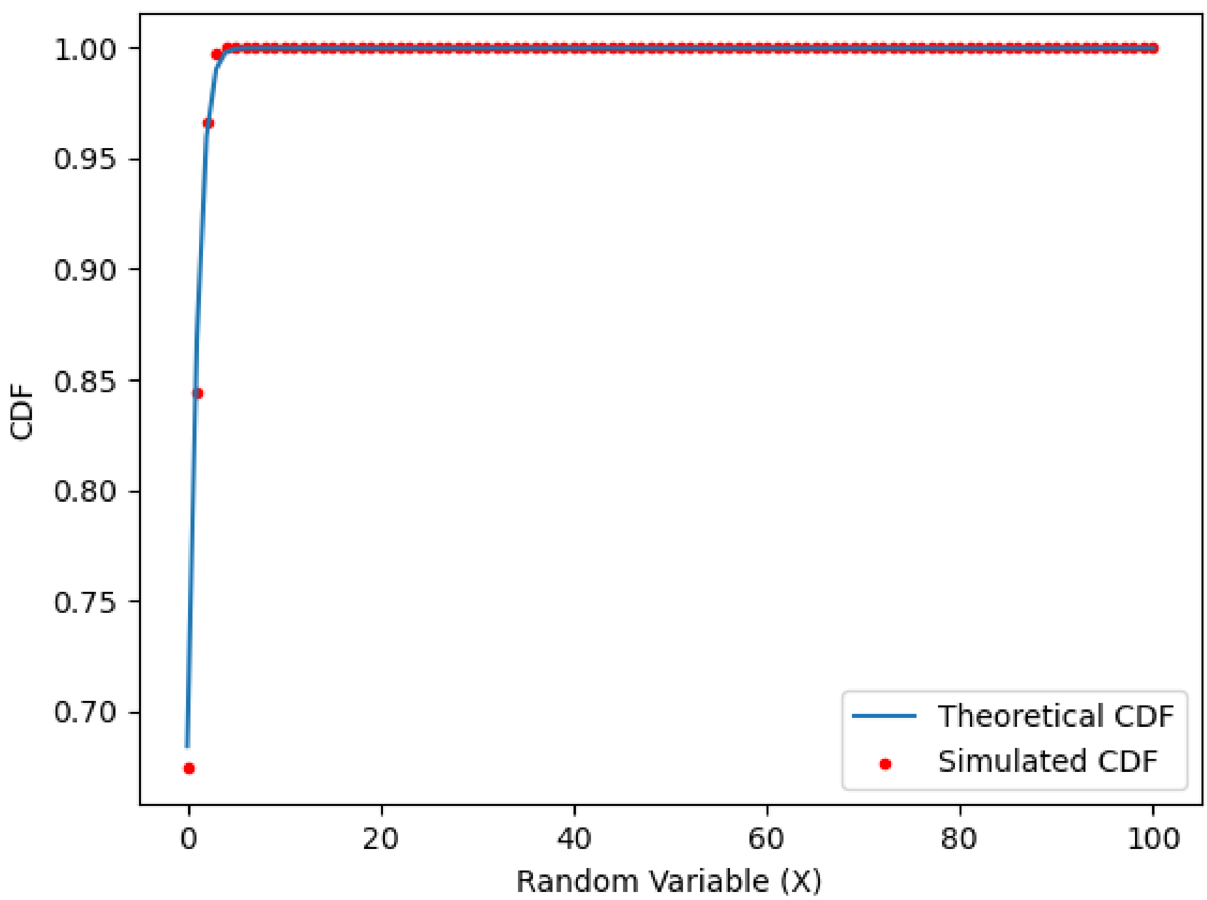

Fig. 2. plot of corcumcirclr O and points A, B and C.