Question : Let $N(t)_{t\geq 0}$ be a Poisson process with rate 1. Consider the following statements.

(a) $\Pr(N(3) = 3|N(5) = 5) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$

(b) If S_5 denotes the time of occurrence of the 5^{th} event for the above Poisson process,then $E(S_5|N(5) = 3) = 7$

Which of the above statements is/are true?

(i) only (a)
(ii) only (b)
(iii) Both (a) and (b)
(iv) Neither (a) and (b)

(GATE ST 2023)

Solution:

Parameter	Values	Description
X	$N(t_1)$	poisson
Y	$N(t_2)$	random
$X + Y$	$N(t_1 + t_2)$	variables

TABLE 4
TABLE 1

(a) Using the Poisson probability formula,

$$\Pr(N(t) = k) = Po(t; k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{1}$$

here λ is 1

$$\Pr(N(t) = k) = \frac{(t)^k e^{-t}}{k!} \tag{2}$$

$$\tag{3}$$

X and Y are independent Poisson random variables, then $X + Y$ is also Poisson

$$\Pr(X = k, X + Y = n) = pr(X = k, Y = n - k) \tag{4}$$

$$= \frac{(t_1)^k}{k!} e^{-t_1} \frac{(t_2)^{n-k}}{(n-k)!} e^{-t_2} \tag{5}$$

$$= e^{-(t_1+t_2)} \left(\frac{(t_1 + t_2)^n}{n!}\right) {}^nC_k \left(\frac{t_1}{t_1 + t_2}\right)^k \left(\frac{t_2}{t_1 + t_2}\right)^{n-k} \tag{6}$$

$$\Pr(X + Y = n) = e^{-(t_1+t_2)} \left(\frac{(t_1 + t_2)^n}{n!}\right) \tag{7}$$

From conditional probability, from the equations (6) and (7)

$$\Pr(X = k|X + Y = n) = \frac{\Pr(X = k, Y = n - k)}{\Pr(X + Y = n)} \tag{8}$$

$$= {}^nC_k \left(\frac{t_1}{t_1 + t_2}\right)^k \left(\frac{t_2}{t_1 + t_2}\right)^{n-k} \tag{9}$$

For the given question,

Parameter	Values
t_1	3
t_1	5

TABLE (a)
TABLE 1

$$\Pr\left(N(3)=3|N(5)=5\right) = {}^5C_3\left(\frac{3}{2+3}\right)^3\left(\frac{2}{2+3}\right)^2 \tag{10}$$

$$= {}^5C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2 \tag{11}$$

Hence statement (a) is true.

Generation of poisson Random Variable from uniform in C language

(i) Define the Poisson Random Variable Generator Function:
In your program, define the poissonRandomVariable function to generate Poisson random variables with a given lambda parameter

(ii) ”lambda” is the mean parameter for the Poisson distribution, representing the average rate of events in the given interval.
L is exp(-lambda), where exp is the exponential function. This value represents the probability of having zero events in the interval.
The function enters a loop that continues until p is less than or equal to L.

(iii) rand() / (double)RAND_MAX:
This generates a random variable

(iv) the Main Function:
”numSamples” controls how many random samples you want to generate.

(v) srand(time(NULL)):
The ”srand(time(NULL))” line seeds the random number generator using the current time to ensure different random sequences each time you run the program.