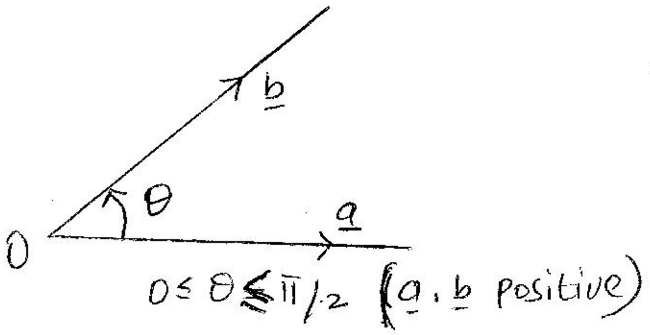
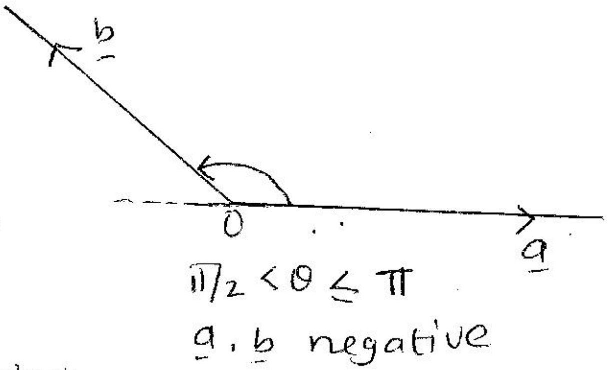
1.6 PRODUCT OF VECTORS

(1) The Dot or Scalar product

The scalar product (dot product) of two vectors $\underline{a}$ and $\underline{b}$, denoted by $\underline{a} \cdot \underline{b}$ (read $\underline{a}$ dot $\underline{b}$), is defined as $|\underline{a}|\,|\underline{b}|\cos\theta$; where θ is the angle between the vectors $\underline{a}$ and $\underline{b}$. That is, $\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\cos\theta;\ 0 \le \theta \le \pi$.





Properties of scalar products.

1. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ - commutative law for dot product.
2. $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$ - distributive law.
3. $\lambda(\underline{a} \cdot \underline{b}) = (\lambda\underline{a}) \cdot \underline{b} = \underline{a} \cdot (\lambda\underline{b})$; where λ is a scalar.
4. Let $\underline{i}, \underline{j}$ and $\underline{k}$ be usual mutually perpendicular unit vectors in the rectangular co-ordinate system then,
   $\underline{i} \cdot \underline{i} = 1,\ \underline{j} \cdot \underline{j} = 1,\ \underline{k} \cdot \underline{k} = 1,\ \underline{i} \cdot \underline{j} = 0,\ \underline{i} \cdot \underline{k} = 0$ and $\underline{j} \cdot \underline{k} = 0$.
5. If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then,
   (a) $\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2 = |\underline{a}|^2$
   (b) $\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$
   (c) The angle θ between $\underline{a}$ and $\underline{b}$ is given by,
   $$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\underline{a}|\,|\underline{b}|}$$
6. $\underline{a} \cdot \underline{b} = 0$ if either
   (a) $\underline{a} = 0$ or $\underline{b} = 0$
   or
   (b) $\underline{a}$ is perpendicular to $\underline{b}$.
7. If $\underline{a}$ and $\underline{b}$ are parallel vectors then $\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|$.

Example (1): If $\underline{a} = 2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{b} = 3\underline{i} - 5\underline{j} + 7\underline{k}$ then find $\underline{a} \cdot \underline{b}$.

$$\begin{aligned}\text{For : } \underline{a} \cdot \underline{b} &= (2\underline{i} + 3\underline{j} - 4\underline{k}) \cdot (3\underline{i} - 5\underline{j} + 7\underline{k}) \\ &= 2 \cdot 3 + 3 \cdot (-5) + (-4) \cdot 7 \\ &= 6 - 15 - 28 \\ &= -37.\end{aligned}$$

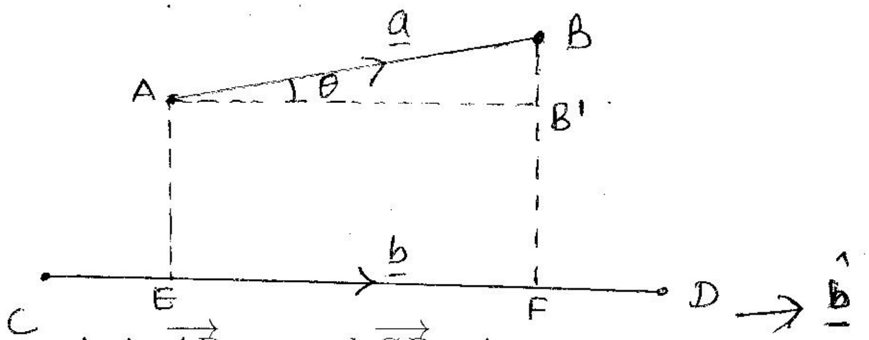
Example (2): Find the angle between $\underline{a} = \underline{i} - 2\underline{j} + 2\underline{k}$ and $\underline{b} = 2\underline{i} + 3\underline{j} + 6\underline{k}$.

$$\begin{aligned}\text{For : Since } \underline{a} \cdot \underline{b} &= |\underline{a}|\,|\underline{b}| \cos\theta \\ \Rightarrow \cos\theta &= \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|} \\ |\underline{a}| &= \left[1^2 + (-2)^2 + 2^2\right]^{\frac{1}{2}} = 3 \\ |\underline{b}| &= \left[2^2 + 3^2 + 6^2\right]^{\frac{1}{2}} = 7 \\ \text{and } \underline{a} \cdot \underline{b} &= 1 \cdot 2 + (-2) \cdot 3 + 2 \cdot 6 \\ &= 2 - 6 + 12 \\ &= 8. \\ \cos\theta &= \frac{8}{3 \cdot 7} = \frac{8}{21} \\ \theta &= \cos^{-1}\left(\frac{8}{21}\right)\end{aligned}$$

Example (3): If $\underline{a} \cdot \underline{b} = 0$ and if $\underline{a}$ and $\underline{b}$ are not zero, show that $\underline{a}$ is perpendicular to $\underline{b}$.

Example (4): Prove that the projection of $\underline{a}$ on $\underline{b}$ is equal to $\underline{a} \cdot \hat{\underline{b}}$; where $\hat{\underline{b}}$ is a unit vector in the direction of $\underline{b}$.

For:



Let $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{CD} = \underline{b}$.

$$\begin{aligned}\text{The projection of } \underline{a} \text{ on } \underline{b} &= EF = AB' \\ &= |\underline{a}| \cos\theta \\ &= \underline{a} \cdot \hat{\underline{b}}\end{aligned}$$

Exercise: Prove that,

1. $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$.
2. $(\underline{a} + \underline{b}) \cdot (\underline{c} + \underline{d}) = \underline{a} \cdot \underline{c} + \underline{a} \cdot \underline{d} + \underline{b} \cdot \underline{c} + \underline{b} \cdot \underline{d}$.

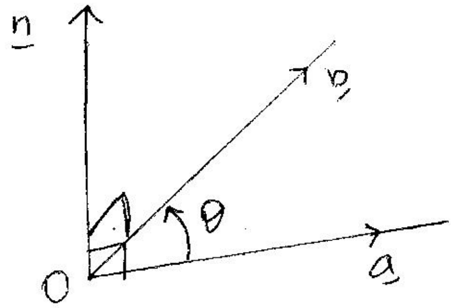
Example (5): Determine the value of a so that $\underline{a} = 4\underline{i} - 3\underline{j} + 2a\underline{k}$ and $\underline{b} = \underline{i} - 2\underline{j} - 5\underline{k}$ are perpendicular.

For : $\underline{a}$ and $\underline{b}$ are perpendicular.

$$
\begin{aligned}
\Rightarrow \underline{a} \cdot \underline{b} &= 0 \\
(4\underline{i} - 3\underline{j} + 2a\underline{k}) \cdot (\underline{i} - 2\underline{j} - 5\underline{k}) &= 0 \\
4 + 6 - 10a &= 0 \\
a = 1.&
\end{aligned}
$$

(2) The cross or vector product

The vector product (cross product) of the two vectors $\underline{a}$ and $\underline{b}$ is a vector $\underline{c} = \underline{a} \times \underline{b}$ (reads $\underline{a}$ cross $\underline{b}$). The magnitude of $\underline{a} \times \underline{b}$ is defined as the product of the magnitudes of $\underline{a}$ and $\underline{b}$ and the sine of the angle θ between them. The direction of the vector $\underline{c} = \underline{a} \times \underline{b}$ is perpendicular to the plane of $\underline{a}$ and $\underline{b}$ and such that $\underline{a}, \underline{b}$ and $\underline{c}$ form a right-handed system.
That, $\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta\ \underline{n};\ 0 \leq \theta \leq \pi$,
where $\underline{n}$ is a unit vector indicating the direction of $\underline{a} \times \underline{b}$.



Here $\underline{a}, \underline{b}$ and $\underline{n}$ form a right-handed system.

Properties of cross product

1. $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$ (Commutative law for cross product fails).
2. $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$ (Distributive law)
3. $\lambda\,(\underline{a} \times \underline{b}) = (\lambda \underline{a}) \times \underline{b} = \underline{a} \times (\lambda \underline{b})$; λ is a Scalar.

4. Let $\underline{i}, \underline{j}$ and $\underline{k}$ be mutually perpendicular unit vectors along X, Y and Z axes then,
$\underline{i} \times \underline{j} = \underline{k}$, $\underline{j} \times \underline{k} = \underline{i}$ and $\underline{k} \times \underline{i} = \underline{j}$ and
$\underline{i} \times \underline{i} = 0 = \underline{j} \times \underline{j} = \underline{k} \times \underline{k}$.

5. If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then,

$$\begin{aligned} \underline{a} \times \underline{b} &= \left(a_1\underline{i} + a_2\underline{j} + a_3\underline{k}\right) \times \left(b_1\underline{i} + b_2\underline{j} + b_3\underline{k}\right) \\ &= \left(a_2b_3 - b_2a_3\right)\underline{i} + \left(a_3b_1 - a_1b_3\right)\underline{j} + \left(a_1b_2 - a_2b_1\right)\underline{k} \\ &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \end{aligned}$$

6. $\underline{a} \times \underline{b} = 0$ if either

   (a) $\underline{a} = 0$ or $\underline{b} = 0$
   or

   (b) $\underline{a}$ is parallel to $\underline{b}$.

7.

$$\begin{aligned} \underline{a} \times \underline{b} &= |\underline{a}|\,|\underline{b}| \sin\theta\, \underline{n} \\ \text{but } \underline{b} \times \underline{a} &= |\underline{b}|\,|\underline{a}| \sin\theta\,(-\underline{n}) \\ &= -\,|\underline{a}|\,|\underline{b}| \sin\theta \underline{n} \\ &= -\left(\underline{a} \times \underline{b}\right) \\ \therefore \underline{a} \times \underline{b} &= -\left(\underline{a} \times \underline{b}\right). \end{aligned}$$

Example (1): If $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$ and $\underline{b} = 2\underline{i} + 5\underline{j} - 3\underline{k}$. Find,

(a) $\underline{a} \times \underline{b}$

(b) $\underline{b} \times \underline{a}$

(c) $\left(\underline{a} + \underline{b}\right) \times \left(\underline{a} - \underline{b}\right)$.

Example (2): Show that $|\underline{a} \times \underline{b}|^2 + |\underline{a} \cdot \underline{b}|^2 = |\underline{a}|^2\,|\underline{b}|^2$.

Example (3): If $\underline{a} \times \underline{b} = 0$ and if $\underline{a}$ and $\underline{b}$ are non-zero, Show that $\underline{a}$ is parallel to $\underline{b}$.

Example (4): Prove that the area of the triangle with sides $\underline{a}$ and $\underline{b}$ is $\frac{1}{2}\,|\underline{a} \times \underline{b}|$.

Example (5): Prove that the area of a parrallelogram with sides $\underline{a}$ and $\underline{b}$ is $|\underline{a} \times \underline{b}|$.

Example (6): Find the area of the triangle having vertices at A $(1, 3, 2)$, B $(2, -1, 1)$, C.$(-1, 2, 3)$.

Example (7): Determine a unit vector perpendicular to the plane of $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$.

Exercise(1) Prove the sine rule for a triangle by using cross product.

Exercise (2): Using the dot product of vectors, Prove the following theorems:

(a) Pythagarus theorem for a rightangled triangle.

(b) Cosine rule for a triangle.