

Chapter 1

Vector Algebra

In this chapter, we will study some of the basic concept about vectors, various operations (addition, subtraction, multiplication-product) on vectors, and their algebraic and geometric properties. These properties, when considered together give a full realization to the concept of vectors, and lead to their vital applicability in various areas like Mathematics, Physics and Engineering.

Vector algebra is a mathematical structure formed by a collection of vectors.

1.1 Vectors and scalars

Physical quantities such as mass, temperature and work are measured by numbers referred to some chosen unit. These numbers are called **Scalars**.

Other quantities exist such as displacement, velocity, acceleration and force, which require for their complete specification a direction as well as a magnitude. These quantities are called **Vectors**.

Definition 1.1.1: (Vector)

A vector is a quantity having both magnitude and direction.

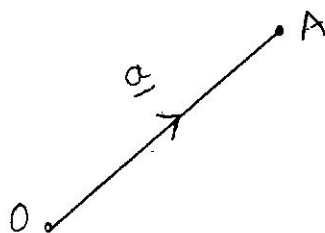
Example: displacement, velocity, acceleration, and force.

Definition 1.1.2: (Scalar)

A scalar a is a quantity having magnitude but no direction.

Example: mass, temperature and work.

Graphically, a vector is a directed line segment. In other words, a vector is represented by an arrow \overrightarrow{OA} defining the direction, the magnitude of the vector being indicated by the length of the line segment.



The tail end O of the arrow is called the origin or initial point of the vector, and the head A is called the terminal point.

- A vector is represented by a small alphabet with line under it, as \underline{a} in the figure. (Also a vector is represented by \overrightarrow{OA} , \overrightarrow{OB} , ..., etc.).
- The magnitude of a vector \overrightarrow{OA} is defined by the length (OA) of the vector and is denoted by $|\overrightarrow{OA}|$ or $|\underline{a}|$.
That is, $|\overrightarrow{OA}| = |\underline{a}| = OA$.

Definition 1.1.3: (Zero vector / Null vector)

If the magnitude of a vector is zero then the vector is called a zero vector or Null vector. It has no specific direction and is denoted by $\underline{0}$. ('A' coincides with 'O').

Definition 1.1.4: (Unit Vector)

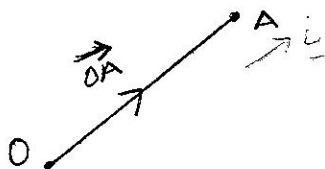
If the magnitude of a vector is unity (one) then that vector is called a unit vector.

If \underline{a} is a vector

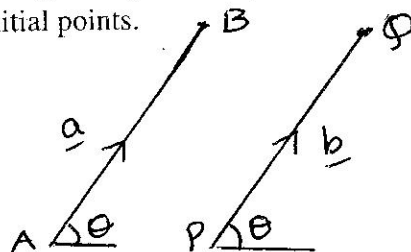
then $\frac{\underline{a}}{|\underline{a}|}$ is a unit vector along the vector \underline{a} .

Usually the letters \underline{i} , \underline{j} and \underline{k} referred for unit vectors.

Note: Let \underline{i} be the unit vector in the OA direction then $\overrightarrow{OA} = OA \underline{i}$

**Definition 1.1.5: (Equal vectors)**

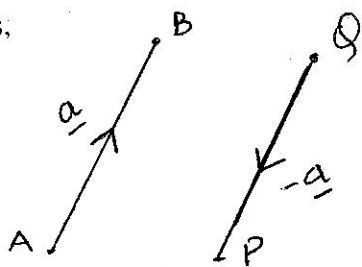
Two vectors \underline{a} and \underline{b} are equal if they have same magnitude and direction regardless of the position of their initial points.



- $\underline{a} = \underline{b}$ if and only if
- (i) $AB = PQ$
 - (ii) AB parallel to PQ
 - (iii) sense $A \rightarrow B = \text{sense } P \rightarrow Q$

A vector having direction opposite to that of a given vector \underline{a} but having same magnitude is denoted by $-\underline{a}$

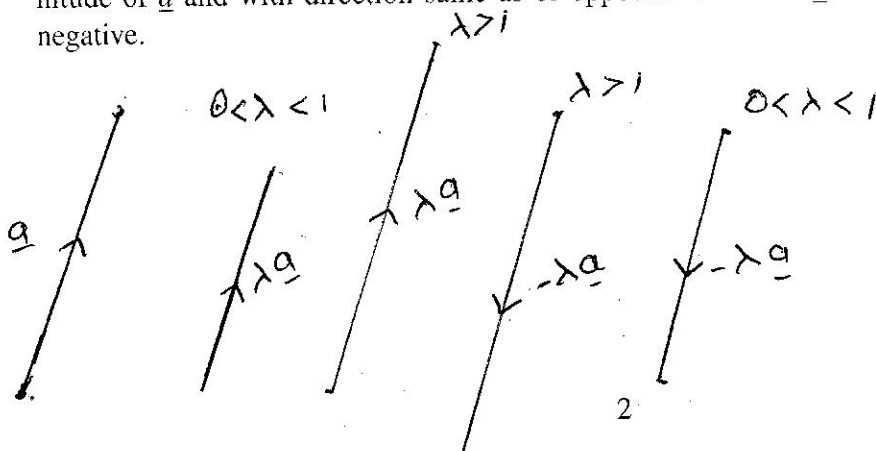
That is,



Here, length $AB = \text{length } PQ$.

1.1.1 Multiplication of vectors by scalars

Let \underline{a} be a vector and m be a scalar then the vector $m\underline{a}$ with magnitude of $|m|$ times the magnitude of \underline{a} and with direction same as or opposite to that of \underline{a} according as m is positive or negative.

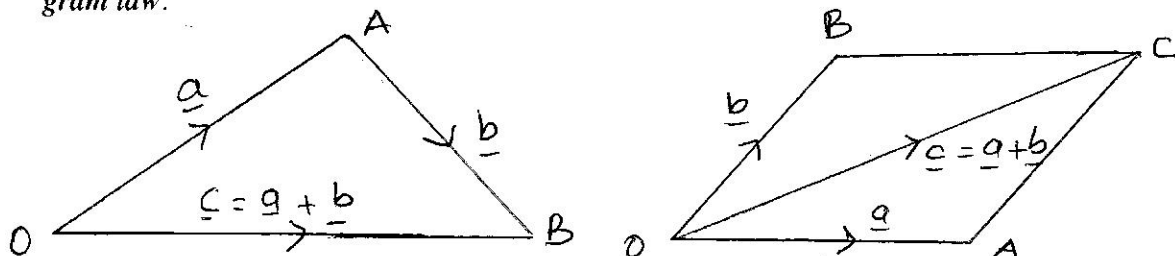


1.1.2 Vector Addition

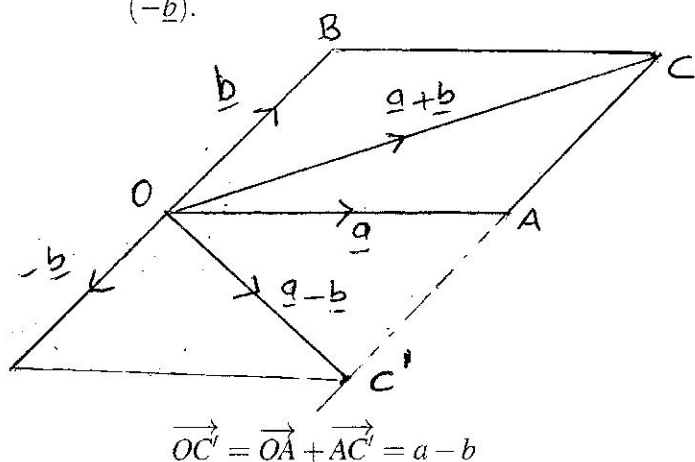
If \underline{a} and \underline{b} are given vectors then the sum or resultant of vectors \underline{a} and \underline{b} is a vector \underline{c} such that $\underline{c} = \underline{a} + \underline{b}$.

Two vectors \underline{a} and \underline{b} may be added graphically either by drawing both vectors from a common origin and completing the parallelogram or by beginning the second vector from the head of the first and completing the triangle.

The diagonal vector starting from the common origin is the sum $\underline{a} + \underline{b}$, this is called **parallelogram law**.



Note: The difference of vectors \underline{a} and \underline{b} represented by $\underline{a} - \underline{b}$ can be defined as the sum of \underline{a} and $(-\underline{b})$.



1.1.3 Laws of vector algebra

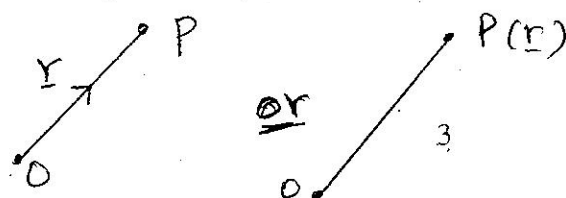
If \underline{a} , \underline{b} and \underline{c} are vectors and λ and μ are scalars, then

1. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ - Commutative law for addition.
2. $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$ - Associative law for addition.
3. $\lambda \underline{a} = \underline{a} \lambda$ - Commutative law for multiplication.
4. $\lambda(\mu \underline{a}) = (\lambda \mu) \underline{a} = \mu(\lambda \underline{a})$ - Associative law for multiplication.
5. $(\lambda + \mu) \underline{a} = \lambda \underline{a} + \mu \underline{a}$ - Distributive law.
6. $\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$ - Distributive law.

These laws enable us to treat vector equations in the same way as ordinary algebraic equations. For example, if $\underline{a} + \underline{b} = \underline{c}$ then $\underline{a} = \underline{c} - \underline{b}$.

Definition 1.1.6: (Position Vector)

Let O be a fixed point in space. Let P be any variable point and Let $\vec{OP} = \underline{r}$ then \underline{r} is the position vector of P with respect to the point O .

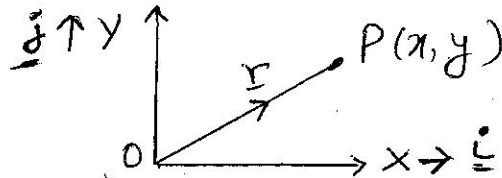


1.2 Vectors in Cartesian coordinate system

1. In Two dimensional Cartesian coordinates

Let us consider a two dimensional Cartesian coordinates system, obtained by introducing two mutually perpendicular axes, labeled X and Y, with the same unit of length on both axes.

Let \underline{i} be the unit vector parallel to X axis, in the positive X direction, and \underline{j} be the unit vector parallel to Y axis, in the positive Y direction.



Any vector \overrightarrow{OP} with initial point at origin O can be written in the form $\overrightarrow{OP} = \underline{r} = x\underline{i} + y\underline{j}$, where (x, y) is the Coordinate of the point P.

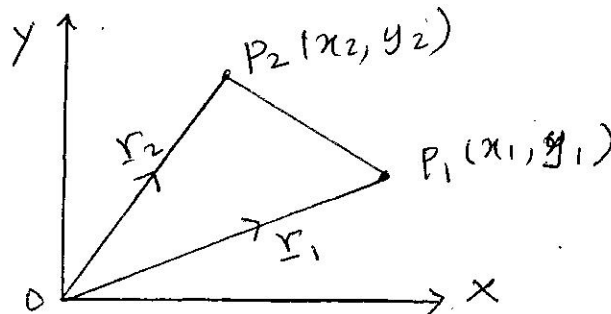
We call $x\underline{i}$ and $y\underline{j}$ are component vectors and x and y are components of vector \underline{r} in the X and Y direction and

$$|\overrightarrow{OP}| = |\underline{r}| = OP = (x^2 + y^2)^{\frac{1}{2}}$$

Note: The vector \underline{r} can also write in the ordered pair form $\underline{r} = x\underline{i} + y\underline{j} = (x, y)$.

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points in the XY plane then the vector represented by the directed line segment P_1P_2 (initial point P_1 , terminal point P_2) is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j}.$$



Since $\overrightarrow{OP_1} = x_1\underline{i} + y_1\underline{j}$ and $\overrightarrow{OP_2} = x_2\underline{i} + y_2\underline{j}$

By triangular law,

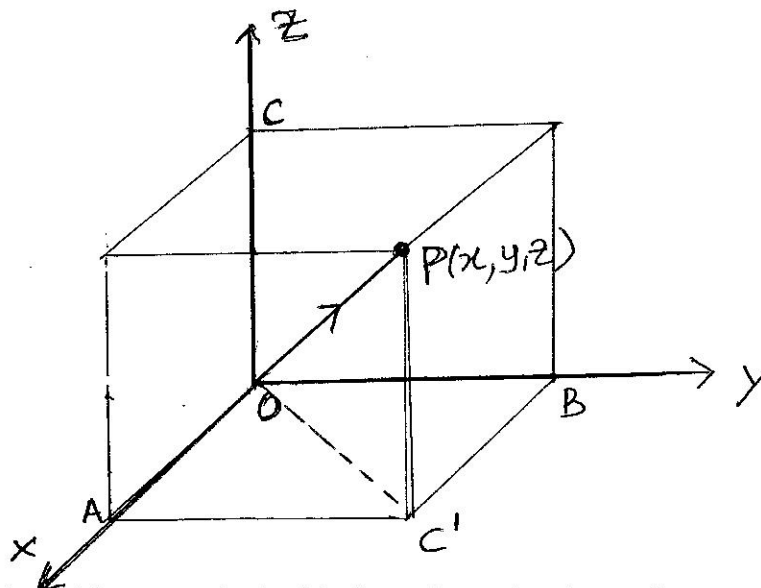
$$\begin{aligned} \overrightarrow{OP_1} + \overrightarrow{P_1P_2} &= \overrightarrow{OP_2} \\ \overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1} \\ &= (x_2\underline{i} + y_2\underline{j}) - (x_1\underline{i} + y_1\underline{j}) \\ &= (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} \\ \text{or } \overrightarrow{P_1P_2} &= \underline{r_2} - \underline{r_1}, \text{ where } \underline{r_1} = \overrightarrow{OP_1} \text{ and } \underline{r_2} = \overrightarrow{OP_2} \end{aligned}$$

$$\text{and } |\overrightarrow{P_1P_2}| = P_1P_2 = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{\frac{1}{2}}.$$

2. In three dimensional Cartesian coordinates

Let us consider a three dimensional Cartesian coordinate system, obtained by introducing three mutually perpendicular axes, labeled X, Y and Z with the same unit of length along all three axes.

Let \underline{i} , \underline{j} and \underline{k} be unit vectors in the positive X, Y and Z directions, respectively.



Let $P \equiv (x, y, z)$ be any point in this three dimensional coordinate system. Then $\overrightarrow{OA} = x\underline{i}$, $\overrightarrow{OB} = y\underline{j}$, and $\overrightarrow{OC} = z\underline{k}$.

$$\begin{aligned}\overrightarrow{OC'} &= \overrightarrow{OA} + \overrightarrow{AC'} = \overrightarrow{OA} + \overrightarrow{OB} = x\underline{i} + y\underline{j} \\ \overrightarrow{OP} &= \overrightarrow{OC'} + \overrightarrow{C'P} = x\underline{i} + y\underline{j} + z\underline{k} \quad [\because \overrightarrow{C'P} = \overrightarrow{OC} = z\underline{k}] \\ \underline{r} &= x\underline{i} + y\underline{j} + z\underline{k}.\end{aligned}$$

That is, Any vector in three dimensional coordinate system is of the form

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

or simply $\underline{r} = (x, y, z)$,

We call x, y and z are components of \underline{r} and $|\underline{r}| = |\overrightarrow{OP}| = OP$

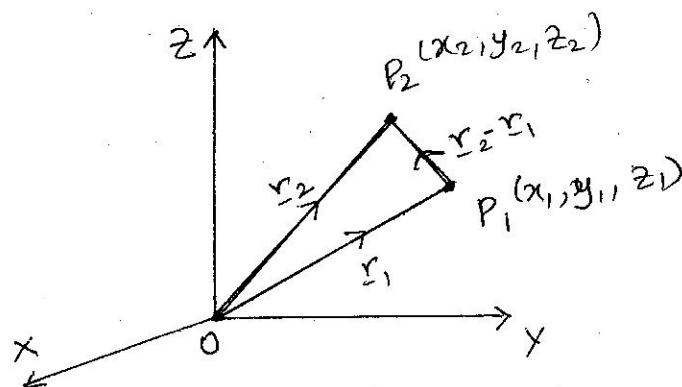
$$\begin{aligned}\text{But, } OP^2 &= OC'^2 + C'P^2 \\ &= x^2 + y^2 + z^2 \\ OP &= (x^2 + y^2 + z^2)^{\frac{1}{2}}\end{aligned}$$

$$\text{Therefore } |\underline{r}| = |\overrightarrow{OP}| = OP = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

Note: A unit vector in the direction of $\overrightarrow{OP} = \frac{\overrightarrow{OP}}{OP} = \frac{\underline{r}}{OP} = \frac{x\underline{i} + y\underline{j} + z\underline{k}}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}.$

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points in the space (three dimensional coordinate system), then the vector $\overrightarrow{P_1P_2}$ can be written as

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}.$$



Let $\underline{r}_1 = \overrightarrow{OP_1} = x_1\underline{i} + y_1\underline{j} + z_1\underline{k}$ and $\underline{r}_2 = \overrightarrow{OP_2} = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$

$$\begin{aligned} \text{But } \overrightarrow{OP_2} &= \overrightarrow{OP_1} + \overrightarrow{P_1P_2} \\ \overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1} \\ &= (x_2\underline{i} + y_2\underline{j} + z_2\underline{k}) - (x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) \end{aligned}$$

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

$$\text{or } \overrightarrow{P_1P_2} = \underline{r}_2 - \underline{r}_1.$$

$$\text{and } |\overrightarrow{P_1P_2}| = P_1P_2 = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{\frac{1}{2}}.$$

Example 1: Let $\underline{r}_1 = 2\underline{i} + 4\underline{j} + 5\underline{k}$, $\underline{r}_2 = 5\underline{i} - 2\underline{j} + 7\underline{k}$ and $\underline{r}_3 = 3\underline{i} - 2\underline{j} - 4\underline{k}$, determine

(a) $4\underline{r}_1 - 3\underline{r}_2 + \underline{r}_3$.

(b) $|4\underline{r}_1 - 3\underline{r}_2 + \underline{r}_3|$.

Example 2: Determine the vector \overrightarrow{AB} for each of the following pairs of points

(a) $A(3, 7, 2)$ and $B(9, 12, 5)$

(b) $A(4, 1, 0)$ and $B(3, 4, -2)$

For each of the vectors found in (a) and (b), determine a unit vector in the direction of \overrightarrow{AB}

Solution:

1.

$$\begin{aligned} \text{(a) } 4\underline{r}_1 - 3\underline{r}_2 + \underline{r}_3 &= 4(2, 4, 5) - 3(5, -2, 7) + (3, -2, -4) \\ &= (-4, 20, -5) \\ &= -4\underline{i} + 20\underline{j} - 5\underline{k} \end{aligned}$$

$$\begin{aligned} \text{(b) } |4\underline{r}_1 - 3\underline{r}_2 + \underline{r}_3| &= \sqrt{(-4)^2 + (20)^2 + (-5)^2} \\ &= \sqrt{16 + 400 + 25} \\ &= \sqrt{441} \end{aligned}$$

$$= 21.$$