MECH 539 Assignment 1

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1 Question 1

In this problem, I solve the 1 Dimensional linear advection equation,

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x}$$

using the upwind, Lax, Lax-Wendroff, Leap-Frog, and MacCormack schemes and discuss the differences between the various schemes when we choose $\delta t = 1.0$ and when $\delta t = 0.5$ using the following initial condition:

$$u = \frac{1}{2}(1 + \tanh[250(x - 20)]$$

We choose a 41 gridpoint mesh, and initially take $\delta x = 1$ and compute to $t = 10$. The programming is done with MATLAB and the code that was used to derive the following results has been uploaded along with this document on myCourses for your reference. The following graphs were produced for the different schemes when we take the time step ($\delta t = 1.0$), the last graph shows a summary of all the schemes in one graph. The exact solution for the linear advection equation is obtained by using,

$$u(x) = u(x - v\delta t)$$

where, 'v' is the velocity of the wave which in our case is $\frac{1}{2}$.

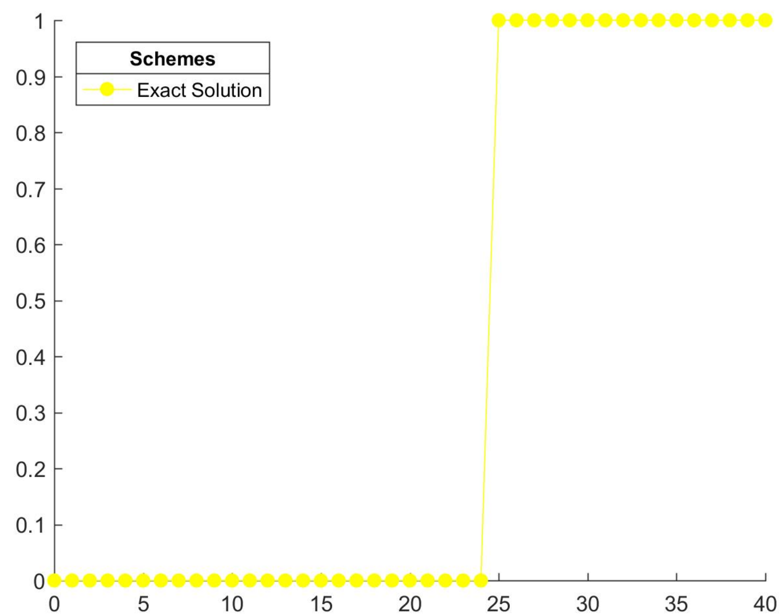

Figure 1: Exact Solution

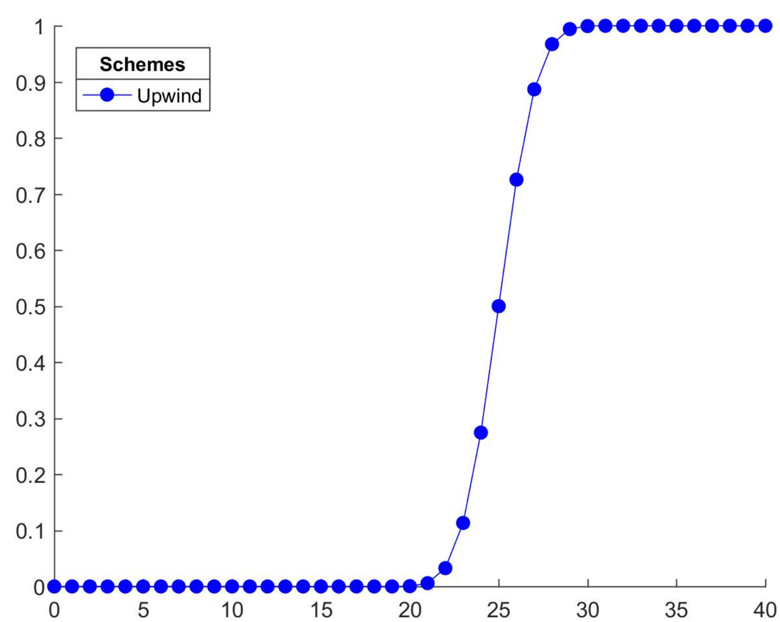

Figure 2: Upwind Scheme

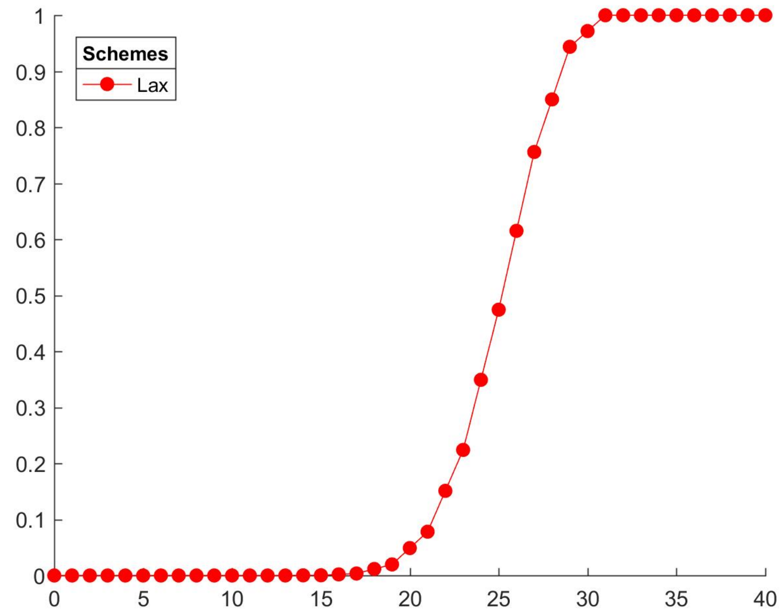

Figure 3: Lax Scheme

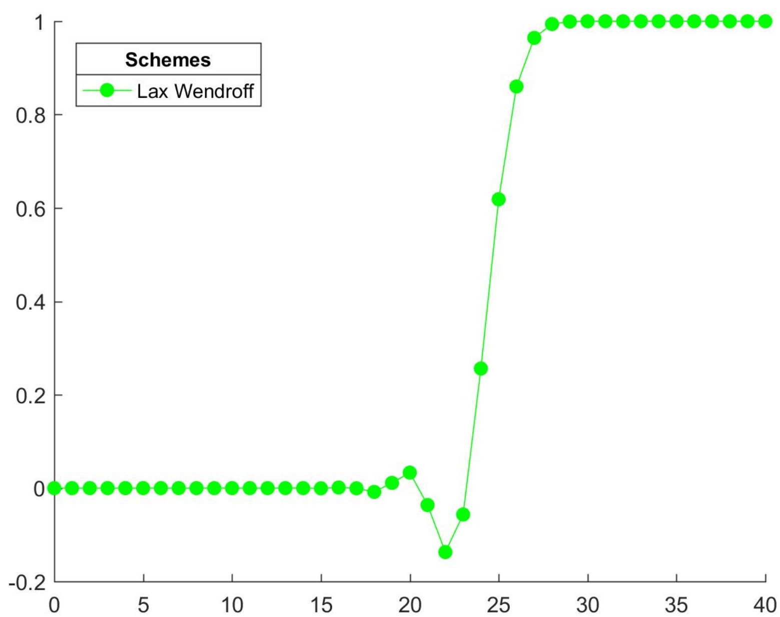

Figure 4: Lax Wendroff Scheme

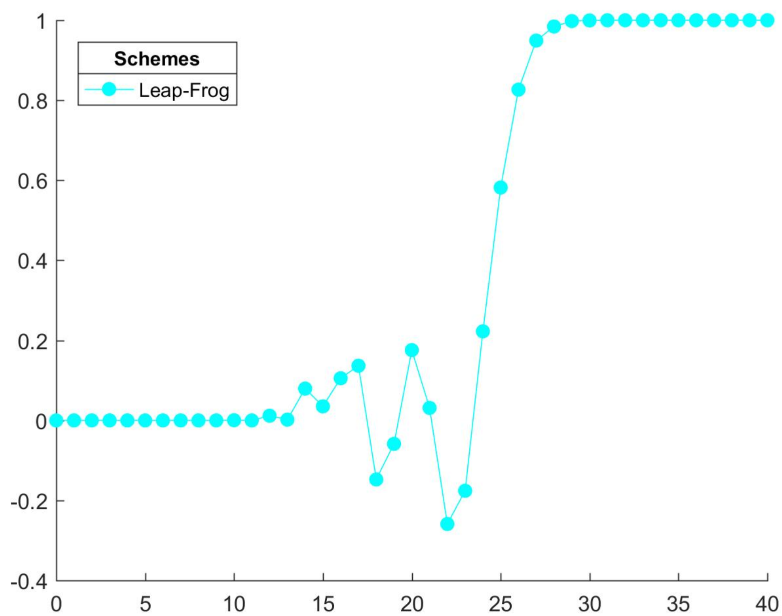

Figure 5: Leap-Frog Scheme

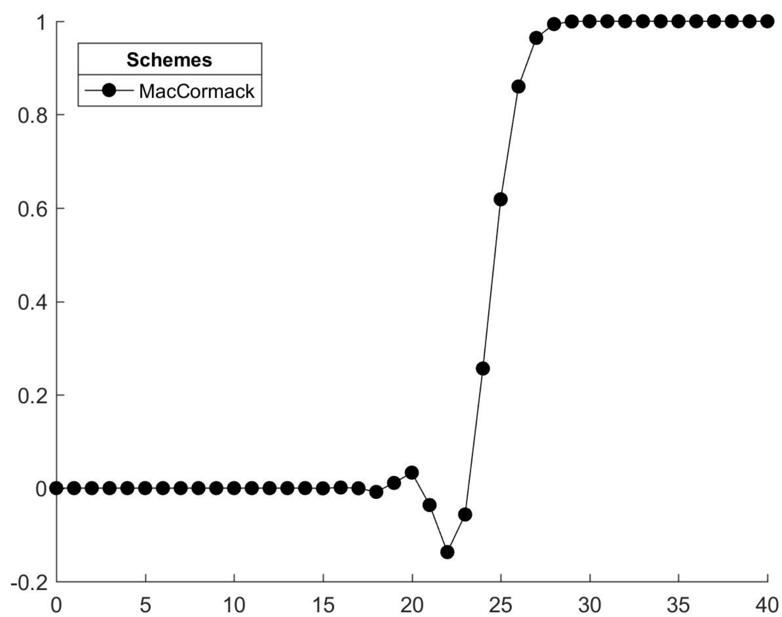

Figure 6: MacCormack Scheme

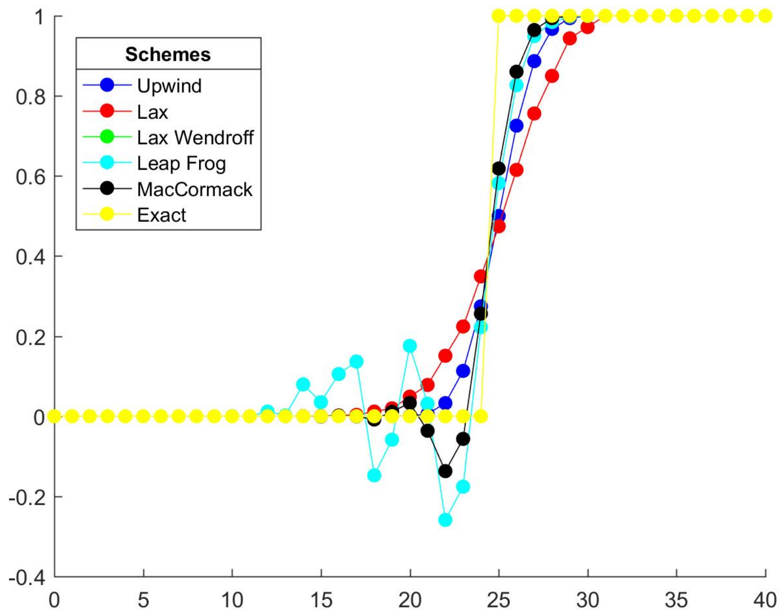

Figure 7: Comparison of the various schemes

Now, if we reduce the time step of the equation by using $\delta t = 0.5$ instead of using $\delta t = 1.0$, we get the following plot. I have only plotted the overall comparison between the different schemes here in order to limit the pages of this report, but the individual schemes can be easily obtained by simply commenting out the plot functions for that scheme in the MATLAB code.

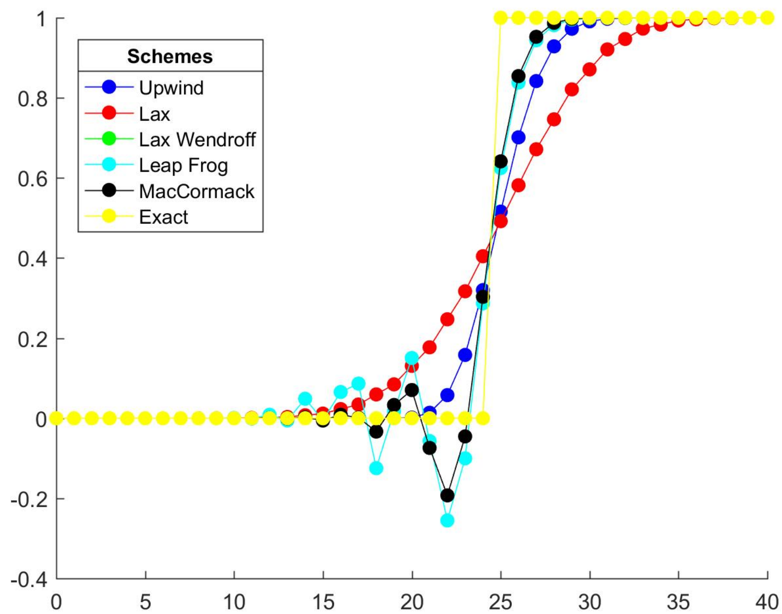

Figure 8: Comparison of the various schemes when $\delta t = 0.5$

Now, if we are to compare the various schemes used to the exact solution, we can immediately notice that the 'Lax Scheme' has a lot of dissipation as the curve has deviated quite a bit from the exact solution. The 'Upwind Scheme' has a comparatively lower dissipation of energy when compared to the 'Lax Scheme'. On the other hand, the 'MacCormack Scheme' has a very low dissipation of energy (the Black line on the graph above represents the MacCormack scheme) as its line is very close to the exact solution, but we notice at the bottom that there is a significant dispersion of energy. The 'Leap-Frog Scheme' is very similar to MacCormack, but has a higher dispersion of energy than compared to that of MacCormack. The 'Lax Wendroff Scheme' is not even visible in the graph above because it is so similar to the graph produced by the MacCormack scheme (i.e. the MacCoemack scheme overlaps the graph produced by the Lax Wendroff scheme).

2 Question 2

In this question we perform a grid study over space and time on the Lax and MacCormack schemes. We shall obtain the results at four conditions: 1. $\delta t = 1.0 and \delta x = 1.0$ 2. $\delta t = 1.0 and \delta x = 0.5$ 3. $\delta t = 0.5 and \delta x = 1.0$ 4. $\delta t = 0.5 and \delta x = 0.5$ Then these are compared with that of the Exact Solution. These results are obtained by simply changing the values of 'dx' and 'dt' at the start of my MATLAB code, and other values are automatically updated. The results are shown below:

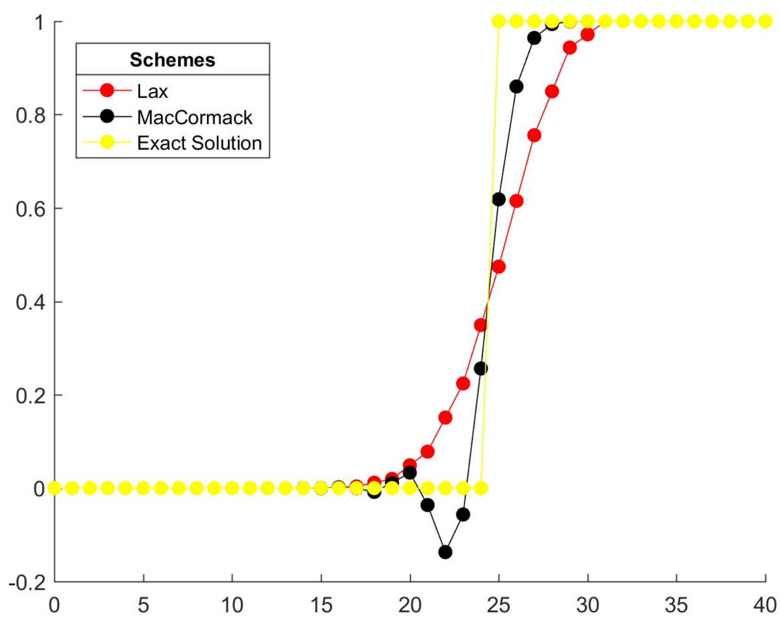

Figure 9: $\delta t = 1.0$ and $\delta x = 1.0$

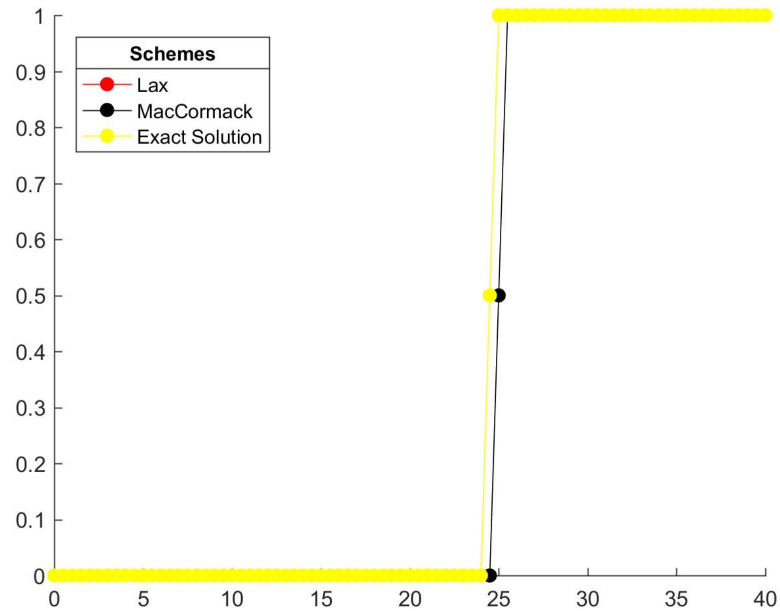

Figure 10: $\delta t = 1.0$ and $\delta x = 0.5$

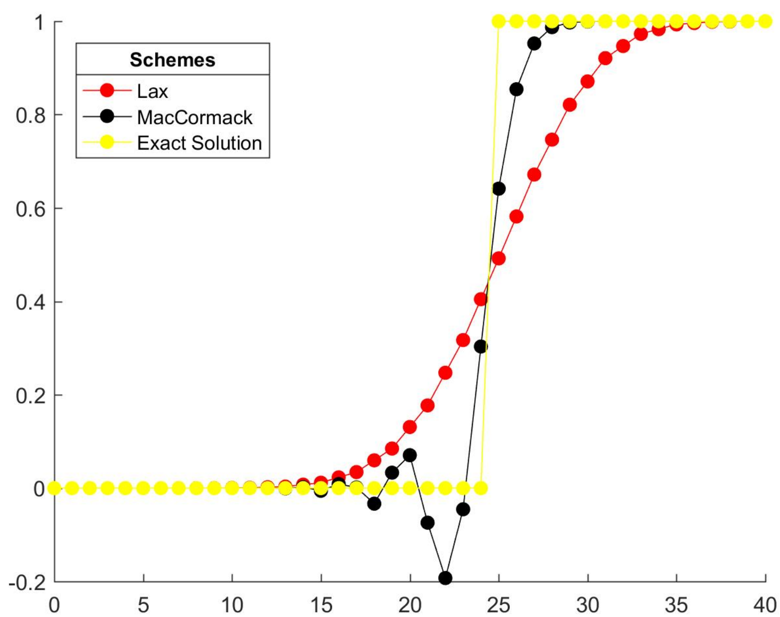

Figure 11: $\delta t = 0.5$ and $\delta x = 1.0$

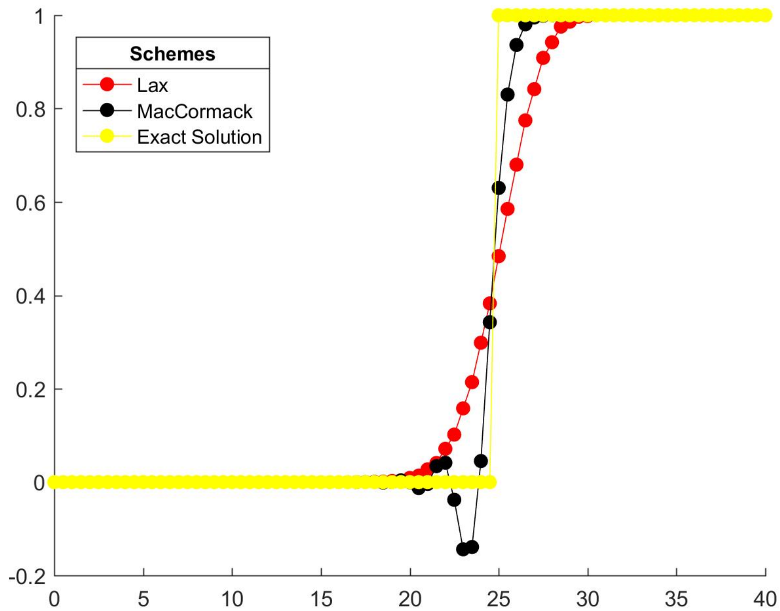

Figure 12: $\delta t = 0.5$ and $\delta x = 0.5$

We notice from the graphs above that if δt is kept constant and δx is decreased, the results from the schemes converges towards the exact solution.

If δt is kept constant and δx is decreased to half then the results almost converge to the exact solution and MacCormack and Lax give the exact same graphs.

Also we notice that if we don't decrease δx and only decrease δt, the schemes especially the Lax schemes dissipates energy even more and is farther away from the exact solution.

When both δt and δx are reduced at a similar ratio, then we notice that the schemes seem to converge towards the exact solution.

3 Question 3

Here, I plotted the log of the maximum difference in the solution of the various schemes versus that of the exact solution, as we decrease the value of dx, i.e. make the grid finer. The following graph was obtained:

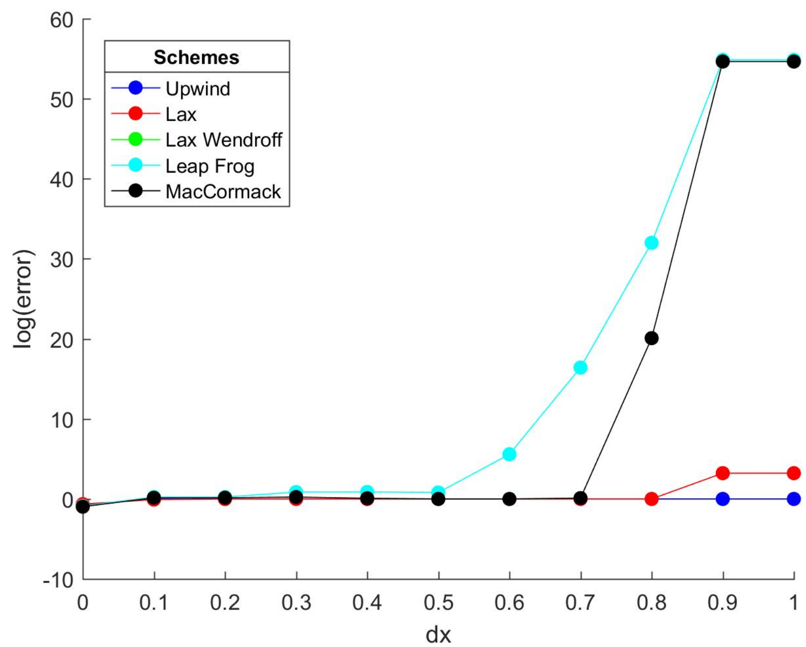

Figure 13: Maximum error compared to the exact solution for various schemes

4 Question 4

Here, we will derive the stability condition for the Leap Frog Scheme and the Lax Wendroff Scheme.

4.1 Leap-Frog Scheme

The Finite Difference Method is given by,

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} = 0$$

Taking,

$$\epsilon(x,t) = \Sigma b_m(t)e^{ik_m x}$$

Substituting this in the FDM equation we get,

For any mode m,

$$\frac{e^{a(t+\delta t)}e^{ik_m x} - e^{a(t-\delta t)}e^{ik_m x}}{2\delta t} + c\frac{e^{at}e^{ik_m(x+\delta x)} - e^{at}e^{ik_m(x-\delta x)}}{2\delta x}$$

Simplifying the above equation we end up with a quadratic equation over $e^{a\delta t}$, whose roots are given by,

$$e^{a\delta t} = \mid -ri\sin\beta \pm \sqrt{1 - r^2\sin^2\beta} \mid$$

where,

$$r = \frac{c\delta t}{\delta x}$$

and

$$\beta = k_m \delta x$$

So now if we take the modulus, we see that:

$$e^{a\delta t} = 1$$

So it can be said that this scheme is unconditionally stable.

4.2 Lax-Wendroff Scheme

The Lax-Wendroff Scheme is given by,

$$u_j^{n+1} = u_j^n - \frac{c\delta t}{2\delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{c^2\delta t^2}{2\delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

Taking,

$$\epsilon(x,t) = \Sigma b_m(t)e^{ik_m x}$$

Substituting this in the FDM equation we get,
For any mode m,

$$e^{a\delta t} = 1 - \frac{c\delta t}{2\delta x}(e^{ik_m\delta x} - e^{-ik_m\delta x} + \frac{c^2\delta t^2}{2\delta x^2}(e^{ik_m\delta x} - 2 - e^{-ik_m\delta x}$$

Substituting, $r = \frac{c\delta t}{\delta x}$ and $\beta = k_m\delta x$ in the above equation and further simplifying we get,

$$e^{a\delta t} = 1 + r^2(\cos\beta - 1) - ir\sin\beta$$

$$\mid e^{a\delta t} \mid = \sqrt{1 + r^4(\cos\beta - 1)^2 + 2r^2(\cos\beta - 1) + r^2\sin^2\beta} \leq 1$$

Further simplifying,

$$r^2(1 - r^2)(1 - \cos\beta)^2 \geq 0$$

Therefore,

$$1 - r^2 \geq 1$$

$$r^2 \leq 1$$

$$-1 \leq r \leq 1$$

This is the stability condition for the Lax Wendroff scheme.