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→ Sigma algebra

Consider a universal set Ω and a coll. of subset of Ω, let E, F denote subsets in this collection. This collection of subsets forms a field $\mathcal{M}$ if.

1) $\phi \in \mathcal{M}$, $\Omega \in \mathcal{M}$
2) if $E \in \mathcal{M}$ and $F \in \mathcal{M}$ then $E \cup F \in \mathcal{M}$ and $EF \in \mathcal{M}$

from if follows that $E_1, E_2, \ldots E_n$ belongs to $\mathcal{M}$ so do.

$$\bigcup_{i=1}^{n} E_i \in \mathcal{M} \text{ and}$$

$$\bigcap_{i=1}^{n} E_i \in \mathcal{M}$$

3) if $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$.

A sigma field $\mathcal{F}$ is a field that is closed under any countable set of unions, intersection and combination.

Then if $E_1 \ldots E_m \ldots$ belong to $\mathcal{F}$.

$$\left[\bigcup_{i=1}^{\infty} E_i \text{ or } \bigcap_{i=1}^{\infty} E_i\right] \checkmark_{y}$$

Borel field

<u>Axiomatic def. of probability</u>

Probability is a set function $P(\cdot)$ that assigns to every event $E \in \mathcal{F}$ a number $P[E]$ called the probability of E such that

$P(E) \geq 0$

$P[\Omega] = 1$

$P[E \cup F] = P(E] + P(F]$ if $EF = \phi$

- $P[\phi] = 0$

$P(E) = 1 - P[E^c]$

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$$P[E \cup F] = P(E) + P(F) - P[EF]$$

Ex- The Experiment consists of tossing a coin once.

Hence

$$\Omega = \{H, T\}$$

The σ field of that consists of following :

$\{H\}; \{T\}, \Omega, \phi$. with the coin assumed fair we have.

$$P[\{H\}] = P[\{T\}] = \frac{1}{2}$$

$$P[\Omega] = 1$$

$$P[\phi] = 0$$

Conjunction Graphs

Good reference unit

PSP

→ joint, conditional and total probability

Assume the following Exp:

$P[A/B]$
↳ Conditional Prob

A is the Event that on any particular day, the temp $\geq 10°C$.

B is the Event that the amount of precipitation $\geq 5mm$

C is the Event on any particular day A and B occur.

$P(C) = P(AB)$, is the joint probab of the Event A and B

let n_i, denote the no. of days on which Event i occ.

∴ Over a thousand days period ($n = 1000$) obs. made.

$P(A) = \frac{n_A}{n} = \frac{811}{1000} = 0.811$

$n_A = 811$

$n_B = 306$

$n_{AB} = 283$

$P(B) = \frac{n_B}{n} = 0.306$

$P(AB) = \frac{n_{AB}}{n} = 0.283$

now consider the ratio $\frac{n_{AB}}{n_A}$,

this is the relative freq. with which event AB occurs when event A occurs.

It is the freq. of time that the amount of fere. $\geq 5mm$, on those when temp $\geq 10^\circ c$.

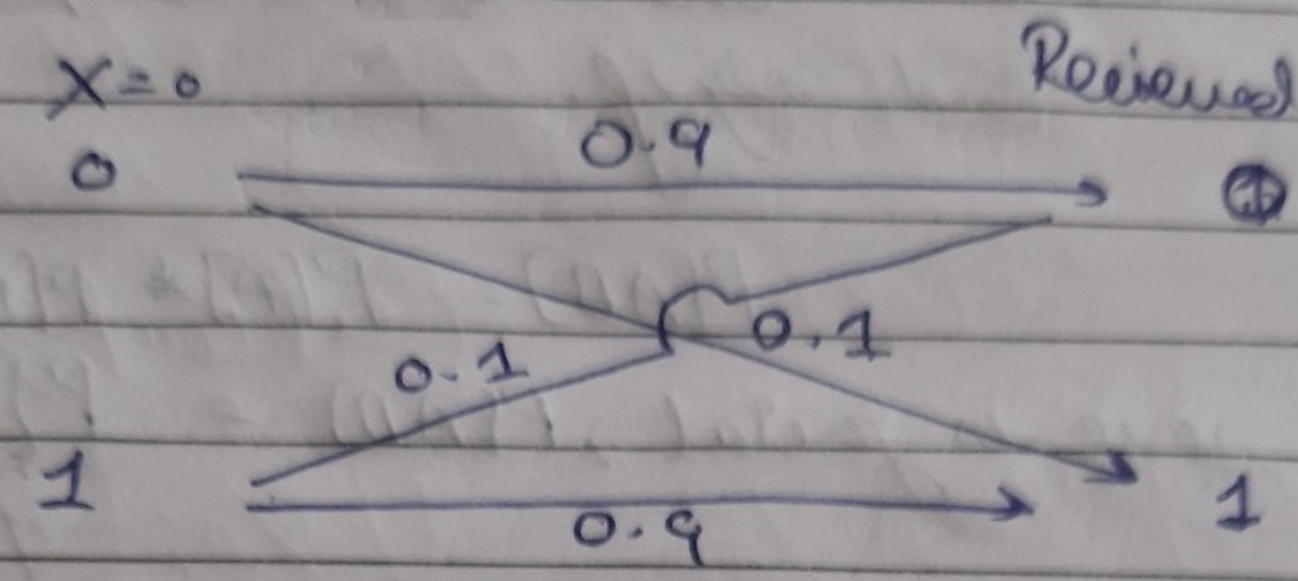
$$\frac{n_{AB}}{n_A} = \frac{n_{AB}/n}{n_A/n} = \frac{P(A \cap B)}{P(A)}$$

$$P[B/A] \triangleq \frac{P[AB]}{P[A]}, \quad P[A] > 0$$

$P[B/A] = \frac{P[AB]}{P[A]}$, $P[A] > 0$

i.e prob that B occurs given that A has occurred.

Q.



Probability of transmission of zero and one is = $\frac{1}{2}$.

→ $P[AB] = ?$

$P[Y=1/X=1] = 0.9$ $\quad$ $P[Y=1/X=0] = 0.1$

$P[Y=0/X=1] = 0.1$ $\quad$ $P[Y=0/X=0] = 0.9$

$P[X=0] = 0.5$ $\quad$ $P[X=1] = 0.5$

$P[X=0, Y=0] = P[Y=0/X=0] * P[X=0] = 0.45$

$P[X=0, Y=1] = P[Y=1/X=0] * P[X=0] = 0.05$

$P[X=1, Y=0] = P[Y=0/X=1] * P[X=1] = 0.05$

$P[X=1, Y=1] = P[Y=1/X=1] * P[X=1] = 0.45$

Independent Event

1) Two Events $A \in \mathcal{F}$, $B \in \mathcal{F}$ with $P(A) > 0$, $P(B) > 0$ are said to be independent if and only if.

$$P[AB] = P[A] * P[B]$$

since in general, $P[AB] = \underbrace{P[B/A]}_{P(B)} * P[A] = \underbrace{P[A/B]}_{P(A)} * P[B]$

$$P[A/B] = P(A)$$
$$P[B/A] = P(B)$$

If A and B are independent Event then Outcome of B should have no Effect on the prob. of A and vice-versa.

B) Three Events A, B and C defined on S, and having non-zero probability are they are independent iff.

→ Eg 8

unconditional probability.

P(B) of an event B in terms of the sum of weighted, conditional probab.

Let $A_1, A_2, A_3 \ldots A_n$ be n mutually exclusive events such that $\bigcup_{i=1}^{n} A_i = \Omega$, let P be any prob defined on i=1 the probab. space of the A_i's. Then with $P[A_i] \neq 0$ all i;

$$P[B] = P[B/A_1] * P[A_1] + \ldots + P[B/A_n] * P[A_n]$$

Some times P(B) is called the "avg." or total probability of B.

$P(A/B) = \frac{\quad}{P(B)}$

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<u>Bayes Theorem</u> : Let A_i, $i = 1 \ldots n$ be a set of disjoint and exhaustive events defined on a probability space S then $\bigcup_{i=1}^{n} A_i = \Omega$.

$A_i A_j = \phi$ for $i \neq j$. With B any event defined on S with $P(B) > 0$ and $P[A_i] \neq 0$ all i.

$$P[A_j/B] = \frac{P[B/A_j]\, P[A_j]}{\sum_{i=1}^{n} P[B/A_i]\, P[A_i]}$$

$P(A_j/B)$

$P[B]$

$P[A_j/B]$ = a posteriori prob of A_j given B,

$P[B/A_j]$ = a priori prob of B given A_j,

$P[A_j]$ = a priori probability

Distribution Function.

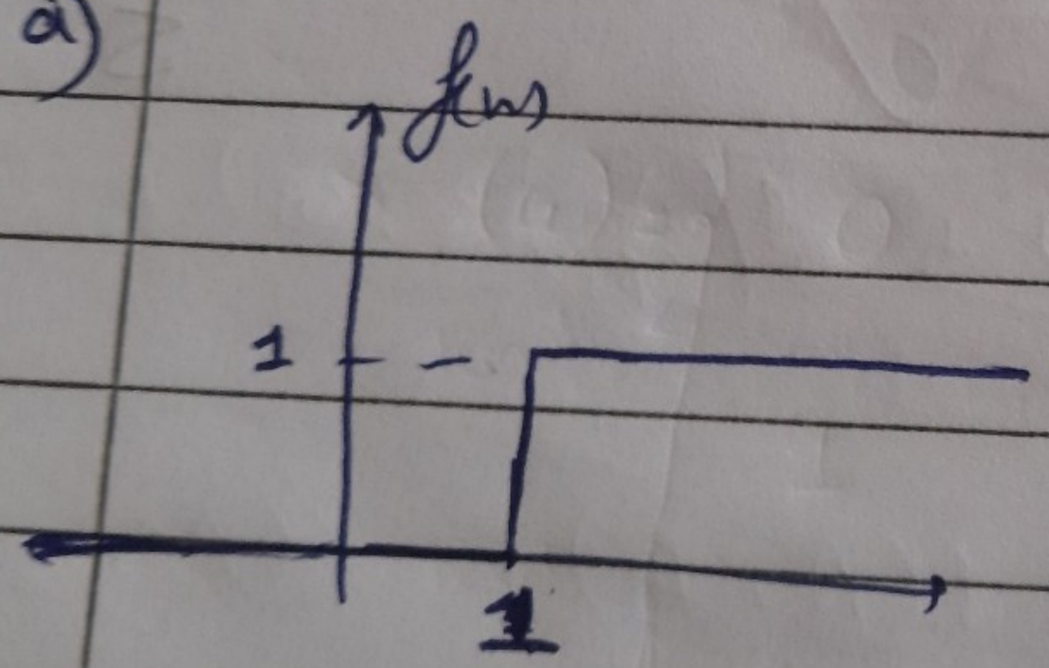
Def:

A real valued fn F satisfying

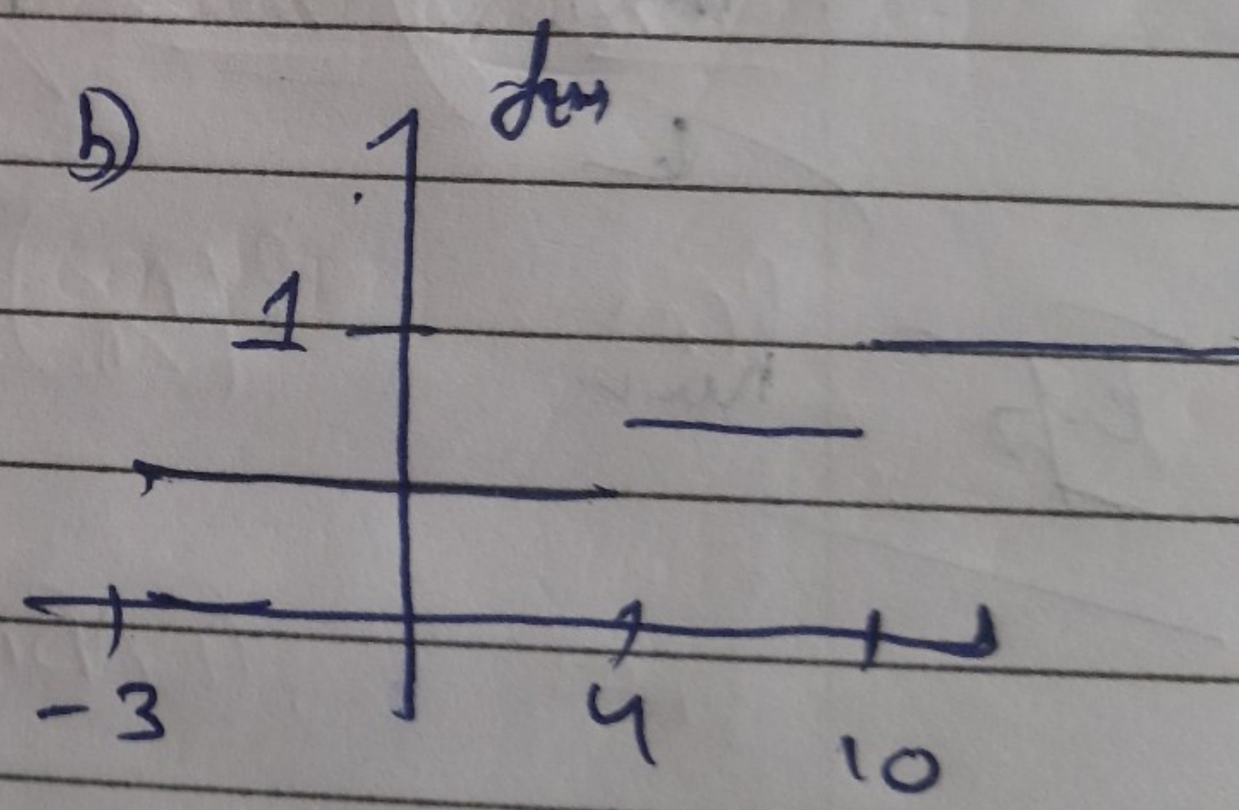
1) $0 \le F(x) \le 1 \quad \forall \; -\infty < x < \infty$
2) Monotonically increasing fn in x i.e if $x_1 < x_2$ then $f(x_1) \le f(x_2)$
3) $\lim_{x \to -\infty} f(x) = 0$ & $\lim_{x \to +\infty} f(x) = 1$
4) F is (right) continuous fn.

Then F is called distribution function

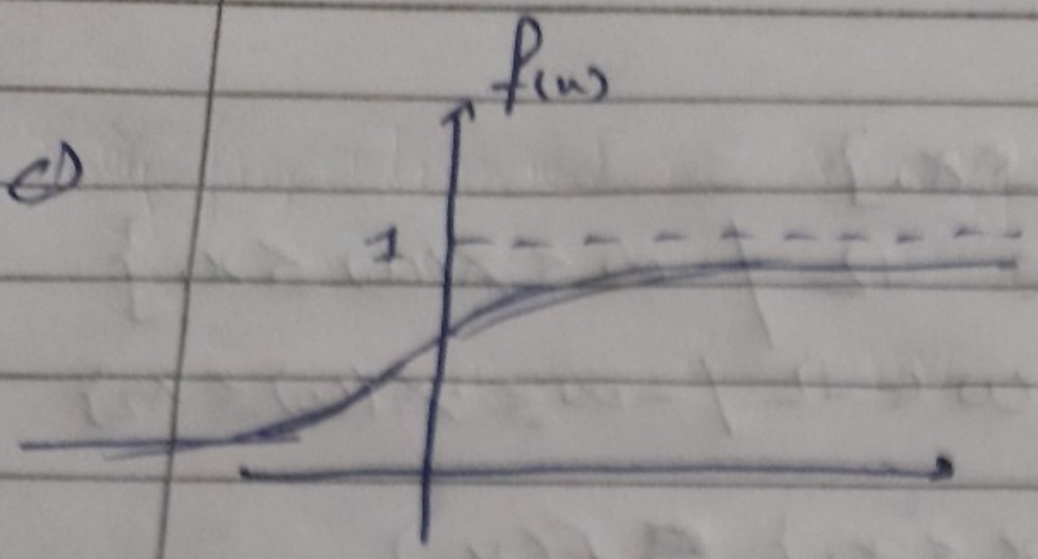
a)



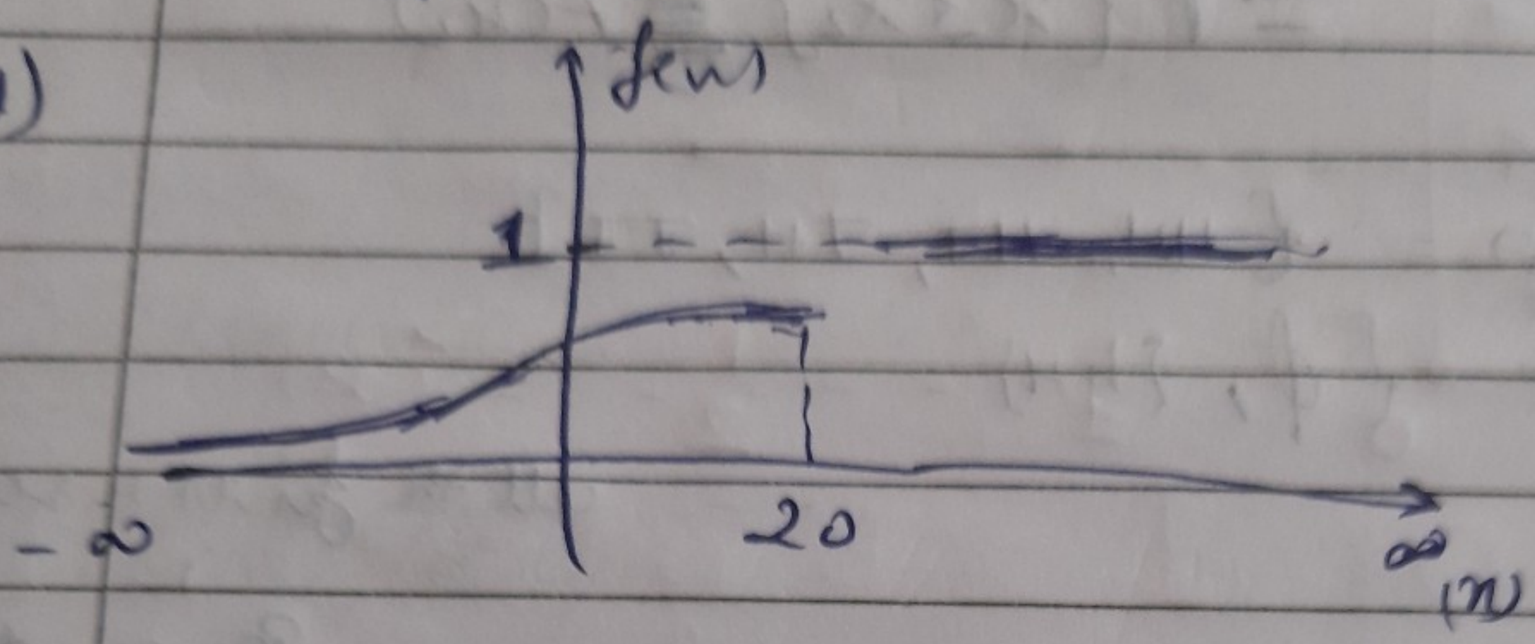
b)



c)



d)



→ Cumulative distribution function. (CDF)

$$F_X(x) = P\{X \le x\},\ x \in \mathbb{R}$$

or

$$F(x) = P\{\omega \in \Omega \mid X(\omega) \le x\}$$

i) Let $x_1 < x_2$

Then, $\underbrace{(-\infty, x_1] \subset [-\infty, x_2]}$

(One set is contained in the other set)

$F(x_1) = P\{X \le x_1\}$

$P\{\omega \in \Omega \mid -\infty < X(\omega) \le x_1\}$

$\le P\{\omega \in \Omega \mid -\infty \le X(\omega) \le x_2\}$

$\le P\{X \le x_2\} = F(x_2)$

Ex → $\Omega = \{HH, HT, TH, TT\}$

$\mathcal{F} = \{\phi, \{HH\}, \ldots\ldots$

— all the finite subsets

Power set of Ω

X / No. of ~~Heads~~ Heads observed

$$X(\omega) = \begin{cases} 0 & : \omega = \{TT\} \\ 1 & : \omega = \{HT\} \text{ or } \{TH\} \\ 2 & : \omega = \{HH\} \end{cases}$$

Head

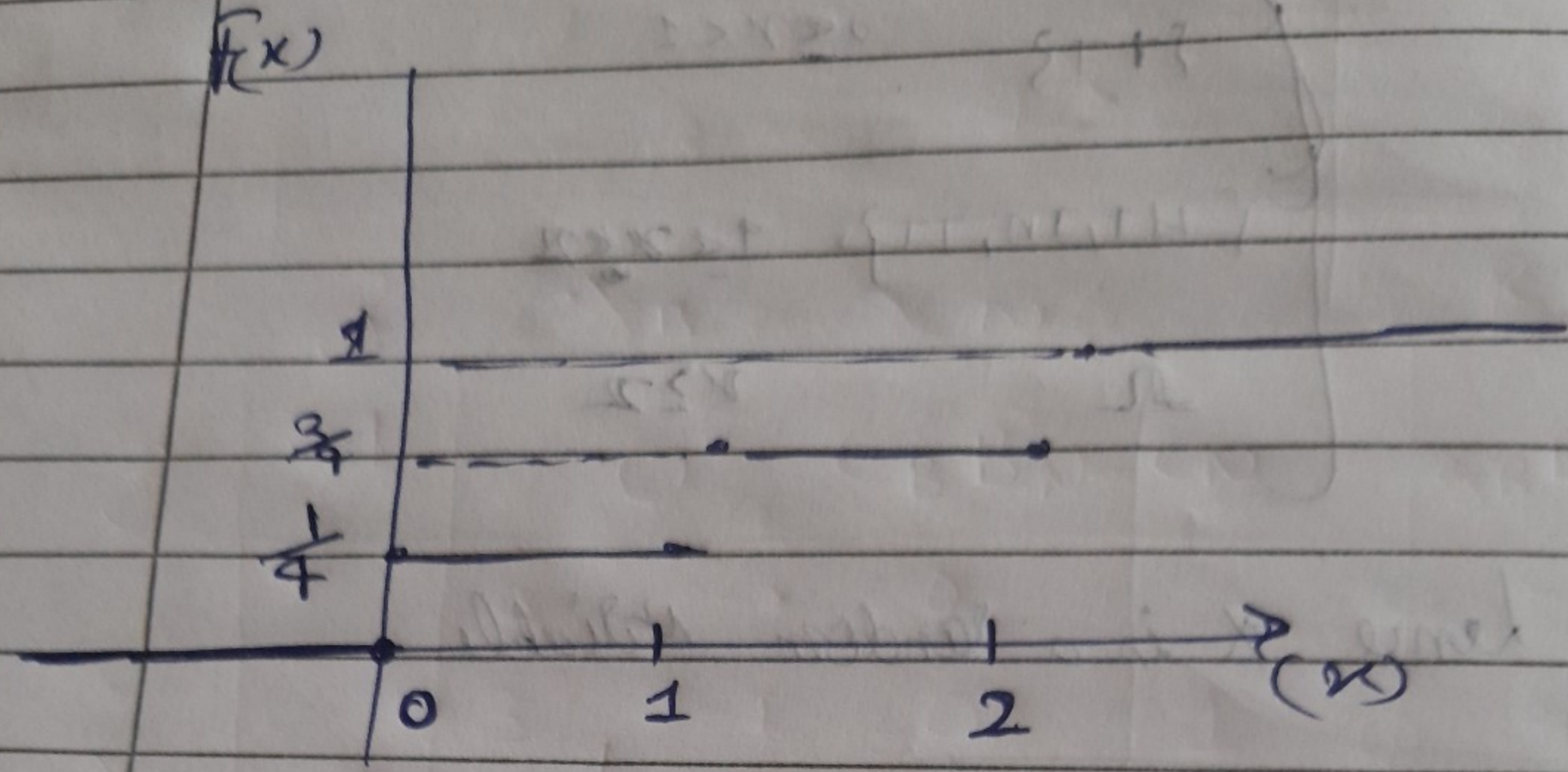
Head

$$X\{[-\infty, x]\} = \begin{cases} \phi & x \le 0 \\ \{TT\} & 0 \le x < 1 \\ \{HT, TH, TT\} & 1 \le x \le 2 \\ \Omega & x \ge 2 \end{cases}$$

Hence X is a random variable.

$$F_X = P\{X \le x\}$$

$$= \begin{cases} 0 & -\infty < x \le 0 \\ P\{\omega / X(\omega) \le x\} = P\{TT\} & 0 \le x < 1 \\ \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} & \\ P(TT) + P(HT) + P(TH) & 1 \le x < 2 \\ \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} & \\ P(TT) + P(HT) + P(TH) + P(HH) & x \ge 2 \\ \frac{4}{4} = 1 & \end{cases}$$



f(x) is a distribution fn.

$F(x) = P\{X \le x\}$ is the CDF of R.V X.

Random variable are of diff. types like:

a) Discrete Random Variable
b) Continuous Random Variable

Discrete Type Random variable

A Random variable X is said to be a discrete of Random Variable if it has countable no. of discontinuity.

Let the discontinuities a $x_1, x_2, \ldots$

$$P(X=x_i) = P\{\omega / x(\omega) = x_i\}, \quad i=1,2,$$
$$\omega \in \Omega$$

Probability mass funct.

$$F(x) = \sum_{x_i \le x} P(X=x_i) \quad ; \quad -\infty < x < \infty$$

the $x_i = 1, 2, \ldots$ are called jump points

$P(X=x_i) > 0$ jump value

Probability mass fnc:

Discrete type RV is defined as

$$P(x) = P(X = x_i) \quad i = 1, 2, \ldots, x \in \mathbb{R}$$

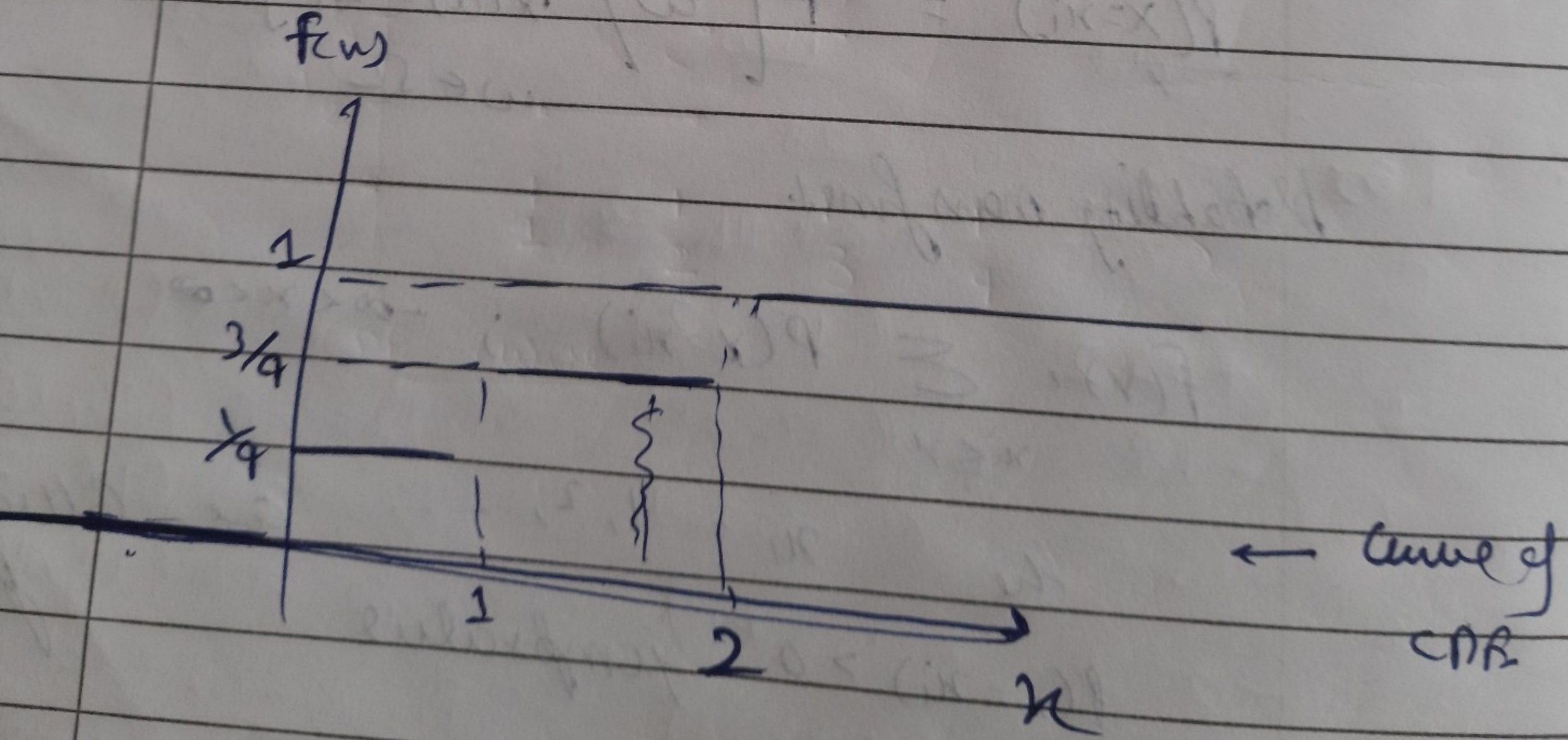
It has foll. properties

1) $P(x) \geq 0 \ \forall x$
2) $\sum_x P(x) = 1$

X : # of heads obtained.

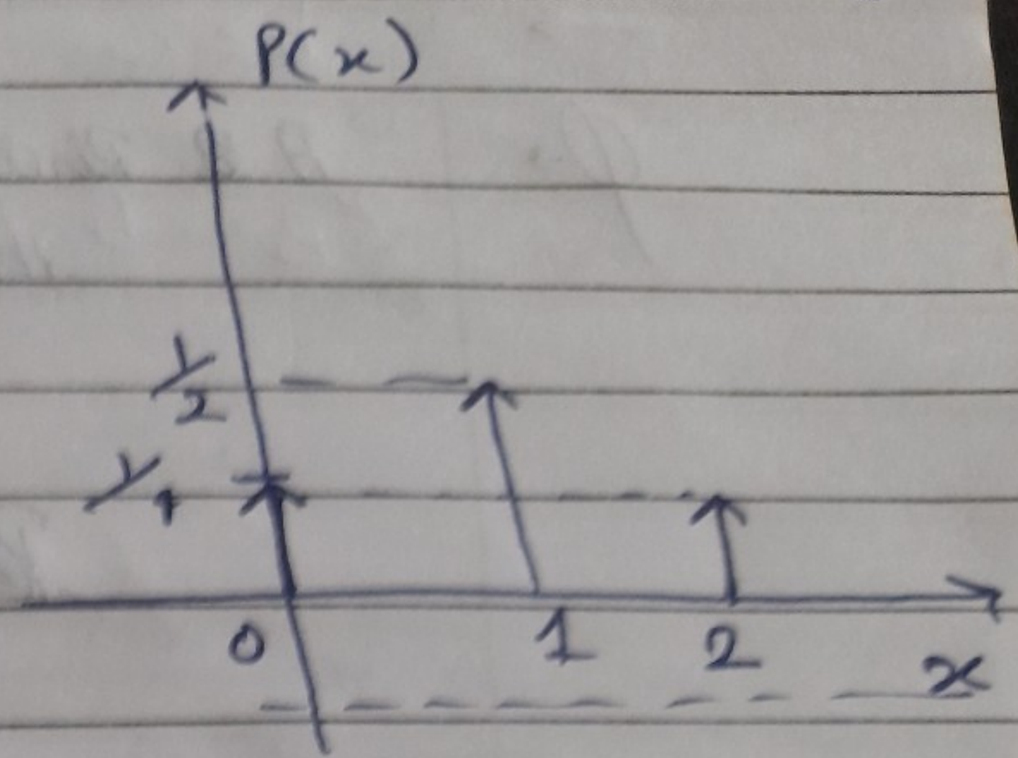
$$F(x) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq x \leq 1 \\ 3/4 & 1 \leq x \leq 2 \\ 1 & 2 \leq x \leq \infty \end{cases}$$

CDF



HT TH HH

→ $P(x) = \begin{cases} 1/4 & x=0 \\ 1/2 & x=1 \\ 1/4 & x=2 \\ 0 & \text{otherwise.} \end{cases}$



→

Q→ verify the function $P(x) = \begin{cases} \frac{3}{4}\cdot\left(\frac{1}{4}\right)^x & \text{when } x \text{ is } 0, 1, \\ 0 & \text{otherwise} \end{cases}$

$\frac{3}{4}\cdot\left(\frac{1}{4}\right)^1 + ?$

$\frac{3}{4}\left(\frac{1}{4} + \left(\frac{1}{4}\right) - - -\right)$

$\frac{3}{4}\left(\frac{1}{1-\frac{1}{4}}\right)$

$\frac{3}{4}\left(\frac{4}{3}\right) = 1$ Hence proved

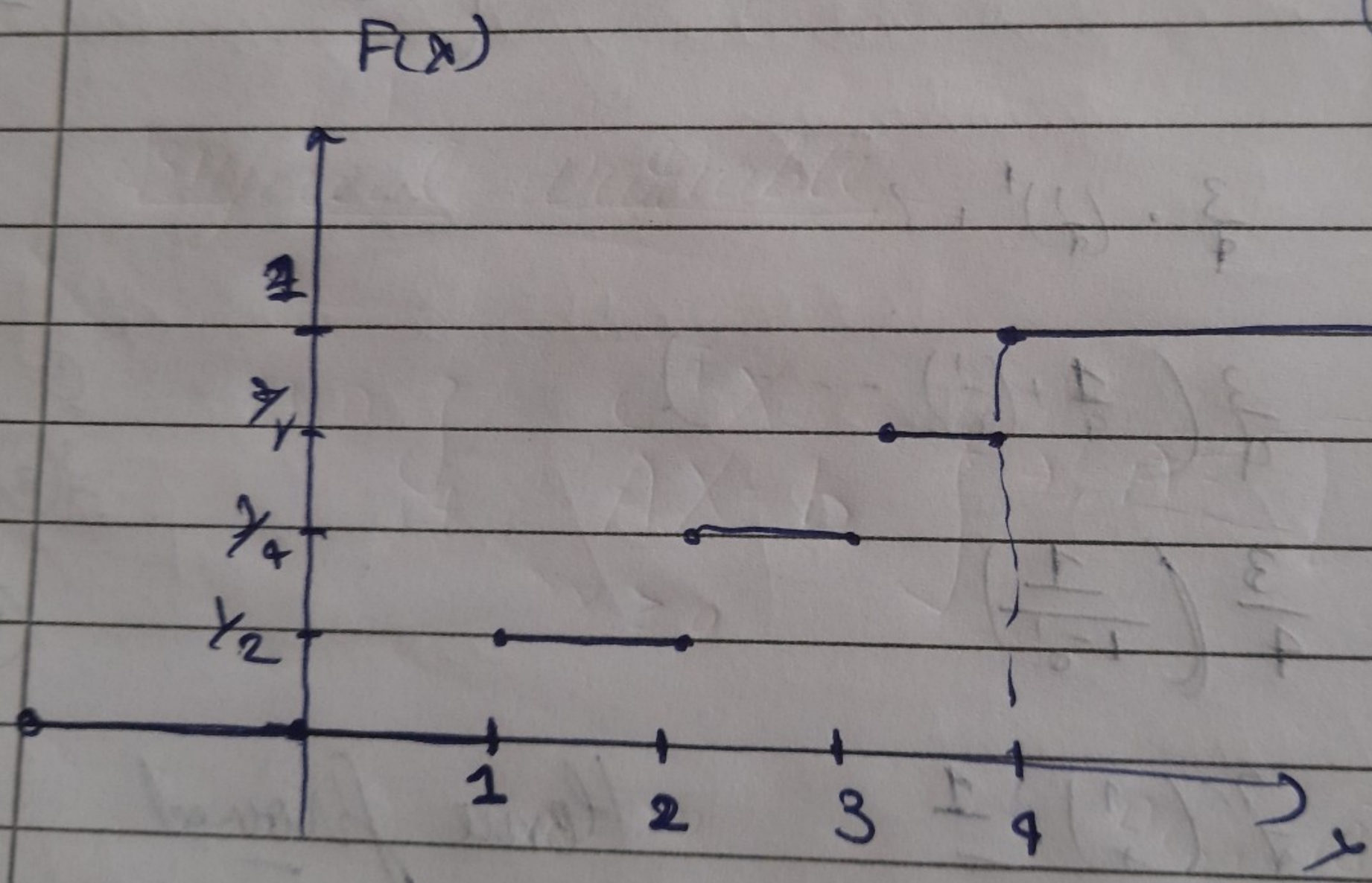
→ $P(x=2) = \frac{3}{4}\cdot\frac{1}{16} =$

$P(x \le 2) = \frac{3}{4}\cdot 1 + \frac{3}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{16} =$

$P(x \ge 1)$ $\frac{1}{4}$

Q. A Discrete R.V X has a following Probability mass fonc.

$$P(x) = \begin{cases} 1/2 & x = 1 \\ 1/4 & x = 2 \\ 1/8 & x = 3 \\ 1/8 & x = 4 \end{cases}$$



CDF Curve

PSP

→ Continuous Type of Random Variable.

→ A RV x is said to be continuous type RV if its CDF is a continuous function of x

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad -\infty < x < \infty$$

where $f(t)$ is called probability density fun of the continuous type R.V, X.

it satisfies the following properties:-

$$f(t) \geq 0 \quad \forall\, t \in \mathbb{R}.$$

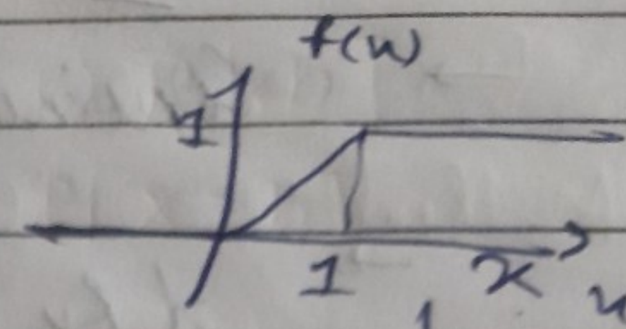
$$\int_{-\infty}^{\infty} f(t)\,dt = 1$$

$$\boxed{f(x) = \frac{dF(x)}{dx}}$$ — CDF

PDF

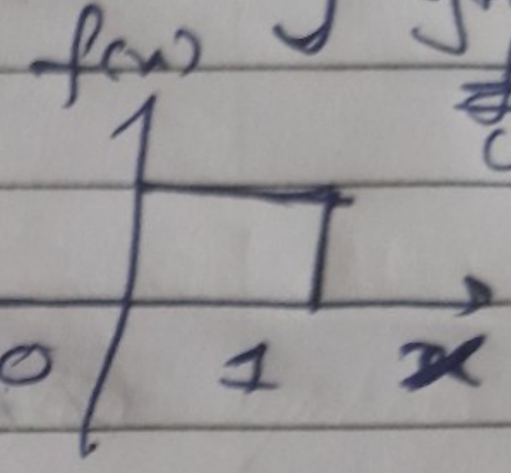
Ex → Let X be a RV with CDF given by-

$$F(x) = \begin{cases} 0 & x<0 \\ x & 0 \le x \le 1 \\ 1 & 1 \le x < \infty \end{cases}$$



derivative of this ~~fun~~ curve

$$f(x) = \begin{cases} 0 & x<0,\ x>1 \\ 1 & 0<x<1 \end{cases}$$



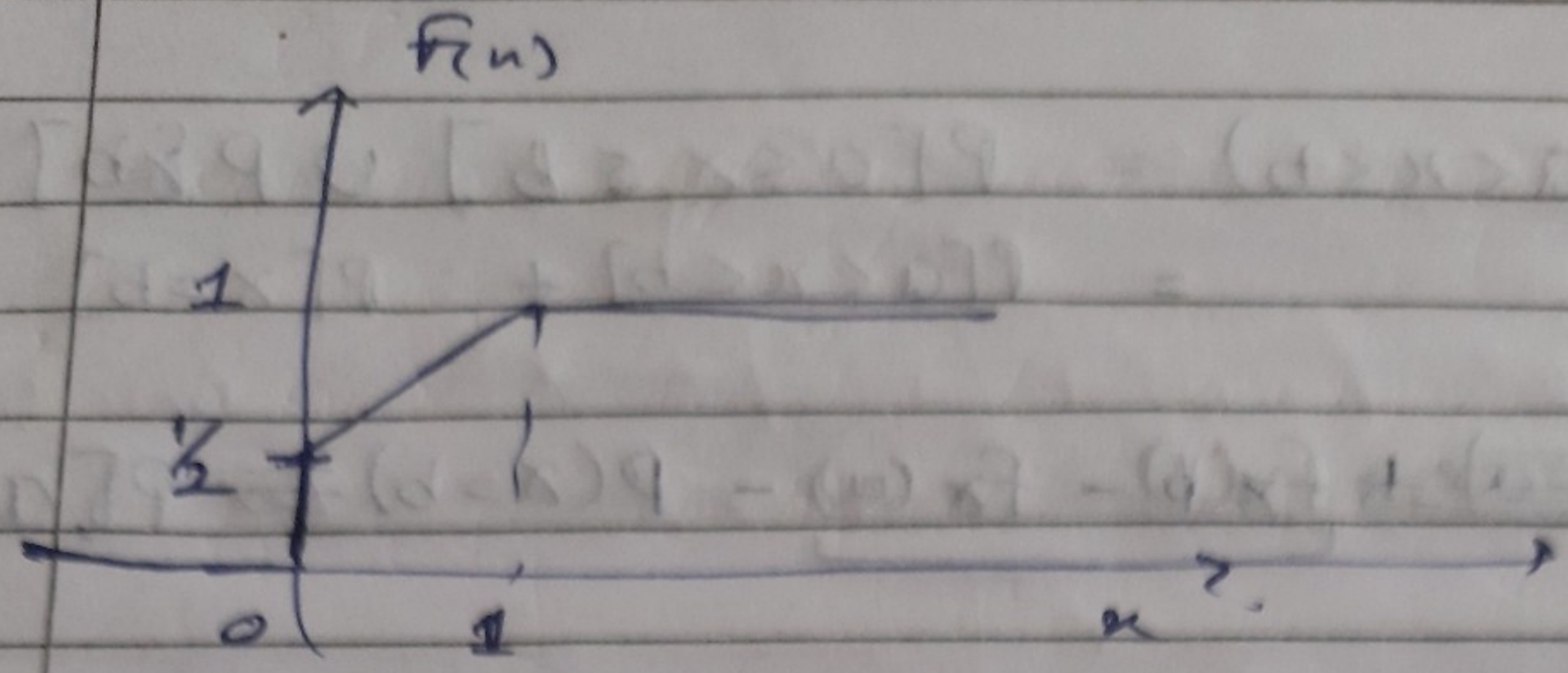
<u>Mixed Type RV</u>

A RV x is said to be mixed type RV if its cdf has both countable numbers of discontinuous and continuous in some interval.

$$\sum_i P(X=x_i) + \int_{-\infty}^{\infty} f(t)\,dt = 1$$

Ex

$$F(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ \frac{1}{2}+\frac{x}{2} & 0 \le x < 1 \\ 1 & 1 \le x < \infty \end{cases}$$



X is a mixed type RV.

<u>Remarks</u>

$P(a < x \leq b) = F(b) - F(a)$

$P(a \leq X \leq b) = F(b) - F(a) + P(X=a)$

Q- a) $P(a \leq x \leq b) = P(X=a) + F_X(b) - F_X(a)$

$\Downarrow$

$P(X=a) \cup \{P(a < X \leq b) \cup P(b)$

$\cancel{P(X=a) + P(}$

$P(X=a) + F_X(b) - F_X(a)$

b) $P(a < X < b) = F_X(b) - F_X(a) - P(X=b$

$$P(a<x\leq b) = P[a<x\leq b] \cup P[x=b]$$

$$= P[a<x<b] + P[x=b]$$

$$\underbrace{F_X(b) - F_X(a)} - P(X=b) = P[a<x\leq b]$$

~~An information source generates symbols~~
~~Random~~

$P(a) = \frac{1}{2}$ $\quad P(b) = \frac{1}{4}$ $\quad P(c) = P(d) = \frac{1}{8}$

Q → Mean and variance

Mean or Expectation

Let X be a RV ~~defs~~ defined on $(\Omega, \mathcal{F}, P)$ probab space, if X is a discrete type RV with values $x_1, x_2, x_3, \ldots$

it is said that X has Expected value if

$$\sum_i |x_i| P(X = x_i) < \infty$$

The Expected value of RV X,

$$\boxed{E(X) = \sum_i x_i P(X = x_i)}$$

if X is a continuous type RV ~~then~~ with pdf $f(x)$ it is said that X has the Expected value if

$$\int_{-\infty}^{\infty} |x_i| \, f(x)\,dx < \infty$$

The Expected value of the Random variable X is given by :

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Ex Let X be a continuous type RV with Pdf

$$f(x) = \frac{1}{\pi(1+x^2)} \quad , \quad -\infty < x < \infty$$

Now :

$$\int_{-\infty}^{\infty} |x| \, f(x)\,dx = \frac{2}{\pi}\int_{0}^{\infty} \frac{x}{(1+x^2)}\,dx$$

$$= \infty$$

E(x) does not Exist

Let X be a RV with n^{th} order, moment about the origin value exists.

n^{th} moment $\mu_n' = E[X^n]$; $n = 1, 2 \ldots$

$\Downarrow$

$\mu_1' = E[X] \rightarrow$ mean

$\mu_2' = E[X^2]$

$Var(X) = \mu_2 = E[(X-\mu)^2]$

$= E(X^2) - \mu^2$

$$\boxed{Var(X) = \mu_2' - \mu_1'^2}$$

Q. Let X be a cont. type random variable with pdf

$$\left\{ f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right\} \rightarrow \text{gaussian distribut}$$

$$-\infty < x < \infty$$

$$\sigma > 0$$

→ $\mu_1 = E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$= \mu$

$E[x^2] = \int_{-\infty}^{\infty} x f(x) dx = \sigma^2 + \mu^2$

$var[x] = \mu^2 + \sigma^2 - \mu^2$

$= \sigma^2$

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Generating function →

1. Probability Generating fn.
2. moment gen. func.
3. Characteristics function.

Defn Probability Generating fnct :

Let X be a non-negative, integer-valued RV with $P_k = P(X=k)$, $k=0,1,2\ldots$

Then $G_X(s) = \sum_{k=0}^{\infty} P(X=k)s^k$, $|s| \le 1$

Real variable

Defn Moment generating function →

Let X be a RV such that $E(e^{tx})$ exists for t in some interval including zero.

Then $M_X(t) = E[e^{tX}] = 1 + t\,E[X] + \frac{t^2}{2!}E[X^2] + \cdots \frac{t^n}{n!}E(X^n) + \cdots$

1) $E[x^n] = \left.\dfrac{d^n M_x(t)}{dt^n}\right|_{t=0}$, $n = 1, 2, \ldots$

n^{th} order moment about the origin

2) If x and y are having $M_x(t) = M_y(t)\ \forall t$. Then x and y are having the same distribution.

3) <u>Characteristic function</u> →

Let X be a Random variable

$$\varphi_x(t) = E(e^{itx}) \qquad \varphi_x : \mathbb{R} \to \mathbb{C}$$

$$E(e^{itx}) = \int_{-\infty}^{\infty} |e^{itx}| f(x)\,dx \leq \int_{-\infty}^{\infty} f(x)\,dx = 1$$

→ $\psi_x(0) = E[1] = 1$

→ $|\psi_x(t)| \leq 1 \quad \forall t$

if $E[X^n]$ exist, then

$$E[X^n] = \frac{1}{i^n} \times \left(\frac{d^n \psi_x(t)}{dt^n}\right)\Big|_{t=0}$$

n^{th} order moment about the origin

Q. Let X be a RV with Pdf.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$\int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$\boxed{M_x(t) = e^{\mu t + t^2\sigma^2/2}}$$

Various Standard Distribution

Some common Discrete Distribution

1. Bernoulli's distribution
2. Binomial distribution
3. Poisson distribution

A RV X is called Bernoulli's RV with parameter 'p', if its Pmf is given by →

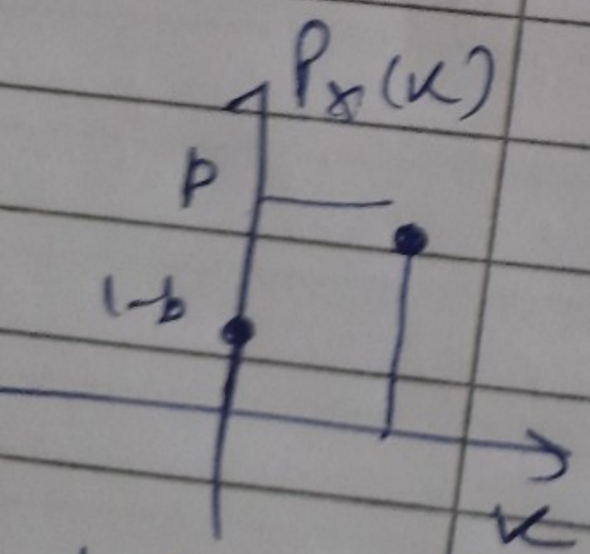
$$p_X(k) = P(X=k) = p^k (1-p)^{1-k}, \quad k \in \{0, 1\}$$

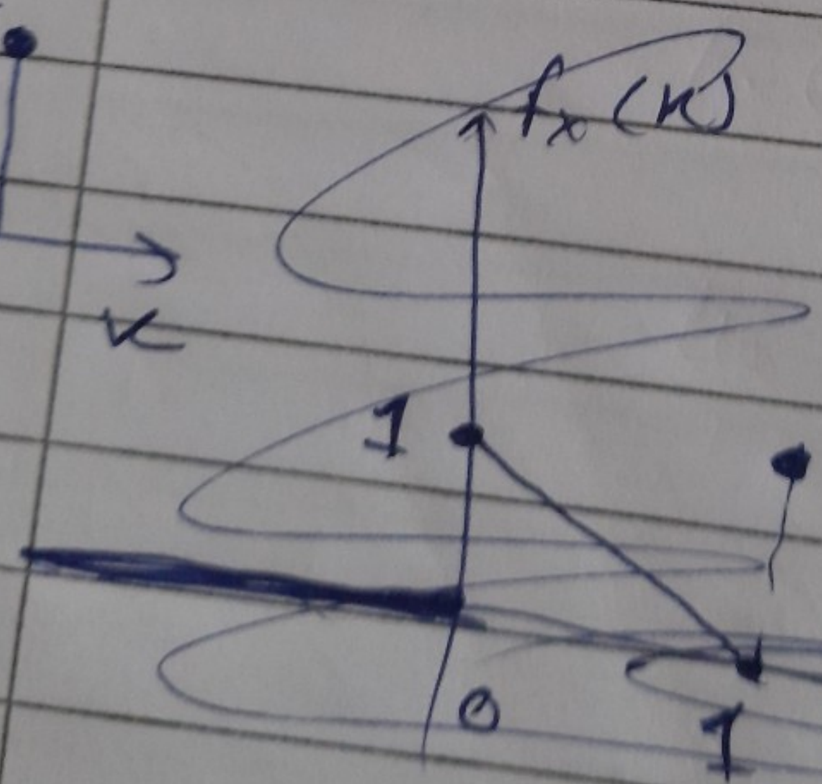
when $0 \le p \le 1$

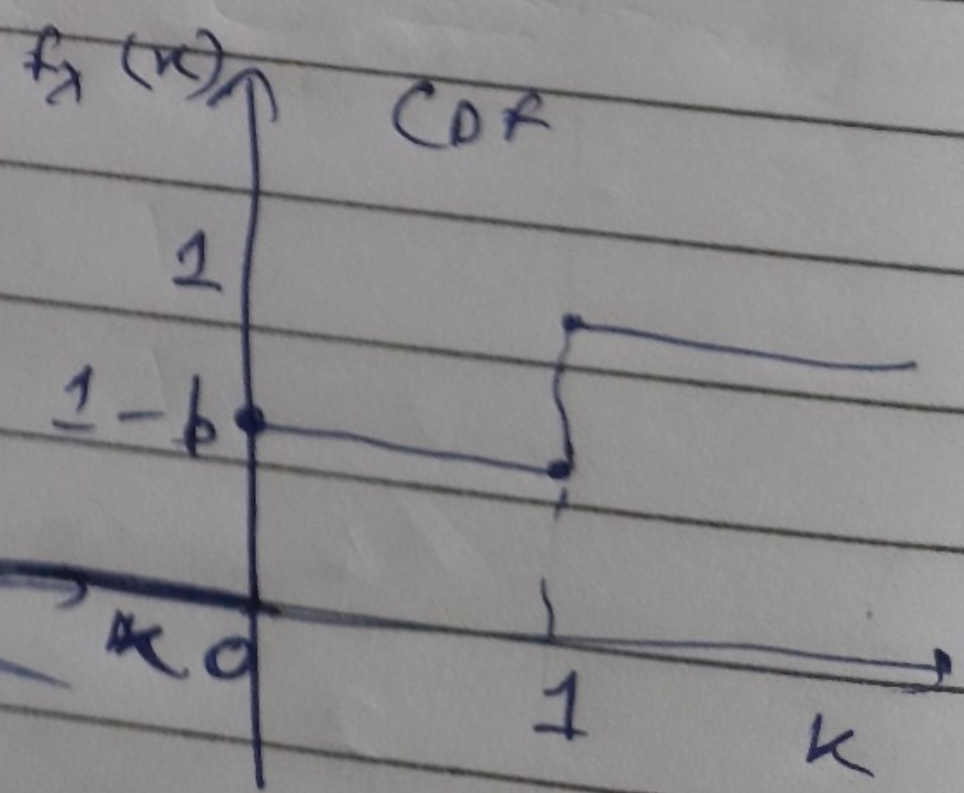
CDF

$$f_X(x) = \begin{cases} 0 & , x<0 \\ 1-p & , 0 \le x < 1 \\ 1 & , x \ge 1 \end{cases}$$

Pmf







$\mu_x = E(x) = p$

$\sigma_x^2 = Var(x) = p(1-p)$

→ Binomial distribution

A RV x is a Binomial distributed RV with parameters (n, p) if its pmf is given by :

$$p_x(k) = P(x=k) = {}^nC_k \, p^k (1-p)^{n-k}$$

$k \in 0, 1, 2 \ldots n$

$0 \leq p \leq 1$

The CDF of X is

$$F_x(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

The mean and variance of the Binomial RV are :

$\mu_x = E(x) = np$

$\sigma_x^2 = Var(x) = np(1-p)$

A binomial RV X is associated with some exp. in which n independent Bernoulli's trials are performed and X represents the number of success that occur in the n trials.

$\therefore$ Bernoulli's RV is just a Binomial RV with parameter $(1, p)$.

Poisson Distribution.

A RV X is called Poisson RV with Parameter λ $(\lambda > 0)$ if its Pmf is given by

$$\boxed{p_X(k) = P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, \ldots n.}$$

The cdf of X is.

$$F_X(x) = e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} \quad 0 \le x \le n$$

$$\mu_X = E(X) = \lambda$$
$$\sigma_X^2 = Var(X) = \lambda$$

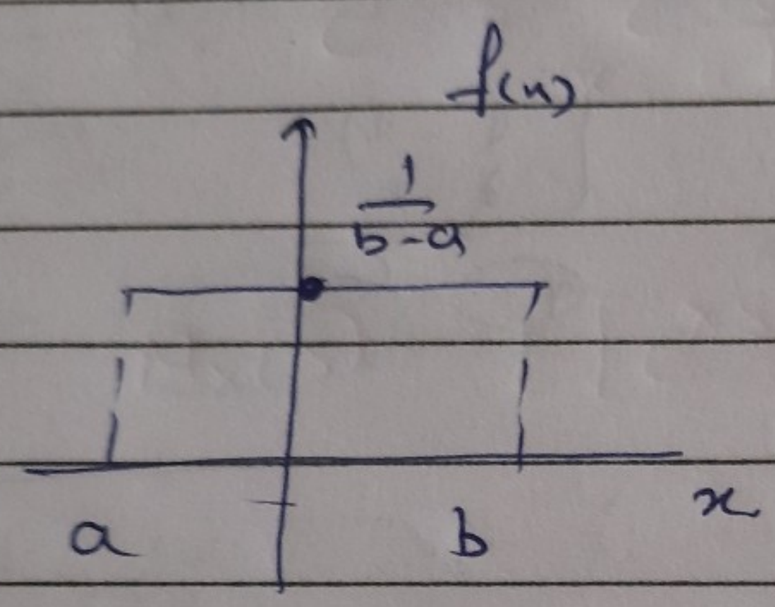
__/__/__

Some Imp. Continuous distribution

→ ① continuous uniform distribution : $X \sim U(a,b)$

→ Pdf

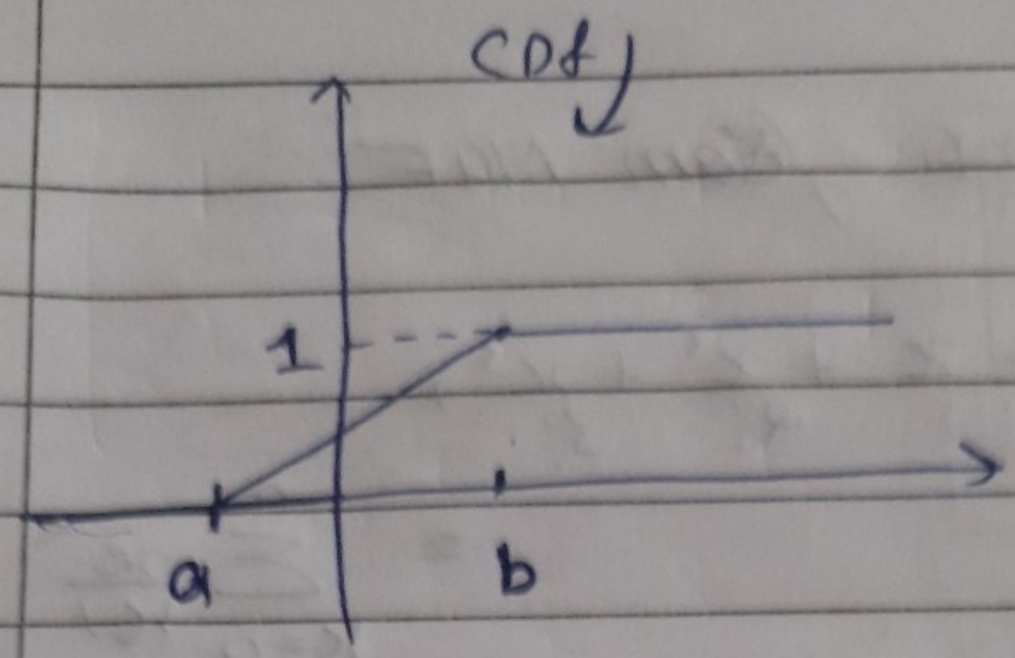
$$f_X(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0 & , o.w \end{cases}$$



$$P(X \le x) \rightarrow = F_X(x) = \int_{-\infty}^{x} f(t)\,dt \Rightarrow \int_{-\infty}^{a} 0 + \int_{a}^{x} \frac{1}{b-a}\,dt + \int_{b}^{a} 0$$

$$\frac{1}{b-a}[t]_a^x = \boxed{\frac{x-a}{b-a}}$$

→



$$E[x] = \int_{-\infty}^{\infty} x f(x) dx = \frac{a+b}{2} = \underline{\text{mean}}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \frac{b^2+ab+a^2}{3}$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(a-b)^2}{12}$$

→ <u>MGF</u>

$$M_x(s) = E[e^{sx}] = \int_a^b e^{sx} \frac{1}{b-a} dx$$

$|s|<1$

$$= \frac{1}{s(b-a)} [e^{sb} - e^{sa}]$$

→ finding moments from MGF

Expansion → $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \ldots$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\therefore e^{sx} = \sum_{k=0}^{\infty} \frac{(sx)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k s^k}{k!}$$

$$M_x(s) = E[e^{xs}] = \sum_{k=0}^{\infty} E[x^k] \frac{s^k}{k!}$$

Ex: if $y \sim U(0,1)$, find $E[y^k]$ using MGF $M_y(s)$.

A:

$$E[e^{sy}] = \int_0^1 e^{sy}\, dy = \frac{e^s - 1}{s}$$

$$\cancel{\sum_{k=0}^{\infty} E[y^k] \frac{s^k}{k!} = \frac{e^s - 1}{s}}$$

Expansion of e^s

$$= \frac{1}{s}\left(\sum_{k=0}^{\infty} \frac{s^k}{k!} - 1\right)$$

$$= \frac{1}{s} \sum_{k=1}^{\infty} \frac{s^k}{k!}$$

$$\sum_{k=1}^{\infty} \frac{s^{k-1}}{k!}$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{k+1}\right) \frac{s^k}{k!}$$

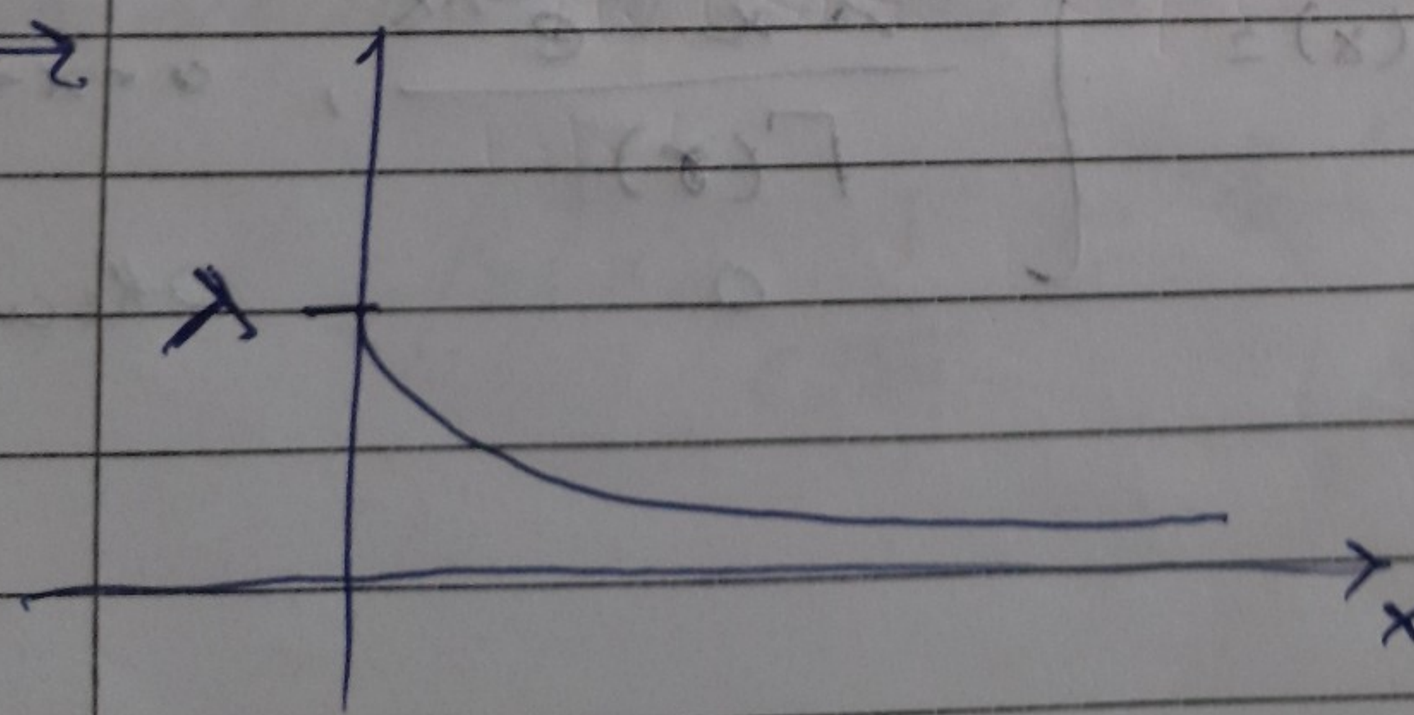
$$\boxed{\text{we can see that } E[y^k] = \frac{1}{k+1}}$$

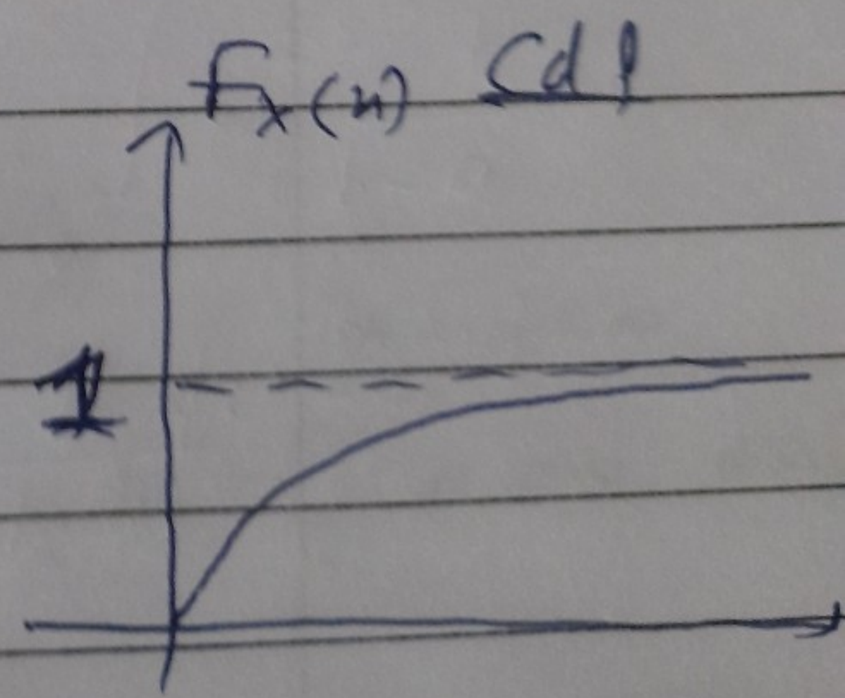
$$\boxed{E(x^k) = \frac{d^k \, m_x(s)}{ds^k}\Big|_{s=0}}$$ Other method

2) Exponential distribution.

Pdf $f(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty \\ 0 & ; \text{o.w} \end{cases}$

$\lambda \geq 0$





1) $E(x) = \frac{1}{\lambda}$

2) $E(x^2) = \frac{2}{\lambda^2}$

$Var(x) = \frac{1}{\lambda^2}$

→ $M_x(s) = E[e^{sx}] = \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx$

$= \frac{1}{1-\frac{s}{\lambda}}, \quad s < \lambda$

$= \frac{\lambda}{\lambda - s}$

→ Gamma distribution

$$f_x(x) = \begin{cases} \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Ill.

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→ where $\lambda > 0$ (real +ve no.) and r is +ve real number

$$\Gamma(r) = \int_0^t t^{r-1} e^{-t}\, dt, \quad r > 0$$

when $r = n$ (+ve integer.)

$$\Gamma(n) = (n-1)!$$

a) when $r = 1$

$$f_X(x) = \lambda e^{-\lambda x}$$

$$X \sim Exp(\lambda)$$

b) $E(x) = \frac{r}{\lambda}$

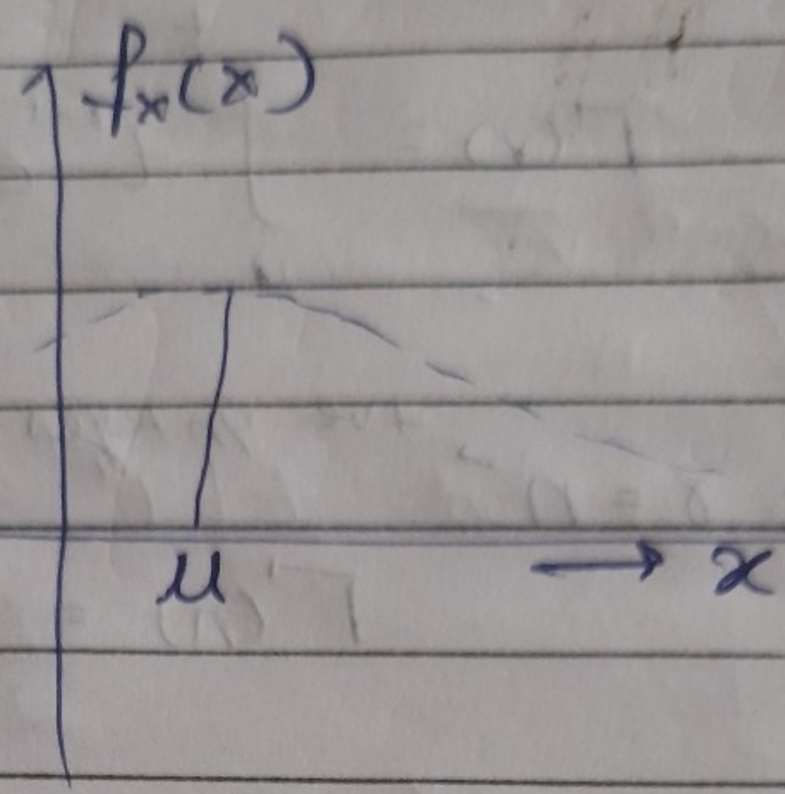
$var(x) = \frac{r}{\lambda^2}$

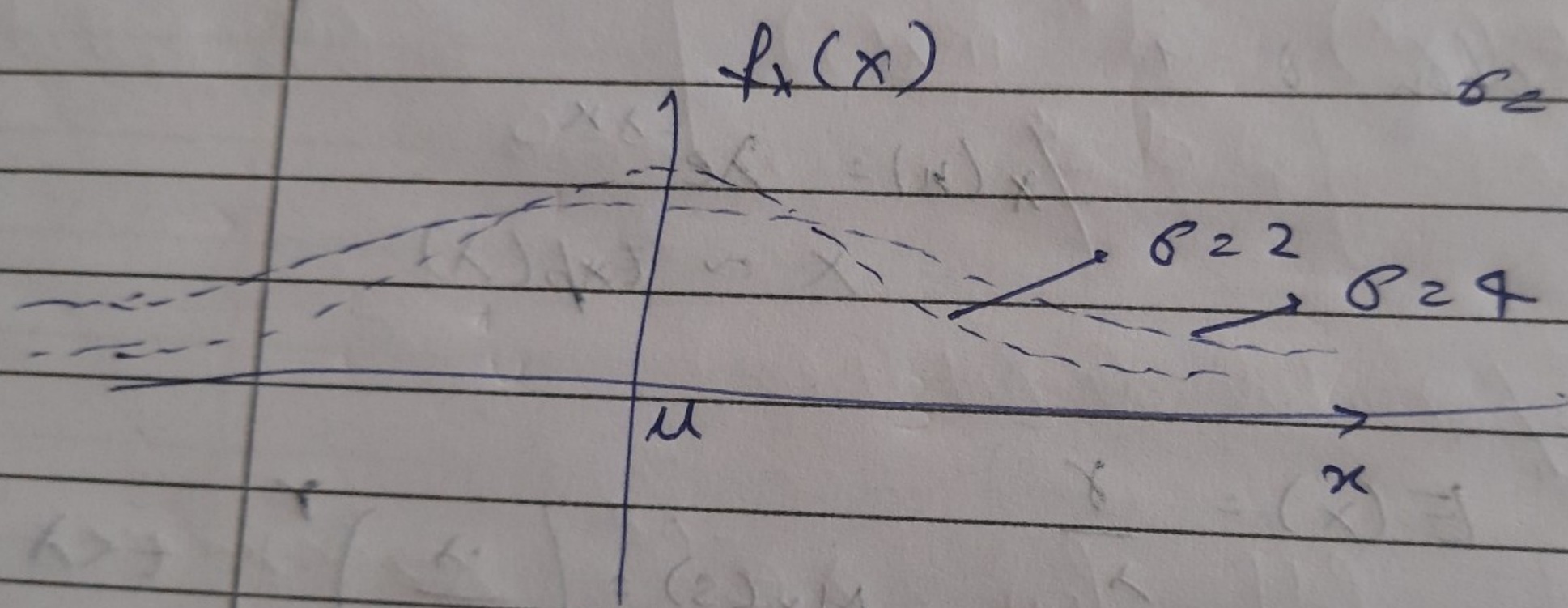
$$M_X(s) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

- Normal distribution

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty,$$

→ $X \sim N(\mu, \sigma^2)$





σ = standard deviation

when $\mu = 0$ and $\sigma^2 = 1$ } standard normal distribution

$Z \sim N(0,1)$

__/__/__

$X \sim N(\mu, \sigma^2)$

$\rightarrow \quad P(a < X \leq b) = \int_a^b f_X(x)\,dx.$

$P\left(\frac{a-\mu}{\sigma} < \frac{x-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$

$P(a_1 < Z \leq b_1)$

$\rightarrow \phi(z) = \int_{-\infty}^{z} f_Z(t)\,dt$

cdf for std. normal dist. $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt.$

$= \phi(b_1) - \phi(a_1)$

$*$

$f_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = F_X(x)$

$\rightarrow$

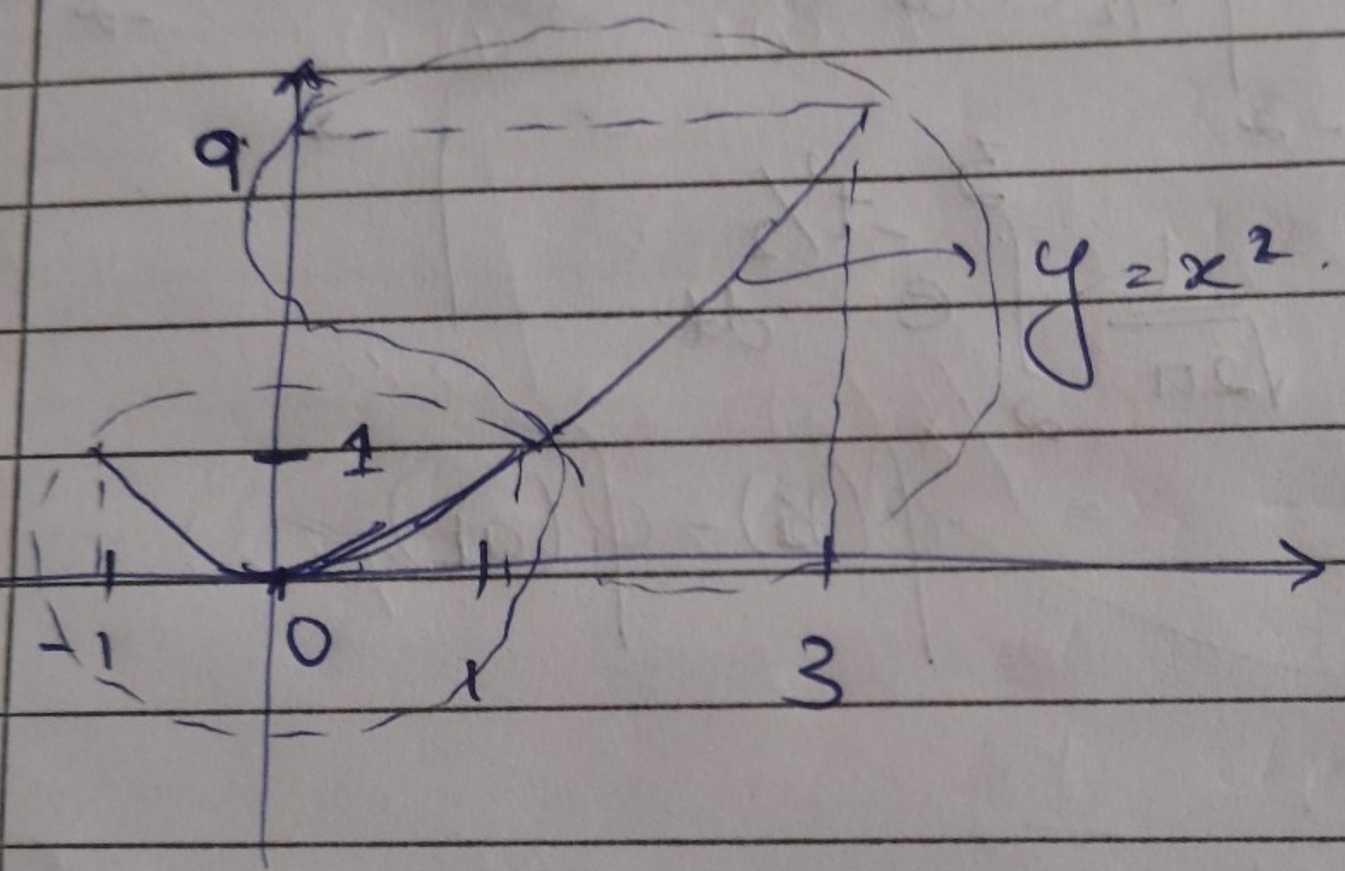
Q. A number is randomly chosen b/w [1,3] Calculate the probability that the first digit of the right side of decimal pt is 5.

$$f_X(x) = \begin{cases} \left(\frac{1}{2}\right) & a < x < b \\ 0 & \text{o.w} \end{cases}$$

(unif $\frac{1}{b-a}$)

Q- A pt. X is chosen random & in interval $[-1, 3]$ find the Pdf of $y = x^2$.

$$f_X(x) = \begin{cases} \frac{1}{4}, & -1 < x < 3 \\ 0, & o.w \end{cases}$$



$$CDF\ Y = P(Y \le y) = P(X^2 \le y)$$

$$P(-\sqrt{y} \le X \le \sqrt{y})$$

$$\int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\, dx = \frac{\sqrt{y}}{2}$$

for $y \in [1, 9)$

$$F_X(y) = P(Y \le y)$$

$$= P(X^2 \le Y)$$

$$= P(X \le \sqrt{y})$$

$$= \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{1} \frac{1}{4}\,dx + \int_{1}^{\sqrt{y}} \frac{1}{4}\,dx$$

$$= \frac{1}{2} + \frac{\sqrt{y}-1}{4}$$

$$\boxed{f_X(x) = \frac{d\,F_X(x)}{dx}}$$

$\rightarrow$

$$f_X(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & 0 < y < 1 \\ \dfrac{1}{8\sqrt{y}} & 1 < y < 9 \\ 0 & \text{o.w.} \end{cases}$$

→ Q Let x be a random variable.

$$f_x(x) = \frac{1}{\sqrt{\pi}} \left(e^{[-x^2+x-a]} \right), \quad -\infty < x < \infty$$

find a?

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-x^2+x-a}\,dx = 1$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x^2-x+\frac{1}{4}+a-\frac{1}{4})}\,dx = 1$$

$$\cancel{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left((x-\frac{1}{2})^2-(\frac{1}{2})^2\right)}}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x-\frac{1}{2})^2} \cdot e^{-(a-}$$

$\frac{2\pi}{\sigma}$

$$e^{-\left(a-\frac{1}{4}\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} e^{-\left(x-\frac{1}{2}\right)^2} dx = 1$$

by comparing with normal distribution

$$\mu = \frac{1}{2}, \quad \sigma^2 = \frac{1}{2}$$

value of normal distribution function from $-\infty$ to ∞ is 1

$$e^{-\left(a-\frac{1}{4}\right)} = 1$$

$$-\left(a-\frac{1}{4}\right) = \ln 1$$

$$= 0$$

$$\boxed{a = \frac{1}{4}}$$

→ joint Event / joint variable.

Random vector (n-dimensional Random variables)

→ joint pmf / pdf of n-dimensional R.V

Let (x_1, x_2) be a 2-D discrete type Random variable.

with cdf $F(x_1, x_2) = F(x_1 = x_1, x_2 = x_2)$

$\therefore$ pmf $= P_{x_1, x_2}(x_1, x_2) = P(x_1 = x_1, x_2 = x_2)$

= is called the joint pmf of RV (x_1, x_2).

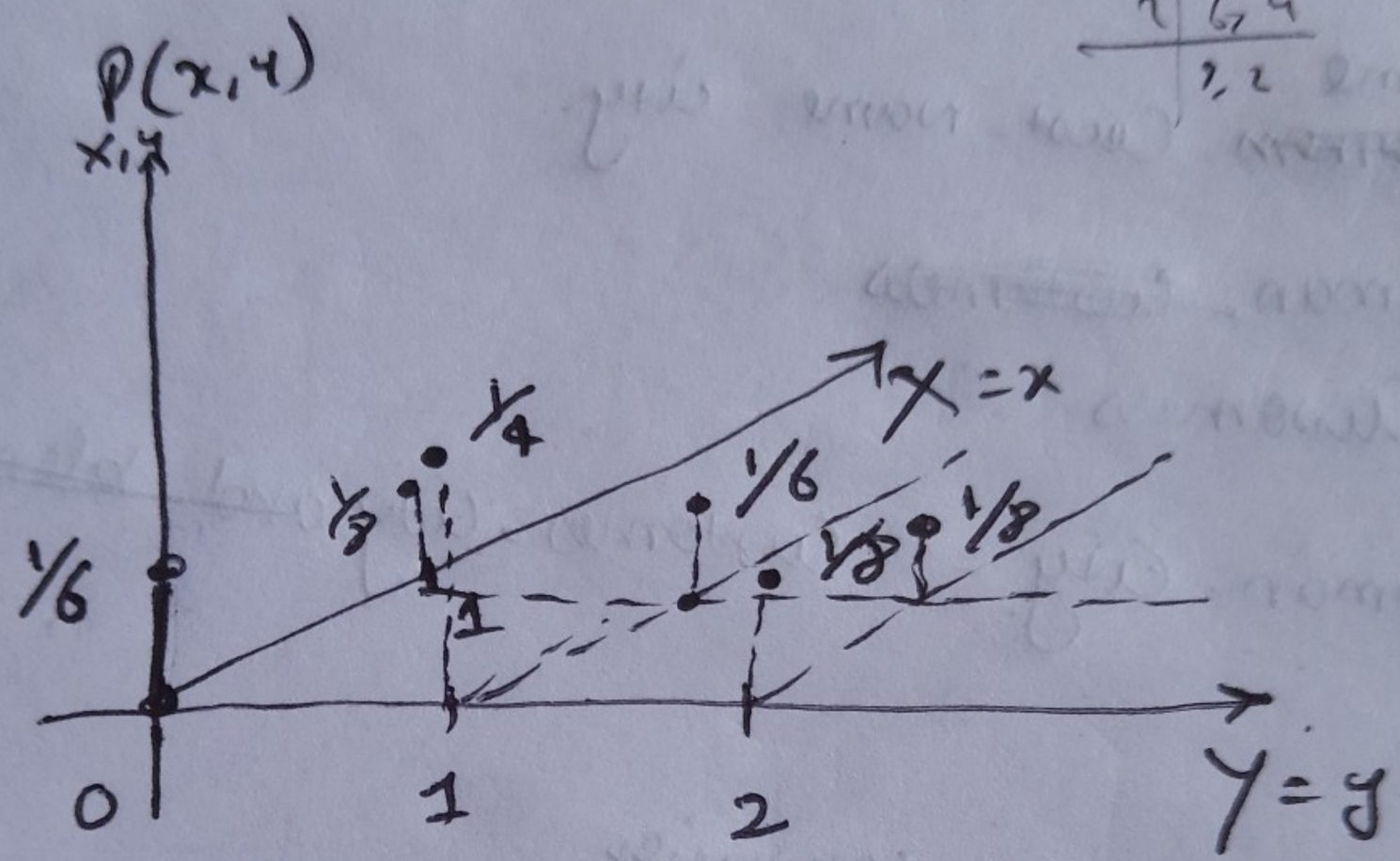
→ $0 \le P(x_1, x_2) \le 1 \quad \forall \; x_1, x_2$

→ $\sum_{x_2} \sum_{x_1} P(x_1, x_2) = 1$

Let x and y be RV of ~~continuous~~ discrete type with joint ~~pdf~~ pmf

x \ y	y = 0	y = 1	y = 2
x = 0	1/6	1/4	1/8
x = 1	1/8	1/6	1/6

→



i) find $P(x=0, y \le 1)$

Sol $P(x=0, y \le 1) = P_{xy}(0,0) + P_{xy}(0,1)$

$$= \frac{1}{6} + \frac{1}{4} = \boxed{\frac{5}{12}}$$

Marginal distribution

→ $P_{x_1}(x_1) = \sum_{x_2} P(x_1, x_2) \rightarrow$ pmf of x_1

Illy $P_{x_2}(x_2) = \sum_{x_1} P(x_1, x_2) \rightarrow$ pmf of x_2

Cos → Nam No [illegible] / M. [illegible]

→ joint cdf of (X_1, X_2)

$$F(x_1, x_2) = P(X \le x_1, X_2 \le x_2)$$

$$-\infty < x_1 < \infty$$
$$-\infty < x_2 < \infty$$

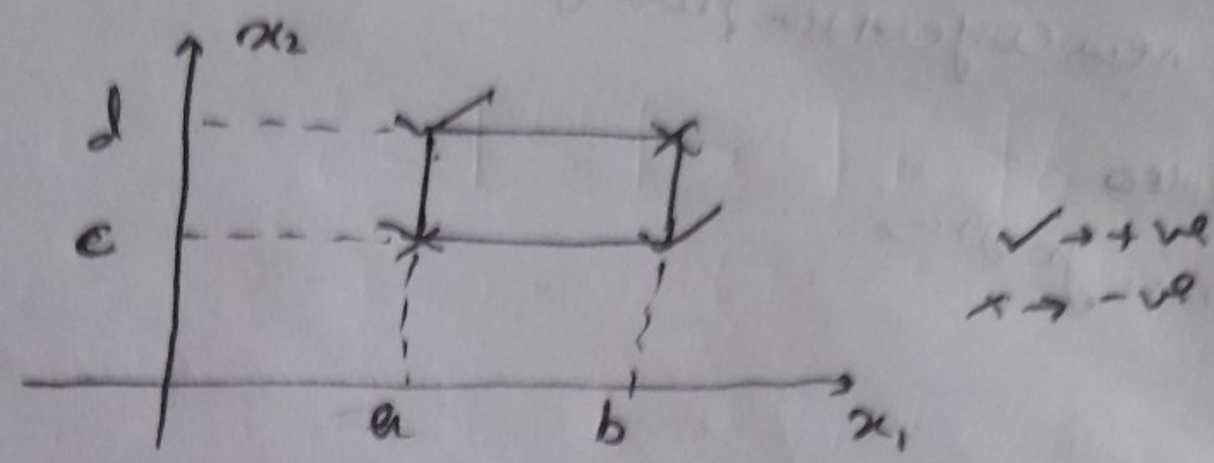
it satisfies

i) $F(x_1, x_2)$ is a non-decreasing and continuous increasing function from the right w.r.t each coordinate

2) $\lim_{\substack{x_1 \to \infty \\ x_2 \to \infty}} [F(x_1, x_2)] = 1$; $\lim_{x_2 \to -\infty} F(x_1, x_2) = 0$

; $\lim_{x_1 \to -\infty} F(x_1, x_2) = 0$

3) for every (a,c) ; (b,d) with $a<b$; $c<d$.
we have $F(b,d) - F(b,c) - F(a,d) + F(a,c) \geq 0$

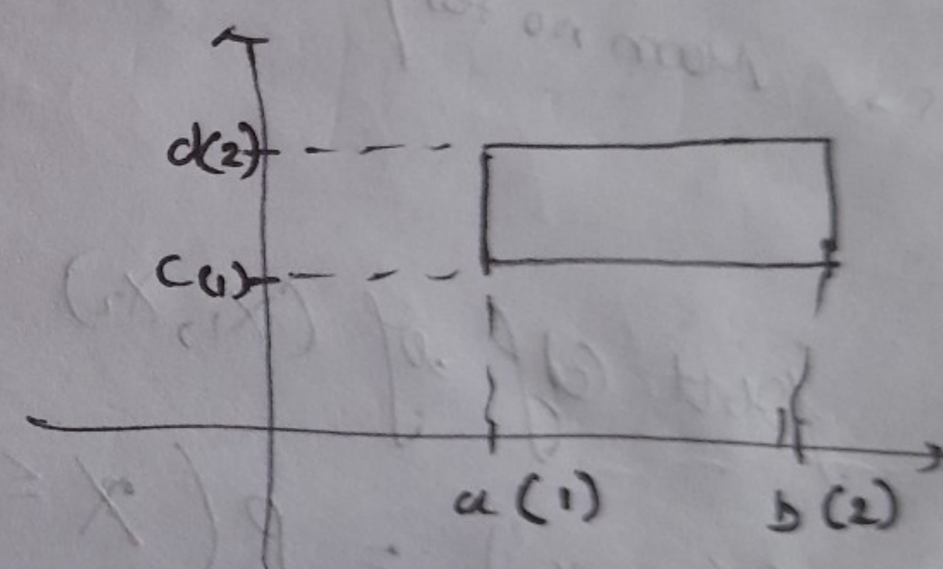


<u>Problem</u> → $F(x,y) = \begin{cases} 1-e^{-(x+y)} & 0 \leq x < \infty, 0 \leq y < \infty \\ 0, & 0,0 \end{cases}$

Can this fn be a joint cdf of a bivariate random variable (x,y).

<u>Sol</u>. $x_2 = y_2 = 2$

$x_1 = y_1 = 1$



$$F(2,2) - F(1,2) - F(2,1) + F(1,1)$$

Find it you will get ≤ 0
hence it can't be joint cdf.

Joint probability Density function

→ Let X and Y be continuous type RV with joint distribution F.

joint prob. density fn is defined as:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial^2 F(x,y)}{\partial y \partial x}$$

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(t,s)\, dt\, ds$$

↳ $\boxed{\text{Pdf}}$ or $\boxed{\text{Joint pdf}}$

$$F_X(x) = \int_{-\infty}^{x} f(t)\, dt$$

marginal distribution

→ $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

→ $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

H.W

Q→ Let X and Y be two jointly continuous Random variable with joint Pdf.

$$f_{XY}(x,y) = \begin{cases} x^2 + cy^2, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{o.w} \end{cases}$$

a) find the constant c

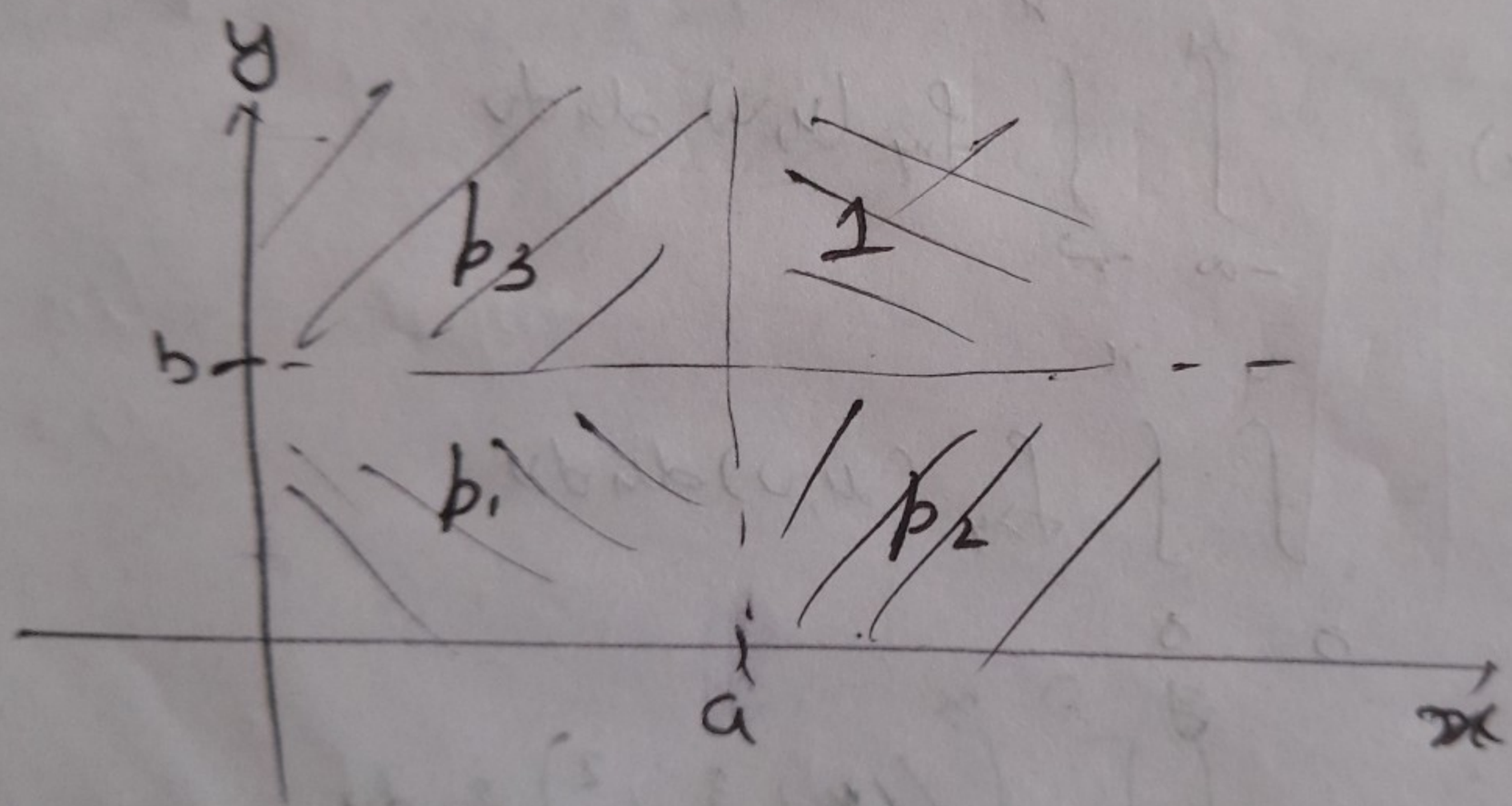
b) find $P(0 \le x \le \frac{1}{2},\ 0 \le y \le \frac{1}{2})$

c) find $f_X(x)$ and $f_Y(y)$

$F_{xy}(x,y) = 1$

The joint cdf of a bivariate RV (x,y) is given by

$$F_{xy}(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ p_1, & 0 \le x \le a,\ 0 \le y < b \\ p_2, & x \ge a,\ 0 \le y < b \\ p_3, & 0 \le x < a,\ y \ge b \\ 1, & x \ge a,\ x \ge b \end{cases}$$



a) find Marginal cdf of ~~x and y~~ X and Y

b)

The marginal cdf of X is given by :-

$$F_X(x) = F_{XY}(x, \infty) = \begin{cases} 0, & x < 0 \\ p_1, & 0 \le x < a \\ 1, & x \ge a \end{cases}$$

$$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} 0, & y < 0 \\ p_2, & 0 \le y < b \\ 1, & y \ge b \end{cases}$$

Q → The joint pdf of bivariate RV (X, Y) is

$$f_{XY}(x,y) = \begin{cases} kxy, & 0 < x < 1,\ 0 < y < 1 \\ 0, & o.w \end{cases}$$

a) find the value of k

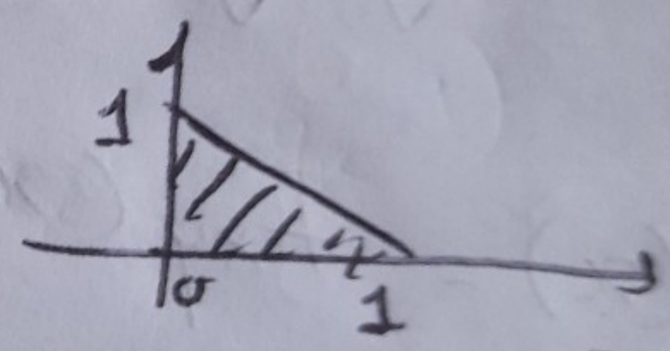
b) find marginal pdf $f_X(x)$ and $f_Y(y)$

c) find $P(x+y<1)$

Sol. →

a) $\underline{\underline{k = 8}}$.

(c) $P(x+y<1)$



$$P(x+y<1) = \int_0^1 \int_0^{1-y} 4xy\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} \qquad \Downarrow \qquad 1/6$$

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