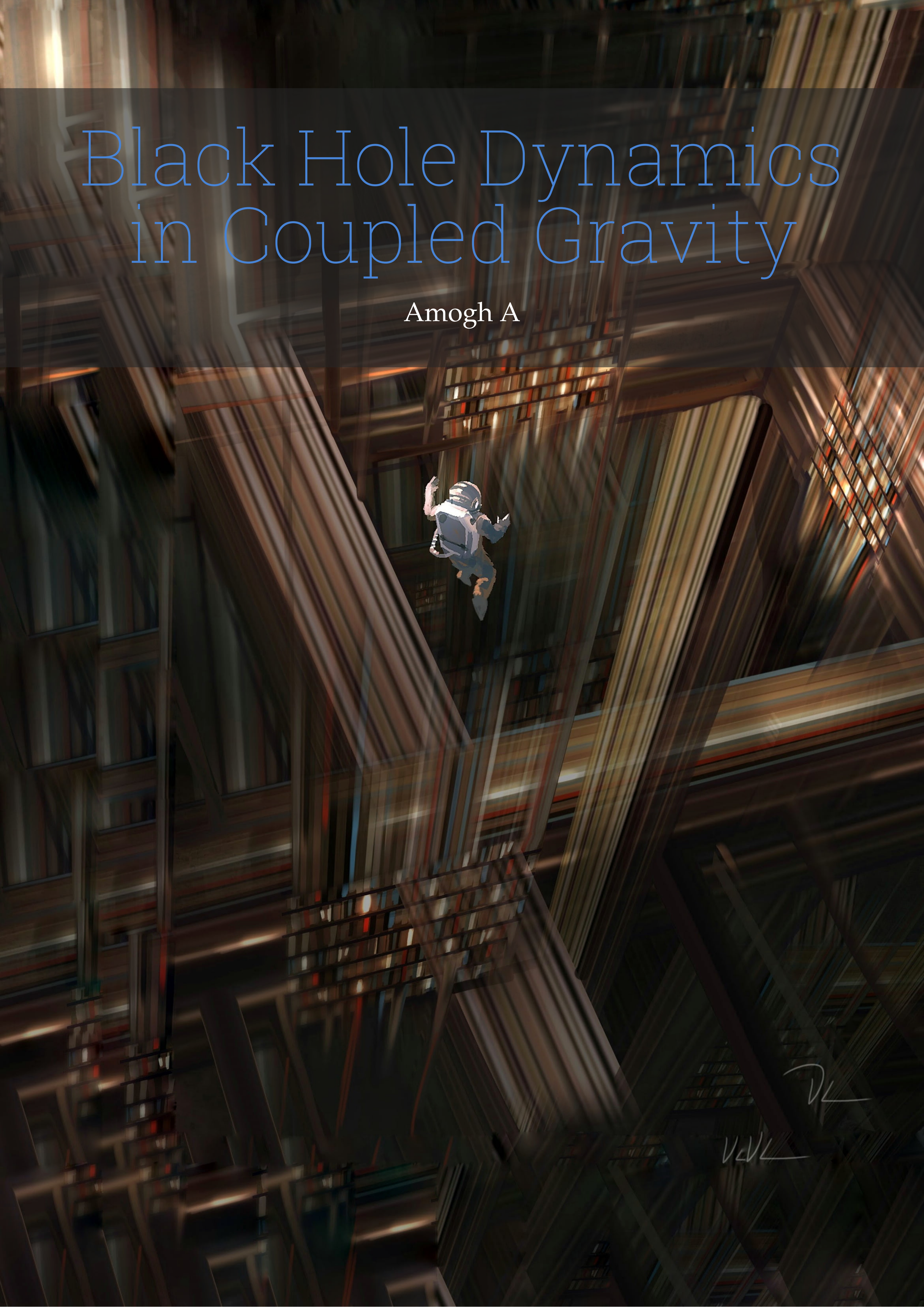
# Black Hole Dynamics in Coupled Gravity

Amogh A



A PROJECT REPORT ON

# Black Hole Dynamics in Coupled Gravity

by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prepared in the partial fulfilment of
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Department: Department of Physics

# Certificate

This is to certify that the thesis entitled, "*Black Hole Dynamics in Coupled Gravity*" and submitted by Amogh A ID No. `2022B5AA0890H` in partial fulfilment of the requirements of the study project PHY F266 embodies the work done by him under my supervision.

*Supervisor*
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# Declaration of Authorship

I, Amogh A, declare that this document titled, *'Black Hole Dynamics in Coupled Gravity'* and the work presented in it are my own. I confirm that:

- this work was done wholly or mainly while in candidature for the study project course PHY F266 at this university.
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- where I have consulted the published work of others, this is always clearly attributed.
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this report is entirely my own work.
- I have acknowledged all main sources of help.
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: Amogh A

Date: 30th April, 2025

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# 1

# World Before Einstein

> "I can calculate the motion of heavenly bodies but not the madness of people."
>
> Sir Isaac Newton

## 1.1. Newton's Dictatorship

Newtonian gravity is pretty simple: every body in the universe has an intrinsic gravitational mass which permits it to exert a force on another masses. This is universal. A 10 gram pen in my pocket exerts an attractive force on me and a neutron star billions of light years away. The force is always attractive, proportional to the masses interacting and inversely proportional to the distance between them squared. We can neatly summarise this fact in an equation:

$$F_{\mathrm{N}} = -G\frac{M_1 M_2}{r^2}$$

where $G$ is the universal gravitational constant and has the value[1] $6.67430(15) \times 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$. But something does not feel right about that. How can my pen exert a force on the neutron star billions of light years away through a vacuum and who is the messenger of this force? This exact question bugged Newton himself. Even though his theory was spectacularly successful in explaining the observed motion of planets and their moons, the thought of "*action at a distance*" made Newton deeply uncomfortable.

[1] This data is obtained from the NIST Database on Constants and can be accessed here: `https://physics.nist.gov/cgi-bin/cuu/Value?bg`.

In their private correspondence, Newton confesses to Richard Bentley:

> That one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it. [1]

Before we tear apart Newtonian gravity and study its replacement, let us take a moment to appreciate the genius of Newton. In 1781, William Herschel discovered Uranus, which was the furthest object in the solar system then. But soon astronomers discovered a problem. The orbit of Uranus was anomalous, but a solution was offered. Urban Le Verrier, a French astronomer and mathematician, hypothesised that if a planet were to exist in a particular region of the sky with a particular set of properties, it could explain Uranus. Miraculously, a set of German astronomers discovered Neptune in a single night since they had a prediction of where to look. Solar eclipse calculations were done a precision which rivals modern computer based ones and astronomers worked out the relative distances between the Earth and other objects in the solar system. Perhaps the single greatest achievement of Newtonian gravity is the fact that it can capture Kepler's laws in a simple equation.

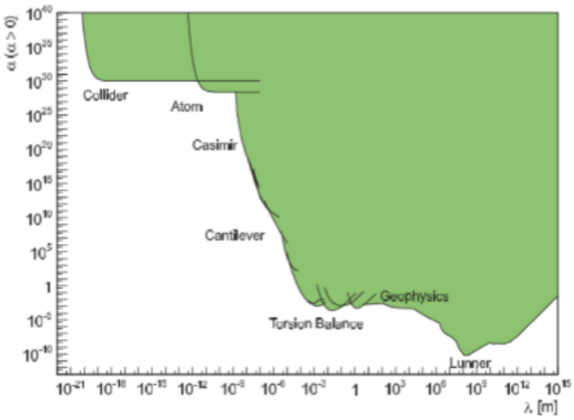



**Figure 1.1:** Figure taken from [3]. Plot showing the agreement of Newtonian gravity prediction with experiment at different length scales. Lower the y-value, higher the accuracy (95% confidence interval).

While Newtonian gravity is successful in explaining the observed motion of planets around the sun and the moons around their planets, towards the late 19th century it was observed that the orbit of Mercury had an anomaly. The perihelion of Mercury was precessing at a constant rate which could not be explained by Newton's laws. Moreover, people noted that light was "bending" when passing by the sun and using a strictly Newtonian model, their calculations for the deviation and the observed deviation was off by a factor of 2.

The stage was set for Einstein. 1905 was a miracle year where the world saw 4 brilliant papers from Einstein which changed the world forever. But after this, Einstein went back and had a thought. He came back in 1916 and reformulated the whole approach to gravity from the ground up. The new theory he was proposing did not even remotely match the simplicity of Newton's laws. General relativity uses almost all of the mathematics which had developed at that point from differential geometry, to calculus and from tensor analysis to differential equations. To even understand the physical predictions of GR, it is of utmost importance we understand the mathematical machinery behind it.

# 2

# Differential Geometry

> “Out of nothing I have created a strange new universe."
>
> Nicholas Bolyai

## 2.1. Need for Tensors

Everyone has an intuitive understanding of flat and curved spaces. It must be quantified in a rigorous way before it can be applied to study something physical. The hero of GR is the metric tensor. The laws of GR are co-ordinate invariant. It should not matter which co-ordinates are being employed to study the universe and this simple fact is at the heart of tensor analysis. Thus, mathematicians spent significant time and effort rigorously studying the behaviour of objects which are co-ordinate invariant.

**Definition 1** *A tensor is an object that describes a multilinear relationship between sets of algebraic objects related to a vector space.*

Scalars remain invariant under co-ordinate transformations. This is by construction. Temperature at a point should not change if one switches from Cartesian to elliptical co-ordinates.

$$S_x = S_y \qquad X \to Y$$

Contravariant vectors are vectors which transform in the opposite sense of their basis. Loosely speaking, a scaling of the basis vector by $a$ scales the components of the vector by $1/a$. More formally, a contravariant vector is an object that transforms like

$$V_Y^m = \frac{\partial Y^m}{\partial X^n} V_X^n \qquad X \to Y$$

where the transformation rule is given by

$$\begin{aligned} Y_1 &= X_1(x_1, x_2, x_3, \dots, x_n) \\ Y_2 &= X_2(x_1, x_2, x_3, \dots, x_n) \\ &\vdots \\ Y_n &= X_n(x_1, x_2, x_3, \dots, x_n). \end{aligned}$$

Covariant vectors are vectors which transform in the same sense as their basis. A scaling of $a$ of the basis also scales the vector by $a$. An intuitive example for this is the gradient vector $\nabla\vec{V}$. Covariant vectors transform like

$$W_{Y(m)} = \frac{\partial X^m}{\partial Y^n} W_{X(n)} \qquad X \to Y$$

for the same transformation rule given for contravariant vectors. It is by convention that covariant indices are in the subscript and contravariant indices are in the superscript.

A tensor can have any number of covariant and contravariant components denoted by $m$ and $n$. The rank of the tensor is simply $m+n$. It is also denoted as a rank $\binom{m}{n}$ tensor. Such a tensor has the following transformation rule:

$$T^{abc\dots}_{\alpha\beta\gamma\dots(Y)} = \left(\frac{\partial Y^a}{\partial X^{a'}}\frac{\partial Y^b}{\partial X^{b'}}\frac{\partial Y^c}{\partial X^{c'}}\dots\right)\left(\frac{\partial X^\alpha}{\partial Y^{\alpha'}}\frac{\partial X^\beta}{\partial Y^{\beta'}}\frac{\partial X^\gamma}{\partial Y^{\gamma'}}\dots\right)T^{a'b'c'\dots}_{\alpha'\beta'\gamma'\dots(X)} \qquad X \to Y.$$

Note that it is possible to construct a completely co-ordinate invariant system to study objects by analysing the flow on the manifold but that is out of the scope of this research.

## 2.2. Riemannian Geometry

To describe the geometry in a quantitative way, the concept of tensor derivatives has to be introduced. But before that, there are some desirable properties that these yet-to-be introduced objects must satisfy:

1. $\mathbf{D}_{\frac{\partial}{\partial x^\lambda}}(\mathbb{T}) = \mathbb{T}$, where $\mathbb{T}$ is a rank $(m,n)$ tensor. The derivative of a tensor must be another tensor of the same type.

| Object | Components of the Object | Co-ordinate Induced Basis Vector | Transformation Rule |
|---|---|---|---|
| Scalar (0 tensor) | Invariant under co-ordinate transformation | | |
| Contravariant Vector (1,0 tensor) | $V^{\mu}_{(X)}(X)$ | $e_{\mu} = \dfrac{\partial}{\partial x^{\mu}}$ | $V^{m}_{Y} = \dfrac{\partial Y^{m}}{\partial X^{n}} V^{n}_{X}$ |
| Covariant Vector (0,1 tensor) | $W^{\mu}_{(X)}(X)$ | $e^{\mu} = dx^{\mu}$ | $W_{Y(m)} = \dfrac{\partial X^{m}}{\partial Y^{n}} W_{X(n)}$ |
| Contravariant Tensor (2,0 tensor) | $T^{mn}(X)$ | $e_{\mu} \otimes e_{\nu} = \dfrac{\partial}{\partial x^{\mu}} \otimes \dfrac{\partial}{\partial x^{\nu}}$ | $T^{mn}_{Y} = \dfrac{\partial Y^{m}}{\partial X^{a}} \dfrac{\partial Y^{n}}{\partial X^{b}} T^{ab}_{X}$ |
| Convariant Tensor (0,2 tensor) | $T_{mn}(X)$ | $e^{\mu} \otimes e^{\nu} = dx^{\mu} \otimes dx^{\nu}$ | $T_{mn} = \dfrac{\partial X^{m}}{\partial Y^{a}} \dfrac{\partial X^{n}}{\partial Y^{b}} T_{ab}$ |
| Mixed Tensor (1,1 tensor) | $T^{m}_{n}(X)$ | $e_{\mu} \otimes e^{\nu} = \dfrac{\partial}{\partial x^{\mu}} \otimes dx^{\nu}$ | $T^{m}_{n} = \dfrac{\partial Y^{m}}{\partial X^{a}} \dfrac{\partial X^{b}}{\partial Y^{n}} T^{a}_{b}$ |

**Table 2.1:** Tensor transformations till rank 2.

2. $\mathbf{D}_{\frac{\partial}{\partial x^{\lambda}}}(f) = \dfrac{\partial f}{\partial x^{\mu}}$. The derivative operator acting on a scalar function should result in the normal partial derivative.
3. $\mathbf{D}(\mathbb{T} + \mathbb{S}) = \mathbf{D}\mathbb{T} + \mathbf{D}\mathbb{S}$. The derivative operator should be distributive.
4. $\mathbf{D}(\mathbb{T} \otimes \mathbb{S}) = \mathbf{D}\mathbb{T} \otimes \mathbb{S} + \mathbb{T} \otimes \mathbf{D}\mathbb{S}$. The derivative operator must follow the chain rule of normal derivatives.
5. $\mathbf{D}(\phi\mathbb{T}) = \dfrac{\partial \phi}{\partial x^{\mu}}\mathbb{T} + \phi\dfrac{\partial \mathbb{T}}{\partial x^{\mu}}$, where $\phi$ is any scalar field.
6. $\mathbf{D}(\mathbb{V}) = \dfrac{\partial \mathbb{V}^{\mu}}{\partial x^{\lambda}} \dfrac{\partial}{\partial x^{\mu}} + \mathbb{V}^{\mu}\Gamma^{\beta}_{\mu\lambda}\left(\dfrac{\partial}{\partial x^{\beta}}\right)$

This **D** operator is the covariant derivative as described in Riemannian geometry.

Before we proceed further, we should make sure that the laws and equations in one co-ordinate is equally valid in another co-ordinate.

**Theorem 2.2.1** *A zero tensor in one co-ordinate is also zero in all other co-ordinates.*

Transformation for a rank (1,0) tensor is given by: $T^{mn}_{Y} = \dfrac{\partial Y^{m}}{\partial X^{a}} \dfrac{\partial Y^{n}}{\partial X^{b}} T^{ab}_{X}$. Here, $T^{ab}_{X}$ is identically zero, thus $T^{mn}_{Y}$ is also zero for all co-ordinate transformations.

**Theorem 2.2.2** *A valid tensor equation in one co-ordinate is also valid in all other co-ordinates.*

Let the tensor equation be $G_{\mu\nu} = T_{\mu\nu}$. Subtract $T_{\mu\nu}$ from both sides and on the right side we have a zero tensor. It has been proved that a zero tensor in one co-ordinate is zero in all co-ordinates. Thus, a valid tensor equation in one co-ordinates is valid in all other co-ordinates. Now that we have the covariant derivative, we wish to find the covariant derivative of a covariant vector.

$$\begin{aligned}
\mathbf{D}_m(V^n W_n) &= \partial_m(V^n W_n) \\
&= (\partial_m V^n)W_n + V^n \partial_m W_n \\
(\mathbf{D}_m V^n)W_n + V^n \mathbf{D}_m W_n &= (\partial_m V^n)W_n + V^n \partial_m W_n \\
W_n \partial_m V^n + \Gamma^n_{mq} V^q W_n + V^n \mathbf{D}_m W_n &= (\partial_m V^n)W_n + V^n \partial_m W_n \\
V^n \mathbf{D}_m W_n &= V^n \partial_m W_n - \Gamma^q_{mn} V^n W_q \\
\mathbf{D}_m W_n &= \partial_m W_n - \Gamma^q_{mn} W_q
\end{aligned}$$

Thus, for any covariant component in the tensor, we subtract off from the partial derivative the tensor times its Christoffel symbol. Now the covariant derivative of the metric tensor is identically 0. This property is called metric compatibility. We also have the fact that the Christoffel symbol is symmetric under the interchange of the lower two indices. That is $\Gamma^n_{mq} = \Gamma^n_{qm}$. Now, from metric compatibility we have

$$\begin{aligned}
\partial_m g_{rs} - \Gamma^q_{mr} g_{qs} - \Gamma^q_{ms} g_{rq} &= 0, \\
\partial_r g_{sm} - \Gamma^q_{rs} g_{qm} - \Gamma^q_{rm} g_{sq} &= 0, \\
\partial_s g_{mr} - \Gamma^q_{sm} g_{qr} - \Gamma^q_{sr} g_{mq} &= 0.
\end{aligned}$$

Subtracting the first from the sum of the last two, we have

$$\boxed{\Gamma^q_{mr} = \frac{1}{2} g^{qk} \left(\partial_m g_{rk} + \partial_r g_{mk} - \partial_k g_{mr}\right).}$$

Thus, for any given metric tensor, we can calculate the corresponding Christoffel symbols using the above equation.

Another very powerful tensor which needs to be studied is the Riemann curvature tensor. This is typically a (1,3) tensor and it completely captures the curvature of the space given its metric.

$$\begin{aligned}
(\mathbf{D}_m \mathbf{D}_s - \mathbf{D}_s \mathbf{D}_m)V^q &= R^q{}_{msr} V^r, \\
\mathbf{D}_m \mathbf{D}_s V^q &= \partial_m(\mathbf{D}_s V^q) + \Gamma^q_{mp}(\mathbf{D}_s V^p), \\
\mathbf{D}_s \mathbf{D}_m V^q &= \partial_s(\mathbf{D}_m V^q) + \Gamma^q_{sp}(\mathbf{D}_m V^p), \\
(\mathbf{D}_m \mathbf{D}_s - \mathbf{D}_s \mathbf{D}_m)V^q &= \left[\partial_m(\partial_s V^q + \Gamma^q_{sr} V^r) + \Gamma^q_{mp}(\partial_s V^p + \Gamma^p_{sr} V^r)\right] \\
&\quad - \left[\partial_s(\partial_m V^q + \Gamma^q_{mr} V^r) + \Gamma^q_{sp}(\partial_m V^p + \Gamma^p_{mr} V^r)\right], \\
&= (\partial_m \Gamma^q_{sr} - \partial_s \Gamma^q_{mr} + \Gamma^q_{mp}\Gamma^p_{sr} - \Gamma^q_{sp}\Gamma^p_{mr})V^r, \\
R^q{}_{msr} &= \partial_s \Gamma^q_{mr} - \partial_m \Gamma^q_{sr} + \Gamma^q_{sp}\Gamma^p_{mr} - \Gamma^q_{mp}\Gamma^p_{sr}.
\end{aligned}$$

We can construct a lower rank tensor from the Riemann curvature tensor by contracting it with the metric tensor. The result is a rank (0,2) tensor called the Ricci curvature tensor. It is given by

$$R_{ab} = R_{ab\alpha\beta}g^{\alpha\beta},$$

and a scalar can be constructed from this Ricci tensor by contracting it again with the metric:

$$R = R_{ab}g^{ab}.$$

## 2.3. Flat and Curved Space

With all the heavy mathematical framework in our arsenal, we are now ready to talk quantitatively about the curvature of a given space. For a given metric, first compute its Christoffel symbols and then using them, compute the Riemann curvature tensor. From that obtaining the Ricci curvature tensor and the Ricci scalar is a straightforward task. Now, with all 3 tensors, what does it mean to be in a flat space?

**Definition 2** *A "flat" space is a manifold whose Riemann curvature tensor is identically zero.*

Thus, for a flat space, the Ricci tensor and Ricci scalar are also zero. From definition 2.4 it is very straightforward to classify manifolds into flat and curved. If the Riemann tensor is not zero, then the manifold is not flat. It does not matter in which co-ordinate system you view the manifold, the Riemann tensor will be either be identically zero or not.

We use the Riemann tensor for determining the curvature for two reasons:

1. With enough training the metric itself can be used to tell if the manifold is flat or curved. But for the same manifold, the metric looks different in different co-ordinate systems. For example, in two dimensions the flat space metric in Cartesian and polar co-ordinates if given by

   $$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

   and for 3D, it is given by

   $$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

   .

2. For genuinely curved manifolds, the curvature can change from one point to another and in general relativity, it can change from one instance of time to

another. For complicated metrics, it may not be possible to tell by looking if the manifold is flat or curved and the Ricci scalar is a measure of deviation from flatness.

3. In Minkowski spacetime, flat space defaults to two inertial observers moving relative to each other. Thus, in flat spacetime, all the laws of Special Relativity applies: length contraction, and time dilation. If the big globe is analogous to GR, then the zoomed in flat world is analogous to SR.

# 3

# Field Theory from First Principles

"Wir mussen wissen. Wir werden wissen."

David Hilbert

The three most successful theories ever written down, in chronological order, are

1. Maxwell's laws of electrodynamics,
2. Einstein's theory of general relativity, and
3. Dirac's equation of relativistic quantum mechanics.

All three theories are examples of *field theories*, where the fundamental object is a field and everything happens on the field because of some disturbance in the field. Field theory[1] is currently the most fundamental and experimentally verified theory which can describe the dynamics of systems for any length scale. The derivation of the equations governing the fields themselves are done by varying a fundamental object called the *action lagrangian*. We shall now see how to derive the three equations by varying their respective action lagrangians.

## 3.1. Maxwell's Equations from the Maxwell Lagrangian

The Maxwell action for the electromagnetic field is given by

$$S_{\text{EM}} = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu},$$

[1]While I acknowledge that string theory also exists, no experimental evidence of strings have been found as of writing this report. Thus, fields are **the** fundamental object.

where the field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where $A_{\mu\nu}$ is the electromagnetic 4-potential. Varying the action with respect to the potential $A_\mu$:

$$\delta S_{\mathrm{EM}} = -\frac{1}{2}\int d^4x\sqrt{-g}F^{\mu\nu}\delta F_{\mu\nu}.$$

Using $\delta F_{\mu\nu} = \nabla_\mu \delta A_\nu - \nabla_\nu \delta A_\mu$, integrating by parts, and discarding boundary terms, we get

$$\nabla_\mu F^{\mu\nu} = 0.$$

This is the source-free Maxwell equation.

## 3.2. Dirac Equation from the Dirac Lagrangian

The Dirac Lagrangian density is given by:

$$\mathcal{L}_{\mathrm{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi,$$

where $\psi$ is the Dirac spinor, $\bar{\psi} = \psi^\dagger\gamma^0$ is the adjoint spinor, $\gamma^\mu$ are the gamma matrices, and $m$ is the mass of the particle.

To derive the Dirac equation, we vary the action

$$S = \int d^4x\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

with respect to $\bar{\psi}$:

$$\delta S = \int d^4x \left(\delta\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\right) = 0.$$

Since this must hold for all variations $\delta\bar{\psi}$, we obtain the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

Similarly, varying with respect to $\psi$ gives the adjoint equation:

$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0.$$

This derivation confirms that the Dirac equation describes the dynamics of a spin-$\frac{1}{2}$ particle.

## 3.3. Variation of the Einstein-Hilbert Action

To derive the field equations, we vary the action with respect to the metric $g^{\mu\nu}$:

$$\delta S = \frac{1}{2\kappa}\int d^4x\,\delta(\sqrt{-g}R) - \frac{1}{2\kappa}\int d^4x\,\delta(2\Lambda\sqrt{-g}) + \delta S_{\text{matter}}.$$

Using the well-known identity

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu},$$

we obtain

$$\delta(2\Lambda\sqrt{-g}) = 2\Lambda\delta\sqrt{-g} = \Lambda\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}.$$

For the Ricci scalar term, we use the Palatini identity:

$$\delta(\sqrt{-g}R) = \sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \text{total divergence}.$$

Ignoring the total divergence term, the variation simplifies to

$$\delta S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} + \delta S_{\text{matter}}.$$

## 3.4. Vacuum Einstein Field Equations

The action for classical Einstein gravity with a cosmological constant $\Lambda$ is given by

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(R - 2\Lambda) + S_{\text{matter}},$$

where:

1. $g$ is the determinant of the metric tensor $g_{\mu\nu}$,
2. $R$ is the Ricci scalar,
3. $\Lambda$ is the cosmological constant,
4. $\kappa = 8\pi G$ (in natural units where $c = 1$),
5. $S_{\text{matter}}$ is the action for the matter fields.

The Einstein-Hilbert action without a cosmological constant and without matter is given by

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R.$$

Varying the action with respect to the metric $g^{\mu\nu}$:

$$\delta S = \frac{1}{2\kappa} \int d^4x \delta(\sqrt{-g} R).$$

Using the Palatini identity,

$$\delta(\sqrt{-g} R) = \sqrt{-g}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} + \text{total divergence},$$

ignoring the total divergence term, we obtain the vacuum Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$

## 3.5. Complete Einstein Field Equations

Setting $\delta S = 0$ for arbitrary variations $\delta g^{\mu\nu}$, we obtain

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

where the stress-energy tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}.$$

Rearranging, we arrive at the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu}.$$

## 3.6. Einstein Field Equations with Maxwell and Dirac Terms

The action including both the Einstein-Hilbert term, the Maxwell term, and the Dirac term is given by

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R - \frac{1}{4} \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu} + \int d^4x \sqrt{-g} \bar{\psi}(i\gamma^\mu \nabla_\mu - m)\psi.$$

Varying with respect to $g^{\mu\nu}$:

$$\delta S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} - \frac{1}{4} \int d^4x \sqrt{-g} T^{\mu\nu}_{\text{EM}} \delta g^{\mu\nu} + \delta S_{\text{Dirac}}.$$

The energy-momentum tensor of the electromagnetic field is

$$T^{\mu\nu}_{\text{EM}} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}.$$

The energy-momentum tensor of the Dirac field is given by

$$T^{\mu\nu}_{\text{Dirac}} = \frac{i}{2}\left(\bar{\psi}\gamma^{(\mu}\nabla^{\nu)}\psi - \nabla^{(\mu}\bar{\psi}\gamma^{\nu)}\psi\right).$$

Setting $\delta S = 0$ for arbitrary variations, the Einstein field equations with Maxwell and Dirac terms are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\left(T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{Dirac}}\right).$$

# 4

# Einstein's Coup

"Before I came here today I was confused about this topic. Having listened to your lecture I am still confused. But on a higher level."

Enrico Fermi

## 4.1. "One Small Step For Man"

All the mathematics in the previous section was known in Einstein's time. What no one did with the mathematics was to produce gravity. Einstein's theory in a nutshell is: matter causes spacetime to curve and gravity is the result of the curvature of spacetime. Thus, the Einstein field equations are (a complete derivation of the EFE is in chapter 4):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu},$$

where $T_{\mu\nu}$ is the energy-momentum tensor which captures the matter fields. It is a $4 \times 4$ tensor and it is broken down as follows:

$$T_{\mu\nu} = \begin{bmatrix} \boxed{T_{00}} & \boxed{T_{01}} & \boxed{T_{02}} & \boxed{T_{03}} \\ \boxed{T_{10}} & \boxed{T_{11}} & \boxed{T_{12}} & \boxed{T_{13}} \\ \boxed{T_{20}} & \boxed{T_{21}} & \boxed{T_{22}} & \boxed{T_{23}} \\ \boxed{T_{30}} & \boxed{T_{31}} & \boxed{T_{32}} & \boxed{T_{33}} \end{bmatrix}.$$

Energy density , momentum density , energy flux , shear stress , and pressure are captured in the energy-momentum tensor. The reason why I said they are the field *equations* is because it is written compactly in Einstein notation. In fact, they are a set of 10 coupled non-linear partial differential equations which have to be solved all at once if we have to make predictions about the curvature of spacetime given the matter distribution and it also enables us to talk about the motion of the very matter distribution in the calculated curved spacetime.

# 5

# Deriving the Metric Tensors

> "Black Holes are where *God* divided by zero."
>
> Albert Einstein

## 5.1. The Schwarzschild Metric

### A Brief Historical Note

Einstein presented his theory of general relativity in 1915 to the Prussian Academy of Sciences but he thought that a solution of the field equations was not possible. A few hundred kilometers away on the German front, a 40 year old Karl Schwarzschild who was volunteering for the army calculating artillery trajectories took a walk in Einstein's universe and found the first exact solution to EFE. He did so by assuming that there is just one small spherically symmetric and static mass located in the universe by itself which allowed him to simplify the mathematics by a lot.

### 5.1.1. Deriving the Schwarzschild Metric

Einstein's field equations in vacuum are given by:

$$R_{\mu\nu} = 0.$$

For a spherically symmetric, static metric, we assume the general form:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of a 2-sphere. Computing the nonzero components and substituting them into the Ricci tensor equations yields:

$$R_{tt} = \frac{A''}{2B} - \frac{A'B'}{4B^2} + \frac{A'}{2rB} = 0, \tag{5.1}$$
$$R_{rr} = -\frac{A''}{2A} + \frac{A'B'}{4AB} + \frac{B'}{rB} = 0, \tag{5.2}$$
$$R_{\theta\theta} = 1 - \frac{1}{B} - \frac{rB'}{2B^2} + \frac{rA'}{2AB} = 0. \tag{5.3}$$

Adding the first two equations, we obtain:

$$\frac{A'}{A} + \frac{B'}{B} = 0 \Rightarrow AB = \text{constant}.$$

At large $r$, $A \longrightarrow 1$ and $B \longrightarrow 1$, because the metric at $\infty$ should be the Minkowski metric, we set the constant to 1, giving:

$$B = \frac{1}{A}.$$

From the $R_{\theta\theta}$ equation:

$$1 - \frac{1}{B} - \frac{rB'}{2B^2} + \frac{rA'}{2AB} = 0.$$

Substituting $B = 1/A$, this simplifies to the differential equation:

$$rA' + A - 1 = 0.$$

Solving this by separation of variables:

$$\frac{dA}{dr} = \frac{1 - A}{r},$$

which integrates to:

$$A = 1 - \frac{C}{r}.$$

The constant $C$ is determined by requiring that the solution matches Newtonian gravity in the weak field limit, leading to $C = 2GM$. Thus:

$$A = 1 - \frac{2GM}{r}.$$

Substituting $A$ and $B$ into the metric:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.}$$

This is the Schwarzschild metric, describing the spacetime outside a static, spherically symmetric mass distribution.

## 5.2. The Kerr Metric

### 5.2.1. A 50 Year Delay

Much of the mid 1900s saw technological developments at a rate never seen before. Most of the efforts by scientists and engineers was to rapidly advance the technology so that their country was in a better position to fight the two world wars which happened in that era. Thus, areas like quantum mechanics, nuclear and particle physics, radar, flight mechanics and signal processing saw unprecedented fundamental and technological developments. This is to say that general relativity and astrophysics took a backseat in mid 20th century physics.

The Kerr metric describes the spacetime geometry around a rotating, uncharged mass. It is an exact solution to the Einstein field equations in vacuum ($T_{\mu\nu} = 0$). While the physics and mathematics of a static mass was understood to some degree of confidence by the 1960s, no one had been able to derive the metric around a spinning mass for over 50 years. Physicists thought that black holes as predicted by Schwarzschild and Einstein were simply too evil to exist as the theory suggested nonsensical results like time stoppage at $r_s$ and infinite curvature at the singularity. Solutions offered by Martin Kruskal and George Szekeres and proofs of imminent star collapse by Robert Oppenheimer, Richard Tolman and George Volkoff led the community to accept the Schwarzschild metric as physical but no one had observed anything like that yet.

It was only in 1963 when Roy Kerr found an exact solution to EFE for an isolated charge neutral spinning mass. Many, including Einstein, doubted the results stemming from his theory namely, black holes, though it was not called a black hole back then. It was only after significant efforts to understand the Kerr BH and its astrophysical implications and the first theoretical photograph of a black hole did we start appreciating the physics of GR.

### 5.2.2. Deriving the Kerr Metric

We seek a stationary, axisymmetric solution to the vacuum Einstein equations. The metric must respect the following properties:

1. Stationarity: No explicit dependence on the time coordinate $t$.
2. Axisymmetry: No explicit dependence on the azimuthal coordinate $\phi$.
3. Asymptotic flatness: The metric reduces to the Minkowski metric at large distances.

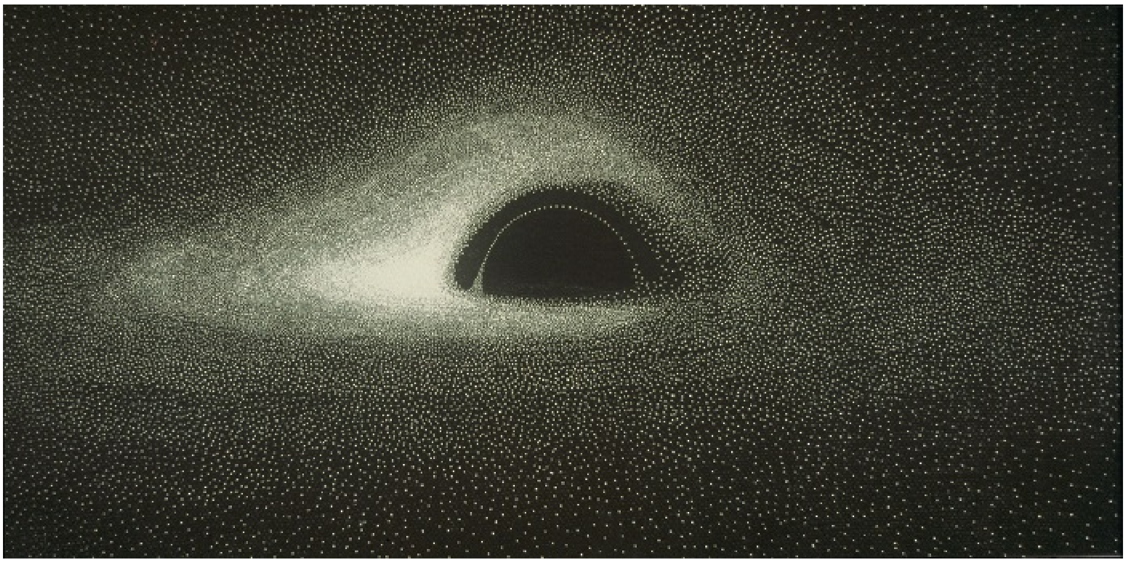


**Figure 5.1:** First ever photo of a black hole with an accretion disk generated by Jean-Pierre Luminet using a primitive computer, mathematics, and ink.

4. Rotating source: Characterized by a mass parameter $M$ and an angular momentum parameter $a$.

To describe a rotating mass, we introduce Boyer-Lindquist coordinates $(t, r, \theta, \phi)$, transforming from spherical to ellipsoidal coordinates:

$$x = \sqrt{r^2 + a^2} \sin\theta \cos\phi, \quad y = \sqrt{r^2 + a^2} \sin\theta \sin\phi, \quad z = r\cos\theta. \tag{5.4}$$

Here, $a$ is the spin parameter, defined as $a = J/M$ where $J$ is the angular momentum per unit mass. The corresponding line element in Boyer-Lindquist coordinates is assumed to be:

$$ds^2 = -Adt^2 + Bdr^2 + Cd\theta^2 + Dd\phi^2 + Edtd\phi. \tag{5.5}$$

A general axisymmetric metric in Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ takes the form:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2.$$

Due to stationarity and axisymmetry, there exist two Killing vectors:

$$\xi^\mu = (1, 0, 0, 0), \quad \eta^\mu = (0, 0, 0, 1).$$

Thus, the metric components are functions of $r$ and $\theta$ only. A reasonable ansatz is:

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_1}dr^2 + e^{2\mu_2}d\theta^2.$$

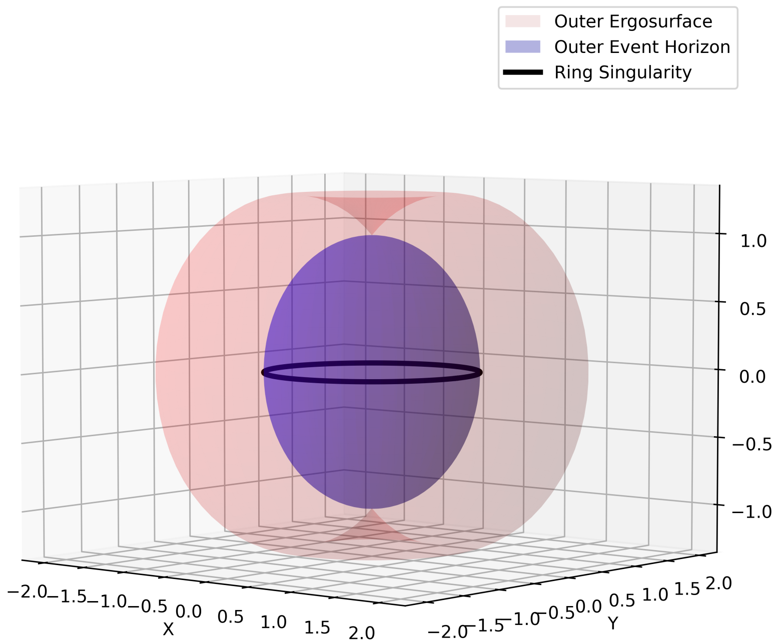



**Figure 5.2:** 3D view of a Kerr BH with spin parameter 0.998. Plot shows the outer ergosurface, outer event horizon and the ring singularity as if it could be seen. Plot also shows the polar caps of the Kerr BH which is not present in the Schwarzschild BH because of its stationarity.

where $\nu, \psi, \mu_1, \mu_2$, and $\omega$ are functions of $r$ and $\theta$. The vacuum Einstein equations are given by $R_{\mu\nu} = 0$. Solving these equations step by step leads to:

$$g_{tt} = -\left(1 - \frac{2Mr}{\rho^2}\right), \tag{5.6}$$

$$g_{t\phi} = -\frac{2Mar\sin^2\theta}{\rho^2}, \tag{5.7}$$

$$g_{rr} = \frac{\rho^2}{\Delta}, \tag{5.8}$$

$$g_{\theta\theta} = \rho^2, \tag{5.9}$$

$$g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\rho^2}\right)\sin^2\theta, \tag{5.10}$$

where we define:

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{5.11}$$

$$\Delta = r^2 - 2Mr + a^2. \tag{5.12}$$

Thus, the Kerr metric is given by:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \\ &\left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2. \end{aligned} \tag{5.13}$$

When plotted, the Kerr geometry looks like figure 5.2.

# 6

# Black Hole Dynamics in Einstein Gravity

"Cooper, there's no time for caution!"

CASE, *Interstellar, 2014*

## 6.1. Deriving Orbital Equations

Owing to the lack of symmetry in the Kerr black hole, the nature of the orbits for a random trajectory is very complicated and finding an analytical solution is almost impossible. Thus, many numerical and approximation tools have been developed to simulate the particle trajectories of particles around black holes. It is very clear that planar motion is possible only on the equatorial plane and this is of significant interest to astrophysicists as motion on the equatorial plane corresponds to a BH accretion disc.

Starting from the Hamilton–Jacobi equation for the action $\mathcal{A}$, given by

$$g^{ab}\partial_a\mathcal{A}\partial_b\mathcal{A} = -m^2,$$

where $m$ is the mass of the particle, we can find the equations governing the particle trajectories. Since the spacetime is spherically asymmetric possessing only axial symmetry, the partial differential equations can be separated out into two $t$ and $\phi$ equations. Since the Kerr metric possesses a Killing tensor, we can also separate out the $r$ and $\theta$ dependencies into two equations, giving us 1 for each spacetime coordinate.Therefore we use the ansatz

$$\mathcal{A} = -\mathcal{E}_0 t + L\phi + \mathcal{A}_r(r) + \mathcal{A}_\theta(\theta).$$

Substituting into the Hamilton–Jacobi equations we find that $\mathcal{A}_r(r)$ and $\mathcal{A}_\theta(\theta)$ satisfy the ordinary differential equations:

$$\left(\frac{d\mathcal{A}_\theta}{d\theta}\right)^2 + \left(a\mathcal{E}_0 \sin\theta + \frac{L}{\sin\theta}\right)^2 + a^2 m^2 \cos^2\theta = K,$$

$$\left(\frac{d\mathcal{A}_r}{dr}\right)^2 + \frac{1}{\Delta}\left[(r^2+a^2)\mathcal{E}_0 - aL\right]^2 + m^2 r^2 = -K,$$

where $K$ is a separation constant. Differentiating the action with respect to $m^2$, $\epsilon_0$, $L$, and $K$, and massaging the equations, we get

$$\begin{aligned}
m\frac{dt}{d\tau} &= -\frac{r_g r a}{\rho^2\Delta}L + \frac{\mathcal{E}_0}{\Delta}\left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta\right)\\
m\frac{d\phi}{d\tau} &= \frac{L}{\Delta\sin^2\theta}\left(1 - \frac{r_g r}{\rho^2}\right) + \frac{r_g r a}{\rho^2\Delta}\mathcal{E}_0\\
m^2\left(\frac{dr}{d\tau}\right)^2 &= \frac{1}{\rho^4}\left[(r^2+a^2)\mathcal{E}_0 - aL\right]^2 - \frac{\Delta}{\rho^4}(K + m^2 r^2)\\
m^2\left(\frac{d\theta}{d\tau}\right)^2 &= \frac{1}{\rho^4}(K - a^2 m^2\cos^2\theta) - \frac{1}{\rho^4}\left(a\mathcal{E}_0\sin\theta - \frac{L}{\sin\theta}\right)^2.
\end{aligned}$$

I have used one of the numerical methods to solve the geodesic equations as implemented in the KerrGeoPy library[4]. The complete code used to generate these plots can be found at my GitHub repository `https://github.com/AmoghA4/BlackHoleDynamics`.

## 6.2. Stable Photon Orbits

Solving these equations numerically is very computationally expensive and requires a lot of computer processor time. I have created two 30 second animations of a photon being tracked in its orbit around a Kerr BH with parameters used in figure 6.2(A) and figure 6.2(B) can be accessed here.

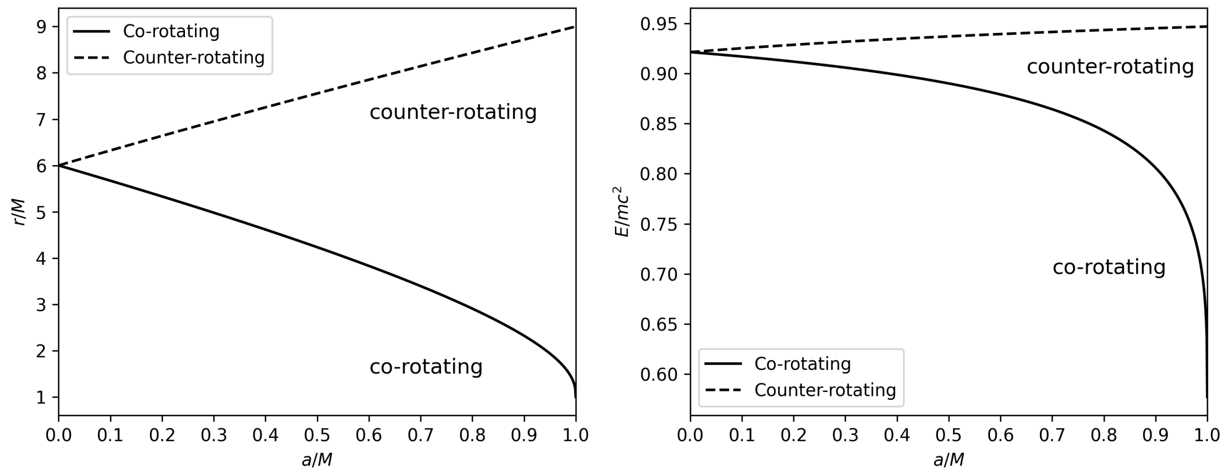
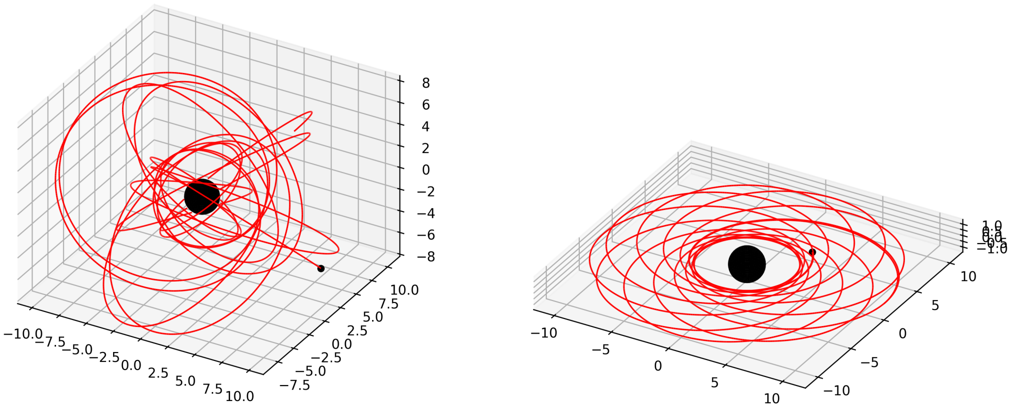



**Figure 6.1:** Radius and binding energy of the ISCO for a particle in an equatorial orbit around a Kerr black hole as a function of the spin parameter $a$. Plot shows the results for particles in prograde and retrograde motion in equatorial orbit.

## 6.3. Unstable Plunging Photon Orbits

For a given BH, all orbits are not necessarily stable. Orbits can be unstable in two ways.
1. Black Hole as a gravitational lens: An incoming photon at just the right energy and initial inclination can be redirected by the black hole acting as a giant cosmic lens. There is a magical distance from the centre of the BH where the photon orbits the black hole in a way that it neither enters the BH nor necessarily shoots off to infinity. This so called photon sphere is a special place as if you were to stand here (you can never as only massless objects can exist here but let us pretend we do not care), you could theoretically see the back of your own head.
2. Black Hole as a cosmic vacuum cleaner: An unlucky photon coming at just the right angle can be permanently trapped inside the BH with no hope of ever leaving it. This is why the black hole appears "*black*".



**(a)** Figure 1: Particle faces no restrictions on what plane it can orbit in.

**(b)** Figure 2: Particle is confined to move in the equatorial plane only.

**Figure 6.2:** Photon orbits around a Kerr BH with $a = 0.9$, orbital semi-latus rectum $p = 6$, and orbital eccentricity $e = 0.5$; figure 1 has no restrictions on orbital plane while figure 2 restricts particle to equatorial plane.

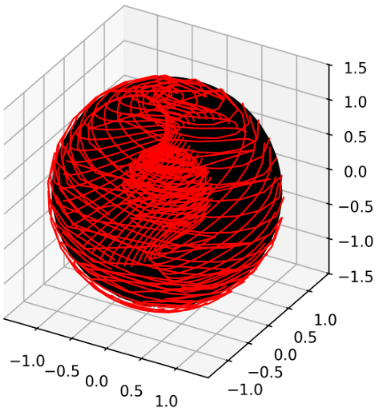


**Figure 6.3:** Plot of a photon orbit which is unstable. Photon eventually gets sucked into the BH and can never exit.

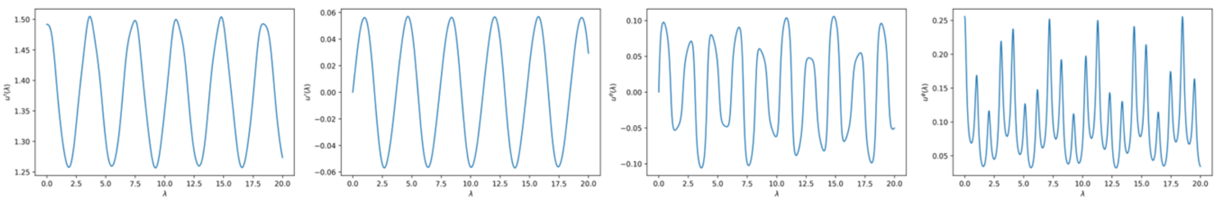


**Figure 6.4:** 4-velocity of particle moving around Kerr BH whose trajectory is shown in Figure 6.2(B) versus proper time.

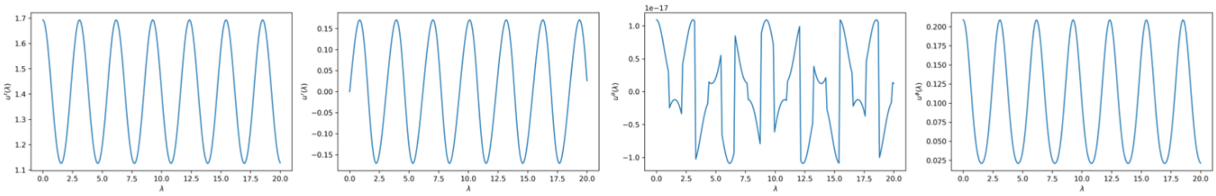


**Figure 6.5:** 4-velocity of particle moving around Kerr BH whose trajectory is shown in Figure 6.2(A) versus proper time

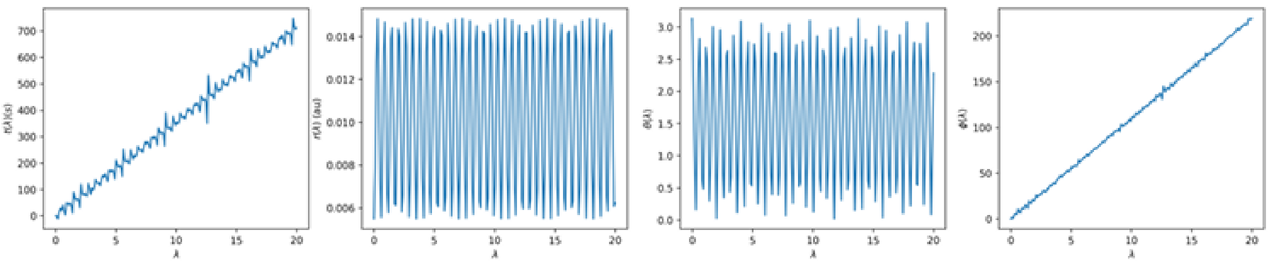



**Figure 6.6:** Plot of 4-vector versus proper time for a photon in an unstable orbit around a Kerr BH with same parameters used for Figure 6.1. The photon ends up being "sucked" by the BH, never to see the light of day again

# 7

# Modified Gravity Theories

> "The world is something like a great game being played by the gods, and we are observers of the game. We do not know what the rules of the game are; all we are allowed to do is to watch the playing. Of course, if we watch long enough, we may eventually catch on to a few of the rules. The rules of the game are what we mean by fundamental physics."
>
> Richard P. Feynman

## 7.1. Why Modify?

Einstein's general relativity is a model to understand the universe. Like all models, it was proposed to replace an older model in light of new evidence, and like all models it too has flaws. Einstein's gravity has famously failed to incorporate dark matter and dark energy. Towards the later end of the 20th century, astronomers and astrophysicists observed a very strange discrepancy in the galaxy rotation curved and could not settle on the value of Hubble constant.

More fundamentally, the Einstein-Hilbert action in Einstein's gravity is only linear in the Ricci scalar. It is absolutely possible that at higher energies, or lower length scales (or both), higher orders of $R$ are present making Einstein's model just a linear approximation. A well known model which accounts for the expanding universe is the

Starobinsky model and is an extension of Einstein gravity and has a cubic term in $R$:

$$S = \frac{1}{2}\int d^4x \left(R + \frac{R^2}{6M}\right).$$

Unless a complete and successful quantum theory of gravity is completed, one can only speculate what the “complete" description of gravity

## 7.2. Minimally Coupled Gravity

Let us formulate the MOG action and field equations in a very simple and straightforward way. Consider $\chi = 1/G$ where $\chi$ is a scalar field and $G$ is the coupling strength of gravity [2, 6]. The MOG action is given by (the metric signature is $(+,-,-,-)$ and units with $c = 1$):

$$S = S_G + S_\phi + S_M,$$

where

$$S_G = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\chi R + \frac{\omega_M}{\chi}\nabla^\mu\chi\nabla_\mu\chi + 2\Lambda\right),$$

and

$$S_\phi = \int d^4x\sqrt{-g}\left(-\frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \frac{1}{2}\mu^2\phi^\mu\phi_\mu\right).$$

$S_M$ is the matter action and $J^\mu$ is the current matter source of the vector field $\phi_\mu$. Moreover, $\nabla_\mu$ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$, $B_{\mu\nu} = \partial_\mu\phi_\nu - \partial_\nu\phi_\mu$ and $\omega_M$ is a constant. The Ricci tensor is

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}.$$

Expand $G$ by $G = G_N(1+\alpha)$, $\Lambda$ is the cosmological constant and $\mu$ is the effective running mass of the spin 1 graviton vector field.

Variation of the matter action $S_M$ yields

$$T^M_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \quad J^\mu = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta\phi_\mu}.$$

Varying the action with respect to $g_{\mu\nu}$, $\chi$ and $\phi_\mu$, we obtain the field equations:

$$G_{\mu\nu} = -\frac{\omega_M}{\chi^2}\left(\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\chi\nabla_\alpha\chi\right) - \frac{1}{\chi}(\nabla_\mu\chi\nabla_\nu\chi - g_{\mu\nu}\Box\chi) + \frac{8\pi}{\chi}T_{\mu\nu},$$

$$\nabla_\nu B^{\mu\nu} + \mu^2\phi^\mu = J^\mu,$$

$$\Box\chi = \frac{8\pi}{(2\omega_M + 3)} T,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R$, $\Box = \nabla^\mu \nabla_\mu$, $J^\mu = \kappa\rho_M u^\mu$, $\kappa = \sqrt{G_N \alpha}$, $\rho_M$ is the density of matter and $u^\mu = dx^\mu/ds$. The energy-momentum tensor is

$$T_{\mu\nu} = T^M_{\mu\nu} + T^\phi_{\mu\nu} + g_{\mu\nu}\frac{\chi\Lambda}{8\pi},$$

where

$$T^\phi_{\mu\nu} = -\left(B_\mu{}^\alpha B_{\alpha\nu} - \frac{1}{4}g_{\mu\nu}B^{\alpha\beta}B_{\alpha\beta} + \mu^2\phi_\mu\phi_\nu - \frac{1}{2}g_{\mu\nu}\phi^\alpha\phi_\alpha\right),$$

and $T = g^{\mu\nu}T_{\mu\nu}$. The final equation is

$$2\chi\Box\chi - \nabla_\mu\chi\nabla^\mu\chi = \frac{R}{\omega_M},$$

by substituting for $R$.

## 7.3. Equations of Motion

The equation of motion for a massive test particle in MOG has the covariant form:

$$m\left(\frac{du^\mu}{ds} + \Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta\right) = q_g B^\mu{}_\nu u^\nu, \tag{7.1}$$

$\Gamma^\mu{}_{\alpha\beta}$ denote the Christoffel symbols. Moreover, $m$ and $q_g$ denote the test particle mass $m$ and gravitational charge $q_g = \sqrt{\alpha G_N}m$, respectively. A massless photon has no gravitational charge, so photons travel on null geodesics:

$$\frac{dk^\mu}{ds} + \Gamma^\mu{}_{\alpha\beta}k^\alpha k^\beta = 0, \tag{7.2}$$

where $k^\mu$ is the photon momentum and $k^2 = k^\mu k_\mu = 0$. Note that for $q_g/m = \sqrt{\alpha G_N}$ the equation of motion for a massive test particle *satisfies the (weak) equivalence principle*, leading to the free fall of particles in a homogeneous gravitational field, although the free-falling particles do not follow geodesics.

In the weak field region, $r \gg 2GM$, the spherically symmetric field $\phi_\mu$, with effective mass $\mu$, is approximated by the Yukawa potential:

$$\phi_0 = -Q_g\frac{\exp(-\mu r)}{r}, \tag{7.3}$$

where $Q_g = \sqrt{\alpha G_N}M$ is the gravitational charge. The radial equation of motion is then given by

$$\frac{d^2r}{dt^2} + \frac{GM}{r^2} = \frac{q_g Q_g}{m}\frac{\exp(-\mu r)}{r^2}(1 + \mu r). \tag{7.4}$$

Since $q_g Q_g/m = \alpha G_N M$, the modified Newtonian acceleration law for a point particle can be written as

$$a_{\rm MOG}(r) = -\frac{G_N M}{r^2}[1 + \alpha - \alpha \exp(-\mu r)(1 + \mu r)]. \tag{7.5}$$

This reduces to Newton's gravitational acceleration in the limit $\mu r \ll 1$.

## 7.4. MOG Black Holes

For the matter free $\phi_\mu$ field-vacuum case, the field equations are given by

$$G_{\mu\nu} = -\frac{\omega_M}{\chi^2}\left(\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\chi\nabla_\alpha\chi\right) - \frac{1}{\chi}(\nabla_\mu\chi\nabla_\nu\chi - g_{\mu\nu}\Box\chi) + \frac{8\pi}{\chi}T^\phi_{\mu\nu},$$

$$\Box\chi = \frac{8\pi}{(2\omega_M + 3)}T^\phi,$$

$$\nabla_\nu B^{\mu\nu} + \mu^2\phi^\mu = 0,$$

where

$$T^\phi \equiv g^{\mu\nu}T^\phi_{\mu\nu} = \mu^2\phi^\mu\phi_\mu.$$

The metric is stationary and axisymmetric whose Killing vectors are:

$$\begin{aligned} \xi^i &= (1,0,0,0) \quad \text{timelike Killing vector field,} \\ \psi^i &= (0,0,0,1) \quad \text{rotational Killing vector field.} \end{aligned}$$

The total mass, angular momentum and total charge is given by

$$M_{ADM} = -\frac{1}{8\pi}\int_S \epsilon_{ijkl}\nabla^k\xi^l = (1+\alpha)M \equiv M_\alpha\,,$$

$$J = \frac{1}{16\pi}\int_S \epsilon_{ijkl}\nabla^k\psi^l = M(1+\alpha)\,a \equiv M_\alpha\,a\,,$$

$$4\pi Q = \frac{1}{2}\int_S \epsilon_{ijkl}B^{kl}.$$

The gravitational field energy momentum tensor is

$$T^\phi_{\mu\nu} = -\left(B_\mu{}^\alpha B_{\alpha\nu} - \frac{1}{4}g_{\mu\nu}B^{\alpha\beta}B_{\alpha\beta}\right),$$

and we have $T^{\phi} = g^{\mu\nu}T^{\phi}_{\mu\nu} = 0$ giving the equation:

$$\Box\chi = 0.$$

From [5], The metric for a static spherically symmetric black hole is given by:

| No. | $a$ | Range of $\alpha$ |
|---|---|---|
| 1 | 0.3 | $0 \leq \alpha \leq 10.111$ |
| 2 | 0.4 | $0 \leq \alpha \leq 5.250$ |
| 3 | 0.5 | $0 \leq \alpha \leq 3.0$ |
| 4 | 0.6 | $0 \leq \alpha \leq 1.777$ |
| 5 | 0.7 | $0 \leq \alpha \leq 1.040$ |
| 6 | 0.8 | $0 \leq \alpha \leq 0.562$ |
| 7 | 0.9 | $0 \leq \alpha \leq 0.234$ |
| 8 | 0.99 | $0 \leq \alpha \leq 0.020$ |

**Table 7.1:** The range of the deformation parameter $\alpha$ corresponding to different values of the spin parameter $a$ is shown for the Kerr-MOG BH. Here the value of the mass parameter $M_{\alpha}$ is unity.

$$ds^2 = \left(1 - \frac{2G_N(1+\alpha)M}{r} + \frac{\alpha(1+\alpha)G_N^2M^2}{r^2}\right)dt^2 - \left(1 - \frac{2G_N(1+\alpha)M}{r} + \frac{\alpha(1+\alpha)G_N^2M^2}{r^2}\right)^{-1}dr^2 - r^2d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The metric reduces to the Schwarzschild solution when $\alpha = 0$. The axisymmetric rotating black hole has the metric solution including the spin angular momentum $J = Ma$:

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - a dt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2,$$

where

$$\Delta = r^2 - 2G_N(1+\alpha)Mr + a^2 + \alpha(1+\alpha)G_N^2M^2, \quad \rho^2 = r^2 + a^2\cos^2\theta.$$

# 8

# Black Hole Dynamics in MOG

"Time changes everything."
"That's what people say; it's not true. Doing things changes things. Not doing things leaves things exactly as they were."

Gregory House, M.D.

## 8.1. Geodesics of Massive Particles in Kerr-MOG BH

Consider a unit mass particle orbiting in the background of a Kerr-MOG BH. The Hamilton-Jacobi equation guiding geodesic motion in this spacetime with the metric tensor $g^{ij}$ is given by

$$2\frac{\partial S}{\partial \tau} = -g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j},$$

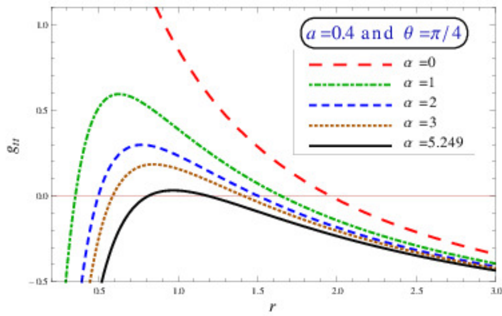
where $S$ denotes Hamilton's principal function, and $\tau$ is an affine parameter along the geodesics. For this BH background, the Hamilton's principal function $S$ can be separated as

$$S = \frac{1}{2}\tau - Et + L\phi + S_r(r) + S_\theta(\theta),$$

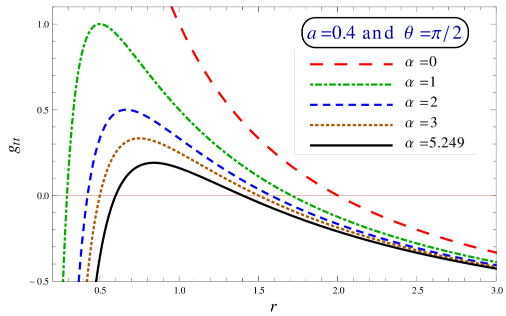
where $S_r$ and $S_\theta$ are functions of $r$ and $\theta$, respectively. The constants $E$ and $L$ correspond to conserved energy and angular momentum per unit mass and are given by the following equations

$$E = \frac{\tilde{E}}{m} = -g_{ij}\xi^i U^j,$$

$$L = \frac{\tilde{L}}{m} = g_{ij}\psi^i U^j.$$

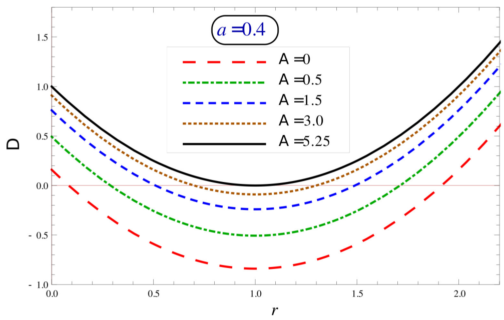



**(a)** Figure 1: Particle initially started off at $\theta = \pi/4$ angle.

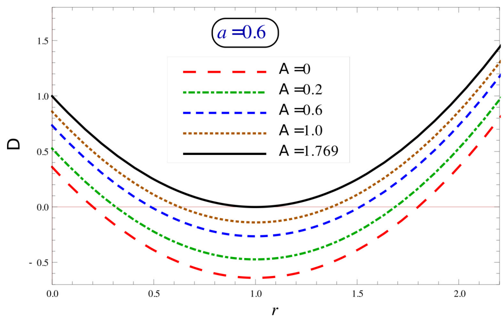



**(b)** Figure 2: Particle is confined to move in the equatorial plane only.

**Figure 8.1:** Here, the solid (black) line corresponds to the extremal Kerr-MOG black hole and the large-dashed (red) line to the Kerr BH (i.e., when $\alpha = 0$)[5].




**(a)** Figure 1: Particle initially started off at $\theta = \pi/4$ angle.




**(b)** Figure 2: Particle is confined to move in the equatorial plane only.

**Figure 8.2:** These plots show the behavior of the event horizon ($\Delta = 0$) vs $r$ for different values of the deformation parameter $\alpha$[5].

# 9

# Discussion and Conclusion

"I am smart enough to know that I am dumb."

Richard P. Feynman

## 9.1. What We Know

Starting from Newton's gravity, we argued that due a lack of agreement with experimental observations, a newer theory of gravity was needed. We built the necessary tools to understand gravity as it is best described today. We saw how Einstein was able to *extract* gravity from geometry and make it make sense. We started from the very basics of field theory and clawed our way into classical Einstein gravity. We then saw how photons orbit a Kerr BH in this universe for different initial conditions. We saw that certain photon orbits are stable while some are not; and in what way the instability arises physically.

We then understood that Einstein, though extremely clever, was not telling us the whole story and had to modify his theory to better suit the observations. While in this modified universe, we derived the metric for a rotating black hole and saw how massive particles orbit around them for different parameters.

## 9.2. What We Don't Know

We have no idea if gravity has an effect in the quantum world. Newton's gravity completely falls apart when applied to *very* heavy or *very* energetic objects and works best when dealing with planets and stars. We have *no* idea what gravity looks like in the quantum realm: is it as strong as Newton predicts or is it $10,000\times$ stronger, or

$100,000,000\times$ weaker? We don't know. Physicists have been working for decades now trying to marry gravity and quantum mechanics but they are not agreeing to it as of now. The elusive graviton is still at large. Maybe one day when a complete picture of gravity is painted, Sir Isaac Newton can finally rest in his grave.

# 10

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