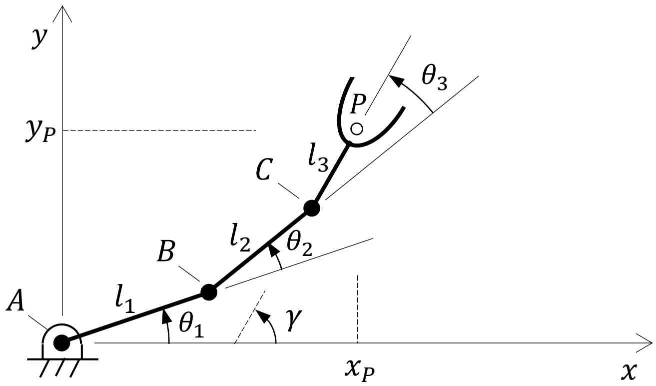
3-b) Kinematic Analysis

Generally speaking, kinematic analysis involves determination of unknown angles and lengths when given certain other angles and lengths together with certain mechanism constants. Loosely speaking, there are “inputs” to a mechanism and “outputs”, where inputs usually refer to where you are putting power into the mechanism and outputs usually referring to where you take power away from the mechanism.

Forward analysis is where you specify what the input values (angles, lengths) are and where you solve for what the corresponding output values (angles, lengths) are. Note there can be multiple solutions from a mathematical standpoint and also a practical standpoint.

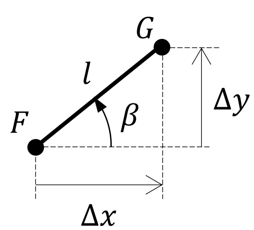
Reverse analysis is where you specify what the output values are and solve for the corresponding input values. Note again there can be multiple solution sets.

The 3R robot. Point P is gripper point and γ is orientation of gripper. Robot joint angles are $\theta_1, \theta_2, \theta_3$.



Shorthand notation:
$c_1 = \cos\theta_1$
$s_1 = \sin\theta_1$
$c_{12} = \cos(\theta_1 + \theta_2)$
$s_{12} = \sin(\theta_1 + \theta_2)$
$c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$
$s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$

Notes
- "dot" notation used for time-rate-of-change.
- Angles are four quadrant
- Lengths are $\pm$
- Angles, lengths have references (use one arrow for "destination" of angle)
- Important to know how to project using triangles

Projections using triangle	
	$l\ is + going\ from\ F\ to\ G.$ $\beta\ measures\ from\ x - axis.$ $\Delta x = l\cos\beta$ $\Delta y = l\sin\beta$ $\Delta x, \Delta y\ remain\ valid\ when\ \beta\ is\ in\ any\ quadrant.$ $\Delta x, \Delta y\ remain\ valid\ when\ l\ is\ negative.$

3R Forward analysis (Position)

Given: $\theta_1, \theta_2, \theta_3$
Find: x_P, y_P, γ
Constants: l_1, l_2, l_3

(3-1)
$$\gamma = \theta_1 + \theta_2 + \theta_3$$
$$x_P = l_1\, c_1 + l_2\, c_{12} + l_3\, c_{123}$$
$$y_P = l_1\, s_1 + l_2\, s_{12} + l_3\, s_{123}$$

Comments: all entries on right-hand sides are known. The x_P, y_P entries are obtained by using the "Projections using triangle" figure three times.

3R Forward Analysis (Velocity):

Given: $\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$
Find: $\dot{x}_P, \dot{y}_P, \dot{\gamma}$
Constants: l_1, l_2, l_3

Simplify differentiate the above relations:

(3-2)
$$\dot{\gamma} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$
$$\dot{x}_P = -l_1\, s_1\, \dot{\theta}_1 - l_2\, s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - l_3\, s_{123}\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)$$
$$\dot{y}_P = l_1\, c_1\, \dot{\theta}_1 + l_2\, c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + l_3\, c_{123}\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)$$

3R Reverse analysis (Position)

Given: x_P, y_P, γ
Find: $\theta_1, \theta_2, \theta_3$
Constants: l_1, l_2, l_3

To do this, first note that:
$\cos\gamma = c_{123}$
$\sin\gamma = s_{123}$.
So, c_{123}, s_{123} are actually givens.

Rewrite the above x_P, y_P equations from (3-1) in this form:

$$x_P - l_3\, c_{123} = l_1\, c_1 + l_2\, c_{12}$$
$$y_P - l_3\, s_{123} = l_1\, s_1 + l_2\, s_{12}.$$

These can be written

$$x_P^* = l_1\, c_1 + l_2\, c_{12}$$

(3-3)

$$y_P^* = l_1\, s_1 + l_2\, s_{12},$$

where

$$x_P^* = x_P - l_3\, c_{123} = x_P - l_3 \cos\gamma$$

$$y_P^* = y_P - l_3\, s_{123} = y_P - l_3 \sin\gamma$$

are actually knowns and represent the x, y coordinates of point C in the $3R$ figure. Now, recall and expand the following:

$$c_{12} = \cos(\theta_1 + \theta_2) = c_1\, c_2 - s_1\, s_2$$
$$s_{12} = \sin(\theta_1 + \theta_2) = s_1\, c_2 + c_1\, s_2.$$

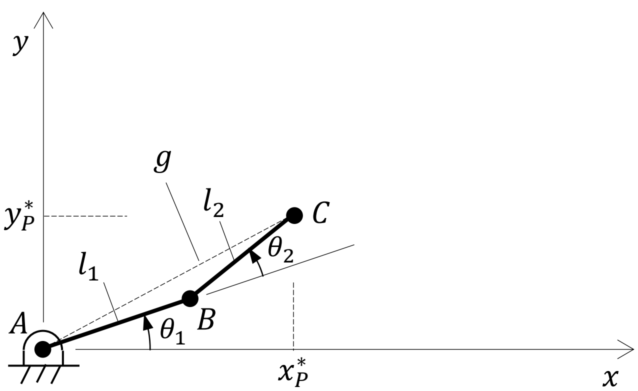
Accordingly substitute into (3-3) above:

$$x_P^* = l_1\, c_1 + l_2\, (c_1\, c_2 - s_1\, s_2)$$

(3-4)

$$y_P^* = l_1\, s_1 + l_2\, (s_1\, c_2 + c_1\, s_2).$$

Put the above x_P^*, y_P^* into view:



Consider distance from A to point C, and you get $g = \sqrt{(x_P^*)^2 + (y_P^*)^2}$, or $g^2 = (x_P^*)^2 + (y_P^*)^2$. Since x_P^* and y_P^* are known, then g and g^2 are known.

Now squaring both sides of (3-4), adding, and simplifying yields

$$(x_P^*)^2 + (y_P^*)^2 = (c_1^2 + s_1^2)\,(l_1^2 + l_2^2\,(c_2^2 + s_2^2) + 2\, l_1\, l_2\, c_2),$$

or

(3-5) $$g^2 = l_1^2 + l_2^2 + 2\, l_1\, l_2\, c_2.$$

Equation (3-5) represents the cosine law for the exterior angle θ_2 of the triangle ABC. (A minus sign on the last term would have made it a cosine law for the corresponding interior angle.)

This equation is probably one of the most used equations in the study of reverse analyses (position).

Rearranging (3-5) yields the important result:

(3-6)
$$c_2 = \frac{g^2 - l_1^2 - l_2^2}{2\, l_1\, l_2}.$$

From (3-6), a unique $\cos\theta_2$ can be determined, since all parameters on right-hand side are known.

It is critically important to remember angles are four quadrant variables.

Knowing just the $\cos\theta_2$ alone does not fully specify θ_2. However, if you know the $\sin\theta_2$ then the four quadrant arc tangent gives you θ_2.

In this case, there are two solutions for $\sin\theta_2$ which are

Elbow Up:	$s_2 = +\sqrt{1 - c_2^2}$
Elbow Down:	$s_2 = -\sqrt{1 - c_2^2}$

Once you have c_2, s_2 and hence θ_2, then you can get c_1, s_1 by rewriting (3-4) into this form:

(3-7)
$$\begin{bmatrix} x_P^* \\ y_P^* \end{bmatrix} = \begin{bmatrix} l_1 + l_2\, c_2 & -l_2\, s_2 \\ l_2\, s_2 & l_1 + l_2\, c_2 \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}.$$

The equation (3-7) is linear in c_1, s_1 and this yields a unique pair c_1, s_1 which establishes θ_1.

Knowing $\theta_1, \theta_2, \gamma$, then from (3-1) $\theta_3 = \gamma - \theta_1 - \theta_2$.

Now, all angles $\theta_1, \theta_2, \theta_3$ are known and there are two sets of solutions (elbow up and elbow down).

3R Reverse analysis (Velocity)

Given: $\dot{x}_P, \dot{y}_P, \dot{\gamma}$
Also given: x_P, y_P, γ and a specified solution $\theta_1, \theta_2, \theta_3$
Find: $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$
Constants: l_1, l_2, l_3

Rewrite (3-2) into the following form:

(3-8)
$$\begin{bmatrix} \dot{x}_P \\ \dot{y}_P \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -l_1\, s_1 - l_2\, s_{12} - l_3\, s_{123} & -l_2\, s_{12} - l_3\, s_{123} & -l_3\, s_{123} \\ l_1\, c_1 + l_2\, c_{12} + l_3\, c_{123} & l_2\, c_{12} + l_3\, c_{123} & l_3\, c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix},$$

which is a linear system of equations having the only unknowns of $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$. This can be written again in this form:

(3-9) $$\widehat{D} = [J]\,\underline{\dot{\theta}}\,,$$

where the 3×1 array

$$\widehat{D} = \begin{bmatrix} \dot{x}_P \\ \dot{y}_P \\ \dot{\gamma} \end{bmatrix}$$

is known, where the 3×3 matrix

$$[J] = \begin{bmatrix} -l_1\, s_1 - l_2\, s_{12} - l_3\, s_{123} & -l_2\, s_{12} - l_3\, s_{123} & -l_3\, s_{123} \\ l_1\, c_1 + l_2\, c_{12} + l_3\, c_{123} & l_2\, c_{12} + l_3\, c_{123} & l_3\, c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

is known, and where the 3×1 array

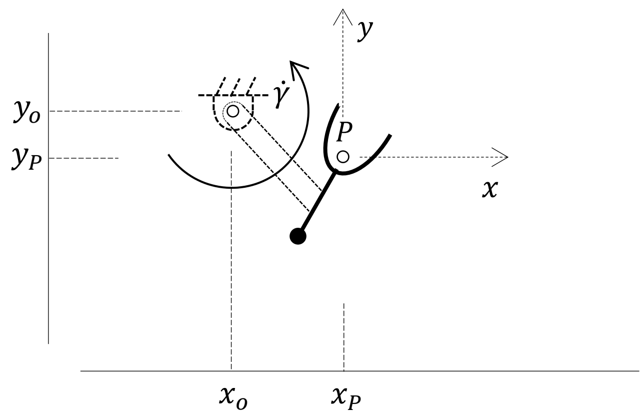
$$\underline{\dot{\theta}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

represents the unknowns. The solution is simply the inverse of (3-9):

(3-10) $$\underline{\dot{\theta}} = [J]^{-1}\ \widehat{D}.$$

Notes:

- If points ABC form a triangle (are not collinear), then any desired $\dot{x}_P, \dot{y}_P, \dot{\gamma}$ can be achieved. This is because $[J]$ is invertible in this case.
- If points ABC are collinear, then $[J]$ is not invertible and not all $\dot{x}_P, \dot{y}_P, \dot{\gamma}$ can be achieved. This is a kinematic singularity. (Transitory Mobility.)
- The matrix $[J]$ is called the "Jacobian" since if you take time out of the equation then the matrix relates differentials
$$\begin{bmatrix} \delta x_P \\ \delta y_P \\ \delta \gamma \end{bmatrix} = [J]\ \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\theta_3 \end{bmatrix},$$
where the δ denotes the differential (or small variation).
- $\widehat{D}$ is the "velocity state" of the gripper (twist coordinates, if you remember those).
- If you remember twist coordinates, then rewrite $\widehat{D}$, factoring out the scalar $\dot{\gamma}$,
$$\widehat{D} = \dot{\gamma}\ \begin{bmatrix} +y_o \\ -x_o \\ 1 \end{bmatrix},$$
where $x_o = -\dot{y}_P/\dot{\gamma}$ and $y_o = \dot{x}_P/\dot{\gamma}$. The coordinates x_o, y_o locate the instant center about which the gripper is turning relative to ground at speed $\dot{\gamma}$ and this location is with respect to a parallel coordinate system located at P.



- A translation where $\dot{\gamma} = 0$ can be realized by conceptually taking x_o, y_o to infinity. The twist coordinates look like this:

$$\widehat{D}_T = \begin{bmatrix} \dot{x}_P \\ \dot{y}_P \\ 0 \end{bmatrix},$$

which can be rescaled:

$$\widehat{D}_T = v_P \begin{bmatrix} c \\ s \\ 0 \end{bmatrix},$$

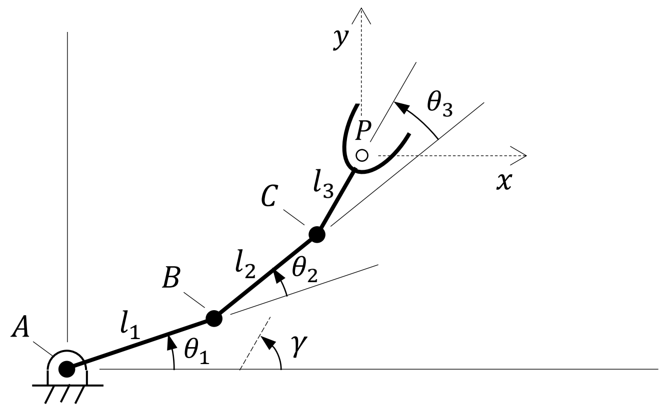
where $v_P = \sqrt{\dot{x}_P^2 + \dot{y}_P^2}$ and where $c = \dot{x}_P / v_p$, $s = \dot{y}_P / v_p$ are the direction cosines of the direction of sliding.

- Using this method, let us rewrite (3-8) in this way

(3-8')
$$\begin{bmatrix} \dot{x}_P \\ \dot{y}_P \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} +y_A & +y_B & +y_C \\ -x_A & -x_B & -x_C \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix},$$

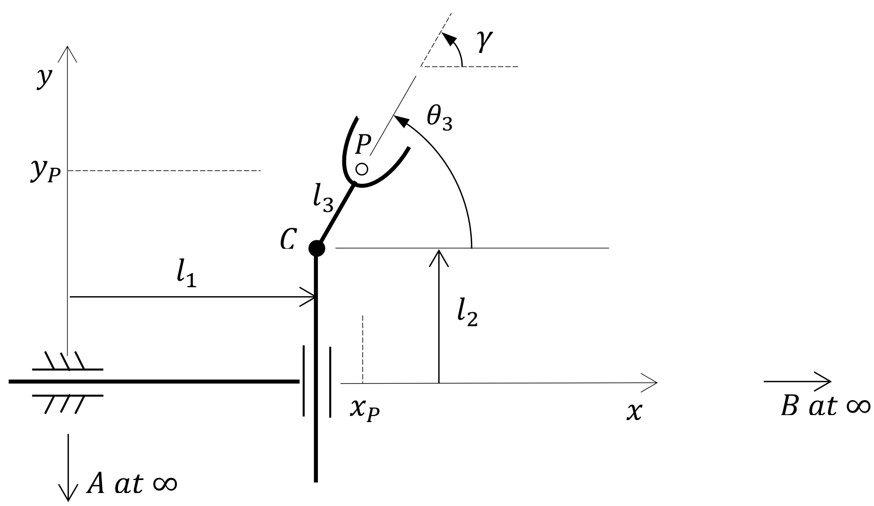
where the columns are coordinates that locate the points A, B, C from the coordinate system located at P, shown below, which is parallel to the original coordinate system, and where

$$x_C = -l_3\, c_{123}$$
$$y_C = -l_3\, s_{123}$$
$$x_B = x_C - l_2\, c_{12}$$
$$y_B = y_C - l_2\, s_{12}$$
$$x_A = x_B - l_1\, c_1$$
$$y_A = y_B - l_1\, s_1.$$



- Since the Jacobian relies on the locations of the three points A, B, C, it makes sense that if it is singular, then those three points are collinear and do not form a triangle.

The 2PR robot. Point P is gripper point and γ is orientation of gripper. Robot joint variables are l_1, l_2, θ_3, and their derivatives are $\dot{l}_1, \dot{l}_2, \dot{\theta}_3$. The figure is shown with $\theta_1 = 0, \theta_2 = 90°, l_3$ being constants.

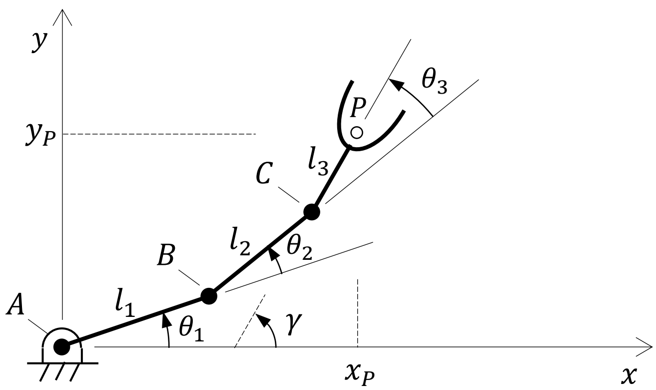


Given the above, the equivalent (3-8) is

(3-11)
$$\begin{bmatrix} \dot{x}_P \\ \dot{y}_P \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 & +y_C \\ 0 & 1 & -x_C \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{\theta}_3 \end{bmatrix}.$$

In the figure, take note that C remains in same place and that A is at ∞ in the negative y direction, and that B is at ∞ in the positive x direction. This shows that A, B, C is a valid triangle, projectively speaking, and $[J]$ is invertible (obviously).

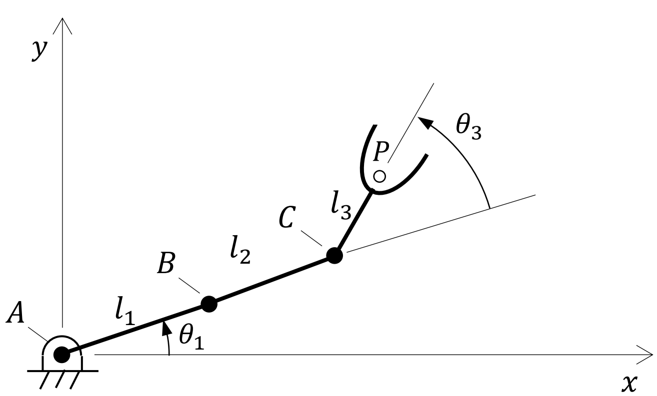
More on Mobility: special cases and singularities.



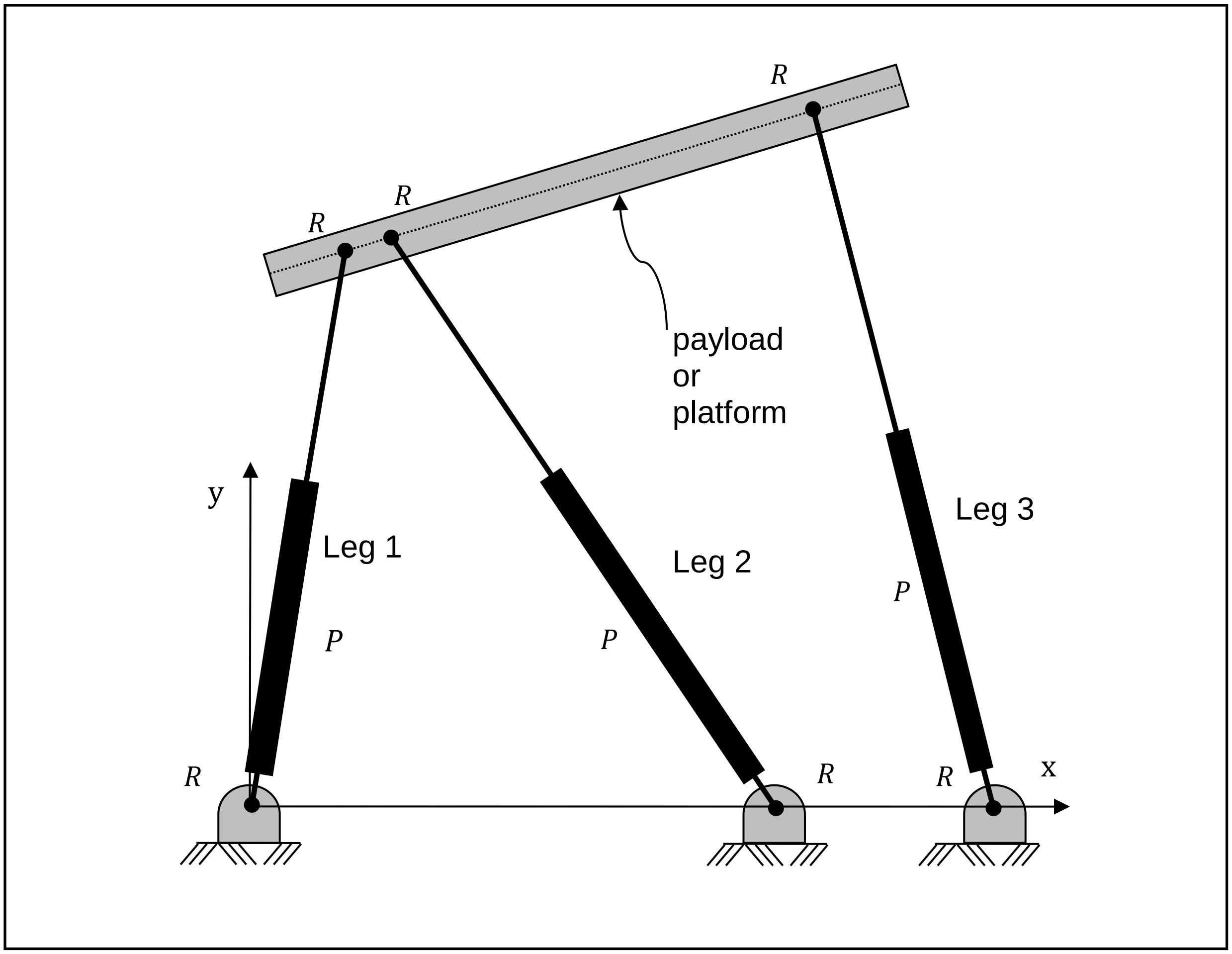
Mobility of open serial chain above: $n = 4, g = 3, f_1 = f_2 = f_3 = 1$.

$$M = 3(n - g - 1) + \sum_{i=1}^{g} f_i = 3(4 - 3 - 1) + (1 + 1 + 1) = 3$$

A *kinematic singularity* occurs when $\theta_2 = 0$. Then all three revolutes lie on the same line and the real Mobility is actually 2. See above for the associated Jacobian. (Transitory Mobility.)



Parallel mechanism (Platform).



The above parallel (planar) mechanism consists of three RPR serial chains that act in parallel to manipulate a "platform".

Mobility of platform above: $n = 3 \times 2 + 2, g = 3 \times 3, f_i = 1$.

$$M = 3(n - g - 1) + \sum_{i=1}^{g} f_i = 3(8 - 9 - 1) + (9) = 3$$

In this type of mechanism, the legs (P) are actuated. In general if you lock the three legs, you can support any load applied to the platform. Note in general, the lines of action of the three legs form a triangle (vertices may go out of view).

A ***static singularity*** occurs when the lines of action of the three legs meet in a point. In this case, locked legs cannot resist any load that has a component which does not go through the point of coincidence. In this case the mechanism gains mobility! (It can move even with locked legs.) (Transitory Mobility.)

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