EXAMPLE 2.2-2: Dynamics of a Two-Link Planar Elbow Arm

In Example A.2-2 are given the kinematics for a two-link planar RR arm. To determine its dynamics, examine Fig. 2.2-4, where we have assumed that the link masses are concentrated at the ends of the links. The joint variable is

$$q = [\theta_1 \quad \theta_2]^T \tag{1}$$

and the generalized force vector is

$$\tau = [\tau_1 \quad \tau_2]^T \tag{2}$$

with τ_1 and τ_2 torques supplied by the actuators.

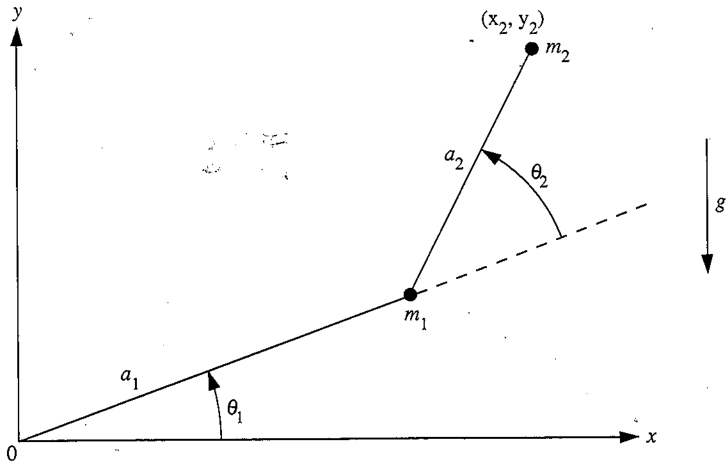

FIGURE 2.2-4 Two-link planar RR arm.

a. Kinetic and Potential Energy

For link 1 the kinetic and potential energies are

$$K_1 = \tfrac{1}{2} m_1 a_1^2 \dot{\theta}_1^2 \tag{3}$$

$$P_1 = m_1 g a_1 \sin \theta_1. \tag{4}$$

For link 2 we have

$$x_2 = a_1 \cos \theta_1 + a_2 \cos (\theta_1 + \theta_2) \tag{5}$$

$$y_2 = a_1 \sin \theta_1 + a_2 \sin (\theta_1 + \theta_2) \tag{6}$$

$$\dot{x}_2 = -a_1 \dot{\theta}_1 \sin \theta_1 - a_2(\dot{\theta}_1 + \dot{\theta}_2) \sin (\theta_1 + \theta_2) \tag{7}$$

$$\dot{y}_2 = a_1 \dot{\theta}_1 \cos \theta_1 + a_2(\dot{\theta}_1 + \dot{\theta}_2) \cos (\theta_1 + \theta_2), \tag{8}$$

so that the velocity squared is

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = a_1^2 \dot{\theta}_1^2 + a_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2a_1 a_2(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos \theta_2. \tag{9}$$

Therefore, the kinetic energy for link 2 is

$$K_2 = \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_2 a_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 a_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 a_1 a_2(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos \theta_2. \tag{10}$$

The potential energy for link 2 is

$$P_2 = m_2 g y_2 = m_2 g[a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2)]. \tag{11}$$

b. Lagrange's Equation

The Lagrangian for the entire arm is

$$\begin{aligned} L &= K - P = K_1 + K_2 - P_1 - P_2 \\ &= \tfrac{1}{2} (m_1 + m_2)\, a_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos \theta_2 \\ &\quad - (m_1 + m_2)\, g a_1 \sin \theta_1 - m_2 g a_2 \sin (\theta_1 + \theta_2). \end{aligned} \tag{12}$$

The terms needed for (2.2-14) are

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)\, a_1^2 \dot{\theta}_1 + m_2 a_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 a_1 a_2 (2\dot{\theta}_1 + \dot{\theta}_2) \cos \theta_2$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} &= (m_1 + m_2)\, a_1^2 \ddot{\theta}_1 + m_2 a_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 a_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) \cos \theta_2 \\ &\quad - m_2 a_1 a_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) \sin \theta_2 \end{aligned}$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2) g a_1 \cos \theta_1 - m_2 g a_2 \cos (\theta_1 + \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 a_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 a_1 a_2 \dot{\theta}_1 \cos \theta_2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} = m_2 a_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 a_2 \ddot{\theta}_1 \cos\theta_2 - m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \sin \theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \sin \theta_2 - m_2 g a_2 \cos(\theta_1 + \theta_2).$$

Finally, according to Lagrange's equation, the arm dynamics are given by the two coupled nonlinear differential equations

$$\begin{aligned}\tau_1 = {} & [(m_1 + m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2 \cos \theta_2]\, \ddot{\theta}_1 \\ & + [m_2a_2^2 + m_2a_1a_2 \cos \theta_2]\, \ddot{\theta}_2 - m_2a_1a_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \sin \theta_2 \\ & + (m_1 + m_2)\, ga_1 \cos \theta_1 + m_2ga_2 \cos (\theta_1 + \theta_2)\end{aligned} \tag{13}$$

$$\begin{aligned}\tau_2 = {} & [m_2a_2^2 + m_2a_1a_2 \cos \theta_2]\, \ddot{\theta}_1 + m_2a_2^2\ddot{\theta}_2 + m_2a_1a_2\dot{\theta}_1^2 \sin \theta_2 \\ & + m_2ga_2 \cos (\theta_1 + \theta_2).\end{aligned} \tag{14}$$

c. Manipulator Dynamics

Writing the arm dynamics in vector form yields

$$\begin{aligned}& \begin{bmatrix} m_1 + m_2)\, a_1^2 + m_2a_2^2 + 2m_2a_1a_2 \cos \theta_2 & m_2a_2^2 + m_2a_1a_2 \cos \theta_2 \\ m_2a_2^2 + m_2a_1a_2 \cos \theta_2 & m_2a_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \\ & + \begin{bmatrix} -m_2a_1a_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \sin \theta_2 \\ m_2a_1a_2\dot{\theta}_1^2 \sin \theta_2 \end{bmatrix} + \begin{bmatrix} (m_1 + m_2)\, ga_1 \cos \theta_1 + m_2ga_2 \cos (\theta_1 + \theta_2) \\ m_2ga_2 \cos (\theta_1 + \theta_2) \end{bmatrix} \\ & = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.\end{aligned} \tag{15}$$

These manipulator dynamics are in the standard form

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau, \tag{16}$$

with $M(q)$ the inertia matrix, $V(q,\dot{q})$ the Coriolis/centripetal vector, and $G(q)$ the gravity vector. Note that $M(q)$ is symmetric.