Assignment 7

Damped Harmonic Oscillations

Problem Statement

A classic second order differential equation is that which describes, the damped harmonic oscillations.

The goal is to solve the equation (homogeneous) and (inhomogeneous) and plot the solutions.

See the equations given below

In[758]:=

```
y''[t] + γ * y'[t] + (ω0)^2 * y[t] == 0
y''[t] + γ * y'[t] + (ω0)^2 * y[t] == a0 * Cos[ω * t]
```

Out[758]=

$$\omega 0^2\, y[t] + \gamma\, y'[t] + y''[t] == 0$$

Out[759]=

$$\omega 0^2\, y[t] + \gamma\, y'[t] + y''[t] == a0\, \mathrm{Cos}[t\, \omega]$$

- Here γ, $\omega 0$, a0 and ω are suitable constants.
- Solve both equations after choosing appropriate numerical values for γ, $\omega 0$, a0 and ω, and plot the solutions.
- The values of γ, $\omega 0$, a0 and ω determine the nature of solutions and you must correlate your choice and the solutions obtained.

In[760]:=

```
Important Note :
  Since this is a staright forward problem late submissions may be penalized.
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Solving the first differential equation

In[760]:=

```
deqn1 := y''[t] + γ * y'[t] + (ω0)^2 * y[t] == 0
Print["Using initial position as 0 and
   initial velocity as 5, we get after differentiation:-"]
incs1 = {y[0] == 0, y'[0] == 5};
soln1 = DSolve[{deqn1, incs1}, y[t], t];
soln1[[1]]
```

Using initial position as 0 and initial velocity as 5, we get after differentiation:-

Out[764]=

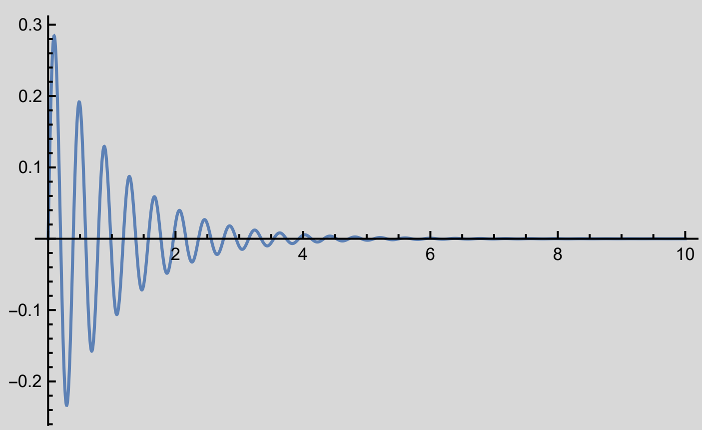
$$\left\{y[t] \to -\frac{5\left(e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\,\omega 0^2}\right)} - e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\,\omega 0^2}\right)}\right)}{\sqrt{\gamma^2-4\,\omega 0^2}}\right\}$$

For underdamped case, γ < 2 ω0 :-

In[765]:=

```
solPlt1 = NDSolve[{deqn1, incs1} /. {γ → 2, ω0 → 16}, y, {t, 0, 25}];
a = Plot[Evaluate[y[t] /. solPlt1], {t, 0, 10}, PlotRange → All]
Print["We can see the amplitude die down as expected."]
```

Out[766]=



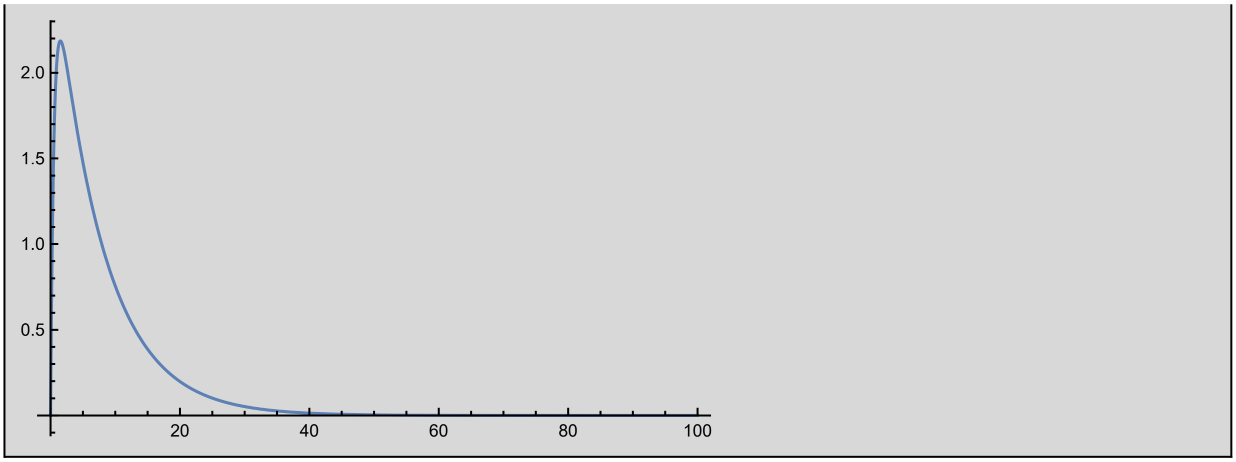
We can see the amplitude die down as expected.

For overdamped case, γ > 2 ω0 :-

In[768]:=

```
solPlt2 = NDSolve[{deqn1, incs1} /. {γ → 2, ω0 → 1/2}, y, {t, 0, 100}];
Plot[Evaluate[y[t] /. solPlt2], {t, 0, 100}, PlotRange → All]
```

Out[769]=

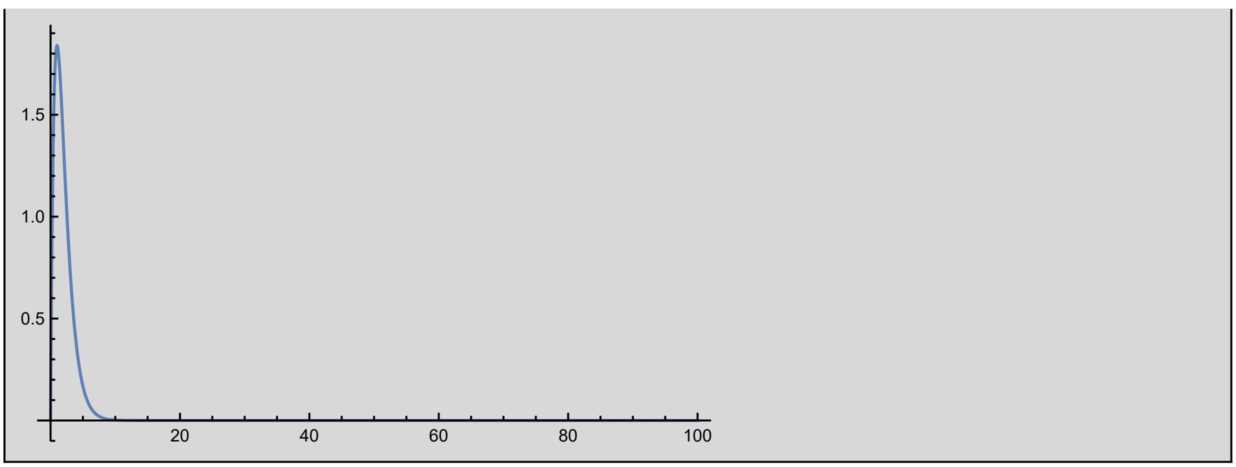


For critically damped case, γ = 2 ω0 : -

In[770]:=

```
solPlt2 = NDSolve[{deqn1, incs1} /. {γ → 2, ω0 → 1}, y, {t, 0, 100}];
Plot[Evaluate[y[t] /. solPlt2], {t, 0, 100}, PlotRange → All]
```

Out[771]=



Solving the second differential equation

In[772]:=

```
deqn2 := y''[t] + γ * y'[t] + (ω0)^2 * y[t] == a0 * Cos[ω * t]
Print["On differentiation:- "]
soln2pure = DSolve[deqn2, y[t], t];
soln2pure[[1]]
Print["Using initial position as 0 and initial velocity as 5, we get:-"]
incs2 = {y[0] == 0, y'[0] == 5};
soln2 = DSolve[{deqn2, incs2}, y[t], t];
soln2[[1]]
```

On differentiation:-

Out[775]=

$$\left\{y[t] \to e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} c_1 + e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} c_2 - \frac{4\, a0 \left(-\omega^2 \mathrm{Cos}[t\,\omega] + \omega0^2 \mathrm{Cos}[t\,\omega] + \gamma\,\omega\, \mathrm{Sin}[t\,\omega]\right)}{\left(\gamma^2 + 2\omega^2 - 2\omega0^2 + \gamma\sqrt{\gamma^2-4\omega0^2}\right)\left(-\gamma^2 - 2\omega^2 + 2\omega0^2 + \gamma\sqrt{\gamma^2-4\omega0^2}\right)}\right\}$$

Using initial position as 0 and initial velocity as 5, we get:-

Out[779]=

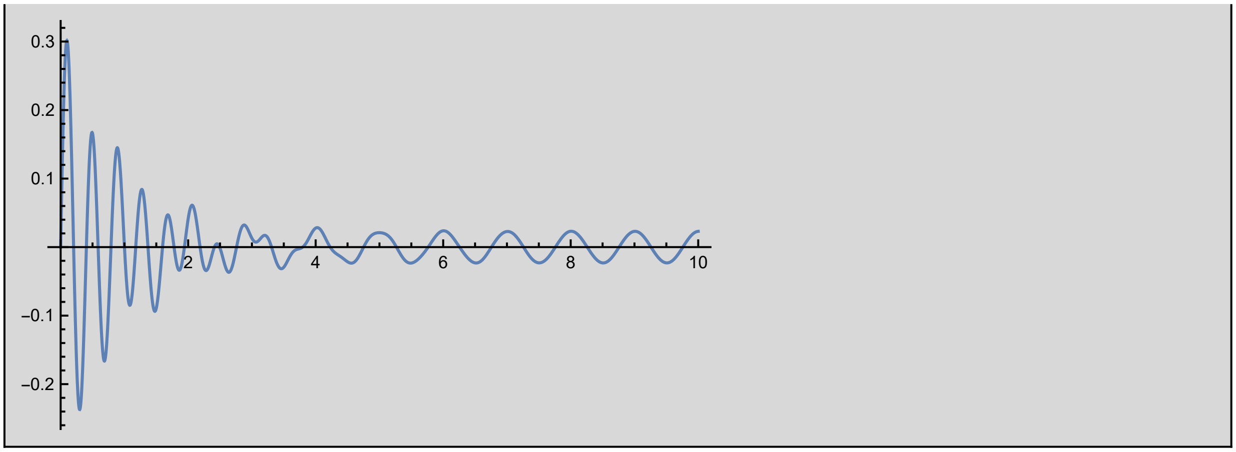
$$\Big\{y[t] \to -\Bigg(\Bigg(2\Bigg(a0\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \gamma\,\omega^2 - a0\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \gamma\,\omega^2 -$$
$$10\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \gamma^2\omega^2 + 10\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \gamma^2\omega^2 - 10\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \omega^4 +$$
$$10\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \omega^4 + a0\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \gamma\,\omega0^2 - a0\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \gamma\,\omega0^2 +$$
$$20\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \omega^2\omega0^2 - 20\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \omega^2\omega0^2 - 10\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \omega0^4 +$$
$$10\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \omega0^4 + a0\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \omega^2\sqrt{\gamma^2-4\omega0^2} +$$
$$a0\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \omega^2\sqrt{\gamma^2-4\omega0^2} - a0\, e^{\frac{1}{2} t \left(-\gamma-\sqrt{\gamma^2-4\omega0^2}\right)} \omega0^2\sqrt{\gamma^2-4\omega0^2} -$$
$$a0\, e^{\frac{1}{2} t \left(-\gamma+\sqrt{\gamma^2-4\omega0^2}\right)} \omega0^2\sqrt{\gamma^2-4\omega0^2} - 2\, a0\,\omega^2\sqrt{\gamma^2-4\omega0^2}\,\mathrm{Cos}[t\,\omega] +$$
$$2\, a0\,\omega0^2\sqrt{\gamma^2-4\omega0^2}\,\mathrm{Cos}[t\,\omega] + 2\, a0\,\gamma\,\omega\sqrt{\gamma^2-4\omega0^2}\,\mathrm{Sin}[t\,\omega]\Bigg)\Bigg)\Bigg/$$
$$\left(\sqrt{\gamma^2-4\omega0^2}\left(\gamma^2 + 2\omega^2 - 2\omega0^2 + \gamma\sqrt{\gamma^2-4\omega0^2}\right)\left(-\gamma^2 - 2\omega^2 + 2\omega0^2 + \gamma\sqrt{\gamma^2-4\omega0^2}\right)\right)\Bigg)\Big\}$$

For underdamped case, γ < 2 ω0 : -

In[780]:=

```
solPltDash1 =
  NDSolve[{deqn2, incs2} /. {γ → 2, ω0 → 16, a0 → 5, ω → 2 Pi}, y, {t, 0, 25}];
Plot[Evaluate[y[t] /. solPltDash1], {t, 0, 10}, PlotRange → All]
```

Out[781]=

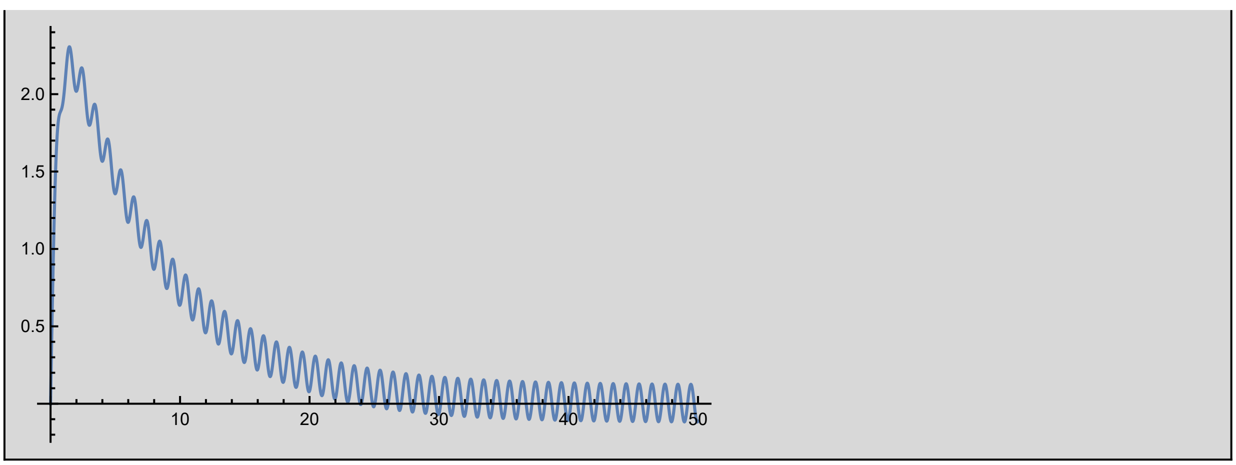


For overdamped case, γ > 2 ω0 : -

In[782]:=

```
solPltDash2 = NDSolve[{deqn2, incs2} /. {γ → 2, ω0 → 1/2, a0 → 5, ω → 2 Pi}, y, {t, 0, 50}];
Plot[Evaluate[y[t] /. solPltDash2], {t, 0, 50}, PlotRange → All]
```

Out[783]=

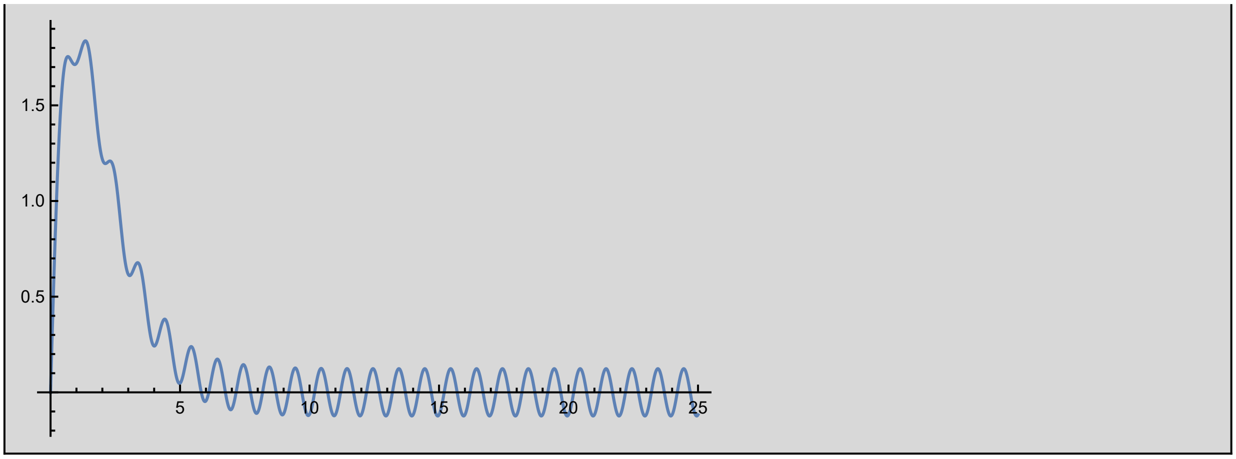


For critically damped case, γ = 2 ω0 : -

In[784]:=

```
solPltDash3 = NDSolve[{deqn2, incs2} /. {γ → 2, ω0 → 1, a0 → 5, ω → 2 Pi}, y, {t, 0, 25}];
Plot[Evaluate[y[t] /. solPltDash3], {t, 0, 25}, PlotRange → All]
```

Out[785]=



Comments

In the graphs of the damped oscillator, the amplitude dies down to zero after some time due to the inverse exponential factor.

However, in all graphs of the damped driven oscillator, we notice that the force completely dominates over the initial vibrations after some time.
This is as expected because the initial vibrations die down due to the inverse exponential factor. However, the driven force continues to oscillate without diminishing.

Extra Points to be noted

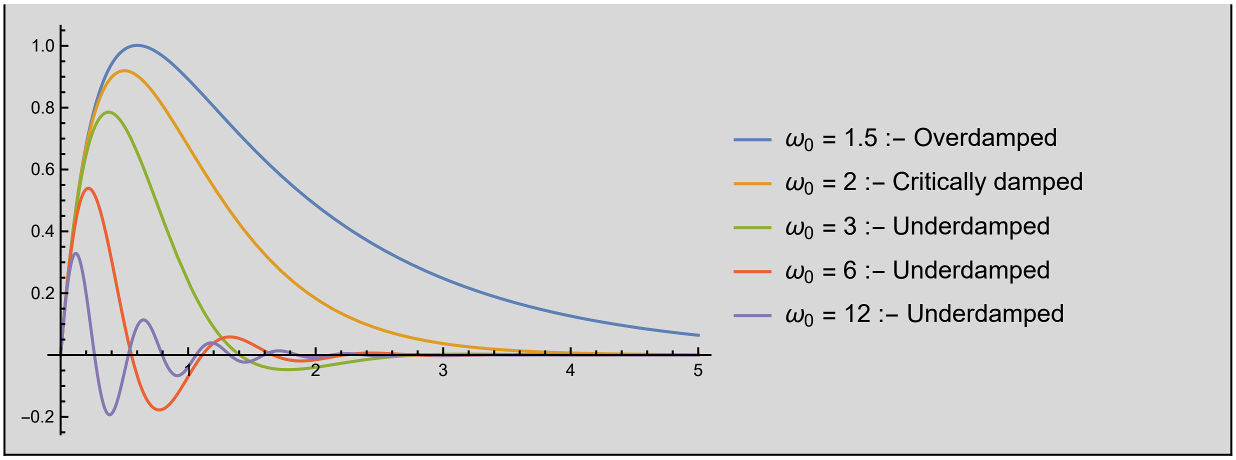
NDSolve has been used to plot graphs here. However, we can use DSolve as well, which makes it easier to plot curves for multiple values of constants γ or $\omega0$ in the same graph.

Plotting graphs keeping γ = 4 as constant, with varying ω_0 : –

In[786]:=

```
f[t_, γ_, ω0_] := y[t] /. soln1[[1]]
f[t, γ, ω0];
tab = Table[f[t, γ, ω0] /. γ → 4, {ω0, {1.5, 2.001, 3, 6, 12}}];
plt = Plot[Evaluate[tab], {t, 0, 5}, PlotRange → All,
  PlotLegends → {"ω0 = 1.5 :- Overdamped", "ω0 = 2 :- Critically damped",
    "ω0 = 3 :- Underdamped", "ω0 = 6 :- Underdamped", "ω0 = 12 :- Underdamped"}]
```

Out[789]=

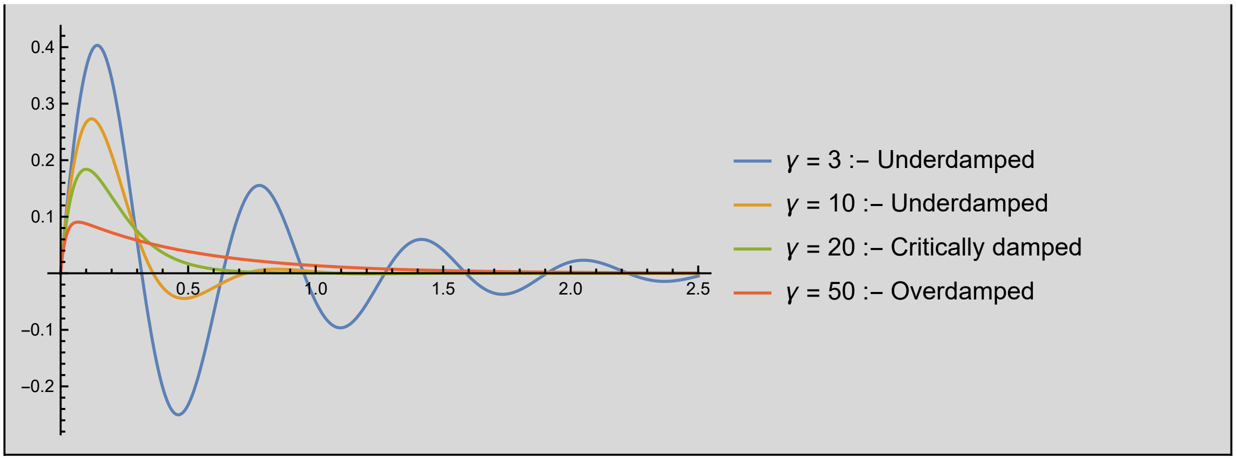


Plotting graphs keeping $\omega_0 = 10$ as constant, with varying γ : –

In[790]:=

```
f[t_, γ_, ω0_] := y[t] /. soln1⟦1⟧
f[t, γ, ω0];
tab = Table[f[t, γ, ω0] /. ω0 → 10, {γ, {3, 10, 20.001, 50}}];
plt = Plot[Evaluate[tab], {t, 0, 2.5}, PlotRange → All,
  PlotLegends → {"γ = 3 :– Underdamped", "γ = 10 :– Underdamped",
    "γ = 20 :– Critically damped", "γ = 50 :– Overdamped"}]
```

Out[793]=



Note that we can also plot multiple curves with different initial conditions (with fixed or changing γ and ω_0) on the same graph easily using DSolve.